# On the lack of semimartingale property<sup>\*</sup>

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#### Abstract

In this work we extend the characterization of semimartingale functions in Çinlar et al. (1980) to the non-Markovian setting. We prove that if a function of a semimartingale remains a semimartingale, then under certain conditions the function must have intervals where it is a difference of two convex functions. Under suitable conditions this property also holds for random functions. As an application, we prove that the median process defined in Prokaj et al. (2011) is not a semimartingale. The same process appears also in Hu and Warren (2000) where the question of the semimartingale property is raised but not settled.

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#### 1. Introduction

Let B be a Brownian motion and suppose that  $(D_t(x))_{t\geq 0,x\in[0,1]}$  satisfies the stochastic differential equation

$$dD_t(x) = D_t(x) \land (1 - D_t(x))dB_t = \sigma(D_t(x))dB_t, \quad D_0(x) = x.$$
(1)

This two parameter process was analyzed in Prokaj et al. (2011) in detail and played an important role in the construction that led to the solution of the drift hiding problem. Hu and Warren (2000) considers

$$dG_t(x) = dB_t + \beta \operatorname{sign}(G_t(x))dt, \quad G_0(x) = x, \quad x \in \mathbb{R}.$$
(2)

When  $\beta = 1/2$  then G is a transformed version of D, G(p(x)) = p(D(x)), where p is the Lamperti transformation associated to  $\sigma$ , that is  $p' = 1/\sigma$ , p(1/2) = 0. The case  $\beta > 0$  is just the matter of scaling both time and space. The aim of this paper is to show that  $(D_t^{-1}(1/2))_{t\geq 0}$  is not a semimartingale. As  $(G_t^{-1}(0))_{t\geq 0}$  is the same

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as  $p^{-1} \circ D^{-1}(1/2)$ , this result also confirms the expectation of Hu and Warren, see the remark after Proposition 1.2 on pp. 288. Their motivation came from Bass and Burdzy (1999);  $G_t(x) - B_t$  is the special case of the bifurcation model of that paper.

Our argument uses the fact that both D and G can be viewed as a stochastic flow in the sense of Kunita (1986). For  $t \ge 0$  denote  $B_{s,t} = B_{s+t} - B_s$ , which is a Brownian motion that starts evolving at time s. Let  $D_{s,t}(x)$  be the solution of (1) starting from xand driven by  $(B_{s,t})_{t\ge 0}$ . Then  $D_{s+t} = D_{s,t} \circ D_s$  and  $D_{s+t}^{-1}(1/2) = D_s^{-1}(D_{s,t}^{-1}(1/2))$  for any  $s, t \ge 0$ .

Suppose that  $m_t = D_t^{-1}(1/2), t \ge 0$  is a semimartingale in the filtration of B denoted by  $(\mathcal{F}_t)_{t\ge 0}$ . Fix a positive time point s, then the same is true for  $(m_{s,t} = D_{s,t}^{-1}(1/2))_{t\ge 0}$ in the filtration of  $(B_{s,t})_{t\ge 0}$ , denoted by  $(\mathcal{F}_{s,t})_{t\ge 0}$ . Put  $\mathcal{G}_t = \mathcal{F}_s \lor \mathcal{F}_{s,t}$  for all t, then  $\mathcal{G}$  is an initial enlargement of  $(\mathcal{F}_{s,t})_{t\ge 0}$  with an independent  $\sigma$ -algebra  $\mathcal{F}_s$ , so  $(m_{s,t})_{t\ge 0}$ remains a semimartingale in  $\mathcal{G}$ .

On the other hand if m is a semimartingale in  $\mathcal{F}$ , then since  $\mathcal{G}_t = \mathcal{F}_{s+t}$ , the process  $m_{s+t} = D_s^{-1}(m_{s,t}), t \geq 0$  is also a semimartingale in  $\mathcal{G}$ , which is obtained by substituting a  $\mathcal{G}$ -semimartingale into a random  $\mathcal{G}_0$  measurable function. By the main result of Section 3 it means that  $D_s^{-1}$  must be a semimartingale function for the process  $(m_{s,t})_{t\geq 0}$  almost surely. We give a necessary condition for this in Section 2, and analysing  $D_s$  and  $m_{s,t}$  in detail in Section 5 we conclude that  $D_s^{-1}$  is almost surely not a semimartingale function for  $(m_{s,t})_{t\geq 0}$ .

In Section 4 we give a simple proof in a similar fashion for a result of Rogers and Walsh (1991). Throughout the paper all  $\sigma$ -algebras are augmented with null sets and each filtration is assumed to be right continuous.

#### 2. Semimartingale functions

We call a function F a semimartingale function for X if F(X) is a semimartingale. Semimartingale functions are characterized in the Markovian setting by Çinlar et al. (1980). First, we give a simple proof of their result for Brownian semimartingale functions using the embedded random walk of the Brownian motion. This argument, with some slight modification, provides a necessary condition for F being a semimartingale function for a continuous semimartingale X even for the case when F is random.

A central part of the argument is a characterization of differences of convex functions. We prove below in Proposition 6.3 that if  $\limsup_{\delta \to 0} \mu_{\delta,F}(I) < \infty$  for an interval I, then F is a difference of two convex functions on I, where

$$\mu_{\delta,F}(H) = \sum_{k:k\delta \in H} \frac{1}{\delta} |(\Delta^{\delta} F)(k\delta)|,$$

and  $(\Delta^{\delta} F)(x) = F(x+\delta) + F(x-\delta) - 2F(x)$  is a kind of discrete Laplace operator. To motivate this condition, observe that when F is twice continuously differentiable then the limit is the total variation of F' on H. Since in the argument we consider a single F, we also use the notation  $\mu_{\delta}$  for the measure  $\mu_{\delta,F}$ .

**Theorem 2.1.** Suppose B is a Brownian motion and  $F : \mathbb{R} \to \mathbb{R}$  is a continuous function. Then F(B) is a semimartingale if and only if F is a difference of two convex functions. *Proof.* The sufficiency of the condition in the statement is just the Itô-Tanaka formula. So we only show the necessity.

In the proof we use the embedded random walk of the Brownian motion B with step size  $\delta$ . That is, we define the stopping times

$$\tau_0^{\delta} = 0, \quad \tau_{n+1}^{\delta} = \inf\left\{t \ge \tau_n^{\delta} : \left|B_t - B_{\tau_n^{\delta}}\right| \ge \delta\right\}, \quad n \ge 0.$$

With this choice, the sequence  $S_n^{\delta} = B_{\tau_n^{\delta}}$ ,  $n \ge 0$  forms a symmetric random walk with step size  $\delta$ .

The discrete Itô formula gives us that

$$F(S_n^{\delta}) = F(0) + \sum_{k=0}^{n-1} \frac{F(S_k^{\delta} + \delta) - F(S_k^{\delta} - \delta)}{2\delta} (S_{k+1}^{\delta} - S_k^{\delta}) + \frac{1}{2} \sum_{k=0}^{n-1} (F(S_k^{\delta} + \delta) + F(S_k^{\delta} - \delta) - 2F(S_k^{\delta})).$$

It can be also written in the Itô–Tanaka form

$$\begin{split} F(S_n^{\delta}) &= F(0) + \sum_{k=0}^{n-1} \frac{F(S_k^{\delta} + \delta) - F(S_k^{\delta} - \delta)}{2\delta} (S_{k+1}^{\delta} - S_k^{\delta}) \\ &+ \frac{1}{2} \sum_{r \in \mathbb{Z}} (F((r+1)\delta) + F((r-1)\delta) - 2F(r\delta)) \ell^{\delta}(r\delta, \tau_n^{\delta}) \end{split}$$

where  $\ell^{\delta}(x,t) = \sum_{k:\tau_k^{\delta} < t} \mathbb{1}_{(S_k^{\delta} = x)}$  denotes the number of visits of  $S^{\delta}$  to the site x before t.

Suppose now that F(B) is a semimartingale with decomposition F(B) = M + A, where M is a local martingale and A is a process of finite variation. Denote V the total variation process of A. Then V has continuous sample paths taking finite values. Consider the following stopping times

$$\rho_K = \inf\{t \ge 0 : \max(V_t, |B_t|) \ge K\},$$

and for a fixed  $\delta$ 

$$\eta_K = \inf \big\{ \tau_k^\delta : \tau_k^\delta \ge \rho_K \big\}.$$

We later choose K to be sufficiently large. From the relation

$$\mathbb{E}\Big(F(S_{k+1}^{\delta}) - F(S_k^{\delta}) \mid \mathcal{F}_{\tau_k^{\delta}}\Big) = \frac{1}{2} \big(F(S_k^{\delta} + \delta) + F(S_k^{\delta} - \delta) - 2F(S_k^{\delta})\big)$$

and that  $\tau_k^\delta < \eta_K$  happens exactly when  $\tau_k^\delta < \rho_K$  we get

$$\begin{split} \mathbb{1}_{(\tau_{k}^{\delta} < \eta_{K})} \frac{1}{2} \Big( F(S_{k}^{\delta} + \delta) + F(S_{k}^{\delta} - \delta) - 2F(S_{k}^{\delta}) \Big) \\ &= \mathbb{1}_{(\tau_{k}^{\delta} < \rho_{K})} \mathbb{E} \Big( F(S_{k+1}^{\delta}) - F(S_{k}^{\delta}) | \mathcal{F}_{\tau_{k}^{\delta}} \Big) \\ &= \mathbb{1}_{(\tau_{k}^{\delta} < \rho_{K})} \mathbb{E} \Big( F(B_{\tau_{k+1}^{\delta}}) - F(B_{\tau_{k+1}^{\delta} \wedge \rho_{K}}) + F(B_{\tau_{k+1}^{\delta} \wedge \rho_{K}}) - F(B_{\tau_{k}^{\delta} \wedge \rho_{K}}) \Big| \mathcal{F}_{\tau_{k}^{\delta}} \Big) \\ &= \mathbb{1}_{(\tau_{k}^{\delta} < \rho_{K})} \mathbb{E} \Big( F(B_{\tau_{k+1}^{\delta}}) - F(B_{\tau_{k+1}^{\delta} \wedge \rho_{K}}) \Big| \mathcal{F}_{\tau_{k}^{\delta}} \Big) + \mathbb{E} \Big( A_{\tau_{k+1}^{\delta} \wedge \rho_{K}} - A_{\tau_{k}^{\delta} \wedge \rho_{K}} \Big| \mathcal{F}_{\tau_{k}^{\delta}} \Big) . \end{split}$$

Estimating the increment of A with that of V and using the boundedness of  $F(B^{\eta_K})$  this leads to

$$\begin{split} \mathbb{1}_{(\tau_{k}^{\delta} < \eta_{K})} \frac{1}{2} \Big| F(S_{k}^{\delta} + \delta) + F(S_{k}^{\delta} - \delta) - 2F(S_{k}^{\delta}) \Big| \\ & \leq \mathbb{E} \Big( V_{\tau_{k+1}^{\delta} \land \rho_{K}} - V_{\tau_{k}^{\delta} \land \rho_{K}} \Big| F_{\tau_{k}^{\delta}} \Big) + 2c(K, \delta) \mathbb{P} \Big( \tau_{k+1}^{\delta} \ge \rho_{K} > \tau_{k}^{\delta} \Big| \mathcal{F}_{\tau_{k}^{\delta}} \Big), \end{split}$$

where

$$c(K,\delta) = \sup_{|x| < K + \delta} |F(x)|.$$

Taking expectation of the sum we get for  $\delta < 1$  that

$$\mathbb{E}\left(\sum_{r\in\mathbb{Z}}\frac{1}{2}|F((r+1)\delta)+F((r-1)\delta)-2F(r\delta)|\ell^{\delta}(r\delta,\eta_K)\right)\leq K+2c(K,1).$$

A simple calculation shows that

$$\mathbb{E}\Big(L^{r\delta}_{\tau^{\delta}_{k+1}} - L^{r\delta}_{\tau^{\delta}_{k}}\Big) = \delta \mathbb{E}\Big(\mathbbm{1}_{(S^{\delta}_{k} = r\delta)}\Big),$$

hence

$$\mathbb{E}\left(L_{\rho_{K}}^{r\delta}\right) \leq \mathbb{E}\left(L_{\eta_{K}}^{r\delta}\right) = \delta \mathbb{E}\left(\ell^{\delta}(r\delta,\eta_{K})\right).$$

From this we obtain that

$$\sum_{r\in\mathbb{Z}}\frac{1}{2\delta}|F((r+1)\delta) + F((r-1)\delta) - 2F(r\delta)|\mathbb{E}(L_{\rho_K}^{r\delta}) \le K + 2c(K)$$

Finally, let I be a bounded interval. Since  $\rho_K \to \infty$  as  $K \to \infty$ , there is a K such

$$\inf_{x\in I} \mathbb{E} \left( L^x_{\rho_K} \right) > 0.$$

To see that this is really the case, note that  $x \mapsto \mathbb{E}(L_{\rho_K \wedge 1}^x) = \mathbb{E}(|B_{\rho_K \wedge 1} - x| - |x|)$  is a continuous function for all K and as K goes to infinity they tend to the everywhere positive function  $x \mapsto \mathbb{E}(L_1^x) = \mathbb{E}(|B_1 - x| - |x|)$  in a pointwise increasing manner, so by virtue of the Dini lemma the convergence is uniform on compacts.

Then

$$\mu_{\delta}(I) = \sum_{r:r\delta \in I} \frac{1}{\delta} |F((r+1)\delta) + F((r-1)\delta) - 2F(r\delta)| \le 2 \frac{K + 2c(K)}{\inf_{x \in I} \mathbb{E}(L_{\rho_K}^x)} < \infty.$$

Since this upper estimate is independent of  $\delta \in (0, 1)$ , we obtain that

$$\sup_{0<\delta<1}\mu_{\delta}(I)<\infty,$$

which together with Proposition 6.3 proves the necessity of our condition.

Next, let X be a continuous semimartingale, and suppose that F is locally Lipschitz. The next theorem provides a necessary condition for a deterministic function F to be a semimartingale function for a continuous semimartingale X.

**Theorem 2.2.** Let X be a continuous semimartingale in the filtration  $\mathcal{F}$ , denote the local time of X by  $(L_t^x)_{x \in \mathbb{R}, t \geq 0}$  and put

$$H(X) = \left\{ x \in \mathbb{R} : \lim_{t \to \infty} \min(\mathbb{E}(L_t^x), \mathbb{E}(L_t^{x-})) > 0 \right\}.$$

Suppose that F is locally Lipschitz continuous.

If F(X) is a semimartingale in  $\mathfrak{F}$ , then each point in H(X) has a neighborhood I such that F is the difference of convex functions on I.

In particular, if F is continuously differentiable, then each point in H(X) has a neighborhood I such that the total variation of F' is finite on I.

*Proof.* We are going to use a similar argument as in the case of the Brownian motion. Suppose that  $X = M^X + A^X$ , where  $M^X$  is a continuous local martingale and  $A^X$  is a process of finite variation. Denote  $V^X$  the total variation process of  $A^X$ . As F(X) is a semimartingale,

$$F(X_t) = F(X_0) + M_t + A_t,$$

where M is a local martingale in the filtration  $\mathcal{F}$  with continuous sample paths starting from zero and A is a process of finite variation also starting from zero. Denote the total variation process of A by V. Let us consider the following stopping time

$$\rho_K = K \wedge \inf \{ t \ge 0 : \max(V_t, V_t^X, |X_t|, [M]_t) \ge K \},\$$

where K will be chosen sufficiently large. With this choice the local martingale part  $(M^X)^{\rho_K}$  of the stopped process  $X^{\rho_K}$  is a true martingale and  $V^{\rho_K}$  is integrable.

Since we want to apply the discrete Ito formula to the embedded random walk, we modify X after  $\rho_K$ . By enlarging the probability space we can assume that there is a Brownian motion B independent of  $\mathcal{F}_{\infty}$ . Let

$$X_t = X_{t \wedge \rho_K} + B_t - B_{t \wedge \rho_K},$$
  
$$\tilde{\mathcal{F}}_t = \mathcal{F}_t \lor \mathcal{F}_t^B.$$

Shortly, we can assume that for each  $\delta > 0$  the stopping times

$$\begin{aligned} \tau_0^{\delta} &= \inf \left\{ t \ge 0 \ : \ \tilde{X}_t \in \delta \mathbb{Z} \right\}, \\ \tau_{k+1}^{\delta} &= \inf \left\{ t \ge 0 \ : \ \left| \tilde{X}_t - \tilde{X}_{\tau_k^{\delta}} \right| = \delta \right\}, \quad k \ge 0, \\ \eta_K &= \inf \left\{ \tau_k^{\delta} \ : \ \tau_k^{\delta} \ge \rho_K \right\}, \end{aligned}$$

are all almost surely finite. As in the case of the Brownian motion  $S_k^{\delta} = X_{\tau_k^{\delta}}$  is a random walk on the lattice  $\delta \mathbb{Z}$ , although not necessarily symmetric. As in the proof of Theorem 2.1

$$\mathbb{1}_{(\tau_k^{\delta} < \eta_K)} \frac{1}{2} \left( F(S_k^{\delta} + \delta) + F(S_k^{\delta} - \delta) - 2F(S_k^{\delta}) \right)$$
  
=  $\mathbb{1}_{(\tau_k^{\delta} < \rho_K)} \mathbb{E} \left( F(S_{k+1}^{\delta}) - F(S_k^{\delta}) - \frac{F(S_k^{\delta} + \delta) - F(S_k^{\delta} - \delta)}{2\delta} (S_{k+1}^{\delta} - S_k^{\delta}) \middle| \mathcal{F}_{\tau_k^{\delta}} \right).$ 

This identity follows from the fact that the left hand side is  $\mathfrak{F}_{\tau_k^\delta}$  measurable. In this case, however,

$$\mathbb{1}_{(\tau_k^{\delta} < \rho_K)} \mathbb{E} \Big( S_{k+1}^{\delta} - S_k^{\delta} \, \Big| \, \mathcal{F}_{\tau_k^{\delta}} \Big) = \mathbb{E} \Big( A_{\tau_{k+1}^{\delta} \wedge \rho_K}^X - A_{\tau_k^{\delta} \wedge \rho_K}^X \, \Big| \, \mathcal{F}_{\tau_k^{\delta}} \Big),$$

and

$$\begin{split} & \mathbb{1}_{(\tau_{k}^{\delta} < \rho_{K})} \mathbb{E} \Big( F(S_{k+1}^{\delta}) - F(S_{k}^{\delta}) \ \Big| \ \mathcal{F}_{\tau_{k}^{\delta}} \Big) \\ &= \mathbb{1}_{(\tau_{k}^{\delta} < \rho_{K})} \mathbb{E} \Big( F(\tilde{X}_{\tau_{k+1}^{\delta}}) - F(\tilde{X}_{\tau_{k+1}^{\delta} \land \rho_{K}}) \ \Big| \ \mathcal{F}_{\tau_{k}^{\delta}} \Big) + \mathbb{E} \Big( F(\tilde{X}_{\tau_{k+1}^{\delta} \land \rho_{K}}) - F(\tilde{X}_{\tau_{k}^{\delta} \land \rho_{K}}) \ \Big| \ \mathcal{F}_{\tau_{k}^{\delta}} \Big). \end{split}$$
Here

Here

$$\mathbb{E}\Big(F(\tilde{X}_{\tau_{k+1}^{\delta}\wedge\rho_{K}})-F(\tilde{X}_{\tau_{k}^{\delta}\wedge\rho_{K}})\ \Big|\ \mathcal{F}_{\tau_{k}^{\delta}}\Big)=\mathbb{E}\Big(A_{\tau_{k+1}^{\delta}\wedge\rho_{K}}-A_{\tau_{k}^{\delta}\wedge\rho_{K}}\ \Big|\ \mathcal{F}_{\tau_{k}^{\delta}}\Big).$$

From these we get that

$$\mathbb{E}\left(\mathbb{1}_{(\tau_{k}^{\delta}<\rho_{K})}\frac{1}{2}\left|F(S_{k}^{\delta}+\delta)+F(S_{k}^{\delta}-\delta)-2F(S_{k}^{\delta})\right|\right)$$

$$\leq c(K,\delta)\mathbb{E}\left(V_{\tau_{k+1}^{\delta}\wedge\rho_{K}}^{X}-V_{\tau_{k}^{\delta}\wedge\rho_{K}}^{X}\right)+\mathbb{E}\left(V_{\tau_{k+1}^{\delta}\wedge\rho_{K}}-V_{\tau_{k}^{\delta}\wedge\rho_{K}}\right)+2C(K,\delta)\mathbb{P}\left(\tau_{k}^{\delta}<\rho_{K}\leq\tau_{k+1}^{\delta}\right)$$

where

$$c(K,\delta) = \sup\left\{ \left| \frac{F(y) - F(x)}{y - x} \right| : |x|, |y| \le K + \delta, x \ne y \right\}, \quad C(K,\delta) = \max_{|x| \le K + \delta} |F(x)|.$$

After summation

$$\sum_{r \in \mathbb{Z}} \frac{1}{2} |F((r+1)\delta) + F((r-1)\delta) - 2F(rk)| \mathbb{E}\left(\ell^{\delta}(r\delta, \rho_K)\right)$$
$$\leq K(1 + 2c(K, \delta)) + 2C(K, \delta) < \infty.$$
(3)

To finish the proof we use Lemma 6.1 below, which for each  $x \in H(X)$  provides us a K > 0, a non-empty open interval I and  $\delta_0 > 0$ , such that

$$\inf_{\delta \in (0,\delta_0)} \inf_{r:r\delta \in I} \delta \mathbb{E} \left( \ell^{\delta}(r\delta, \rho_K) \right) > 0.$$

Then a rearrangement of (3) gives

$$\limsup_{\delta \to 0} \mu_{\delta}(I) < \infty,$$

which by Proposition 6.3 proves the claim.

#### 3. Random semimartingale function

Throughout this section (S, S) will be a measurable space and  $\xi$  will be a random variable taking values in S. As it can cause no confusion, for the sake of brevity, we use "for almost all  $z \in S$ " instead of the correct but longer form "for  $\mathbb{P} \circ \xi^{-1}$  almost

all  $z \in S$ ". We will consider random variables (and processes) of the form  $\tilde{X}(\xi)$  where  $\tilde{X} : \tilde{\Omega} = \Omega \times S \to \mathbb{R}$ . The goal here is to show a version of Theorem 2.2 when the function F is also random. For the precise formulation we replace the single variable function F with a parametric version  $F : \mathbb{R} \times S \to \mathbb{R}$ , and consider the process  $(F(X_t, \xi))_{t\geq 0}$ . This has the form  $(\tilde{X}_t(\xi))_{t\geq 0}$  with  $\tilde{X}_t(z) = F(X_t, z)$ . Suppose now that  $\tilde{X}(\xi)$  is a semimartingale in the enlarged filtration  $\mathcal{F}^{\xi}$ , where  $\mathcal{F}^{\xi}_t = \mathcal{F}_t \vee \sigma(\xi)$  and  $\xi$  is independent of  $\mathcal{F}_{\infty}$ . Note that the right continuity of  $\mathcal{F}$  is inherited to  $\mathcal{F}^{\xi}$  by the independence of  $\mathcal{F}_{\infty}$  and  $\xi$ . We show in Theorem 3.1 below that in this case  $\tilde{X}(z)$  is also a semimartingale in  $\mathcal{F}$  for almost all  $z \in S$ . So, if  $(F(X_t, \xi))_{t\geq 0}$  is a semimartingale in  $\mathcal{F}^{\xi}$ , then  $F(X_t, z)$  is a semimartingale in  $\mathcal{F}$  for almost all z.

## **Theorem 3.1.** Let $\mathfrak{F}$ be a filtration such that $\xi$ is independent of $\mathfrak{F}_{\infty}$ .

Suppose that  $\tilde{X} : [0, \infty) \times \tilde{\Omega} \to \mathbb{R}$  is such that  $\tilde{X}(\xi)$  is a continuous semimartingale in  $\mathcal{F}^{\xi}$ . Then  $\tilde{X}(z)$  is a semimartingale in  $\mathcal{F}$  for almost all  $z \in S$ .

**Corollary 3.2.** Let  $\mathfrak{F}$  be a filtration such that  $\xi$  is independent of  $\mathfrak{F}_{\infty}$ . Suppose that X is a continuous semimartingale in  $\mathfrak{F}$ ,  $F : \mathbb{R} \times S \to \mathbb{R}$  is locally Lipschitz in the first variable and  $(F(X_t,\xi))_{t\geq 0}$  is a semimartingale in  $\mathfrak{F}^{\xi}$ .

Then each point in  $\overline{H}(X)$  has a neighborhood I(z) for almost all  $z \in S$  such that  $x \mapsto F(x, z)$  is the difference of convex functions on I(z).

In particular, if F is continuously differentiable in its first variable for almost all z, then the following holds for almost all z: each point in H(X) has a neighborhood I(z)such that the total variation of  $x \mapsto \partial_x F(x, z)$  is finite on I(z).

We start with preliminary lemmas. In the next lemmas for a  $\sigma$  algebra  $\mathcal{A}$  the family of  $\mathcal{A}$  measurable functions is denoted by  $m(\mathcal{A})$ .

**Lemma 3.3.** Let  $\mathcal{A}$  be a  $\sigma$ -algebra of events and  $X \in m(\mathcal{A} \vee \sigma(\xi))$ . Then  $X = \tilde{X}(\xi)$ , where  $\tilde{X} : \tilde{\Omega} \to \mathbb{R}$  and  $\tilde{X}$  is  $\mathcal{A} \times S$  measurable.

*Proof.* This is a monotone class argument. Let  $\mathcal{H}$  be the collection of random variables having the desired representation property, that is

$$\mathcal{H} = \{ Y \in m(\mathcal{A} \lor \sigma(\xi)) : \exists \tilde{Y} \in m(\mathcal{A} \times S), Y = \tilde{Y}(\xi) \}.$$

 $\mathcal{H}$  is obviously a linear space. Although  $\tilde{Y}$  is not unique one can find  $\tilde{Y}$  for  $Y \in \mathcal{H}$ , such that  $\sup|Y| = \sup|\tilde{Y}|$  and therefore  $\mathcal{H}$  is closed under uniform convergence.

Then let

$$\mathcal{D} = \{ A \in \mathcal{A} \lor \sigma(\xi) : \mathbb{1}_A \in \mathcal{H} \}.$$

 ${\mathcal D}$  is clearly a  $\lambda\text{-system},$  and it is also obvious that  ${\mathcal D}$  extends the  $\pi\text{-system}$ 

$$\mathcal{C} = \{A \cap B : A \in \mathcal{A}, B \in \sigma(\xi)\}$$

which generates  $\mathcal{A} \vee \sigma(\xi)$ . So  $\mathcal{D} = \mathcal{A} \vee \sigma(\xi)$  and  $\mathcal{H}$  contains all bounded  $\mathcal{A} \vee \sigma(\xi)$ measurable random variables. Now if  $X \in m(\mathcal{A} \vee \sigma(\xi))$ , then  $Y = \arctan(X) \in \mathcal{H}$  and  $\tilde{X} = \mathbb{1}_{(|\tilde{Y}| < \pi/2)} \tan(\tilde{Y})$  shows that  $X \in \mathcal{H}$ .

**Lemma 3.4.** Let  $\mathfrak{F}$  be a filtration. Suppose that X is a càdlàg process adapted to the filtration  $(\mathfrak{F}_t^{\xi})_{t\geq 0}$ . Then there is a parametric process  $\tilde{X} : [0,\infty) \times \tilde{\Omega} \to \mathbb{R}$ , such that

- 1. X and  $\tilde{X}(\xi)$  are indistinguishable,
- 2.  $\tilde{X}(z)$  is adapted to  $\mathfrak{F}$  for all  $z \in S$ .

When  $\mathcal{F}_{\infty}$  is independent of  $\xi$  then the sample path properties are inherited by  $\tilde{X}$ , that is

- 3.  $\tilde{X}(z)$  is càdlàg for almost all  $z \in S$ .
- 4. If X have continuous sample paths, the same is true for  $\tilde{X}(z)$  for almost all  $z \in S$
- 5. If X is of finite variation, then the same is true for  $\tilde{X}(z)$  for almost all  $z \in S$ .

*Proof.* Via Lemma 3.3 there are  $\tilde{X}_q \in m(\mathcal{F}_q \times S)$  for  $q \in [0, \infty) \cap \mathbb{Q}$  such that  $\tilde{X}_q(\xi) = X_q$ . Then

$$\tilde{X}_t(\omega, z) = \liminf_{q \searrow t} \tilde{X}_q(\omega, z), \quad t \ge 0$$

defines for each  $t \ge 0$  an  $\mathcal{F}_t \times S$  measurable function. Here we used the right continuity of  $\mathcal{F}$ , which is tacitly assumed. In other words,  $\tilde{X}(z)$  is adapted to  $\mathcal{F}$  for all  $z \in S$ .

From the definition of X we also have that  $X_t(\xi) = X_t$  for all  $t \ge 0$  whenever the sample path of X is càdlàg, which happens almost surely, hence  $\tilde{X}(\xi)$  and X are indistinguishable.

Suppose now that  $\mathcal{F}_{\infty}$  and  $\xi$  are independent. The last part of the claim is the application of the Fubini theorem. Let  $A \subset \Omega$  be the collection of outcomes  $\omega \in \Omega$  such that  $(X_q(\omega))_{q \in \mathbb{Q} \cap [0,\infty)}$  has right limits at each  $t \geq 0$  and left limits at each t > 0. Then, using for example the upcrossing number,  $A \in \sigma(\{X_q : q \in \mathbb{Q} \cap [0,\infty)\})$ . Similarly, let  $g: S \times \Omega \to \{0,1\}$  the indicator that  $(\tilde{X}_q(z))_{q \in \mathbb{Q} \cap [0,\infty)}$  has right and left limits at each t. The function g satisfies  $g(\xi) = \mathbb{1}_A$  and  $g \in m(S \times A)$ . As by assumption  $\mathbb{P}(A) = 1$ , so we have that  $\mathbb{E}(g(z, \cdot)) = 1$  for almost all  $z \in S$ . That is, for almost all  $z, \tilde{X}(z)$ , which is the right continuous extension of  $(\tilde{X}_q(z))_{q \in \mathbb{Q} \cap [0,\infty)}$ , is càdlàg.

The last two parts of the claim go along the same line, so they are left to the reader.  $\Box$ 

**Lemma 3.5.** Let  $\mathcal{G} \subset \mathcal{F}$  be two  $\sigma$ -algebras. Suppose that  $\xi$  is independent of  $\mathcal{F}$ . Using Lemma 3.3 write  $X \in L^1(\mathcal{F} \lor \sigma(\xi))$  and  $Y = \mathbb{E}(X \mid \mathcal{G} \lor \sigma(\xi))$  as  $X = \tilde{X}(\xi)$  and  $Y = \tilde{Y}(\xi)$ . Then  $\tilde{Y}(z) = \mathbb{E}(\tilde{X}(z) \mid \mathcal{G})$  almost surely for almost all  $z \in S$ .

*Proof.* X is integrable, so by the independence of  $\xi$  and  $\mathcal{F}$  we also have that  $\tilde{X}(z)$  is integrable for almost all  $z \in S$ . By the monotone class argument  $\mathbb{E}(\tilde{X}(z)|\mathcal{G})$  has a version which is  $\mathcal{G} \times \mathcal{S}$  measurable, that is there is  $\tilde{Z} \in m(\mathcal{G} \times \mathcal{S})$  such that  $\mathbb{E}(\tilde{X}(z)|\mathcal{G}) = \tilde{Z}(z)$  almost surely for almost all  $z \in S$ .

Let

$$A = \{ \tilde{Z}(\xi) > \tilde{Y}(\xi) \} \quad \text{and} \quad A(z) = \{ \tilde{Z}(z) > \tilde{Y}(z) \}$$

Then  $A \in \mathcal{G} \vee \sigma(\xi)$  and  $\mathbb{1}_A = \mathbb{1}_{A(z)}|_{z=\xi}$ , so by the independence of  $\xi$  and  $\mathcal{F}$ 

$$\mathbb{E}\left( (\tilde{X}(\xi) - \tilde{Y}(\xi)) \mathbb{1}_A \mid \xi \right) = \mathbb{E}\left( (\tilde{X}(z) - \tilde{Y}(z)) \mathbb{1}_{A(z)} \right) \Big|_{z=\xi}$$
$$= \mathbb{E}\left( (\tilde{Z}(z) - \tilde{Y}(z)) \mathbb{1}_{A(z)} \right) \Big|_{z=\xi}.$$

In the last step we used that  $\mathbb{E}(\tilde{X}(z) \mid \mathcal{G}) = \tilde{Z}(z)$ .

Then as  $Y = \mathbb{E}(X \mid \mathcal{G} \lor \sigma(\xi))$  we have that

$$0 = \mathbb{E}((X - Y)\mathbb{1}_A) = \mathbb{E}(g(\xi)), \text{ where } g(z) = \mathbb{E}((\tilde{Z}(z) - \tilde{Y}(z))\mathbb{1}_{A(z)}).$$

Since g is non negative by the choice of A(z), we have that g is zero for almost all z and  $\tilde{Z}(z) \leq \tilde{Y}(z)$  almost surely for almost all z. The other direction is obtained similarly.

So for almost all z the  $\mathcal{G}$  measurable random variables  $\tilde{Y}(z)$ ,  $\tilde{Z}(z)$  and  $\mathbb{E}(\tilde{X}(z) \mid \mathcal{G})$  are almost surely equal, which proves the claim.

**Lemma 3.6.** Let  $\mathfrak{F}$  be a filtration. Suppose that  $\xi$  is independent of  $\mathfrak{F}_{\infty}$  and X is a càdlàg martingale in the filtration  $(\mathfrak{F}_{t}^{\xi})_{t>0}$ .

Then there is a parametric process  $\tilde{X} : [0, \infty) \times \tilde{\Omega} \to \mathbb{R}$  such that  $\tilde{X}(\xi)$  and X are indistinguishable and  $\tilde{X}(z)$  is a martingale in  $\mathfrak{F}$  for almost all z.

*Proof.* Let  $\tilde{X}$  be the parametric process obtained from the application of Lemma 3.4. By Lemma 3.5  $(\tilde{X}_t(z))_{t \in [0,\infty) \cap \mathbb{Q}}$  is martingale in  $(\mathcal{F}_t)_{t \in [0,\infty) \cap \mathbb{Q}}$  for  $z \in H$  where  $\mathbb{P}(\xi \in H) = 1$ . As we have seen in Lemma 3.4  $\tilde{X}(z)$  is a càdlàg process for almost all z, hence for  $z \in H'$ with  $\mathbb{P}(\xi \in H') = 1$ .

As the filtration  $\mathcal{F}$  is right continuous we have for  $z \in H \cap H'$  the càdlàg process X(z) is a martingale in  $\mathcal{F}$ .

The localized version of Lemma 3.6 follows from the next claim.

**Lemma 3.7.** Suppose that  $\tau$  is a stopping time in  $\mathcal{F}^{\xi}$ . Then there is  $\tilde{\tau} : \tilde{\Omega} \to [0, \infty]$  such that  $\tilde{\tau}(\xi) = \tau$  and  $\tilde{\tau}(z)$  is a stopping time in  $\mathcal{F}$  for all  $z \in S$ .

*Proof.*  $\{\tau < t\} \in \mathcal{F}_t^{\xi}$ , so using Lemma 3.3,  $\mathbb{1}_{(\tau < t)} = \tilde{Y}_t(\xi)$  where  $\tilde{Y}_t \in m(\mathcal{F}_t \times S)$ . If we define

$$\tilde{\tau}(z) = \inf \big\{ q \in \mathbb{Q} \cap (0,\infty) \ : \ \tilde{Y}_q(z) \neq 0 \big\}, \quad z \in S,$$

then, as for fixed z the random variable  $\tilde{Y}_q(z) \in m(\mathcal{F}_q)$ , it follows that  $\{\tilde{\tau}(z) < t\} \in \mathcal{F}_t$ for all  $t \geq 0$ , which by the right continuity of  $\mathcal{F}$  implies that  $\tilde{\tau}(z)$  is a stopping time for all  $z \in S$ . It is an easy exercise that

$$\{\tilde{\tau}(\xi) < t\} = \bigcup_{q < t} \{\tilde{Y}_q(\xi) \neq 0\} = \bigcup_{q < t} \{\tau < q\} = \{\tau < t\},\$$

and  $\tilde{\tau}(\xi) = \tau$  follows.

Now we turn to the

Proof of Theorem 3.1.  $\tilde{X}(\xi)$  is a semimartingale in  $\mathcal{F}^{\xi}$ , which means that  $\tilde{X}(\xi) = M + A$ , where M is local martingale and A is a process of finite variation. Then  $M = \tilde{M}(\xi)$  and  $A = \tilde{A}(\xi)$  by Lemma 3.4, moreover both  $\tilde{M}(z)$  and  $\tilde{A}(z)$  inherit the path properties of M and A, respectively, for almost all  $z \in S$ .

Combining Lemma 3.6 and Lemma 3.7 we get that  $\tilde{M}(z)$  is a local martingale in  $\mathcal{F}$  for almost all  $z \in S$ .

Finally, using the Fubini theorem we get that  $\tilde{X}(z)$  and  $\tilde{M}(z) + \tilde{A}(z)$  are indistinguishable for almost all  $z \in S$ .

#### 4. Examples

In the following few examples S is either  $C(\mathbb{R})$  or C([0, s]) for some s > 0, that is, the space of continuous functions. With a suitable metric S is complete separable metric space and its Borel  $\sigma$ -algebra S is the same as a the smallest  $\sigma$ -algebra making all coordinate mapping measurable. We consider two type of examples here

$$F(x,w) = \int_0^x w_s ds \quad \text{or} \quad F(x,w) = \int_0^s \mathbb{1}_{(w(u) < x)} du.$$

In the first version  $x \mapsto F(x,\xi)$  is continuously differentiable regardless of the choice of  $\xi$ , in the second version it is continuously differentiable almost surely, for example, when  $\xi$  is a Brownian motion.

Suppose that  $\xi$  is a random variable taking values in S and independent of the Brownian motion B. Denote  $\mathcal{F}$  the natural filtration of B. Then since  $H(B) = \mathbb{R}$ , for the process  $(F(B_t,\xi))_{t\geq 0}$  to be a semimartingale in  $\mathcal{F}^{\xi}$  it is needed that  $x \mapsto F(x,\xi)$  is the difference of two convex functions almost surely. When  $F(x,\xi)$  is  $C^1$  in x, it simply requires that the sample paths of  $x \mapsto \partial_x F(x,\xi)$  have to have finite total variation on compact intervals.

**Proposition 4.1.** Let *B* and  $\xi$  be two independent standard Brownian motions and let  $\mathfrak{F} = (\mathfrak{F}_t)_{t\geq 0}$  be the natural filtration of *B* and  $\mathfrak{F}^{\xi}$  is the initial enlargement of  $\mathfrak{F}$  with  $\sigma(\xi)$ , that is  $\mathfrak{F}_t^{\xi} = \mathfrak{F}_t \vee \sigma(\xi)$ ,  $t \geq 0$ . Let *F* be the following random function:

$$F(x,\xi) = \begin{cases} \int_0^x \xi_y dy, & x \ge 0\\ 0, & x < 0. \end{cases}$$

Then  $(F(B_t,\xi))_{t>0}$  is not an  $\mathcal{F}^{\xi}$ -semimartingale.

*Proof.* In this example  $\partial_x F(x,\xi) = \xi_x$ , where  $\xi$  is a Brownian motion. Since the total variation of  $\xi$  is almost surely infinite on any non-empty sub-interval of the positive half line,  $x \mapsto F(x,\xi)$  is not a difference of convex functions almost surely and  $(F(B_t,\xi))$  is not a semimartingale.

**Proposition 4.2.** Let  $\xi = B_H$  be a fractional Brownian motion with Hurst index  $H \in (0,1)$  and B be a Brownian motion, independent of  $B_H$ . Let  $\mathfrak{F} = (\mathfrak{F}_t)_{t\geq 0}$  be the filtration generated by B and  $\mathfrak{F}^{\xi}$  be the initial enlargement of  $\mathfrak{F}$  with  $\sigma(B^H)$ , that is  $\mathfrak{F}_t^{\xi} = \mathfrak{F}_t \vee \sigma(B^H)$ ,  $t \geq 0$ . Let F be the following function:

$$F(x,\xi) = \begin{cases} \int_0^x \xi_y dy, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Then  $(F(B_t,\xi))_{t>0}$  is not an  $\mathcal{F}^{\xi}$ -semimartingale.

*Proof.* It is well know that the total variation of the fractional Brownian motion is almost surely infinite on any nonempty sub-interval of the positive half line, see for example Pratelli (2010) and the references therein, and the statement follows the same way as in the case of Proposition 4.1

As an application of Theorem 3.1 we can obtain a preparatory example for the process  $A(t, B_t)$  investigated in Rogers and Walsh (1991). Here A(t, x) is the amount of time spent by the Brownian motion B below the level x up to time t. First, we treat the case where there are two independent Brownian motions involved.

**Proposition 4.3.** Let B and  $\xi$  be two independent standard Brownian motions. For a fixed s > 0 consider the following function

$$F(x,\xi) = \int_0^s \mathbb{1}_{(\xi_u \le x)} du.$$

Let  $\mathfrak{F}$  be the natural filtration of B and  $\mathfrak{F}^{\xi}$  be the initial enlargement of  $\mathfrak{F}$  with  $\sigma(\xi)$ , that is  $\mathfrak{F}_t^{\xi} = \mathfrak{F}_t \vee \sigma(\xi), t \geq 0.$ 

Then  $(F(B_t,\xi))_{t\geq 0}$  is not an  $\mathcal{F}^{\xi}$ -semimartingale.

Proof. By the occupation time formula

$$F(x,\xi) = \int_{-\infty}^{x} Z_y dy,$$

where  $Z_y = L_s^y(\xi)$  denotes the local time profile of  $\xi$  at time s. For a compact interval [a, b] the quadratic variation of Z (see (Revuz and Yor, 1991, Theorem (1.12))) exists and is given by the following formula

$$[Z]_b - [Z]_a = \int_a^b 4Z_y dy.$$

Since  $P(L_s^0(\xi) > 0) = 1$  and  $Z_y = L_s^y(\xi)$  is continuous in y, we can conclude that  $\mathbb{P}\left(\int_a^b Z_y dy > 0\right) = 1$  also holds for any a < 0 < b. From this it easily follows that on an almost sure event  $x \mapsto Z_x$  has infinite total variation on any non-empty interval containing zero. So almost surely F is not a semimartingale function for B and  $(F(B_t, \xi))_{t\geq 0}$  can not be a semimartingale in  $\mathcal{F}^{\xi}$  by Theorem 3.1.

Now we turn to the process  $A(t, B_t)$  of Rogers and Walsh (1991).

**Proposition 4.4.** Let B be a standard Brownian motion and denote  $\mathcal{F}$  its natural filtration. For  $x \in \mathbb{R}$  put

$$A(t,x) = \int_0^t \mathbb{1}_{(B_u \le x)} du.$$

Then  $A(t, B_t)$  is not semimartingale in  $\mathfrak{F}$ .

The result in Rogers and Walsh (1991) actually states more than just the lack of the semimartingale property, namely, they define

$$X_t = A(t, B_t) - \int_0^t L_s^{B_s} dB_s$$

and show that the p order variation of X on [0, 1] is zero when p > 4/3 and infinite when p < 4/3.

*Proof.* Fix s > 0 and for  $t \ge s$  write A as

$$A(t,x) = \int_0^t \mathbb{1}_{(B_u \le x)} du = \int_0^s \mathbb{1}_{(B_u \le x)} du + \int_s^t \mathbb{1}_{(B_u - B_s \le x - B_s)} du.$$

Denote by  $\beta_t = B_{s+t} - B_s$ ,  $t \ge 0$ . Then  $(\beta_u)_{u\ge 0}$  is a Brownian motion independent of  $\mathcal{F}_s.$  With

$$\alpha(t,x) = \int_0^t \mathbb{1}_{(\beta_u \le x)} du$$

we can write for  $t \ge 0$ 

$$A(s+t,x) = A(s,x) + \alpha(t,x-B_s), \quad A(s+t,B_{s+t}) = A(s,B_{s+t}) + \alpha(t,\beta_t).$$
(4)

Suppose now, on the contrary to the claim that  $(A(t, B_t))_{t\geq 0}$  is a semimartingale in the natural filtration of B, that is, in  $\mathcal{F}$ . Then  $(\alpha(t,\beta_t))_{t>0}$  is a semimartingale in the filtration of  $\beta$ . As  $(\mathfrak{G}_t = \mathfrak{F}_{s+t})_{t\geq 0}$  is an initial enlargement of the natural filtration of  $\beta$  with an independent  $\sigma$ -algebra  $\mathcal{F}_s$ , the process  $(\alpha(t, \beta_t))_{t>0}$  is also a semimartingale in  $(\mathcal{G}_t = \mathcal{F}_{s+t})_{t \geq 0}$ . Since  $(A(s+t, B_{s+t}))_{t \geq 0}$  is obviously a semimartingale in  $(\mathcal{G}_t)_{t > 0}$ , we get from the second part of (4) that  $(A(s, B_{s+t}))_{t\geq 0}$  also needs to be a  $(\mathfrak{G}_t)_{t\geq 0}$ semimartingale. But

$$A(s, B_{s+t}) = \int_0^s \mathbb{1}_{(B_u \le B_{t+s})} du = \int_0^s \mathbb{1}_{(B_u - B_s \le B_{t+s} - B_s)} du.$$

Note that  $\xi_u = B_{s-u} - B_s$ ,  $u \in [0, s]$ , which is the time reversal of  $(B_u)|_{u \in [0,s]}$ , is a Brownian motion (on [0, s]), independent of  $(\beta_t)_{t \ge 0}$ . With

$$F(x,\xi) = \int_0^s \mathbb{1}_{(\xi_u \le x)} du = \int_0^s \mathbb{1}_{(B_u - B_s \le x)} du$$

we have

$$A(s, B_{s+t}) = F(\beta_t, \xi).$$

In Proposition 4.3 we already proved that  $(F(\beta_t,\xi))_{t>0}$  is not a semimartingale in the initial enlargement of the natural filtration of  $\beta$  with  $\xi$  which is the same as  $\mathfrak{G}$ . On the other hand, the assumption that  $(A(t, B_t))_{t \ge 0}$  is a semimartingale in  $\mathcal{F}$  would lead to the conclusion that  $(A(s, B_{s+t})_{t\geq 0})$  is semimartingale in  $\mathcal{G}$ . This contradiction proves the claim. 

#### 5. The median process is not a semimartingale

Consider the equation

$$dD_t(x) = \sigma(D_t(x))dB_t, \quad D_0(x) = x, \tag{5}$$

where  $\sigma(x) = x \wedge (1-x)$ . The aim of this section is to show that  $(D_t^{-1}(\alpha))_{t\geq 0}$  is not a semimartingale for  $\alpha \in (0,1)$ , in particular, the conditional median  $m_t = \overline{D}_t^{-1}(1/2)$ lacks the semimartingale property. It is useful to collect some of the properties of D. It

was shown in Prokaj et al. (2011) that  $x \mapsto D_t(x)$  is differentiable for all t almost surely and the space derivative is given by the stochastic exponential of

$$\int_0^t \sigma'(D_s(x)) dB_s$$

provided that  $\sigma$  is Lipshitz continuous. In this case  $\sigma'$  exists Lebesgue almost everywhere and is a bounded function. Note that, it also follows, for example from the positivity of  $D'_t$ , that  $x \mapsto D_t(x)$  is stricly increasing for all t on an almost sure event. We prove below that  $(x,t) \mapsto D'_t(x)$  is continuous whenever  $\sigma$  is Lipshitz and  $\sigma'$  is monotone, for example,  $\sigma(x) = x \land (1-x)$  has this property. In other words, D' is continuous almost surely if  $\sigma$  is a convex (or concave) Lipschitz function, that is  $\sigma(x) = \int_0^x \sigma'$  for some monotone bounded function denoted by  $\sigma'$ . The proof of the continuity in Lemma 5.2 uses the Lamperti representation of D, and at this step the monotonocity of  $\sigma'$  is important. On the other hand, all other parts of our argument only use that  $\sigma'$  is of bounded variation on compact intervals, that is  $\sigma$  is a difference of two convex functions. Even this assumption seems to be too strict. In the proof below we compute the quadratic variation of  $D'_t$  with respect to the space variable x and use that the total variation is infinite if the quadratic variation is non-zero. Finding another way of proving the unboundedness of the total variation would relax the condition imposed on  $\sigma$ .

**Theorem 5.1.** Suppose that  $\sigma$  is a convex (or concave) Lipschitz function, B is a Brownian motion and (D, B) satisfies (5). Let  $\alpha \in \mathbb{R}$  such that  $\sigma(\alpha) \neq 0$  and denote C the connected component of  $\mathbb{R} \setminus \{\sigma = 0\}$  containing  $\alpha$ .

If  $\sigma'$  is not continuous on C, then  $q_t = D_t^{-1}(\alpha)$  is not a semimartingale in the natural filtration of B.

Before the proof we discuss the case  $\sigma(x) = x \wedge (1 - x)$ . Then

$$D_t(x) = \begin{cases} x \exp\{B_t - \frac{1}{2}t\}, & x \le 0, \\ 1 - (1 - x) \exp\{-B_t - \frac{1}{2}t\}, & x \ge 1. \end{cases}$$

So for  $\alpha \notin (0,1)$  the process  $q_t(\alpha) = D_t^{-1}(\alpha)$  can be explicitly given

$$q_t(\alpha) = \begin{cases} \alpha \exp\{-B_t + \frac{1}{2}t\}, & \alpha \le 0, \\ 1 - (1 - \alpha) \exp\{B_t + \frac{1}{2}t\}, & \alpha \ge 1, \end{cases}$$

and from this explicit form we see that it is a semimartingale. On the other hand, C = (0, 1) contains 1/2, the point of discontinuity for  $\sigma'$  and for  $\alpha \in (0, 1)$  the process  $(q_t(\alpha))_{t\geq 0}$  is not a semimartingale by the theorem.

*Proof.* We suppose on the contrary to the claim that q is a semimartingale and show that this leads to a contradiction.

Let s > 0. Then, as it was already indicated in the introduction, we can decompose  $(q_{s+t} = D_{s+t}^{-1}(\alpha))_{t \ge 0}$  as

$$D_{s+t}^{-1}(\alpha) = D_s^{-1}(D_{s,t}^{-1}(\alpha)), \tag{6}$$

where  $D_{s,t}(x)$  is the solution of (5) with B replaced by  $B_{s,t} = B_{s+t} - B_s$  a "Brownian motion that starts evolving at time s". Then  $D_{s+t}(x) = D_{s,t}(D_s(x))$  and this is the rationale behind (6).

As by our assumption  $q = D^{-1}(\alpha)$  is a semimartingale in the filtration of the driving Brownian motion, so is  $(q_{s,t} = D_{s,t}^{-1}(\alpha))_{t\geq 0}$  in the filtration of  $(B_{s,t})_{t\geq 0}$  denoted by  $(\mathcal{F}_{s,t})_{t\geq 0}$ , where  $\mathcal{F}_{s,t} = \sigma(\{B_{s,u} : u \leq t\})$ . Denote by  $\mathcal{F}$  the filtration of B, that is  $\mathcal{F}_t = \sigma(\{B_u : u \leq t\})$ . Then  $(q_{s,t})_{t\geq 0}$  remains a semimartingale in the filtration  $(\mathcal{F}_{s+t})_{t\geq 0}$ as  $\mathcal{F}_{s+t} = \mathcal{F}_s \vee \mathcal{F}_{s,t}$ , that is,  $(\mathcal{F}_{s+t})_{t\geq 0}$  is an initial enlargement of  $(\mathcal{F}_{s,t})_{t\geq 0}$  with an independent  $\sigma$ -algebra  $\mathcal{F}_s$ .

On the other hand, as q is supposed to be a semimartingale in  $\mathcal{F}$ , the process  $(q_{s+t})_{t\geq 0}$  is a semimartingale in  $(\mathcal{F}_{s+t})_{t\geq 0}$ . By (6)  $q_{s+t} = D_s^{-1}(q_{s,t})$  and the random function  $D_s^{-1}$  is a semimartingale function for  $(q_{s,t})_{t\geq 0}$ .

Our necessary condition for  $(D_s^{-1}(q_{s,t}))_{t\geq 0}$  being a semimartingale involves the set  $H((q_{s,t})_{t\geq 0})$  defined in Theorem 2.2. Below we show in Corollary 5.5 that  $H((q_{s,t})_{t\geq 0})$  is dense in C, the connected component of  $\mathbb{R} \setminus \{\sigma = 0\}$  containing  $\alpha$ . In Corollary 5.10 we also show that almost surely there is a non-empty open interval  $I \subset C$  such that the total variation  $(D_s^{-1})'$  is infinite on any non-empty subinterval  $J \subset I$ .

Recall that by Corollary 3.2, if  $q_{s,t}$  and  $D_s^{-1}(q_{s,t})$  are semimartingales, then each point in  $H(q_{s,\cdot})$  must have a neighborhood which may depend on  $\mathcal{F}_s$  and on which the total variation of  $(D_s^{-1})'$  is finite. Points in I do not have this property, and as  $H(q_{s,\cdot})$  is dense in C,  $D_s^{-1}$  can not satisfy the necessary condition of Corollary 3.2. We can conclude that  $q_{s+t} = D_s^{-1}(q_{s,t})$  can not be a semimartingale in  $(\mathcal{F}_{s+t})_{t\geq 0}$ , which would follow from our starting assumption.

## 5.1. Continuity of $(D'_t(x))_{t\geq 0, x\in\mathbb{R}}$

First we want to show the continuity of  $(t, x) \mapsto D'_t(x)$  on  $[0, \infty) \times (\mathbb{R} \setminus \{\sigma = 0\})$ . Recall from Prokaj et al. (2011) that  $x \mapsto D_t(x)$  is absolutely continuous and its derivative can be written as

$$D'_t(x) = \exp\{Z_t(x) - \frac{1}{2}[Z(x)]_t\}, \text{ where } Z_t(x) = \int_0^t \sigma'(D_s(x))dB_s.$$

When  $|\sigma'| \leq L$  this yields the estimate

$$\mathbb{E}(|D'_t(x)|^p) \le \exp\{\frac{1}{2}L^2p(p-1)t\}.$$
(7)

**Lemma 5.2.** Suppose that  $\sigma$  is a convex (or concave) Lipschitz function, B is a Brownian motion and (D, B) satisfies (5). Then

$$Z_t(x) = \int_0^t \sigma'(D_s(x)) dB_s, \quad and \quad [Z(x)]_t = \int_0^t (\sigma'(D_s(x)))^2 ds$$
(8)

have modifications which are continuous functions of (t, x) on  $[0, \infty) \times (\mathbb{R} \setminus \{\sigma = 0\})$ almost surely.

**Remark.** Even in one of the simplest cases the continuity of  $D'_t(x)$  on  $[0, \infty) \times \mathbb{R}$  may fail. Consider  $\sigma(x) = |x|$ , then  $Z_t(x) = \operatorname{sign}(x)B_t$ , which is continuous on  $[0, \infty) \times (\mathbb{R} \setminus \{0\})$  but discontinuous at (t, 0) whenever  $B_t \neq 0$ , that is almost surely for t > 0.

*Proof.* We need to show the continuity of Z on  $[0, \infty) \times C$ , where C is a connected component of  $\mathbb{R} \setminus \{\sigma = 0\}$ . This is obtained via the usual combination of the Burkholder-Davis-Gundy inequality and Kolmogorov's lemma, see for example (Revuz and Yor, 1991,

Chapter VI, proof of Theorem (1.7)). That is we use that

$$\mathbb{E}\left(\sup_{s\leq t} |Z_s(y) - Z_s(x)|^{2p}\right) \leq c_p \mathbb{E}([Z(y) - Z(x)]_t^p) \quad \text{for } x, y \in C,$$

and show that this can be further bounded by  $c|y-x|^p$ . We can assume without loss of generality that  $\sigma'$  is increasing (for the other case replace both  $\sigma, B$  with  $-\sigma, -B$ ). By assumption  $\sigma'$  is bounded, say  $|\sigma'| \leq L$ , so for x < y (hence  $D_s(x) \leq D_s(y)$ ) we have that

$$[Z(y) - Z(x)]_t = \int_0^t (\sigma'(D_s(y)) - \sigma'(D_s(x)))^2 ds$$
  
$$\leq 2L \int_0^t \sigma'(D_s(y)) - \sigma'(D_s(x)) ds.$$
(9)

The right hand side of (9) can be related to the process  $Y_t(x) = h(D_t(x))$ , where  $h: C \to \mathbb{R}$  is such that  $h' = 1/\sigma$ . Sometimes  $h(D_t(x))$  is referred to as the Lamperti representation of the process. Simple calculus with the Itô formula yields that

$$dY_t(x) = dB_t - \frac{1}{2}\sigma'(D_t(x))dt, \quad Y_0(x) = h(x).$$

From this we get that

$$[Z(y) - Z(x)]_t \le 4L(h(y) - Y_t(y) - (h(x) - Y_t(x))).$$

Then

$$Y_t(y) - Y_t(x) = \int_x^y h'(D_t(z))D'_t(z)dz$$

and

$$(Y_t(y) - Y_t(x))^p \le (y - x)^p \frac{1}{y - x} \int_x^y |h'(D_t(z))|^p |D'_t(z)|^p dz$$

To finish the proof let a < b such that  $[a, b] \subset C$ ,  $0 < \delta < \min_{x \in [a, b]} \sigma(x)$  and

$$\tau_{\delta} = \tau = \inf\{t > 0 : \sigma(D_t(z)) < \delta \text{ for some } z \in [a, b]\}.$$

With these choices we get that

$$\mathbb{E}((Y_{t \wedge \tau}(y) - Y_{t \wedge \tau}(x))^p) \le (y - x)^p \frac{1}{\delta^p} e^{\frac{1}{2}L^2(p(p-1))t} \quad \text{if } a < x < y < b,$$

and

$$|h(y) - h(x)|^p \le (y - x)^p \frac{1}{\delta^p}.$$

So, we can conclude that

$$\mathbb{E}([Z(y) - Z(x)]_{t \wedge \tau}^p) \le c_p |y - x|^p,$$

which shows that  $(t, x) \mapsto Z_t(x)$  is continuous on  $[0, \tau) \times (a, b)$  almost surely. Letting  $\delta \to 0$  we have that  $\tau_{\delta} \to \infty$ , so  $(t,x) \mapsto Z_t(x)$  has a continuous modification on 15

 $[0,\infty) \times (a,b)$  whenever  $[a,b] \subset C$ . As C is open interval, this also means that there is a continuous modification on  $[0, \infty) \times C$ .

For  $[Z(x)]_t$  note that by the Schwarz inequality

$$\left( \left[ Z(y) \right]_t^{1/2} - \left[ Z(x) \right]_t^{1/2} \right)^2 \le \left[ Z(x) - Z(y) \right]_t$$

hence the argument above also proves the continuity of  $[Z(x)]_t^{1/2}$  and therefore the continuity of  $(x,t) \mapsto [Z(x)]_t$  on  $[0,\infty) \times \mathbb{R} \setminus \{\sigma = 0\}$ .

## 5.2. Decomposition of $(q_t)_{t\geq 0}$

**Lemma 5.3.** Suppose that  $\sigma$  is Lipschitz continuous, B is a Brownian motion, (D, B)satisfies (5), and  $(x,t) \mapsto D'_t(x)$  is continuous on  $[0,\infty) \times \mathbb{R} \setminus \{\sigma = 0\}$  almost surely. For  $\alpha \in \mathbb{R}$  let  $q_t = D_t^{-1}(\alpha)$ . Then

$$A_t = q_t + \int_0^t (D_s^{-1})'(\alpha)\sigma(\alpha)dB_s$$

is a process of zero energy, that is, the quadratic variation of A exists and  $[A] \equiv 0$ .

*Proof.* When  $\sigma(\alpha) = 0$ , then  $D_t(\alpha) = \alpha$  and  $q_t = A_t = \alpha$  for all  $t \ge 0$ , so the claim is obvious for this case. Hence we may assume that  $\sigma(\alpha) \neq 0$ .

For s < t

$$D_t(q_t) - D_t(q_s) = D_s(q_s) - D_t(q_s) = -\int_s^t \sigma(D_u(q_s)) dB_u.$$

From this

$$A_t - A_s = q_t - q_s + \int_s^t (D_u^{-1})'(\alpha)\sigma(\alpha)dB_u$$
$$= \int_s^t (D_u^{-1})'(\alpha)\sigma(\alpha)dB_u - R(s,t)\int_s^t \sigma(D_u(q_s))dB_u$$

where by the mean value theorem R(s,t) has the following form

$$R(s,t) = \begin{cases} \frac{q_t - q_s}{D_t(q_t) - D_t(q_s)} & q_t \neq q_s \\ \frac{1}{D'_t(q_t)} & q_t = q_s \end{cases} = \frac{1}{D'_t(\vartheta q_t + (1 - \vartheta)q_s)}, \quad \text{for some } \vartheta \in (0,1).$$

The only idea in the calculation is that when s is close to t then  $\sigma(D_u(q_s))$  is close to  $\sigma(\alpha)$  and R(s,t) is approximately  $(D_s^{-1})'(\alpha)$ . Let  $\pi = \{t_0 = 0 < t_1 < \cdots < t_r = t\}$  be a subdivision of [0,t]. Then

$$\sum_{i} (A_{t_{i+1}} - A_{t_i})^2 \le 3(Q_1(\pi) + Q_2(\pi) + Q_3(\pi))$$

where

$$Q_{1}(\pi) = \sum_{i} \left( \int_{t_{i}}^{t_{i+1}} \left( (D_{u}^{-1})'(\alpha) - (D_{t_{i}}^{-1})'(\alpha) \right) \sigma(\alpha) dB_{u} \right)^{2},$$
  

$$Q_{2}(\pi) = \sum_{i} \left( R(t_{i}, t_{i+1}) \int_{t_{i}}^{t_{i+1}} \sigma(\alpha) - \sigma(D_{u}(q_{t_{i}})) dB_{u} \right)^{2},$$
  

$$Q_{3}(\pi) = \sum_{i} \left( \left( (D_{t_{i}}^{-1})'(\alpha) - R(t_{i}, t_{i+1}) \right) \int_{t_{i}}^{t_{i+1}} \sigma(\alpha) dB_{u} \right)^{2}.$$

For  $Q_3$  observe that since  $(t, x) \mapsto (D_t^{-1})'(x)$  is continuous, we have that

$$\max_{i} |(R(t_{i}, t_{i+1}) - (D_{t_{i}}^{-1})'(\alpha)| \to 0, \text{ almost surely as } \max_{i}(t_{i+1} - t_{i}) \to 0,$$

so  $Q_3(\pi) \xrightarrow{p} 0$  as the mesh of the partition  $\pi$  goes to zero.

For  $Q_1, Q_2$  we use the next observation

**Proposition 5.4.** For a subdivision  $\pi = \{t_0 = 0 < t_1 < \cdots < t_r = t\}$  of [0,t] let  $\pi(u) = t_i$  if  $u \in [0,t)$  and  $t_i \leq u < t_{i+1}$ , that is,  $\pi(u)$  is the starting point of the subinterval containing u. Denote mesh $(\pi) = \max_i(t_{i+1} - t_i)$ . Suppose that  $(\varphi(u, v))_{u,v \geq 0}$  is a two parameter process such that  $\varphi(u, v) \mathbb{1}_{(u \geq v)}$  is integrable with respect to the Brownian motion B for each fixed v. If

$$\int_0^t \varphi^2(u, \pi(u)) du \xrightarrow{p} 0, \quad as \operatorname{mesh}(\pi) \to 0, \tag{10}$$

then

$$\sum_{i} \left( \int_{t_i}^{t_{i+1}} \varphi(u, t_i) dB_u \right)^2 \xrightarrow{p} 0, \quad as \operatorname{mesh}(\pi) \to 0.$$

For  $Q_1(\pi)$  we can use

$$\varphi(u, v) = (D_u^{-1})'(\alpha) - (D_v^{-1})'(\alpha)$$

and note that by the continuity of  $(x, u) \mapsto D_u^{-1}(x)$  condition (10) holds and  $Q_1(\pi) \xrightarrow{p} 0$  as  $\operatorname{mesh}(\pi) \to 0$ .

For  $Q_2$  we consider

$$\varphi(u,v) = \sigma(D_u(q_v)) - \sigma(\alpha) = \sigma(D_u(q_v)) - \sigma(D_v(q_v))$$

and now we can refer to the continuity of  $(x, u) \mapsto D_u(x)$  and q to get that condition (10) holds. To see that  $Q_2(\pi) \xrightarrow{p} 0$  we only need to add that

$$\max_{i} R(t_i, t_{i+1}) \to \max_{s \le t} (D_s^{-1})'(\alpha) < \infty, \quad \text{almost surely as } \operatorname{mesh}(\pi) \to 0. \qquad \Box$$

Proof of Proposition 5.4. For a subdivision  $\pi = \{t_0 = 0 < t_1 < \cdots < t_r = t\}$  of [0, t], denote by  $(Y(s, \pi))_{s \ge 0}$  the following process

$$Y(s) = Y(s,\pi) = \int_0^s \varphi(u,\pi(u)) dB_u.$$
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Note that from  $\int_0^t \varphi^2(u, \pi(u)) du \to 0$  it follows that

$$\sup_{s \le t} |Y(s,\pi)| \xrightarrow{p} 0 \quad \text{as mesh}(\pi) \to 0.$$
(11)

By Itô's formula

$$\left(\int_{t_i}^{t_{i+1}} \varphi(u, t_i) dB_u\right)^2 = 2 \int_{t_i}^{t_{i+1}} (Y(u) - Y(t_i))\varphi(u, t_i) dB_u + \int_{t_i}^{t_{i+1}} \varphi^2(u, t_i) du.$$

Hence

$$\sum_{i} \left( \int_{t_i}^{t_{i+1}} \varphi(u, t_i) dB_u \right)^2 = 2 \int_0^t (Y(u) - Y(\pi(u))) \varphi(u, \pi(u)) dB_u + \int_0^t \varphi^2(u, \pi(u)) du.$$

By assumption the time integral tends to zero in probability as  $\operatorname{mesh}(\pi) \to 0$ . For the stochastic integral it is enough to show that

$$\int_0^t (Y(u) - Y(\pi(u)))^2 \varphi^2(u, \pi(u)) du \xrightarrow{p} 0.$$

Here

$$\int_0^t (Y(u) - Y(\pi(u)))^2 \varphi^2(u, \pi(u)) du \le 4 \sup_{u \le t} Y^2(u) \int_0^t \varphi^2(u, \pi(u)) du.$$

Although it is suppressed in the notation both factors in the upper bound depend on  $\pi$ . Nevertheless, it is easy to see that both factor tends to zero in probability and so does their product. For the second factor it is the main assumption, while for the first one it was already observed in (11).

The next claim follows from Lemma 5.3.

**Corollary 5.5.** Suppose that the assumptions of Lemma 5.3 hold and  $\sigma(\alpha) \neq 0$ . Let C be the connected component of  $\mathbb{R} \setminus \{\sigma = 0\}$  containing  $\alpha$ . Then

$$\mathbb{P}\left(\int_0^\infty \mathbb{1}_{(q_s \in (a,b))} d[q]_s > 0\right) > 0 \quad \text{for any } (a,b) \subset C.$$

Especially, if q is a semimartingale, then H(q) is dense in C.

*Proof.* By Lemma 5.3 q has quadratic variation and

$$\int_0^\infty \mathbb{1}_{(q_s \in (a,b))} d[q]_s = \int_0^\infty \mathbb{1}_{(D_s(a) < \alpha < D_s(b))} ((D_s^{-1})'(\alpha)\sigma(\alpha))^2 ds.$$

Since  $((D_s^{-1})'(\alpha)\sigma(\alpha))^2 > 0$ , it is enough to show that the continuous process  $(D_t(x))_{t\geq 0}$ hits  $\alpha$  with positive probability for any x in C. Indeed, the set  $\{s > 0 : D_s(a) < \alpha < D_s(b)\}$  is open, so if not empty, then it has positive Lebesgue measure.

This is an easy martingale argument. Let  $c < \alpha, x < d$  be such that  $[c, d] \subset C$ . Then  $\inf \sigma|_{[c,d]} > 0$  implies that  $\tau = \inf \{t \ge 0 : D_t(x) \notin [c, d] \}$  is almost surely finite, and since

 $\mathbb{E}(D_{\tau}(x)) = x$ , both  $\mathbb{P}(D_{\tau}(x) = c)$  and  $\mathbb{P}(D_{\tau}(x) = d)$  are positive, which implies by the continuity of the sample paths of D(x) that D(x) hits  $\alpha$  with positive probability.

Finally, suppose that q is a semimartingale. Then its local time exists, and for any fixed t the mapping  $x \mapsto L_t^x(q)$  is càdlàg. From the first part of the statement, for  $(a,b) \subset C$  and t large enough  $\int_{(a,b)} L_t^x(q) dx > 0$  with positive probability. This yields that for some  $\varepsilon > 0$  and  $(a',b') \subset (a,b)$  we have  $\inf_{x \in (a',b')} L_t^x(q) > \varepsilon$  with positive probability and  $(a',b') \subset H(q)$ .

### 5.3. Quadratic variation of $D'_t$ in the space variable

**Lemma 5.6.** Suppose that  $\sigma$  is Lipschitz continuous and is the difference of convex functions, B is a Brownian motion, (D, B) satisfies (5), and

$$(x,t) \mapsto Z_t(x) = \int_0^t \sigma'(D_s(x)) dB_s$$

is continuous on  $[0, \infty) \times \mathbb{R} \setminus \{\sigma = 0\}$  almost surely. Then the quadratic variation processes of the random functions  $x \mapsto Z_t(x)$  and  $x \mapsto U_t(x) = [Z(x)]_t$  exist,

$$[Z_t](b) - [Z_t](a) = \int_0^t \sum_{z \in [D_s(a), D_s(b))} (\Delta \sigma'(z))^2 ds \quad and \quad [U_t] \equiv 0,$$

where  $\sigma'$  denotes the left hand side derivative of  $\sigma$ .

Proof.

$$(Z_t(y) - Z_t(x))^2 = 2 \int_0^t (Z_s(y) - Z_s(x))(\sigma'(D_s(y)) - \sigma'(D_s(x)))dB_s + \int_0^t (\sigma'(D_s(y)) - \sigma'(D_s(x)))^2 ds.$$

For a subdivision  $a = x_0 < x_1 < \cdots < x_r = b$ 

$$\sum_{i} (Z_{t}(x_{i+1}) - Z_{t}(x_{i}))^{2}$$

$$= 2 \int_{0}^{t} \sum_{i} (Z_{s}(x_{i+1}) - Z_{s}(x_{i}))(\sigma'(D_{s}(x_{i+1})) - \sigma'(D_{s}(x_{i})))dB_{s}$$

$$+ \int_{0}^{t} \sum_{i} (\sigma'(D_{s}(x_{i+1})) - \sigma'(D_{s}(x_{i})))^{2}ds.$$
(12)

As  $\sigma$  is the difference of convex functions, its left derivative exists everywhere and is of finite total variation on compact intervals. Denote by  $\mu$  the total variation measure of  $\sigma'$ . Then the integrand in the stochastic integral is bounded by

$$\sup_{s \le t, i} |Z_s(x_{i+1}) - Z_s(x_i)| \mu([\min_{s \le t} D_s(a), \max_{s \le t} D_s(b)]).$$

This upper bound goes to zero almost surely by the continuity of  $(s, x) \mapsto Z_s(x)$ , hence the integral with respect to the Brownian motion goes to zero in probability as the mesh of the subdivision goes to zero. For the second integral, note that

$$\sum_{i} (\sigma'(D_s(x_{i+1})) - \sigma'(D_s(x_i)))^2 \to \sum_{z \in [D_s(a), D_s(b))} (\Delta \sigma'(z))^2$$

and

$$\sum_{i} (\sigma'(D_s(x_{i+1})) - \sigma'(D_s(x_i)))^2 \le (\mu([\min_{s \le t} D_s(a), \max_{s \le t} D_s(b)))^2.$$

So by the dominated convergence theorem the time integral in (12) almost surely goes to

$$\int_0^t \sum_{z \in [D_s(a), D_s(b))} (\Delta \sigma'(z))^2 ds$$

as the mesh of the subdivision goes to zero and the formula for the quadratic variation of  $x \mapsto Z_t(x)$  is proved.

The computation for the  $U_t$  is similar

$$(U_t(y) - U_t(x))^2 = 2 \int_0^t (U_s(y) - U_s(x)) ((\sigma')^2 (D_s(y)) - (\sigma')^2 (D_s(x))) ds$$
  
$$\leq 4L \int_0^t |U_s(y) - U_s(x)| \mu([D_s(x), D_s(y))) ds,$$

where L is the Lipschitz constant for  $\sigma$ , that is  $\sup_{r} |\sigma'(r)|$ , and

$$\sum_{i} (U_t(x_{i+1}) - U_t(x_i))^2 \le 4Lt\mu([\min_{s \le t} D_s(a), \max_{s \le t} D_s(b))) \sup_{s \le t, i} |U_t(x_{i+1}) - U_t(x_i)|.$$

By the continuity of  $(s, x) \mapsto U_s(x)$  this upper bound goes to zero almost surely when the mesh of the subdivision goes to zero.

**Lemma 5.7.** Let  $(X_z)_{z \in [a,b]}$  be a process with continuous sample paths whose quadratic variation  $([X]_z)_{z \in [a,b]}$  exists. If  $h : [a,b] \to \mathbb{R}$  is continuously differentiable, then the quadratic variation of h(X) exists and

$$[h(X)](b) - [h(X)](a) = \int_{a}^{b} (h'(X_{z}))^{2} [X](dz).$$
(13)

*Proof.* When h is linear or continuous and piecewise linear, then (13) follows from the definition of the quadratic variation.

For a subdivision  $\pi = \{t_0 = a < t_1 < \cdots < t_r = b\}$  of [a, b] and for a process Y denote

$$Q_{\pi}^{2}(Y) = \sum_{i} (Y_{t_{i+1}} - Y_{t_{i}})^{2}$$

Note that  $Q_{\pi}(Y)$  is the Euclidean norm of the vector  $(Y_{t_1} - Y_{t_0}, \ldots, Y_{t_r} - Y_{t_{r-1}})$ , so by the triangle inequality, for any processes Y, Z and subdivision  $\pi$ , we have

$$|Q_{\pi}(Y) - Q_{\pi}(Z)| \le Q_{\pi}(Y - Z).$$
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Similarly, if  $\tilde{Q}^2(h) = \int_a^b (h'(Z_z))^2 d[Z](dz)$  denotes the right hand side of (13), then

$$|\tilde{Q}(h_1) - \tilde{Q}(h_2)| \le \tilde{Q}(h_1 - h_2)$$

If h is Lipschitz continuous with Lipschitz constant L, then

$$(h(X_t) - h(X_s))^2 \le L^2 (X_t - X_s)^2,$$

so for any subdivision  $\pi$ 

$$Q_{\pi}(h(X)) \le LQ_{\pi}(X).$$

When h is  $C^1$  and  $\eta > 0$ , then there is a continuous piecewise linear function  $h_{\eta}$  such that  $h - h_{\eta}$  is Lipschitz continuous with Lipschitz constant  $\eta$ . Then

$$\begin{aligned} \left| Q_{\pi}(h(X)) - \tilde{Q}(h) \right| \\ &\leq \left| Q_{\pi}(h(X)) - Q_{\pi}(h_{\eta}(X)) \right| + \left| Q_{\pi}(h_{\eta}(X)) - \tilde{Q}(h_{\eta}) \right| + \left| \tilde{Q}(h_{\eta}) - \tilde{Q}(h) \right| \\ &\leq Q_{\pi}((h - h_{\eta})(X)) + \left| Q_{\pi}(h_{\eta}(X)) - \tilde{Q}(h_{\eta}) \right| + \tilde{Q}((h - h_{\eta})) \\ &\leq \eta Q_{\pi}(X) + \left| Q_{\pi}(h_{\eta}(X)) - \tilde{Q}(h_{\eta}) \right| + \eta ([X]_{b} - [X]_{a})^{1/2}. \end{aligned}$$

Now, let  $(\pi_n)$  be a sequence of subdivisions of [a, b] with  $\operatorname{mesh}(\pi_n) \to 0$ . From the previous estimation, using that the middle term and  $Q_{\pi_n}(X) - ([X]_b - [X]_a)^{1/2}$  goes to zero in probability, we get that

$$\limsup_{n} \mathbb{P}\left(\left|Q_{\pi_{n}}(h(X)) - \tilde{Q}(h)\right| > 2\varepsilon\right) \le \inf_{\eta > 0} \mathbb{P}\left(2\eta([X]_{b} - [X]_{a})^{1/2} > \varepsilon\right) = 0.$$

So  $Q_{\pi_n}(h(X)) \xrightarrow{p} \tilde{Q}(h)$  and  $Q^2_{\pi_n}(h(X)) \xrightarrow{p} \tilde{Q}^2(h)$ , which proves the claim.

**Corollary 5.8.** Under the assumptions of Lemma 5.6 the quadratic variation of  $x \mapsto D'_t(x)$  exists and is given by the formula

$$[D'_t](b) - [D'_t](a) = \int_a^b 4(D'_t(x))^2 [Z_t](dx),$$

where

$$[Z_t](x) - [Z_t](a) = \int_0^t \sum_{z \in [D_s(a), D_s(x))} (\Delta \sigma'(z))^2 ds.$$

**Corollary 5.9.** Suppose that the assumptions of Lemma 5.6 hold. Denote by S the set of discontinuity points of  $\sigma'$  and let C be a connected component of  $\mathbb{R} \setminus \{\sigma = 0\}$ .

On an almost sure event the following holds: if

$$(a,b) \subset C$$
 and  $(\min_{s \leq t} D_s(a), \max_{s \leq t} D_s(b)) \cap S \neq \emptyset$ ,

then the total variation of  $D'_t$  over the interval [a, b] is infinite.

**Corollary 5.10.** Suppose that the assumptions of Lemma 5.6 hold. Denote by S the set of discontinuity points of  $\sigma'$  and let C be a connected component of  $\mathbb{R} \setminus \{\sigma = 0\}$ . Suppose that  $S \cap C \neq \emptyset$ 

Then almost surely there exists a non-empty open interval  $I \subset C$  such that the total variation of  $x \mapsto D'_s(x)$  is infinite on any non-empty subinterval of I.

The same is true for  $(D_s^{-1})'$ .

*Proof.* Let  $z \in S \cap C$ . Since  $\sigma(z) \neq 0$ , we have that  $\min_{t \leq s} D_t(z) < z < \max_{t \leq s} D_t(z)$ . By the continuity of  $(t, x) \mapsto D_t(x)$  there is a < z < b, such that  $\min_{t \leq s} D_s(b) < z < \max_{t \leq s} D_t(a)$ . As  $D_t(x)$  is increasing in x from this we have that for any a < c < d < b

$$\min_{t \le s} D_t(c) < \min_{t \le s} D_t(b) < z < \max_{t \le s} D_t(a) < \max_{t \le s} D_t(d).$$

So, by Corollary 5.9 the total variation of  $D'_s$  on (c, d) is infinite.

Finally, since  $(D_s^{-1})' = 1/(D_s' \circ D_s^{-1})$  and  $D_s$  maps C onto C, the image of I under  $D_s$  is a subinterval of C such that the total variation of  $(D_s^{-1})'$  is infinite on any of its non-empty subintervals.

#### 6. Some technical results

**Lemma 6.1.** Let X be a one dimensional continuous semimartingale and for  $x \in \mathbb{R}$ ,  $\delta > 0$  put

$$\begin{aligned} \sigma_0^{x,\delta} &= 0, \\ \sigma_{2k+1}^{x,\delta} &= \inf\left\{t \ge \sigma_{2k}^{x,\delta} : X_t = x\right\}, \quad k \ge 0, \\ \sigma_{2k+2}^{x,\delta} &= \inf\left\{t \ge \sigma_{2k+1}^{x,\delta} : |X_t - x| = \delta\right\}, \quad k \ge 0, \\ \ell^{\delta}(x,t) &= \inf\left\{k : \sigma_{2k+1}^{x,\delta} > t\right\} = \sum_{k \ge 0} \mathbb{1}_{(\sigma_{2k+1} \le t)}. \end{aligned}$$

Suppose that  $(\rho_K)$  is a sequence of stopping times tending to infinity almost surely as  $K \to \infty$  and  $x \in H(X)$ . Then there is a K > 0, a non-empty open interval I containing x and  $\delta_0 > 0$  such that

$$\inf_{\delta \in (0,\delta_0)} \inf_{y \in I} \delta \mathbb{E} \left( \ell^{\delta}(y,\rho_K) \right) > 0.$$

**Remark.** Note that the definition of  $\ell^{\delta}$  in this claim is slightly different from that of used in Theorem 2.1 and 2.2. In this lemma it is defined for  $x \in \mathbb{R}$ , while in the formulation with the embedded random walk it was only defined for  $x \in \delta \mathbb{Z}$ .

We will denote by  $X = X_0 + M + A$  the semimartingale decomposition of X, that is M is local martingale starting from zero and A is process of finite variation. Also, V is used for the total variation process of A.

The proof of Lemma 6.1 is based on the following estimation.

**Proposition 6.2.** With the notation introduced above let

$$\ell^{\delta}_{+}(x,t) = \sum_{k \ge 0} \mathbb{1}_{(X_{\sigma_{2k} > x})} \mathbb{1}_{(\sigma_{2k+1} \le t)}$$

If  $\rho$  is a stopping time such that  $[M]_{\rho}$  and  $V_{\rho}$  are integrable, then

$$\limsup_{\delta \to 0} \sup_{x \in [a,b]} \mathbb{E}\left(\frac{1}{2}L_{\rho}^{x} - \delta \ell_{+}^{\delta}(x,\rho)\right) \leq \sup_{x \in (a,b]} \mathbb{E}\left(\int_{0}^{\rho} \mathbb{1}_{(X_{s}=x)} dV_{s}\right)$$

where  $(L_t^x)_{x \in \mathbb{R}, t \geq 0}$  is the family of local times for X.

Proof of Lemma 6.1.  $\rho_K$  is a sequence of stopping time tending to infinity almost surely. Since  $\ell^{\delta}(x,t)$  is non-decreasing in t, we can clearly assume  $\rho_K$  is bounded, and  $[M]_{\rho_K}$ ,  $V_{\rho_K}$  are integrable for each K. For  $x \in H(X)$ ,  $\mathbb{E}(L^x_{\rho_K})$ ,  $\mathbb{E}(L^{x-}_{\rho_K}) > 0$  for some K sufficiently large.

By our assumptions on  $\rho_K$  the function  $\mathbb{E}(L_{\rho}^x)$  is right continuous, so there is b > xsuch that  $\inf_{y \in [z,b]} \mathbb{E}(L_{\rho}^y) = c \ge \frac{1}{2}\mathbb{E}(L_{\rho}^x) > 0$ . The formula  $\mu(C) = \mathbb{E}(\int_0^{\rho} \mathbb{1}_{(X_s \in C)} dV_s)$  defines a finite measure on the Borel  $\sigma$ -algebra of the real line, whence there are only finitely many y-s, such that  $\mu(\{y\}) > c/4$ , and taking a smaller b if necessary it is possible to achieve that  $\sup_{y \in (x,b]} \mu(\{y\}) < c/4$ . With this choice

$$\liminf_{\delta \to 0} \inf_{y \in [x,b]} \mathbb{E}\big(\delta \ell^{\delta}(y,\rho)\big) \ge \liminf_{\delta \to 0} \inf_{y \in [x,b]} \mathbb{E}\big(\delta \ell^{\delta}_{+}(y,\rho)\big) > \frac{1}{4}c \ge \frac{1}{8} \mathbb{E}\big(L^{x}_{\rho}\big) > 0$$

by Proposition 6.2.

Similar analysis applies to the left hand side, as  $L_t^{-x}(-X) = L_t^{x-}(X)$ . So from the previous argument we get a < x, such that

$$\liminf_{\delta \to 0} \inf_{y \in (a,x]} \mathbb{E} \left( \delta \ell^{\delta}(y,\rho) \right) > \frac{1}{8} \mathbb{E} \left( L_{\rho}^{x-} \right) > 0$$

Then with this K we have a  $\delta_0$ , such that for a < x < b

$$\inf_{y \in (a,b), \, 0 < \delta < \delta_0} \delta \mathbb{E} \left( \ell^{\delta}(y, \rho_K) \right) > 0,$$

which completes the proof.

Proof of Proposition 6.2. Let

$$\varphi_s^{\delta} = \sum_{k \ge 0} \mathbb{1}_{(\sigma_{2k} \le s \le \sigma_{2k+1})} \mathbb{1}_{(X_{\sigma_{2k}} > x)}.$$

Then

$$\int_0^\rho \varphi_s^\delta dX_s = \sum_{k \ge 0} \mathbb{1}_{(X_{\sigma_{2k}} > x)} (X_{\sigma_{2k+1} \land \rho} - X_{\sigma_{2k} \land \rho})$$

and from the definition of the stopping times  $(\sigma_k)_{k\geq 0}$ , we obtain that

$$|X_{\rho} - x|_{+} - |X_{0} - x|_{+} - \int_{0}^{\rho} \varphi_{s}^{\delta} dX_{s} - \delta \ell_{+}^{\delta}(x, \rho) \le 2\delta.$$

Using the Tanaka formula, we have

$$\frac{1}{2}L_{\rho}^{x} = |X_{\rho} - x|_{+} - |X_{0} - x|_{+} - \int_{0}^{\rho} \mathbb{1}_{(X_{s} > x)} dX_{s}$$

The mean difference between  $\frac{1}{2}L_{\rho}^{x}$  and  $\delta \ell_{+}^{\delta}(x,\rho)$  (using that  $s \mapsto \varphi_{s}^{\delta} - \mathbb{1}_{(X_{s} > x)}$  is bounded and  $E[M]_{\rho_{K}} < \infty$ , so the expected value of the integral w.r.t. M vanishes) is

$$\mathbb{E}\left(\frac{1}{2}L_{\rho}^{x}-\delta\ell_{+}^{\delta}(x,\rho)\right) \leq 2\delta + \mathbb{E}\left(\int_{0}^{\rho}\varphi_{s}^{\delta}-\mathbb{1}_{(X_{s}>x)}dX_{s}\right)$$
$$= 2\delta + \mathbb{E}\left(\int_{0}^{\rho_{K}}\varphi_{s}^{\delta}-\mathbb{1}_{(X_{s}>x)}dA_{s}\right).$$

Recall that V is the total variation process of A. We have the following estimate

$$\left| \mathbb{E} \left( \frac{1}{2} L_{\rho}^{x} - \delta \ell_{+}^{\delta}(x, \rho_{K}) \right) \right| \leq 2\delta + \mathbb{E} \left( \int_{0}^{\rho} \left| \varphi_{s}^{\delta} - \mathbb{1}_{(X_{s} > x)} \right| dV_{s} \right)$$
$$\leq 2\delta + \mathbb{E} \left( \int_{0}^{\rho} \mathbb{1}_{(X_{s} \in (x, x + \delta))} dV_{s} \right). \tag{14}$$

Let g and h be the following functions

$$g(y,\delta) = \mathbb{E}\left(\int_0^\rho \mathbb{1}_{(X_s \in (y,y+\delta))} dV_s\right), \quad h(y) = \mathbb{E}\left(\int_0^\rho \mathbb{1}_{(X_s = y)} dV_s\right).$$

For an interval [a, b) let  $(y_n, \delta_n)_{n \ge 1}$  be a sequence for which

$$\lim_{n \to \infty} g(y_n, \delta_n) = \limsup_{\delta \to 0} \sup_{y \in [a, b)} g(y, \delta)$$

Clearly, we can assume (by passing to a subsequence if necessary) that  $\delta_n \to 0$  and  $y_n$  is convergent. Let  $y^* = \lim_{n \to \infty} y_n$ ,  $y^* \in [a, b]$ . We have the following estimation:

$$g(y_n, \delta_n) \leq \mathbb{E}\left(\int_0^\rho \mathbb{1}_{\{X_s \in (\min(y_n, y^*), \max(y_n + \delta_n, y^* + \delta_n))\}} dV_s\right)$$

If  $y_n < y^*$  only for finitely many n, then for n large enough we have  $y_n \ge y^*$  and

$$\mathbb{E}\left(\int_0^{\rho} \mathbb{1}_{\left(X_s \in (\min(y_n, y^*), \max(y_n + \delta_n, y^* + \delta_n))\right)} dV_s\right) \le \mathbb{E}\left(\int_0^{\rho} \mathbb{1}_{\left(X_s \in (y^*, y^* + \delta_n)\right)} dV_s\right),$$

and by the dominated convergence theorem the right hand side tends to 0. If  $y^* = a$ , then this is the case.

If  $y_n < y^*$  for infinitely many n, then

$$\begin{split} \limsup_{\delta \to 0} \sup_{y \in [a,b)} g(y,\delta) &= \lim_{n \to \infty} g(y_n,\delta_n) \\ &\leq \limsup_{n \to \infty} \mathbb{E} \left( \int_0^{\rho} \mathbb{1}_{(X_s \in (\min(y_n,y^*),\max(y_n+\delta_n,y^*+\delta_n)))} dV_s \right) \\ &\leq \mathbb{E} \left( \int_0^{\rho} \mathbb{1}_{(X_s=y^*)} dV_s \right) = h(y^*). \end{split}$$

In this case  $y^* \in (a, b]$ . So in both cases we have

$$\limsup_{\delta \to 0} \sup_{y \in [a,b)} g(y,\delta) \le \sup_{y \in (a,b]} h(y).$$

By (14) this proves the claim.

Recall that

$$\mu_{\delta,F}(H) = \sum_{x \in H \cap \delta \mathbb{Z}} \frac{1}{\delta} |(\Delta^{\delta} F)(x)|$$

where  $(\Delta^{\delta} F)(x) = F(x+\delta) + F(x-\delta) - 2F(x).$ 24 **Proposition 6.3.** Let  $F : \mathbb{R} \to \mathbb{R}$  be a continuous function and I be an open interval. Then F is the difference of convex functions on I if and only if

$$\limsup_{\delta \to 0} \mu_{\delta,F}(J) < \infty, \quad \text{for all compact intervals } J \subset I.$$
(15)

*Proof.* First suppose that (15) holds for I. It is enough to show that F is a difference of convex functions on compact intervals  $J \subset I$ . Using our assumption we define below two sequences of convex functions  $f_n, g_n : J \to \mathbb{R}$  such that  $F - (f_n - g_n)$  is linear on  $J \cap 2^{-n}\mathbb{Z}$  for each n and  $f_{n_k} \to f$ ,  $g_{n_k} \to g$  pointwise for suitable subsequence  $(n_k)$  of the indices. Then f, g are convex and F - (f - g) is a linear function on J, hence F is a difference of convex functions on J.

Let us now define  $f_n, g_n$ . For  $\delta > 0$  and  $H \in \mathcal{B}(J)$  let

$$\tilde{\mu}_{\delta,+}(H) = \sum_{x \in H \cap \delta \mathbb{Z}} \frac{1}{\delta} |(\Delta^{\delta} F)(x)|_{+}, \qquad \tilde{\mu}_{\delta,-}(H) = \sum_{x \in H \cap \delta \mathbb{Z}} \frac{1}{\delta} |(\Delta^{\delta} F)(x)|_{-}.$$

 $\tilde{\mu}_{\delta,+}, \tilde{\mu}_{\delta,-}$  are finite measures on the Borel  $\sigma$ -algebra of J. Since  $\varphi(x,y) = |x-y|_+$  is a convex function in x for each fixed y, we have that

$$f_{\delta}(x) = \int_{J} \varphi(x, y) \tilde{\mu}_{\delta, +}(dy), \quad g_{\delta}(x) = \int_{J} \varphi(x, y) \tilde{\mu}_{\delta, -}(dy)$$

are convex functions (on  $\mathbb{R}$ ). For  $x, y \in \delta \mathbb{Z}$ , we have that

$$\Delta^{\delta}\varphi(x,y) = \varphi(x+\delta,y) + \varphi(x-\delta,y) - 2\varphi(x,y) = \begin{cases} 0 & x \neq y \\ \delta & x = y \end{cases},$$

which yields for  $x \in J \cap \delta Z$  that

$$\frac{1}{\delta}(\Delta^{\delta}f_{\delta})(x) = \mu_{\delta,+}(\{x\}), \quad \frac{1}{\delta}(\Delta^{\delta}g_{\delta})(x) = \mu_{\delta,-}(\{x\})$$

and

$$\frac{1}{\delta}\Delta^{\delta}(F - (f_{\delta} - g_{\delta}))(x) = 0, \text{ for all } x \in J \cap \delta\mathbb{Z}.$$

We obtained that  $F - (f_{\delta} - g_{\delta})$  is linear on  $J \cap \delta Z$ .

We put  $f_n = f_{2^{-n}}$  and  $g_n = g_{2^{-n}}$ . The families of finite measures  $\{\mu_{2^{-n},+} : n \ge 1\}$ and  $\{\mu_{2^{-n},-} : n \ge 1\}$  are tight as J is compact and

$$\sup_{\delta \in (0,1)} \mu_{\delta,\pm}(J) \le \sup_{\delta \in (0,1)} \mu_{\delta,F}(J) < \infty.$$

Then there is a subsequence  $(n_k)$  of the indices such that  $\mu_{n_k,+}$  and  $\mu_{n_k,-}$  converge weakly to some limit. As  $y \mapsto \varphi(x, y)$  is bounded continuous on J, the sequences  $f_{n_k}(x) = \int \varphi(x, y) \mu_{n_k,+}(dy)$  and  $g_{n_k}(x) = \int \varphi(x, y) \mu_{n_k,-}(dy)$  are pointwise convergent on J, which finishes the proof of the sufficiency of (15).

To prove the necessity of (15) it is enough to consider a convex F. Let  $[a, b] \subset I$  and  $\delta$  so small that  $(a - \delta, b + \delta) \subset I$ . As F is convex on I, we have that  $\Delta^{\delta} F(x) \geq 0$  on [a, b]

and

$$\mu_{\delta,F}([a,b]) = \sum_{x \in J \cap \delta Z} \frac{1}{\delta} (\Delta^{\delta} F)(x) = \frac{F(b'+\delta) - F(b')}{\delta} - \frac{F(a') - F(a'-\delta)}{\delta}$$
$$\leq \frac{F(b+\delta) - F(b)}{\delta} - \frac{F(a) - F(a-\delta)}{\delta},$$

where  $a' = \min[a, b] \cap \delta \mathbb{Z}$  and  $b' = \max[a, b] \cap \delta Z$ . In the first step we used that we have a telescoping sum, while in the last step we used that  $a \leq a' \leq b' \leq b$  and the divided difference  $x \mapsto \frac{1}{\delta}(F(x + \delta) - F(x))$  is increasing in x. Now letting  $\delta \to 0$  gives that

$$\limsup_{\delta \to 0} \mu_{\delta,F}([a,b]) \le F'_+(b) - F'_-(a).$$

This upper bound is finite since F is convex on I and  $[a, b] \subset \operatorname{int} I$ .

#### References

- Bass, R.F., Burdzy, K., 1999. Stochastic bifurcation models. Ann. Probab. 27, 50–108. doi:10.1214/aop/1022677254.
- Çinlar, E., Jacod, J., Protter, P., Sharpe, M.J., 1980. Semimartingales and Markov processes. Z. Wahrsch. Verw. Gebiete 54, 161–219. doi:10.1007/BF00531446.
- Hu, Y., Warren, J., 2000. Ray-Knight theorems related to a stochastic flow. Stochastic Process. Appl. 86, 287–305. doi:10.1016/S0304-4149(99)00098-8.
- Kunita, H., 1986. Lectures on stochastic flows and applications. volume 78 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin.
- Pratelli, M., 2010. A Remark on the 1/*H*-Variation of the Fractional Brownian Motion, in: Séminaire de Probabilités XLIII. Springer Berlin Heidelberg, pp. 215–219. doi:10.1007/978-3-642-15217-7\_8.
- Prokaj, V., Rásonyi, M., Schachermayer, W., 2011. Hiding a constant drift. Ann. Inst. Henri Poincaré Probab. Stat. 47, 498–514. doi:10.1214/10-AIHP363.
- Revuz, D., Yor, M., 1991. Continuous martingales and Brownian motion. volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin.
- Rogers, L.C.G., Walsh, J.B., 1991. A(t, B<sub>t</sub>) is not a semimartingale, in: Çinlar, E., Fitzsimmons, P., Williams, R. (Eds.), Seminar on Stochastic Processes, 1990 (Vancouver, BC, 1990). Birkhäuser Boston, Boston, MA. volume 24 of Progr. Probab., pp. 275–283. doi:10.1007/978-1-4684-0562-0\_15.