A NUMERICAL METHOD FOR SOLVING THE EQUATIONS OF MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID

A Thesis submitted for the Doctorate degree in the

Attila József University

September, 1972

by

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ACKNOWLEDGEMENTS

I would like to thank Prof. Dr. Tamas Frey for proposing the topic and for his helpful discussions and comments to me.

Also I would like to thank Prof. Dr. Laszló Kalmár for the facilities he offered to me in using the Cybernetic Laboratory in Szeged and for guiding my studies.

My thanks also to Dr. András Bleyer, in Budapest, Mr. Zsólt Lázár in Szeged, for their valuable assistance.

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INTRODUCTION

This work describes the development of a computer program for the analysis of the time - dependnet incompressible viscous flow problems.

Mathematically, the problem is that of solving numerically a partial differential equation in three variables containing non-linear terms. The most natural boundary conditions to impose, those in which velocities are prescribed at the boundary, are also among those which are the most difficult to handle computationally. These various difficulties are illustrated in the recent work. It was found necessary to approximate the boundary conditions in a way which affected accuracy, and also to take such small time steps for reasons of stability and accuracy that the computer time become excessive.

The method here depends on the use of the primitive variables - i.e the velocities and the pressure - and is applicable to problems in two and three space dimensions. An analytical disscussion of the properties of the method requires a background of numerical analysis for that reason, we have collected the relevant informations, e.g definitions and theormes in chapter I. In chapter II the general method of solving numerically the Navier - Stokes equations for pressure and velocities in Hydrodynamic is presented. Finally chapter III contains a flow chart and Algol 60 Program for solving the test example given at the end of chapter II. Finally at the end of this work a list of the references used will be given.

CHAPTER I

PRELIMINARIES DEFINITIONS AND NOTATIONS

This chapter iincludes the required theoremes, notations and definitions which we shall need throught this work.

§1. NORMS AND MATRICES

The norm of a matrix is a number assigned to the matrix which is in some sense a measure of the magnitude of the matrix. The norm of A, denoted by $\|A\|$ have the following properties.

(1) $\|A\| \geqq 0$, $\|A\| = 0$ if and only if $A = 0$

(2) $\|CA\| = |C| \, \|A\|$ where C is any real number (1-1)

(3) $\|A+B\| \leqq \|A\| + \|B\|$

(4) $\|AB\| \leqq \|A\| \, \|B\|$

among the many possible ways of defining $\|A\|$ which satisfy (1-1) we consider:

$$\|A\|_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\right)^{1/2} \text{, the Euclidean norm}$$

$$\|A\|_S = \max_{i} \left[\lambda_i(AA^T)\right]^{1/2} \text{, the spectral norm}$$

Both of which is defined for any nxn matrix. In the definition of the spectral norm the notation $\lambda_i(AA^T)$ denotes an eigenvalue of AA^T. For vectors we define the norm in the Euclidian sense as,

$$\|x\| = (x^T x)^{1/2} = |x|$$

The spectral radius is defined, by,

$$\rho(A) = \max_i |\lambda_i(A)| \quad \text{the maximum modulus eigenvalue of the matrix } A.$$

Thus $$\|A\| = \left[\rho(AA^T)\right]^{1/2}$$

when A is symmetric, $\|A\| = \rho(A)$

Theorem 1.1

If A is the tridiagonal matrix,

$$\begin{bmatrix} a & b & & & 0 \\ c & a & b & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ 0 & & \cdot & \cdot & b \\ & & & c & a \end{bmatrix}$$ where a, b and c are

real and $bc > 0$, then the eigenvalues of A are givenby,

$$\lambda_m = a + 2\sqrt{bc}\,\cos\frac{m\pi}{n+1}, \quad (m = 1, 2, \ldots, n)$$

Theorem 1.2

For any matrix A, $\|A\| \geq \rho(A)$, if A is symmetric, $\|A\| = \rho(A)$

Def. The matrix A is convergent to zero if the sequance of matrices $A, A^2, A^3, \ldots$ converges to the null matrix 0.

Theorem 1.3

$$\lim_{r\to\infty} A^r = 0 \text{ if } \|A\| < 1$$

proof:

$$\|A^r\| = \|AA^{r-1}\| \leq \|A\|\,\|A^{r-1}\| \leq \|A\|^2\,\|A^{r-2}\| \ldots \leq \|A\|^r.$$

Theorem 1.4

$\lim_{r\to\infty} A^r = 0$ if and only if $|\lambda_i| < 1$ for all eigenvalues λ_i $(i = 1, 2, \ldots, n)$ of A.

proof: Consider the Jordan canonical form of A. A Jordan submatrix of A is of the form,

$$\begin{bmatrix} \lambda_i & & & & 0 \\ 1 & \lambda_i & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & \lambda_i \end{bmatrix}$$

where λ_i is an eigenvalue of A. If the Jordan submatrix is raised to the power r, then the result tends to the null matrix as $r \to \infty$; if and only if $|\lambda_i| < 1$.

Theorem 1.5

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$, more generally if p(x) is a polynomial, the eigenvalues of p(A) are $p(\lambda_1), \ldots, p(\lambda_n)$.

Theorem 1.6

If A is real and symmetric, all eigenvalues and eigenvectors are real. Moreover, eigenvectors corresponding to distinct eigenvalues are orthogonal and the left eigenvector corresponding to the eigenvector x_i is x_i^T.

Theorem 1.7

Any similarity transformation PAP^{-1} applied to A leaves the eigenvalues of the matrix unchanged.

proof: Let λ be an eigenvalue of A and x the associated eigenvector then,

$$Ax = \lambda x, \text{ so that }, PAx = \lambda Px \qquad (1\text{-}2)$$

Let $y = Px$ so that $x = P^{-1}y$,

subistituting in (1.2) gives

$$PAP^{-1}y = \lambda y$$

Thus λ is an eigenvalue of PAP^{-1} and y is the associated eigenvector.

Theorem 1.8

Let $f(\lambda) = |A - \lambda I| = 0$

be the characteristic equation of A then $f(A) = 0$

Theorem 1.9

Given an arbitrary matrix A, there exists a non-singular matrix P, whose elements may be complex, such that

$$P^{-1}AP = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix} \qquad (1\text{-}3)$$

Where J_k , $k = 1, \ldots, M \leq n$ is a matrix with an eigenvalue λ_i of A on its main diagonal and 1'S on the diagonal above the main diagonal.

Thus

$$J_k = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix}$$

Note that a given eigenvalue may appear as the diagonal element of more than one J_k. The matrix in (1-3) is called the Jordan canonical form of A. The determinants

$$|(J_k - \lambda I)| = (\lambda_i - \lambda)^{\nu_k}$$

where ν_k is the order of J_k are called the elementary divisors of A.

§2. THE SUCCESSIVE OVERRELAXATION METHOD FOR SOLVING A SET OF LINEAR ALGEBRIC EQUATIONS

An implicit finite differente formula which approximates a partial differential equation in any number of space variables involves several grid points at the advanced time level. So it is required to find the solution of the equations which arises there. A set of simultanous linear equations, which can be written in the form.

$$P\,\underline{x} = \underline{c} \quad , \quad (|P| \neq 0) \tag{2.1}$$

where P is a square matrix, with no zero on the main diagonal, $\underline{x}$, $\underline{c}$ are vectors requiers to be solved at each time step.

Equation (2.1) can be written in the form

$$A\,\underline{x} = \underline{b} \quad , \quad (|A| \neq 0) \tag{2.2}$$

where $A = DP$, $\underline{b} = D\underline{c}$

and D is a diagonal matrix chosen so that the elements of the principal diagonal of A are unity. A can be written in the form

$$A = I - L - U$$

where I is the unit matrix, and L, U are the lower and upper triangular matrices respectively.

An iterative process is used to solve (2-2). We begin by initial value $\underline{x}_0$ and is successively improved by the iterative process until it is arbitrary close to $\underline{x}$. (2-2) can b written in the form

$$(I-L-U)\underline{x} = \underline{b} \qquad (2\text{-}3)$$

and $\underline{x}_i$, $\underline{x}_{i+1}$ are successive approximate solutions of equation (2-2), then using of equation (2-3) give,

$$I\underline{x}_{i+1} = (L + U)\,\underline{x}_i + \underline{b} \quad ,(i=1,2,\ldots)$$

which is the point Jacobi iterative method, or

$$(I - L)\,\underline{x}_{i+1} = U\underline{x}_i + \underline{b}$$

which is the Gauss-Seidel iterative method. These two methods are special cases of the general iterative process

$$\underline{x}_{i+1} = B\underline{x}_i + \underline{c} \quad , (i = 1, 2,\ldots) \qquad (2\text{-}4)$$

where $B = L + U$ and $(I-L)^{-1}$ in the point Jacobi and Gauss-Seidel processes respectively.

An error in the $i^{\underline{th}}$ iterate is

$$\underline{e}_i = \underline{x}_i - \underline{x} \quad ,(i = 0, 1, 2,\ldots)$$

Then

$$\underline{e}_{i+1} + \underline{x} = B\underline{e}_1 + B\underline{x} + \underline{c}$$

$$\underline{e}_{i+1} = B\underline{e}_i \quad (i = 0, 1, 2,\ldots)$$

then

$$\underline{e}_i = B^i \underline{e}_o \quad , \quad \text{and so,}$$

$$\underline{e}_i \rightarrow 0, \quad \text{as } i \rightarrow \infty, \text{ if } B^i \rightarrow 0$$

where 0 is the null matrix B is convergent (i.e $B^i \rightarrow 0$ as $i \rightarrow \infty$) if and only if $\rho(B) < 1$ (see theorm 1-2).
Thus the iteration (2.4) is convergent if and only if ($\rho(B) < 1$.
Thus to solve equations (2-2) by the method of successive overrelaxation, we introduce,

$$\tilde{\underline{x}}_{i+1} = L\underline{x}_{i+1} + U\underline{x}_i + \underline{b} \qquad (2\text{-}5)$$

where

$$\underline{x}_{i+1} = \omega \tilde{\underline{x}}_{i+1} + (1-\omega)\underline{x}_i \qquad (2\text{-}6)$$

where $\omega > 0$ an arbitrary parameter, independent of i, called the relaxation factor.
Elimination of $\tilde{\underline{x}}_{i+1}$ between (2-5), (2-6), gives,

$$(I - \omega L)\underline{x}_{i+1} = \left[U + (1-\omega)I\right]\underline{x}_i + \omega \underline{b}$$

i.e

$$\underline{x}_{i+1} = (I-\omega L)^{-1}\left[\omega U + (1-\omega)I\right]\underline{x}_i + \omega(I-\omega L)^{-}\underline{b} \qquad (2\text{-}7)$$

This is an iterative method of successive overrelaxation and simillar to (2.4) with

$$B \equiv (I-\omega L)^{-1}\left[\omega U + (1-\omega)I\right]$$

so the method of successive overrelaxation will be convergent if and only if,

$$\rho\left[(I - \omega L)^{-1}\left\{\omega U + (1-\omega)I\right\}\right] < 1$$

we can write

$$H_\omega = (I-\omega L)^{-1}\left\{\omega U + (1-\omega)I\right\}$$

Thus, if λ is an eigenvalue of H_ω , then,

$$|H_\omega - \lambda I| = 0 \qquad (2-8)$$

So, we shall calculate the maximum eigenvalue of H_ω from equation (2-8) and minimize this with respect to ω.

Definition 2.1

A matrix is two-Cyclic if by a siutable permutation of its rows and corresponding columns, it can be written in the form,

$$\begin{bmatrix} I & F \\ G & I \end{bmatrix}$$

where I is a square unit matrix, and F, G are rectangular matrices.

Definition 2.2

A matrix is weakly two-cyclic if by a siutable permutation of its rows and corresponding columns, it can be written in the form

$$\begin{bmatrix} 0 & F \\ G & 0 \end{bmatrix},$$

where 0 is a square null matrix.

Definition 2.3

If the matrix (I-L-U) is two-cyclic, then it is consistently ordered if all the eigenvalues of the matrix

$$\alpha L + \frac{1}{\alpha} U \qquad (\alpha \neq 0)$$

are independent of α.

Thus returning to equation (2-8), it can be written in the form,

$$\left| (I - \omega L)^{-1} \{ I + \omega(U - I) \} - \lambda I \right| = 0$$

$$\left| \{ I + \omega(U - I) \} - \lambda(I - \omega L) \right| = 0$$

$$\left| (U + \omega L) - \frac{\lambda + \omega - 1}{\omega} I \right| = 0$$

$$\left| \lambda^{1/2} (\lambda^{1/2} L + \lambda^{-1/2} U) - \frac{\lambda + \omega - 1}{\omega} I \right| = 0$$

$$\left| \lambda^{1/2} L + \lambda^{-1/2} U) - \frac{\lambda + \omega - 1}{\lambda^{1/2} \omega} I \right| = 0$$

If I - L - U is two-cyclic and consistently ordered, then

$$\left| (L + U) - \frac{\lambda + \omega - 1}{\lambda^{1/2} \omega} I \right| = 0$$

Thus for any eigenvalue λ of the successive overrelaxation H_ω, there corresponds an eigenvalue μ of the point Jacobi matrix (L + U), where

$$\mu = \frac{\lambda + \omega - 1}{\omega \lambda^{1/2}} \qquad (2\text{-}9)$$

Equation (2-9) conects the eigenvalue of the successive overrelaxation matrix with the eigenvalues of the point Jacobi matrix, provided that I-L-U is two-cyclic and consistently ordered.

If the matrix I-L-U is symmetric as well as being two cyclic and consistently ordered, and so (L + U) is symmetric and hence the eigenvalues of (L + U) are real.

Since (L+U) is weakly two-cyclic, its non-zero eigenvalues occure in pairs different in sign.

i.e

$$-\rho(L + U) \leq \mu \leq \rho(L + U)$$

interchanging rows and corresponding columns of (L +U), it can be written as,

$$\begin{bmatrix} O_1 & F \\ G & O_2 \end{bmatrix},$$

where O_1, O_2, are square matrices of order r,s respectively and (L + U) is square matrix of order (r+s). Since the interchanging of rows and columns does not affect the eigenvalues of a matrix, the eigenvalues of (L + U) are given by,

$$\begin{bmatrix} -\mu I_1 & F \\ G & -\mu I_2 \end{bmatrix} = 0$$

where I_1, I_2 are unit matrices of order r and s respectively. Thus

$$\begin{bmatrix} \mu I_1 & F \\ G & \mu I_2 \end{bmatrix} = 0$$

by multiplyaing the first r rows and the last s columns of the determinant by -1. This shows also that $-\mu$ is also an eigenvalues of (L +U). We assume that the point Jacobi method is convergent and hence,

$$0 < \rho(L + U) < 1$$

Also since I - L - U is a consistently ordered two-cyclic matrix, the Gauss-Seidel method is also convergent. From equation (2-9) we consider for a given value of μ in the range

$$0 < \mu \leq \rho(L + U) < 1$$

the two functions of λ

$$f_{\omega}(\lambda) = \frac{\lambda + \omega - 1}{\omega}, \quad g(\lambda) = \mu \lambda^{1/2}$$

These function can be shown in the figure (1), where $f_{\omega}(\lambda)$ is a straight line passing through (1,1) and $g(\lambda)$ is a parabola.

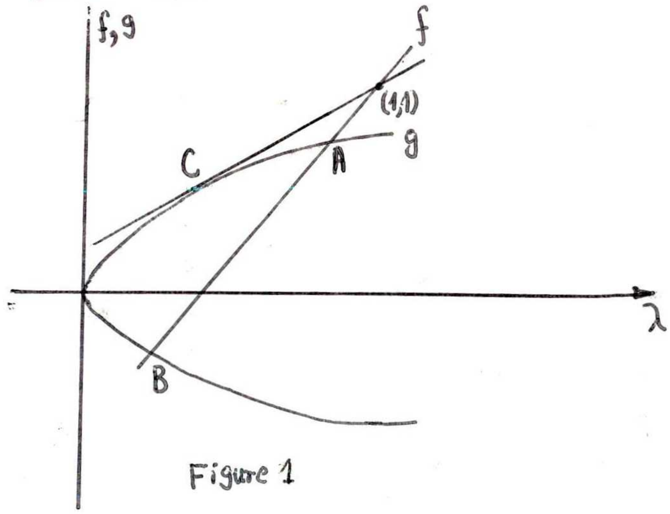

Figure 1

Thus equation (2-9) geometrically represents the intersection of the curves $f_{\omega}(\lambda)$ and $g(\lambda)$ with the two values of λ at the points of intersection A and B given by

$$\lambda^2 + 2\left[(\omega-1) - 1/2\,\mu^2\omega^2\right]\lambda + (\omega-1)^2 = 0$$

i.e

$$\lambda = 1/2\mu^2\omega^2 - (\omega-1) \pm \mu\omega\left[1/4\mu^2\omega^2 - (\omega-1)\right]^{1/2}$$

It is clear that the large abscissa of the two points of intersection decreases with increasing ω, until eventually $f_\omega(\lambda)$ becomes a tangent to $g(\lambda)$ at the point C. Thus

$$1/4\mu^2\omega^2 - \omega + 1 = 0$$

i.e

$$\omega = \frac{1 \pm (1-\mu^2)^{1/2}}{1/2\mu^2}$$

$$\omega = \frac{2}{1 \mp (1-\mu^2)^{1/2}}$$

The range of ω must include $\omega = 1$, and so, we have

$$\tilde{\omega} = \frac{2}{1 + (1-\mu^2)^{1/2}} \qquad (2\text{-}10)$$

if $\omega > \tilde{\omega}$, λ has two cojugate complex roots,

$$\lambda = \frac{1}{2}\mu^2\omega^2 - (\omega - 1) \pm i\mu\omega\left\{(\omega-1) - \frac{1}{4}\mu^2\omega^2\right\}^{1/2}$$

Thus

$$|\lambda| = \omega - 1$$

Thus the minimum value of λ is $\tilde{\lambda} = \tilde{\omega} - 1$

where $\tilde{\omega}$ is given by (2-10) and μ is the eigenvalue of $(L + U)$ in the range

$$0 < \mu \leq \rho(L+U) < 1,$$

since
$$g(\lambda) = \rho(L + U)\lambda^{1/2}$$

is the envelope of all the curves $g(\lambda) = \mu\lambda^{1/2}$, where

$$0 < \mu \leq \rho\,(L + U) < 1,$$

it follows that,

$$\min_{\omega} \rho(H_{\omega}) = \rho(H_{opt}) = \omega_{opt} - 1 \qquad (2\text{-}11)$$

where ω_{opt} is given by,

$$\omega_{opt} = \frac{2}{1 + (1 - \mu^2_{opt})^{1/2}}$$

$$\mu_{opt} = \rho(L + U)$$

Thus we found the value of ω, given by (2-11), which minimizes the maximum modulus eigenvalue of H_{ω}. Also since the point Jacobi method is convergent if $0 < \rho(L + U) < 1$ and so from equation (2-11) it follows that

$$1 < \omega_{opt} < 2$$

and also from (2-10)

$$0 < \rho(H_{\omega_{opt}}) < 1$$

§3. SOME NOTES ABOUT THE ITERATIVE METHOD FOR SOLVING PARTIAL DIFFERENCE EQUATIONS

In the numerical solution by finite differences of boundary value problems involving partial differential equations, one is led to consider linear systems of high order of the form

$$\sum_{j=1}^{n} a_{ij}\, u_j + d_i = 0 \quad (i=1,2,\ldots,n) \qquad (3\text{-}1)$$

where $u_1, u_2, \ldots, u_n$ are unknown and where the real numbers a_{ij} and d_i are known. The coefficients a_{ij} satisfy the conditions

(a) $|a_{ii}| \geqq \sum_{j=1,\, j\neq i}^{n} |a_{ij}|$, and for some i the inequality holds

(b) Given any two nonempty disjoint subsets S and T of W, the set of the first n positive integers such that $S \cup T = W$, there exists $a_{ij} \neq 0$ such that $i \in S$ and $j \in T$. (3-2)

I tcan be shown that the determinant of the matrix $A = (a_{ij})$ does not vanish. Moreover, if the matrix $A^* = (a^*_{ij})$ is symmetric, where $a^*_{ij} = a_{ii} a_{ij} / |a_{ii}|$, $(i,j = 1,2,\ldots,n)$, then A^* is positive definite. For if λ is non positive real numbers, then the matrix $A^* - \lambda I$, where I is the identity matrix, also satisfies (3-2) and hence its determinant can not vanish.

There fore all eigenvalues of $\overset{*}{A}$ are positive, and $\overset{*}{A}$ is positive definite. On the other hand if $\overset{*}{A}$ is positive definite then $a_{ii} \neq 0$, $(i=1,2,\ldots,n)$. An appropriate method for solving equations (3-1) numerically, is that of systematic iteration, which is better for computer. We shall consider linear systems such that either the matrix A satisfies conditions (3-2) or such that the matrix $\overset{*}{A}$ is positive definite. In order to define the iterative methods it is necessary that $a_{ii} \neq 0$ $(i=1,2,\ldots,n)$, we shall assume that $a_{ii} > 0$, $(i=1,2,\ldots,n)$ also the matrix A has property (A): there exist two disjoint subsets S and T of W, the set of the first n integers, such that $S \cup T = W$ and if $a_{ij} \neq 0$ then $i = j$ or $i \in S$ and $j \in T$ or $i \in T$ and $j \in S$. This is the <u>Younge's condition</u> for the matrix A.

A short summery will be given here for the solution of linear systems derived from boundary value-problems, the matrix of which satisfies (3-1) and has property (A).

An iterative method, which converges fastes than the usual methods will given. We assume that the rows and columns of A are arranged in the ordering σ.

$$u_i^{m+1} = \omega\left\{\sum_{j=1}^{i-1} b_{ij} u_j^{m+1} + \sum_{j=i+1}^{n} b_{ij} u_j^{m} + c_i\right\} - (\omega-1) u_i^m$$

$$(m \geq 0,\ i=1,2,\ldots,n) \quad (3\text{-}3)$$

where u_i^0 is arbitrary $(i=1,2,\ldots,n)$, and

$$b_{ij} = \begin{cases} -a_{ij}/a_{ii} & (i \neq j) \\ 0 & (i=j) \end{cases} \tag{3-4}$$

and

$$c_i = -d_i/a_{ii} \qquad (i=1,2,\ldots,n)$$

Equation (3-3) can be written in the form

$$u^{m+1} = L_{\sigma,\omega}[u^m] + f \qquad , m \geqq 0 \tag{3-5}$$

where $u^m = (u_1^m, u_2^m, \ldots, u_n^m)$, $f = (f_1, f_2, \ldots, f_n)$, f is fixed, and $L_{\sigma,\omega}$ is a linear operator. σ denotes the ordering of the equations, and ω is the relaxation factor. This is the method of successive overrelaxation. As we show in §2. that if A has property (A), then there exist certain orderings σ such that for all ω a relation holds between the eigenvalues of $L_{\sigma,\omega}$ and the eigenvalues of the matrix $B = (b_{ij})$ defined by (3-4). If μ denotes the spectral norm of B, i.e the maximum of modulii of the eigenvalues of B, then $L_{\sigma,\omega}$ converges if and only if $\mu < 1$ (the Gauss-Seidel method). Conditions (3.2) imply $\mu < 1$.

If A is assumed to be symmetric and have property (A) then $\mu < 1$ if and only if A is positive definite. The optimum relaxation factor ω_{opt} is given by,

$$\omega_{opt}^2 \mu^2 - 4(\omega_{opt} - 1) = 0 \tag{3-6}$$

$$\omega_{opt} > 2$$

For more details and complete proves of the following theormes see [9].

Theorem 3.1

A matrix A has property (A) if and only if there exists a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ with integral components such that if $a_{ij} \neq 0$ and $i \neq j$ then $|\gamma_i - \gamma_j| = 1$

Theorem 3.2

Let A be an n x n matrix with property (A) and with a consistent ordering of rows and coulmns. If the elements of $A' = (a'_{ij})$ and $A'' = (a''_{ij})$ are defined by

$$a'_{ij} = \begin{cases} a_{ij} & (i \leq j) \\ \lambda a_{ij} & (i > j) \end{cases},$$

$$a''_{ij} = \begin{cases} a_{ij} & (i = j) \\ \lambda^{1/2} a_{ij} & (i \neq j) \end{cases}$$

Then for all λ we have

$$|A'| = |A''|$$

Theorem 3.3

Let A denote a matrix with property (A), and let σ denote a consistent ordering. If $\omega \neq 0$, and if λ is a non-zero eigenvalue of $L_{\sigma,\omega}$ and if μ satisfies,

$$(\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda \qquad (3\text{-}7)$$

Then μ is an eigenvalue of B. On the other hand if μ is an eigenvalue of B, and if λ satisfies (3-7), then λ is an eigenvalue of $L_{\sigma,\omega}$, where B is given by (3-4), and $L_{\sigma,\omega}$ is defined by (3-5).

To prove this theorem we shall need the following Lemma, and corrollaries.

Lemma:

If μ is a k-fold non-zero eigenvalue of B, then $(-\mu)$ is a k-fold eigenvalue of B.

Corrollary 3.1

If μ is an eigenvalue of B, then μ^2 is an eigenvalue of $L_{\sigma,1}$ (Gauss-Siedel), if λ is a non zero eigenvalue of $L_{\sigma,1}$ and if $\mu^2 = \lambda$, then μ is an eigenvalue of B.

Corrollary 3.2

If A is symmetric, then the method of simultanous displacement converges if and only if A is positive definite.

Corrollary 3.3

If A is symmetric, then there exists ω such that $L_{\sigma,\omega}$ converges if and only if A is positive definite.

Theorem 3.4

Let $\bar{\mu}$ and $\bar{\lambda}(\omega)$ denote respectively the spectral norms of B and $L_{\sigma,\omega}$. If ω_{opt} which satisfies

$$\omega_{opt}^2 \mu^2 - 4(\omega_{opt} - 1) = 0 \qquad , \omega_{opt} < 2,$$

where ω_{opt}, the optimum relaxation factor, then the rate of convergence of $L_{\sigma,\omega_{opt}}$ is given by,

$$R(L_{\sigma,\omega_{opt}}) = -2 \log \frac{\mu}{1+(1-\mu^2)^{1/2}}$$

and for all real ω such that $\omega \neq \omega_{opt}$,

$$R(L_{\sigma,\omega}) < R(L_{\sigma,\omega_{opt}})$$

§4. GARABEDIAN METHOD FOR THE ESTIMATION OF THE RELAXATION FACTOR FOR SMALL MESH SIZE

Consider the Laplace difference equation for an unknown function u of two independent variables in a region $\mathcal{D}$ covered by a mesh with h units spaced apart. We use the subscripts p, q to the location of the node points, and superscript n to indicate steps in the relaxation process, so that the method of successive overrelaxation can be described by the equation.

$$4(u^{n+1}_{q,r} - u^{n}_{q,r}) = \omega(u^{n+1}_{q-1,r} + u^{n+1}_{q,r-1} + u^{n}_{q+1,r} + u^{n}_{q,r+1} - 4u^{n}_{q,r}) \quad (4\text{-}1)$$

where ω is the relaxation factor, we express ω in the form,

$$\omega = \frac{2}{1 + Ch} \quad (4\text{-}2)$$

where C is any positive value, and constant, if we rearrange (4-1), we get,

$$\frac{u^n_{q-1,r} + u^n_{q,r-1} + u^n_{q+1,r} + u^n_{q,r+1} - 4u^n_{q,r}}{h^2} =$$

$$\frac{u^{n+1}_{q,r} - u^n_{q,r} - u^{n+1}_{q-1,r} + u^n_{q-1,r}}{h^2} + \frac{u^{n+1}_{q,r} - u^n_{q,r} - u^{n+1}_{q,r-1} + u^n_{q,r-1}}{h^2} +$$

$$+ 2\,C\,\frac{u^{n+1}_{q,r} - u^n_{q,r}}{h} \tag{4-3}$$

by refering the index n as time variable, and that it indicate the location of new net points spaced at time intervals equall to the original mesh size h, it is known that (4-3) is the difference analogue of the hyperbolic partial differential equation.

$$u_{xx} + u_{yy} = u_{xt} + u_{yt} + 2Cu_t \tag{4-4}$$

where u_{xx}, u_{yy} denotes differention with respect to x and y respectively.

Thus for small values of h the convergence of the iterative scheme (4-1) can be investigated by an analysis of the decay of time-dependent terms in the solution of (4-4).

The substitution $s = t + x/2 + y/2$, makes (4-4) in a canonical form,

$$u_{xx} + u_{yy} - \frac{1}{2} u_{ss} - 2Cu_s = 0 \tag{4-5}$$

For a fixed set of boundary conditions, the method of separation of variables gives the representation,

$$u = U_o(x,y) + \sum_{m=1}^{\infty}\left[a_m e^{-(p_m s)} + b_m e^{(-q_m s)}\right]U_m(x,y) \quad (4-6)$$

for the solution of (4-5), where U_o is the steady-state solution, where a_m, b_m are Fourier coefficients, where

$$p_m = 2C-(4C^2-2k_m^2)^{1/2}, \quad q_m = 2C+(4C-2k_m^2)^{1/2} \quad (4-7)$$

where U_m and k_m^2 are the eigenfunctions and eigenvalues of the equation,

$$\nabla^2 U_m + k_m^2 U_m = 0 \quad (4-8)$$

with homogenous boundary conditions,

$$p = Re\left[p_1\right] = Re\left[2C-(4C^2-2k_1^2)^{1/2}\right] \quad (4-9)$$

corresponding to the lowest eigenvalue k_1^2, governs the rate of convergence of the terms on the right in (4-6) with in creasing time t.

By (4-9) the choice of the positive constant C which maximizes p and hence yields the most rapied convergence is clearly $C = k_1/2^{1/2}$, and if A denotes the area of the region $\mathfrak{D}$, it can be shown that,

$$k_1 A^{1/2} \geqq k\,\pi^{1/2} \quad (4-10)$$

where k = 2.405 denotes the first root of the Beesel function of the first kind of order zero. Thus the good approximate formula for the relaxation ω is,

$$\omega = \frac{2}{1+(\pi/2A)^{1/2}\,kh} = \frac{2}{1+3.014\ h/A^{1/2}} \tag{4-11}$$

This approach is given in the case of five-point Laplace difference equation, an approach to nine-point Laplace difference equation can also be given.

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CHAPTER II

A NUMERICAL METHOD FOR SOLVING THE EQUATIONS OF MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID.

I INTRODUCTION

The equations of motion of an incompressible fluid are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + E_i ,$$

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad , \quad \nabla^2 \equiv \sum_j \frac{\partial^2}{\partial x_j^2}$$

where u_i are the velocity components, p is the pressure, ρ_0 is the density, E_i are the components of the external forces per unit mass, ν is the coefficient of the kinematic viscosity, t is the time, $i,j = 1,2,3$ $x_{i,j}$ denotes the space coordinates, the summation convention is used in the equations.

We begin by using the method of dimensionalless analysis, writing

$$u_i' = \frac{u_i}{U} , \quad x_i' = \frac{x_i}{X} , \quad p' = \left(\frac{X}{\rho_0 \nu U} \right) p$$

$$E_i' = \left(\frac{\nu U}{X^2}\right) E_i\ , \quad t' = \left(\frac{\nu}{X^2}\right) t,$$

where U is a reference velocity, and X is a reference lenght, the equations become

$$\frac{\partial u_i}{\partial t} + R\, u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nabla^2 u_i + E_i \qquad (1)$$

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (2)$$

where $R = \frac{UX}{\nu}$ is the Ryenolds number. We now try to introduce a finite difference method for solving these equations in a bounded region $\mathcal{D}$, in either two or three dimentional space. The basic feature of this method lies in the use of equations (1) and (2) rather than higher-order derived equations.
This makes it possible to solve the equations and to satisfy the imposed boundary conditions. We achive adequate computational efficiency, even in problems of three dimensions and space variables.

<u>The princibles of the used method:</u>

Equation (1) can be written in the form

$$\frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i} = \mathcal{F}_i\, u \qquad (1)'$$

where $\mathcal{F}_i$ u depends on u_i and E_i, but not on p, equation (2) can be differentiated to give

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t}\right) = 0 \qquad (2)'$$

The present method can be summerized as follows;
the time t is discretized, and at every time step $\mathcal{F}_i u$ is evaluated, then it is decomposed into the sum of a vector with zero divergence and a vector with zero curl. The component with zero divergence is $\frac{\partial u_i}{\partial t}$ which can be used to obtain u_i at the next time level, and the component with zero curl is $\frac{\partial p}{\partial x_i}$.
This decomposition exists and is uniqually determined whenever the initial value problem for the Navier-Stokes equations is well posed.
The existence and uniquness proofs for the solution of these equations can be seen in[1].
Let u_i, p denote only the solution of (1) but also its discrete approximation, and let $D u = 0$ be a difference approximation to $\frac{\partial u_j}{\partial x_j} = 0$.
It is assumed that at time $t = n\Delta t$ a velocity and pressure fields u_i^n , p^n are given such that $Du^n = 0$.
The method used is to evaluate u_i^{n+1} , p^{n+1} from equation (1) so that $Du^{n+1} = 0$.
Let $Tu_i = bu_i^{n+1} - Bu_i$ approximate $\frac{\partial u_i}{\partial t}$, where b is a constant and Bu_i is a suitable Linear Combination of $u_i^{n-j}, j \geqq 0$. $\left[\text{e.g. } \frac{\partial u^n}{\partial t} = \frac{u^{n+1}-u^{n-1}}{2\Delta t} - \frac{u^{n+1}-2u^n+2u^{n-1}-u^{n-2}}{12\Delta t} + O(\Delta t^4)\right]$.

An auxilairy field u_i^{aux} is first evaluated throught,

$$b\, u_i^{aux} - B\, u_i = F_i\, u \qquad (3)$$

where $F_i u$ approximate $\mathcal{F}_i\, u$. u_i^{aux} differs from u_i^{n+1} because the pressure term and equation (2) have not been taken into account. u_i^{aux} may be evaluated by an implicit scheme, i.e $F_i u$ may depend on u_i^n, u_i^{aux} and intermediate fields, say u_i^{*}, u_i^{**}.

$b\, u_i^{aux} - B\, u_i$ now approximates $\mathcal{F}_i\, u$ within an error which may depend on Δt.

Let $G_i p$ approximates $\frac{\partial p}{\partial x_i}$. To obtain u_i^{n+1} , p^{n+1} it is necessary to perform the decomposition

$$F_i\, u = b u_i^{aux} - B u_i = T\, u_i + G_i\, p^{n+1}, \qquad (3)'$$

$$D(Tu) = 0$$

It is however, assumed that $Du^{n-j} = 0, \quad j \geqq 0$.

Substituting the value of $T\, u_i$ into equation $(3)'$, we obtain

$$u_i^{aux} = u_i^{n+1} + b^{-1}\, G_i p^{n+1} \qquad (4)$$

where $Du_i^{n+1} = 0$, and u_i^{n+1} satisfies the prescribed boundary conditions. Since p^n is usually avaiable and is a good first guess for the values of p^{n+1} , the decomposition (4) is probably best done by iteration. For that purpose, we introduce the following iteration scheme:

$$u_i^{n+1,m+1} = u_i^{aux} - b^{-1} G_i^m p \quad , m \geqq 1 \qquad (5\ a)$$

$$p^{n+1,m+1} = p^{n+1,m} - \lambda D u^{n+1,m+1} \quad , m \geqq 1 \qquad (5\ b)$$

where λ is a parameter, $u_i^{n+1,m+1}$ and $p^{n+1,m+1}$ are successive approximations to $u_i^{n+1,m}$ and $p^{n+1,m}$ and G_i^m p is a function of $p^{n+1,m+1}$ and $p^{n+1,m}$ which converges to $G_i\ p^{n+1}$ as $\left|p^{n+1,m+1} - p^{n+1,m}\right| \xrightarrow[m\to\infty]{} \text{zero}$.

We start by assuming that,

$$p^{n+1,1} = p^n \qquad (5\ c)$$

The iterations (5 a) are to be performed in the interior of $\mathfrak{D}$, and the iterations (5 b) in the interior of $\mathfrak{D}$ and on its boundary.

It is clear that (5 a) tends to (4) if the iterations converge.

G_i^m p is used instead og $G_i p$ in (5 a) to improve the rate of convergence of the iterations. A detailed discussion will be given in a later section.

The form of equation (5 b) was suggested by experience with the artifitial compressibility method [2] , where for the perpose of finding steady state solutions of equations (1) and (2), p was related to u_i by the equation,

$$\frac{\partial p}{\partial t} = \text{const}\ \frac{\partial u_i}{\partial x_j}$$

when for some ℓ and small predetermined constant ε ,

$$\max_{\mathfrak{D}} \left|p^{n+1,\ell+1} - p^{n+1,\ell}\right|$$

we set $\quad u_i^{n+1} = u_i^{n+1,\ell+1} \ , \quad p^{n+1} = p^{n+1,\ell+1}$

The iteration (5) ensure that equation (1) including the pressure term is satisfied inside $\mathfrak{D}$, and equation (2) is satisfied inside $\mathfrak{D}$ and on its boundary.

Now we try to find specific schemes for evaluating u_i^{aux} and specific representations for Du, $G_i p$, $G_i^m p$, many other schemes and representations can be used [7]. The method which will be presented is efficient, and suitable mainly for problems in which the boundary data are smooth and the domain has a simple shape.

<u>Evaluation of u_i^{aux}</u> ,

Schemes for evaluating u_i^{aux}, defined by (3) will be presented here.

Equation (3) represents one step in time for the solution of the equation

$$\frac{\partial u_i}{\partial t} = \mathcal{F}_i u$$

We can use a combined DuFort-Frankel scheme, in which the time and first space derivatives were approximated by centered differences, and a second derivative such as $\frac{\partial^2 u}{\partial x_1^2}$ was replaced by

$$\frac{1}{\Delta x_1^2}\left(u_{q+1}^n + u_{q-1}^n - u_q^{n+1} - u_q^{n-1}\right) \ , \ u_q^n \equiv u(q\Delta x, n\Delta t)$$

This scheme is sutable only when an asymptotic steady solution is sought. It is inaccurate when real time dependence is studied, unless Δt is small.

Our reason for studying this scheme is that, the DuFort-Frankel scheme is explicit and unconditionally stable; it is a natural scheme to use when the nonlinear terms in (1) are differenced in "Conservation-Law" form, i.e $\frac{\partial(u_i u_j)}{\partial x_j}$ rather than $u_j \frac{\partial u_j}{\partial x_i}$, it is found in problems in which the viscosity is not small, it is preferable to use "non-conservative" difference scheme for non-linear terms, and avoid the DuFort-Frankel one. The equation cam be approximated in many ways. But we shall use schemes which are implicit, and accurate to $O(\Delta t) + O(\Delta x^2)$.

Implicit schemes were used because explicit ones requiere, in three space dimenssions that

$$\Delta t < \frac{1}{6} \Delta x^2$$

which is restrictive condition [2]. Also implicit schemes of accuracy higher than $O(\Delta t)$, require the solution of non-linear equations at every time-step, and make it necessary to evaluate u_i^{aux} and u_i^{n+1} simultaneously rather than in succession.

Two schemes will be presented, for both of them,

$$Tu_i \equiv (u_i^{n+1} - u_i^n)/\Delta t \quad ; \quad (b^{-1} \equiv \Delta t,\ Bu_i \equiv u_i^n/\Delta t)$$

(A) In two-dimensional problems, we use a Peaceman-Rachford analogue formula [7]. The implicit form of equation (1) can be written in the form (neglecting the pressure term),

$$\exp\left[-\frac{1}{2}\Delta t(-Ru_1\ D_1 + D_1^2)\right] . \exp\left[-\ \frac{1}{2}\Delta t(Ru_2 D_2 + D_2^2)\right] U_{i(q,r)}^{aux} \equiv$$

$$\exp\left[\frac{1}{2}\Delta t(-Ru_1 D_1 + D_1^2)\right] . \exp\left[\frac{1}{2}\Delta t(-Ru_2 D_2 + D_2^2)\right]\ u_{i(q,r)}^n + E_{i(q,r)}$$

i.e

$$\left[1-\tfrac{1}{2}\Delta t(-Ru_1D_1+D_1^2)\right]\left[1-\tfrac{1}{2}\Delta t\ (-Ru_2D_2+D_2^2)\right] u^{aux}_{i(q,r)} =$$

$$\left[1+\tfrac{1}{2}\Delta t(-Ru_1D_1+D_1^2)\right]\left[1+\tfrac{1}{2}\Delta t(-Ru_2D_2+D_2^2)\right] u^n_{i(qr)} +$$

$$+ E_{i(q,r)}$$

which can be split into two forms, if an intermediate value $\overset{*}{u}{}^{n+1}_i = u_i^{n+\frac{1}{2}} = \overset{*}{u}_i$ is introduced, retaining only the second order terms.

$$\left[1-\tfrac{1}{2}\Delta t(-Ru_1D_1+D_1^2)\right] \overset{*}{u}_{i(q,r)} = \left[1+\tfrac{1}{2}\Delta t(-Ru_2D_2+D_2^2)\right] u^n_{i(q,r)} +$$

$$+ \tfrac{1}{2} E_{i(q,r)}$$

$$\left[1-\tfrac{1}{2}\Delta t(-Ru_2D_2+D_2^2)\right] u^{aux}_{i(q,r)} = \left[1+\tfrac{1}{2}\Delta t(-Ru_1D_1+D_1^2)\right] \overset{*}{u}_{i(q,r)} +$$

$$+ \tfrac{1}{2} E_{i(q,r)}$$

which gives

$$\overset{*}{u}_{i(q,r)} = u^n_{i(q,r)} - R\frac{\Delta t}{4\Delta x_1} u^n_{1(q,r)}(\overset{*}{u}_{i(q+1,r)} - \overset{*}{u}_{i(q-1,r)}) -$$

$$R\frac{\Delta t}{4\Delta x_2} u^n_{2(q,r)}(u^n_{i(q,r+1)} - u^n_{i(q,r-1)}) + \frac{\Delta t}{2\Delta x_1^2}(\overset{*}{u}_{i(q+1,r)} +$$

$$+ \overset{*}{u}_{i(q-1,r)} - 2\overset{*}{u}_{i(q,r)}) + \frac{\Delta t}{2\Delta x_2^2}(u^n_{i(q,r+1)} + u^n_{i(q,r-1)} -$$

$$-2u^n_{i(qr)}) + \frac{\Delta t}{2} E_i \qquad (6\ a)$$

$$u^{aux}_{i(q,r)} = \overset{*}{u}_{i(q,r)} - R\frac{\Delta t}{4\Delta x_1} \overset{*}{u}_{1(q,r)} (\overset{*}{u}_{i(q+1,r)} - \overset{*}{u}_{i(q-1,r)})$$

$$-R\frac{\Delta t}{4\Delta x_2} \overset{*}{u}_{2(q,r)} \left(u^{aux}_{i(q,r+1)} - u^{aux}_{i(q,r-1)}\right) + \frac{\Delta t}{2\Delta x_1^2}\left(\overset{*}{u}_{i(q+1,r)} + \overset{*}{u}_{i(q-1,r)} - 2\overset{*}{u}_{i(q,r)}\right)$$

$$+ \frac{\Delta t}{2\Delta x_2^2}\left(u^{aux}_{i(q,r+1)} + u^{aux}_{i(q,r-1)} - 2u^{aux}_{i(q,r)}\right) + \frac{\Delta t}{2}E_i \qquad (6b)$$

where $D_i = \dfrac{\partial}{\partial x_i}$, $D_i^2 = \dfrac{\partial^2}{\partial x_i^2}$, $i = 1,2$

$$D_i u_m^n = \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x_i}, \quad D_i^2 u_m^n = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}$$

where $\overset{*}{U}_i$ are intermediate fields, $u_{i(q,r)} \equiv u_i(q\Delta x_1, r\Delta x_2)$

(B) In two-dimensional and three-dimensional problems another scheme suggested by Samaraskii[3] will presented in the form,

$$\overset{*}{u}_{i(q,r,s)} = u^n_{i(q,r,s)} - R\frac{\Delta t}{2\Delta x_1} u^n_{1(q,r,s)} (\overset{*}{u}_{i(q+1,r,s)} - \tag{7 a}$$

$$- \overset{*}{u}_{i(q-1r,s)}) + \frac{\Delta t}{\Delta x_2^{\,2}} (\overset{*}{u}_{i(q+1,r,s)} + \overset{*}{u}_{i(q-1,r,s)} - 2\overset{*}{u}_{i(q,r,s,)}$$

$$\overset{**}{u}_{i(q,r,s)} = \overset{*}{u}_{i(q,r,s)} - R\frac{\Delta t}{2\Delta x_2} \overset{*}{u}_{2(q,r,s)} (\overset{**}{u}_{i(q,r+1,s)} -$$

$$- \overset{**}{u}_{i(q,r-1,s)}) + \frac{\Delta t}{\Delta x_2^{\,2}} (\overset{**}{u}_{i(q,r+1,s)} + \overset{**}{u}_{i(q,r-1,s)} -$$

$$- 2\overset{**}{u}_{i(q,r,s)}) \tag{7 b}$$

$$u^{aux}_{i(q,r,s)} = \overset{**}{u}_{i(q,r,s)} - R\frac{\Delta t}{2\Delta x_3} (\overset{**}{u}_{3(q,r,s)} (u^{aux}_{i(q,r,s+1)} -$$

$$- u^{aux}_{i(q,r,s-1)}) + \frac{\Delta t}{\Delta x_3^2} (u^{aux}_{i(q,r,s+1)} + u^{aux}_{i(q,r,s-1)} -$$

$$- 2u^{aux}_{i(q,r,s)}) + \Delta t\, E_{i(q,r,s)} \tag{7 c}$$

where $u_{i(q,r,s)} \equiv u_{i(q\Delta x_1, r\Delta x_2, s\Delta x_3)}$

$$E_{i(q,r,s)} \equiv E_i(q\Delta x_1,\ r\Delta x_2, s\Delta x_3)$$

and $\overset{*}{u}_i$, $\overset{**}{U}_i$ are auxiliary fields.

In symbolic form equations (6) can be written in the form

$$\overset{*}{u}_{i(q,r)} = u^n_{i(q,r)} - \frac{R}{2}\Delta t\, u^n_{1(q,r)} \frac{\partial \overset{*}{u}_{i(q,r)}}{\partial x_1} - R\frac{\Delta t}{2} u^n_{2(q,r)} \cdot \frac{\partial u^n_{i(q,r)}}{\partial x_2} + \frac{\Delta t}{2} \frac{\partial^2 \overset{*}{u}_{i(q,r)}}{\partial x_1^2} + \frac{\Delta t}{2} \frac{\partial^2 u^n_{i(q,r)}}{\partial x_2^2} + \frac{\Delta t}{2} E_i \qquad (8\ a)$$

$$u^{aux}_{i(q,r)} = \overset{*}{u}_{i(q,r)} - R\frac{\Delta t}{2} \overset{*}{u}_{1(q,r)} \frac{\partial \overset{*}{u}_{i(q,r)}}{\partial x_1} - R\frac{\Delta t}{2} \cdot \overset{*}{u}_{2(q,r)} \frac{\partial u^{aux}_{i(q,r)}}{\partial x_2} + \frac{\Delta t}{2} \cdot \frac{\partial^2 u^{aux}_{i(q,r)}}{\partial x_2^2} + \frac{\Delta t}{2} E_i \qquad (8\ b)$$

i.e

$$\left(I + \frac{R}{2}\Delta t\, u^n_{1(q,r)} \frac{\partial}{\partial x_1} - \frac{\Delta t}{2} \frac{\partial^2}{\partial x_1^2}\right) \overset{*}{u}_{i(q,r)} =$$

$$= \left(I - \frac{R}{2}\Delta t\, u^n_{2(q,r)} \frac{\partial}{\partial x_2} + \frac{\Delta t}{2} \frac{\partial^2}{\partial x_2^2}\right) u^n_{i(q,r)} + \frac{\Delta t}{2} E_i$$

$$(I - \Delta t Q_1)\overset{*}{u}_i = (I - \Delta t Q_2) u^n_i + \frac{\Delta t}{2} E_i \qquad (8\ c)$$

where

$$Q_1 = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} - \frac{R}{2} u^n_{1(q,r)} \frac{\partial}{\partial x_1}$$

$$Q_2 = \frac{R}{2} u^n_{2(q,r)} \frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2}$$

also,

$$(I + \frac{R}{2}\ \Delta t\ \overset{*}{u}_{2(q,r)} \frac{\partial}{\partial x_2} + \frac{\Delta t}{2}\ \frac{\partial^2}{\partial x_2{}^2})\ u_i^{aux}\ =$$

$$(\ I - R\ \frac{\Delta t}{2}\ \overset{*}{u}_{1(q,r)} \frac{\partial}{\partial x_1} + \frac{\Delta t}{2} \frac{\partial^2}{\partial x_1{}^2}\ )\ \overset{*}{u}_i + \frac{\Delta t}{2}\ E_i \qquad (9\ a)$$

$$(I - \Delta t\ \overset{*}{Q}_2)\ u_i^{aux} = (I - \Delta t\ \overset{*}{Q}_1)\ \overset{*}{u}_i + \frac{\Delta t}{2}\ E_i \qquad (9\ b)$$

where

$$\overset{*}{Q}_1 = \frac{R}{2}\ \overset{*}{u}_{1(q,r)} \frac{\partial}{\partial x_1} - \frac{1}{2} \frac{\partial^2}{\partial x_1{}^2}$$

$$\overset{*}{Q}_2 = \frac{1}{2} \frac{\partial^2}{\partial x_2{}^2} - \frac{R}{2}\ \overset{*}{u}_{2(q,r)}\ \ \frac{\partial}{\partial x_2}$$

where Q_1, Q_2, $\overset{*}{Q}_1$, $\overset{*}{Q}_2$, involves differentiation with respect to variables x_1, x_2, and I the identity operator.

$$\overset{*}{u}_i = \left[(I - \Delta t Q_2)\ u_i^n + \frac{\Delta t}{2}\ E_i \right]\ (I + \Delta t Q_1) + O(\Delta t^2)$$

$$\overset{*}{u}_i = (I - \Delta t Q_2 + \Delta\ t Q_1)\ u_i^n + \frac{\Delta t}{2}\ E_i + O(\Delta t^2) \qquad (6a)$$

$$u_i^{aux} = \left[(I + \Delta t \overset{*}{Q}_1)\ \overset{*}{u}_i + \frac{\Delta t}{2}\ E_i \right] (I + \Delta t \overset{*}{Q}_2) = (I + \Delta t \overset{*}{Q}_1 +$$

$$+ \Delta t \overset{*}{Q}_2 - \Delta\ t Q_1 - \Delta t Q_2)\ u_i^n + \frac{\Delta t}{2}\ E_i + O(\Delta t^2)$$

$$u_i^{aux} = (I + \Delta\ t \overset{*}{Q}_1 + \Delta t \overset{*}{Q}_2)\ \overset{*}{u}_i + \frac{\Delta t}{2}\ \ E_i + O(\Delta t^2) \qquad (6b)$$

$$u_i^{n+1} = u_i^{aux} - \Delta t\ G_i p^{n+1} = (I + \Delta t \overset{*}{Q}_1 + \Delta t \overset{*}{Q}_2) \Big\{ (I - \Delta t Q_2 +$$

$$+ \Delta t Q_1)\ u_i^n - \frac{\Delta t}{2}\ E_i + O(\Delta t^2) \Big\} - \Delta t G_i p^{n+1}$$

$$u_i^{n+1} = (I - \Delta t Q_2 + \Delta t Q_1 + \Delta t \overset{*}{Q}_1 + \Delta t \overset{*}{Q}_2)\, u_i^n + \frac{\Delta t}{2} E_i +$$

$$+ O(\Delta t^2) - \Delta t G_i p^{n+1}$$

we can set at the boundary

$$\overset{*}{u}_i = (I - \Delta t \overset{*}{Q}_1 - \Delta t \overset{*}{Q}_2)\, u_i^{n+1} - \Delta t E_i + \Delta t \overline{G_i p}$$

$$\text{where } \overline{G_i p} = G_i p + O(\Delta t)$$

at the boundary the normal component of $G_i p$ is approximated by one-sided differences while it is not necessary in the interior of $\mathcal{D}$.

i.e

$$\overset{*}{u}_i = u_i^{n+1} - \Delta t \overset{*}{Q}_1 u_i^{n+1} - \Delta t \overset{*}{Q}_2 u_i^{n+1} - \Delta t E_i + \Delta t \overline{G_i p}$$

$$u_i^{aux} = u_i^{n+1} + \Delta t G_i p$$

But, $\overset{*}{u}_i = (I - \Delta t Q_2 + \Delta t Q_1)\, u_i^n + \frac{\Delta t}{2} E_i + O(\Delta t^2)$

$$\overset{*}{u}_i = \Big[I - \Delta t \Big(\frac{R}{2} u^n_{2(q,r)} \frac{\partial}{\partial x_2} + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} \Big) +$$

$$+ \Delta t \Big(\frac{1}{2} \frac{\partial^2}{\partial x_1^2} - \frac{R}{2} u^n_{1(q,r)} \frac{\partial}{\partial x_1} \Big) \Big] u_i^n + \frac{\Delta t}{2} E_i + O(\Delta t^2)$$

$$\overset{*}{u}_i = u^n_{i(q,r)} - \Delta t \Big\{ \frac{R}{4\Delta x_2} u^n_{2(q,r)} \big(u^n_{i(q,r+1)} - u^n_{i(q,r-1)} \big) +$$

$$+ \frac{1}{2\Delta x_2^2} \big(u^n_{i(q,r+1)} + u^n_{i(q,r-1)} - 2u^n_{i(q,r)} \big) \Big\} + \Delta t \Big\{ \frac{1}{2\Delta x_1^2} \cdot$$

$$\big(u^n_{i(q+1,r)} + u^n_{i(q-1,r)} - 2u^n_{i(q,r)} \big) - \frac{R}{4\Delta x_1} u^n_{1(q,r)} \cdot$$

$$\cdot\ (u^n_{i(q+1,r)} - u^n_{i(q-1,r)})\}\Big] + \frac{\Delta t}{2} E_i + O(\Delta t^2) \qquad (10)$$

also,

$$u_i^{aux} = \Big[(I + \Delta t \overset{*}{Q}_1 + \Delta t \overset{*}{Q}_2 - \Delta t Q_1 - \Delta t Q_2)\, u_i^n + \frac{\Delta t}{2} E_i + O(\Delta t^2)\Big]$$

$$= \Big[I + \Delta t\Big(\frac{R}{2}\overset{*}{u}_1 \frac{\partial}{\partial x_1} - \frac{1}{2}\frac{\partial^2}{\partial x_1^2}\Big) + \Delta t\Big(\frac{1}{2}\frac{\partial^2}{\partial x_2^2} - \frac{R}{2}\overset{*}{u}_2 \frac{\partial}{\partial x_2}\Big) -$$

$$- \Delta t\Big(\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - \frac{R}{2} u_1^n \frac{\partial}{\partial x_1}\Big) - \Delta t\Big(\frac{R}{2} u_2^n \frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2}\Big)\Big] u_i^n +$$

$$+ \frac{\Delta t}{2} E_i + O(\Delta t^2)$$

i.e

$$u_i^{aux} = \Big[u_i^n + \Delta t\Big\{\frac{R}{4\Delta x_1}\overset{*}{u}_1 (u^n_{i(q+1,r)} - u^n_{i(q-1,r)}) - \frac{1}{4\Delta x_1^{\,2}}\ \cdot$$

$$\cdot\ (u^n_{i(q+1,r)} + u^n_{i(q-1,r)} - 2u^n_{i(q,r)})\Big\} + \Delta t\Big\{\frac{1}{4\Delta x_1^{\,2}}\ \cdot$$

$$\cdot\ (u^n_{i(q,r+1)} + u^n_{i(q,r-1)} - 2u^n_{i(q,r)}) - \frac{R}{4\Delta x_2}\overset{*}{u}_2 (u^n_{i(q,r+1)} -$$

$$- u^n_{i(q,r-1)})\Big\} - \Delta t \quad \frac{1}{4\Delta x_1^{\,2}}(u^n_{i(q+1,r)} + u^n_{i(q-1,r)} - 2u^n_{i(q,r)})$$

$$- \frac{R}{4\Delta x_1} u^n_{i(q,r)}(u^n_{i(q+1,r)} - u^n_{i(q-1,r)})\Big\} - \Delta t\Big\{\frac{R}{4\Delta x_2} u^n_{2(q,r)}\ \cdot$$

$$\cdot\ (u^n_{i(q,r+1)} - u^n_{i(q,r-1)}) + \frac{1}{4\Delta x_2^{\,2}}(u^n_{i(q,r+1)} + u^n_{i(q,r-1)} -$$

$$- 2u^n_{i(q,r)})\Big\} + \frac{\Delta t}{2} E_i + O(\Delta t^2) \qquad (11)$$

A similar expressions for scheme (B) can be written in symbolic form, as follows:

$$
\begin{aligned}
(I - \Delta t Q_1)\overset{*}{u}_i &= u_i^n \\
(I - \Delta t Q_2)\overset{**}{u}_i &= \overset{*}{u}_i \qquad\qquad (12) \\
(I - tQ_3)u_1^{aux} &= \overset{**}{u}_i + \Delta t E_i
\end{aligned}
$$

Where I is the identity operator, and Q_ℓ represents differentiations with respeet to x_ℓ only.

It is clear that scheme (6) is accurate to $O(\Delta t^2) + O(\Delta x^2)$ in both cases when $R = 0$ and $R \neq 0$.

If both schemes are to be used in a problem in wich the velocities are known at the boundary, values of $\overset{*}{u}_i$, $\overset{**}{u}_i$, u_i^{aux} at the boundary have to be given in advance so that the several implicit operators can be inverted.

In the case of scheme (12), we have,

$$u_i^{n+1} = (I + \Delta t Q_1 + \Delta t Q_2 + \Delta t Q_3)u_i^n + \Delta t E_i - \Delta t G_i p^n + O(\Delta t^2)$$

$$\overset{*}{u}_i = (I + \Delta t Q_1)u_i^n + O(\Delta t^2)$$

$$\overset{**}{u}_i = (I + \Delta t Q_1 + \Delta t Q_2)\, u_i^n + O(\Delta t^2)$$

$$u_i^{aux} = (I + \Delta t Q_1 + \Delta t Q_2 + \Delta t Q_3)u_i^n + \Delta t\, E_i + O(\Delta t^2)$$

The scheme will be accurate to $O(\Delta t)$ at the boundary if,

$$\overset{*}{u}_i = u_i^{n+1} - \Delta t Q_2 u_i^{n+1} - \Delta t Q_3 u_i^{n+1} - \Delta t\, E_i + \Delta t\, G_i p^n$$

$$u_i^{**} = u_i^{n+1} - \Delta t Q_3 \, u_i^{n+1} + \Delta t \, \overline{G_i p}$$

$$u_i^{aux} = u_i^{n+1} + \Delta t \, G_i p^n$$

where

$$\overline{G_i p^n} = G_i p^n + O(\Delta t)$$

It is clear that more accurate expressions for the auxiliary fields at the boundaries can be used but it needs great programming effort on the computer.

In case of negligible viscosity, i.e. $\nu = 0$, another schemes will used, i.e explicit schemes which accurate to $O(\Delta t^2) + O(\Delta x^2)$, and stable when $\Delta t = O(\Delta x)$.

A scheme we suggested for this case can be given in the explicit form,

$$u_{i(q,r)}^{n+1} = \exp\left[-\Delta t\left(u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2}\right)\right] u_{i(q,r)}^{n} + E_{i(q,r)}$$

$$= \left[1 - \Delta t\left(u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2}\right) + \dots\right] u_{i(q,r)}^{n} + E_{i(q,r)} =$$

$$= u_{i(q,r)}^{n} - \Delta t \left\{ u_{1(q,r)}^{n} \frac{u_{i(q+1,r)}^{n} - u_{i(q-1,r)}^{n}}{2\Delta x_1} \right.$$

$$\left. - \; u_{2(q,r)}^{n} \frac{u_{i(q,r+1)}^{n} - u_{i-q,r-1)}^{n}}{2\Delta x_2} \right\} + E_{i(q,r)}$$

retained only the first order terms, which can be solved to give u_i^{n+1} . Such problem can be discussed later.

The rest of this work will show how we can derive D, G_i^m , and the choice of λ , used in (5 a) and (5 b), so we need some facts about the DuFort - Frankel scheme for heat equation, and its relation to the relaxation method for solving the Laplace equation [7]. Consider the equation,

$$-\nabla^2 u = f \;, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}} \qquad (13)$$

in some domain $\mathcal{D}$, rectangle for example. u is assumed known on the boundary of $\mathcal{D}$, it can be approximate to the equation,

$$- Lu = f \qquad (14)$$

it is clear that L is the five point approximation to the Laplacian, u and f are now m-component vectors.

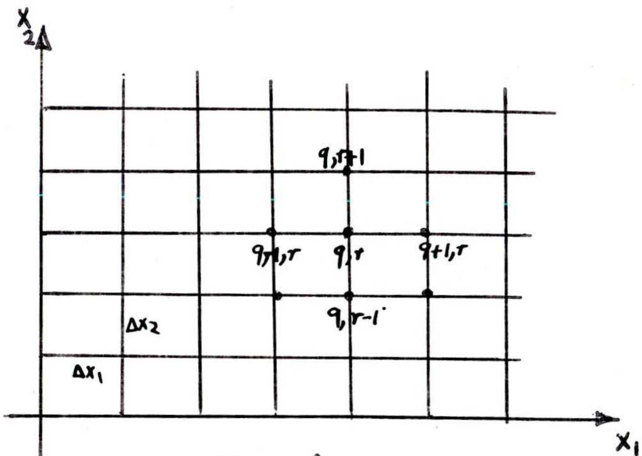


Figure 2

m is the number of internal nodes of the resulting difference equation. For simplicity we assume that the mesh wedthes in the x_1, x_2 directions are equal, i.e.

$\Delta x_1 = \Delta x_2 = \Delta x$, so the operator L is represented by an m x m matrix A.

The matrix A can be written in the form,

$$A = A' - E - E'$$

where A' is diagonal, and E, E' respectively upper and lower triangular matrices. The convergent relaxation scheme for solving (14) is given by,

$$(A' - \omega E)\, u^{n+1} = \left\{(1-\omega)A' + \omega E'\right\} u^{n} + \omega f \qquad (15)$$

and hence $u^{n+1} = (A' - \omega E)^{-1}\left[(1-\omega)A' + \omega E'\right] u^{n} + \omega (A' - \omega E)^{-1} f$

where ω is the relaxation factor, $0 < \omega < 2$, and u^{n} are the successive iterates. [5],[7].

It is known that there is optimal relaxation factor ω_{opt} depends on the fact that Aasatisfies "Young,s condition (A)" [9]. i.e there exists a permutation matrix P such that,

$$P^{-1}AP = \Lambda - N \qquad (15)'$$

where Λ is diagonal, and N has the normal form,

$$\begin{bmatrix} O & G \\ G' & O \end{bmatrix}$$

The zero submatrices here are squere, under this condition ω_{opt} can be determined. The matrix A depends on the order in which the components of u^{n+1} are computed from u^{n}. The changing of that order is equivalent to transforming A into $P^{-1}AP$, where P is a permutation matrix.

The solution (17) is consider to be the steady solution of,

$$\frac{\partial u}{\partial \tau} = \nabla^{2} u + f \qquad (16)$$

The latter equation can be approximated by the DuFort-Frankel scheme,

$$u_{q,r}^{n+1} - u_{q,r}^{n-1} = \frac{2\Delta\tau}{\Delta x^2}\left(u_{q+1,r}^{n} + u_{q-1,r}^{n} + u_{q,r+1}^{n} + u_{q,r-1}^{n} - 2u_{q,r}^{n+1} - 2u_{q,r}^{n-1}\right) + 2\Delta\tau f$$

where

$$u_{q,r}^{n} \equiv u(q\,\Delta x_1,\ r\,\Delta x_2,\ n\,\Delta\tau)$$

which approximates (13), where $\Delta\tau = O(\Delta x)$, we obtain,

$$\left(1 + 4\frac{\Delta\tau}{\Delta x^2}\right) u_{q,r}^{n+1} - \left(1 - 4\frac{\Delta\tau}{\Delta x^2}\right) u_{q,r}^{n-1} = 2\frac{\Delta\tau}{\Delta x^2}\left(u_{q+1,r}^{n} + u_{q-1,r}^{n} + u_{q,r+1}^{n} + u_{q,r-1}^{n}\right) + 2\Delta\tau f \qquad (17)$$

where

$$u_{q,r}^{n} \equiv \frac{1}{2}\left(u_{q,r}^{n+1} + u_{q,r}^{n-1}\right).$$

Clearly $u_{q,r}^{n}$ does not appear in (17) so the calculation splits into two independent calculations on intertwined meshes, one of which can be omitted then we can write,

$$U^{n+1} = \begin{pmatrix} u^{2n} \\ u^{2n+1} \end{pmatrix}, \quad (U^{n+1} \text{ has m components})$$

when we write,

$$\omega = \frac{8\,\Delta\tau/\Delta x^2}{1+4\,\Delta\tau/\Delta x^2} \qquad (18)$$

We see that the iteration (17) reduces to an iteration of the form (15), where the new components of U^{n+1} are calculated in an order such that A has the normal form (15)$'$. [This is clear since, the difference equation of equation (14) can be written in the form,

$$\frac{1}{\Delta x^2}(u^n_{q,r+1} + u^n_{q+1,r} + u^n_{q,r-1} + u^n_{q-1,r} - 4u^n_{q,r}) + f = 0,$$

we can write,

$u^n_{q,r} = \frac{1}{2}(u^{n+1}_{q,r} + u^{n-1}_{q,r})$, hence the equation takes the form,

$$\frac{1}{\Delta x^2}(u^n_{q+1,r} + u^n_{q-1,r} + u^n_{q,r+1} + u^n_{q,r-1} - 2u^{n+1}_{q,r} - 2u^{n-1}_{q,r}) + f = 0 \Big] .$$

Then it is clear that the DuFort - Franke scheme appears to be a particular ordering of the over-relaxation method whose existence is equivalent to Young's condition (A).

The best value of $\Delta\tau$, i.e $\Delta\tau_{opt}$ can be determined from ω_{opt} and equation (18), clarly $\Delta\tau_{opt} = O(\Delta x)$, then for $\Delta\tau = \Delta\tau_{opt}$ the DuFort - Frankel scheme approximate also the equation,

$$\frac{\partial u}{\partial \tau} = \nabla^2 u - 2\left(\frac{\Delta\tau}{\Delta x}\right)^2 \frac{\partial^2 u}{\partial \tau^2} + f$$

see [4].

These remarks can be generalized to problems of more than two space variables. Also it will be noted that, we can approximate equation (16) by explicit method

$$u^{n+1}_{q,r} - u^n_{q,r} = \frac{\Delta\tau}{\Delta x^2}\left(u^n_{q+1,r} + u^n_{q-1,r} + u^n_{q,r+1} + u^n_{q,r-1} - 4u^n_{q,r}\right) + \Delta\tau f \qquad (19)$$

and used as an iteration procedure for solving (14), but the iteration converges only when $\Delta\tau/\Delta x^2 < 1/4$, and converges very slow [4].

The representation of D, G_i, G_i^m and the iteration procedure for determining u_i^{n+1}, p^{n+1}.

For simplicity we shall assume that the domain $\mathcal{D}$ is two-dimensional and rectangular, and the velocities are know at the boundary. Extension to three - dimensional problems is possible, also domains of other shapes can be treated by the help of interpolation procedures. Firstly we define $\mathcal{D}$. Let β denote the boundary of $\mathcal{D}$ and C the set of mesh nodes with a neighbor in β. In $\mathcal{D} - \beta$ we approximate the equation of continuaty by the centered differences, i.e.

$$Du = \frac{1}{2\Delta x_1}\left(u_{1(q+1,r)} - u_{1(q-1,r)}\right) + \frac{1}{2\Delta x_2}\left(u_{2(q,r+1)} - u_{2(q,r-1)}\right) = 0 \tag{20}$$

At the points of β we use second-order one-sided differences, so that Du is accurate to $O(\Delta x^2)$ everywhere. On the boundary line $x_2 = 0$, we have,

$$Du = \frac{2}{\Delta x_2}\left[u_{2(q,2)} - u_{2(q,1)} - \frac{1}{4}\left(u_{2(q,3)} - u_{2(q,1)}\right)\right] + \frac{1}{2\Delta x_1}\left(u_{1(q+1,1)} - u_{1(q-1,1)}\right) = 0, \quad q > 1 \tag{21}$$

$$Du = \frac{2}{\Delta x_2}\left[u_{2(q,2)} - u_{2(q,1)} - \frac{1}{4}\left(u_{2(q,3)} - u_{2(q,1)}\right)\right] + \frac{1}{\Delta x_1}\left(u_{1(q+1,1)} - u_{1(q,1)}\right) = 0, \quad q = 1$$

let (q,r) be a node in $\mathcal{D} - \beta - C$, knowing $u_i^{n+1,m}$, $i = 1, 2$ and $p^{n+1,m}$ we shall evaluate simultanausly

$u^{n+1,m+1}_{1(q+1,r)}$, $u^{n+1,m+1}_{2(q,r+1)}$ and $p^{n+1,m+1}_{q,r}$ using the formula

$$p^{n+1,m+1}_{q,r} = p^{n+1,m}_{q,r} - \lambda Du^{n+1,m+1}$$

with similar expressions at the other boundaries. Clearly equation (20) states that the total flow of the fluid into a rectangle of sides $2\Delta x_1$, $2\Delta x_2$ is zero, while equation (21) does not have this elementry interpolation.

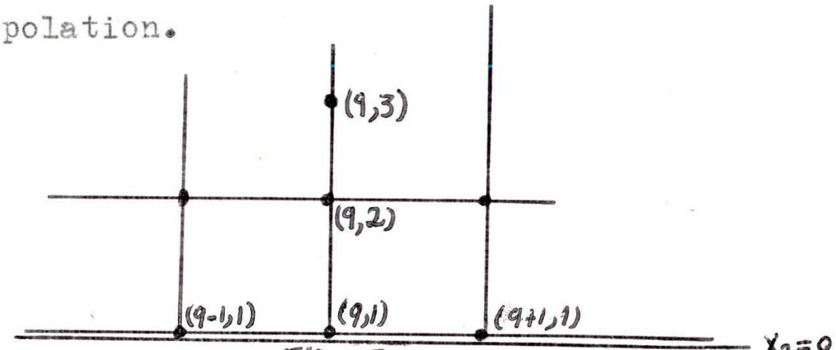


Figure 3

Also we define $G_i p$ at every point of $\mathcal{D} - \beta$ by,

$$G_1 p = \frac{1}{2\Delta x_1}(p_{q+1,r} - p_{q-1,r})$$

$$G_2 p = \frac{1}{2\Delta x_2}(p_{q,r+1} - p_{q,r-1})$$

where $\quad p_{q,r} \equiv p(q\Delta x_1, r\Delta x_2)$

It is clear that $\frac{\partial p}{\partial x}$ is approximated by centered differences. One can use other forms for $G_i p$ and Du. Our purpose now is to perform the decomposition (4). u^{n+1}_i is given on the boundary β, u^{aux}_i is given in $\mathcal{D} - \beta$, also p^{n+1} is to be found in $\mathcal{D}$ (including the boundary) and u^{n+1} in $\mathcal{D} - \beta$, so that in $\mathcal{D} - \beta$

$$u_i^{aux} = u_i^{n+1} + \Delta t G_i p$$

and in $\mathcal{D}$ (including the boundary)

$$Du^{n+1} = 0$$

This must be done using the iterations (5), until now the form of $G_i^m p$ is not specified. At a point (q,r) in $\mathcal{D} - \beta - C$, i.e. far from the boundary, we cun substitute equation (5 a) into equation (5 b) and obtain,

$$p^{n+1,m+1} - p^{n+1,m} = -\lambda Du^{aux} + \Delta t D G^m p \qquad (22)$$

An analogue to this method was used by Harlow and Welch [9] , as follows:

Let Du = 0 approximate $\frac{\partial u_j}{\partial x_j} = 0$, and $G_i p$ approximate $\frac{\partial p}{\partial x_i}$. It is assumed that at time $t = n\Delta t$ velocity fields u_i^n are given, satisfying $Du^n = 0$, then equation (2) can be approximated by

$$u_i^{n+1} = u_i^n + \Delta t L u_i^n - \Delta t Q_i u^n - \Delta t Q_i p^n + \Delta t E_i \qquad (23)$$

wher Lu approximates $\nabla^2 u$, and $Q_i u$ approximates $\frac{\partial u_i u_j}{\partial x_j}$. Performing the operator D on the previous equation, assuming

$$Du^{n+1} = 0, \text{ we have}$$

$$L' p^n = -\frac{Du^n}{\Delta t} + DLu^n - DQu^n + DE_i \qquad (23)'$$

wher $L'p \equiv DGp$ approximates $\nabla^2 p$. This equation is a difference analogue of the equation

$$\nabla^2 p = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u_i u_j + \frac{\partial E_j}{\partial x_j} \qquad (24)$$

which can be obtained from equation (1) by taking its divergence. In view of the definitions of D, G_i^m, and u_i^{aux}, equation (22) is an iteration procedure for solving an analogue of equation (23). In this sense the method used is related to Harlow and Welch like a predictor-corrector method, wheareas Harlow and Welch first determine p^n so that $Du^{n+1} = 0$, a guess will made at the values at u_i^{n+1}, p^{n+1}, and then correct them until the condition $Du^{n+1} = 0$ is satisfied. It is clear that at the points of β or C it is not posible to substitute (5 a) into (5 b) because at the boundary u_i^{n+1} is prescribed, $u^{n+1,m+1} = u^{n+1}$ for all m, (5 a) does not hold and therfore (22) is not true. Near the boundary the iterations (5) provide boundary data for (23)' and ensure that the constraint of incompressibility is satisfied. We proceed as fallows:

We chose $G_i^m p$ and λ such that (22) is rapidly converging iteration for solving (23)'; $G_i^m p$ at the boundary are chosen so that the iteration (5) converges everywhere.

Let (q,r) again be a node in $\mathcal{D} - \beta - C$. $u_i^{n+1,m+1}$ and $p^{n+1,m}$ are assumed known. We shall evaluate simultanously $p_{q,r}^{n+1,m+1}$ and the velocity components involved in the equation $Du^{n+1} = 0$ at (q,r), i.e $u_{1(q\pm 1,r)}^{n+1,m+1}$, $u_{2(q,r\pm 1)}^{n+1,m+1}$.

These velocity components depend on the value of p at (q,r) and on the values of p at the other points. The value of p at (q,r) can taken to be,

$$\frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right)$$

while at the other points we use $p^{n+1,m}$. This leads to the following formula,

$$p^{n+1,m+1}_{q,r} = p^{n+1,m}_{q,r} - \lambda Du^{n+1,m+1} \qquad (25\ a)$$

where Du is given by (20).

$$u^{n+1,m+1}_{1(q+1,r)} = u^{aux}_{1(q+1,r)} - \frac{\Delta t}{2\Delta x_1}\left(p^{n+1,m}_{q+2,r} - \frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right)\right) \qquad (25\ b)$$

$$u^{n+1,m+1}_{1(q-1,r)} = u^{aux}_{1(q-1,r)} - \frac{\Delta t}{2\Delta x_1}\left(\frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right) - p^{n+1,m}_{q-2,r}\right) \qquad (25\ c)$$

$$u^{n+1,m+1}_{2(q,r+1)} = u^{aux}_{2(q,r+1)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,r+2} - \frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right)\right) \qquad (25\ d)$$

$$u^{n+1,m+1}_{2(q,r-1)} = u^{aux}_{2(q,r-1)} - \frac{\Delta t}{2\Delta x_2}\left(\frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right) - p^{n+1,m}_{q,r-2}\right) \qquad (25\ e)$$

It is clear that $G^m_i p \rightarrow G_i p$ [since, $u^{n+1,m+1}_{1(q-1,r)} \rightarrow u^{n+1}_{1(q-1,r)}$

as $p^{n+1,m+1} \rightarrow p^{n+1,m}$. The first equation gives,

$$u^{n+1,m+1}_{1(q-1,r)} = u^{aux}_{1(q-1,r)} - \frac{\Delta t}{2\Delta x_1}\left(\frac{1}{2}\left(p^{n+1,m}_{q,r} + p^{n+1,m}_{q,r}\right) - p^{n+1,m}_{q-2,r}\right.$$

$$u^{n+1}_{1(q-1,r)} = u^{aux}_{1(q-1,r)} - \frac{\Delta t}{2\Delta x_1}\left(p^{n+1,m}_{q,r} - p^{n+1,m}_{q-2,r}\right)$$

$$u^{n+1}_{1(q-1,r)} = u^{aux}_{1(q-1,r)} - \Delta t \frac{\partial p^{n+1}_{q,r}}{\partial x_1} = u^{aux}_{1(q-1,r)} - \Delta t G_1 p^{n+1} \Big] .$$

and similar expressions for the other equations. In C and β these formulae have to be modified. Consider again the boundary line $x_2 = 0$, assume the velocities are prescribed at the boundary i.e $u^{n+1}_{i(q,1)}$ are given, $i = 1, 2$. There are several ways of including that information in the iteration (5). The consistent way would to be set.

$$u^{aux}_{i(q,1)} = u^{n+1}_{i(q,1)} + \Delta t \; G_i p^n$$

and

$$u^{n+1,m+1}_{i(q,1)} = u^{n+1}_{i(q,1)}$$

for the sake of simplicity, we chose an in inconsistent way of treating the boundary, we set $u^{aux}_{i(q,1)} = u^{n+1,m}_{i(q,1)} = u^{n+1,m+1}_{i(q,1)} = u^{n+1}_{i(q,1)}$. This does not affect the values of u^{n+1}_i, it introduce an additional error of $O(\Delta t)$ into the computed ppressure term. Equations (25) can be solved for $p^{n+1,m+1}_{q,r}$ as follows:

$$p^{n+1,m+1}_{q,r} = p^{n+1,m}_{q,r} - \lambda \left[\frac{1}{2\Delta x_1} \left(u^{n+1,m+1}_{1(q+1,r)} - u^{n+1,m+1}_{1(q-1,r)} \right) + \frac{1}{2\Delta x_2} \left(u^{n+1,m+1}_{2(q,r+1)} - u^{n+1,m+1}_{2(q,r-1)} \right) \right]$$

$$= p^{n+1,m}_{q,r} - \lambda \left[\frac{1}{2\Delta x_1} \left\{ u^{aux}_{1(q+1,r)} - \frac{\Delta t}{2\Delta x_1} \left(p^{n+1,m}_{q+2,r} - \frac{1}{2} \left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r} \right) \right) \right. \right.$$

$$\left. - \left(u^{aux}_{1(q-1,r)} - \frac{\Delta t}{2\Delta x_1} \left(\frac{1}{2} \left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r} \right) - p^{n+1,m}_{q-2,r} \right) \right) \right\} +$$

$$\frac{1}{2\Delta x_2}\left\{ u^{aux}_{2(q,r+1)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,r+2} - \frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right)\right) - \right.$$

$$\left.\left(u^{aux}_{2(q,r-1)} - \frac{\Delta t}{2\Delta x_2}\left(\frac{1}{2}\left(p^{n+1,m+1}_{q,r} + p^{n+1,m}_{q,r}\right) - p^{n+1,m}_{q,r-2}\right)\right\}\right]$$

collecting the similar terms we find that;

$$p^{n+1,m+1}_{q,r}(1+\alpha_1+\alpha_2) = (1-\alpha_1-\alpha_2)p^{n+1,m}_{q,r} - \lambda D u^{aux} + \alpha_1\left(p^{n+1,m}_{q+2,r} + p^{n+1,m}_{q-2,r}\right) + \alpha_2\left(p^{n+1,m}_{q,r+2} + p^{n+1,m}_{q,r-2}\right)$$

i.e.

$$p^{n+1,m+1}_{q,r} = (1+\alpha_1+\alpha_2)^{-1}\left[(1-\alpha_1-\alpha_2)p^{n+1,m}_{q,r} - \lambda D u^{aux} + \alpha_1\left(p^{n+1,m}_{q+2,r} + p^{n+1,m}_{q-2,r}\right) + \alpha_2\left(p^{n+1,m}_{q,r+2} + p^{n+1,m}_{q,r-2}\right)\right] \qquad (26\ a)$$

where

$\alpha_i = \lambda\Delta t / 4\Delta x_i^2$, i = 1, 2 , also,

$$D u^{aux}\Big|_{q,r} = \frac{1}{2\Delta x_1}\left(u^{aux}_{1(q+1,r)} - u^{aux}_{1(q-1,r)}\right) + \frac{1}{2\Delta x_2}\left(u^{aux}_{2(q,r+1)} - u^{aux}_{2(q,r-1)}\right)$$

This can be seen to be a DuFort - Frankel relaxation scheme for the solution of (23). The $\Delta\tau$ of the proceeding equations (17) is replaced by $\lambda\frac{\Delta t}{2}$. It is clear that corresponding to $\Delta\tau_{opt}$ or ω_{opt}, we find λ_{opt}. If p were known on β and C, convergence of the iterations (26 a) would follow and $\lambda = \lambda_{opt}$ would lead to fastest convergence.

In β and C formulae (25) are modified by the use of the values of u_i^{n+1} at the boundary.

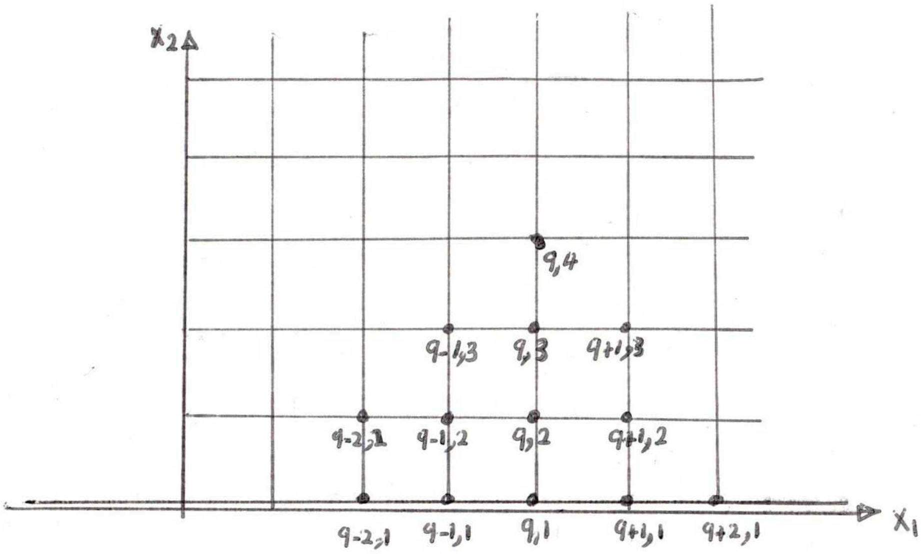


Figure 4

On C,

$$Du\Big|^{n+1,m+1}_{q,2} = \frac{1}{2\Delta x_1}\left(u^{n+1,m+1}_{1(q+1,2)} - u^{n+1,m+1}_{1(q-1,2)}\right) + \frac{1}{2\Delta x_2}\left(u^{n+1,m+1}_{2(q,3)} - u^{n+1,m+1}_{2(q,1)}\right) \qquad (27)$$

$u_i^{n+1,m+1}$ takes the form,

$$u^{n+1,m+1}_{1(q+1,2)} = u^{aux}_{1(q+1,2)} - \frac{\Delta t}{2\Delta x_1}\left(p^{n+1,m}_{q+2,2} - \frac{1}{2}\left(p^{n+1,m+1}_{q,2} + p^{n+1,m}_{q,2}\right)\right) \qquad (28\text{ a})$$

$$u^{n+1,m+1}_{1(q-1,2)} = u^{aux}_{1(q-1,2)} - \frac{\Delta t}{2\Delta x_1}\left(\frac{1}{2}\left(p^{n+1,m+1}_{q,2} + p^{n+1,m}_{q,2}\right) - p^{n+1,m}_{q-2,2}\right), \quad q > 2 \qquad (28\text{ b})$$

$$u^{n+1,m+1}_{1(q-1,2)} = u^{aux}_{1(q-1,2)}, \quad q = 2$$

$$u^{n+1,m+1}_{2(q,3)} = u^{aux}_{2(q,3)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,4} - \frac{1}{2}\left(p^{n+1,m+1}_{q,2} + p^{n+1,m}_{q,2}\right)\right) \qquad (28\text{ c})$$

$$u^{n+1,m+1}_{2(q,1)} = u^{aux}_{2(q,1)} \qquad (28\text{ d})$$

Subestituting equations (28) into (27) we have, using the equation,

$$p^{n+1,m+1}_{q,2} = p^{n+1,m}_{q,2} - \frac{\lambda}{2\Delta x_1} u^{n+1,m+1}_{1(q+1,2)} + \frac{\lambda}{2\Delta x_1} u^{n+1,m+1}_{1(q-1,2)} - \frac{\lambda}{2\Delta x_2} u^{n+1,m+1}_{2(q,3)} + \frac{\lambda}{2\Delta x_2} u^{n+1,m+1}_{2(q,1)}$$

$$p^{n+1,m+1}_{q,2} = p^{n+1,m}_{q,2} - \frac{\lambda}{2\Delta x_1}\left\{u^{aux}_{1(q+1,2)} - \frac{\Delta t}{2\Delta x_1}\left(p^{n+1,m}_{q+2,2} - \frac{1}{2}\left(p^{n+1,m+1}_{q,2} + p^{n+1,m}_{q,2}\right)\right)\right\}$$

$$+ \frac{\lambda}{2\Delta x_1}\left\{u^{aux}_{1(q-1,2)} - \frac{\Delta t}{2\Delta x_1}\left(\frac{1}{2}\left(p^{n+1,m+1}_{q,2} + p^{n+1,m}_{q,2}\right) - p^{n+1,m}_{q-2,2}\right)\right\} -$$

$$-\frac{\lambda}{2\Delta x_2}\left\{ u^{aux}_{2(q,3)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,4} - \frac{1}{2}\left(p^{n+1,m+1}_{q,2} + p^{n+1,m}_{q,2}\right)\right)\right\} + \frac{\lambda}{2\Delta x_2} u^{aux}_{2(q,1)} \quad , q>2$$

i.e.

$$p^{n+1,m+1}_{q,2} = \left(1+\alpha_1+\frac{1}{2}\alpha_2\right)^{-1}\left[\left(1-\alpha_1-\frac{1}{2}\alpha_2\right)p^{n+1,m}_{q,2} - \lambda D u^{aux}\Big|_{q,2} + \alpha_1\left(p^{n+1,m}_{q+2,2}+p^{n+1,m}_{q-2,2}\right) + \alpha_2 p^{n+1,m}_{q,4}\right] \quad , q>2$$

$$p^{n+1,m+1}_{q,2} = \left(1+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}\right)^{-1}\left[\left(1-\frac{\alpha_1}{2}-\frac{\alpha_2}{2}\right)p^{n+1,m}_{q,2} - \lambda D u^{aux} + \alpha_1 p^{n+1,m}_{q+2,2} + \alpha_2 p^{n+1,m}_{q,4}\right] \quad , q=2 \qquad (26\ b)$$

where

$$D u^{aux}\Big|_{q,2} = \frac{1}{2\Delta x_1}\left(u^{aux}_{1(q+1,2)} - u^{aux}_{1(q-1,2)}\right) + \frac{1}{2\Delta x_2}\left(u^{aux}_{2(q,3)} - u^{aux}_{2(q,1)}\right)$$

Similarly the equations for $p^{n+1,m+1}_{q,1}$ on the boundary β (e.g the line $x_2 = 0$) are given as follows,

$$D u^{n+1,m+1}\Big|_{q,1} = \frac{2}{\Delta x_2}\left[\left(u^{n+1,m+1}_{2(q,2)} - u^{n+1,m+1}_{2(q,1)}\right) - \frac{1}{4}\left(u^{n+1,m+1}_{2(q,3)} - u^{n+1,m+1}_{2(q,1)}\right)\right] + \frac{1}{2\Delta x_1}\left(u^{n+1,m+1}_{1(q+1,1)} - u^{n+1,m+1}_{1(q-1,1)}\right)$$

where,

$$u^{n+1,m+1}_{2(q,2)} = u^{aux}_{2(q,2)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,3} - \frac{1}{2}\left(p^{n+1,m+1}_{q,1} + p^{n+1,m}_{q,1}\right)\right) \quad , q>1$$

$$u^{n+1,m+1}_{2(q,2)} = u^{aux}_{2(q,2)} \quad , q=1$$

$$u^{n+1,m+1}_{2(q,1)} = u^{aux}_{2(q,1)}$$

$$u^{n+1,m+1}_{2(q,3)} = u^{aux}_{2(q,3)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,4} - p^{n+1,m}_{q,2}\right) \quad , q>1$$

$$u^{n+1,m+1}_{2(q,3)} = u^{aux}_{2(q,3)} \quad , q=1$$

$$u^{n+1,m+1}_{1(q+1,1)} = u^{aux}_{1(q+1,1)}$$

$$u^{n+1,m+1}_{1(q-1,1)} = u^{aux}_{1(q-1,1)}$$

hence,

$$p^{n+1,m+1}_{q,1} = p^{n+1,m}_{q,1} - \frac{2\lambda}{\Delta x_2} u^{n+1,m+1}_{2(q,2)} + \frac{2\lambda}{\Delta x_2} u^{n+1,m+1}_{2(q,1)} + \frac{2\lambda}{4\Delta x_2} u^{n+1,m+1}_{2(q,3)} - \frac{2\lambda}{4\Delta x_2} u^{n+1,m+1}_{2(q,1)}$$

$$- \frac{\lambda}{2\Delta x_1} u^{n+1,m+1}_{1(q+1,1)} + \frac{\lambda}{2\Delta x_1} u^{n+1,m+1}_{1(q-1,1)}$$

i.e

$$p^{n+1,m+1}_{q,1} = p^{n+1,m}_{q,1} - \frac{2\lambda}{\Delta x_2}\left\{ u^{aux}_{2(q,2)} - \frac{\Delta t}{2\,\Delta x_2}\left(p^{n+1,m}_{q,3} - \frac{1}{2}\left(p^{n+1,m+1}_{q,1} + p^{n+1,m}_{q,1}\right)\right)\right\}$$

$$+ \frac{2\lambda}{\Delta x_2} u^{aux}_{2(q,1)} + \frac{1}{4}\frac{2\lambda}{\Delta x_2}\left\{ u^{aux}_{2(q,3)} - \frac{\Delta t}{2\Delta x_2}\left(p^{n+1,m}_{q,4} - p^{n+1,m}_{q,2}\right)\right\}$$

$$- \frac{2\lambda}{4\Delta x_2} u^{aux}_{2(q,1)} - \frac{\lambda}{2\Delta x_1} u^{aux}_{1(q+1,1)} + \frac{\lambda}{2\Delta x_1} u^{aux}_{1(q-1,1)} \;, \quad q > 1$$

$$p^{n+1,m+1}_{q,1} = (1+2\alpha_2)^{-1}\left[(1-2\alpha_2) p^{n+1,m}_{q,1} - \lambda D u^{aux}_{q,1} + 4\alpha_2\left(p^{n+1,m}_{q,3} - \frac{1}{4}\left(p^{n+1,m}_{q,4} - p^{n+1,m}_{q,2}\right)\right)\right], \quad q > 1 \qquad (26\ c)$$

$$p^{n+1,m+1}_{q,1} = p^{n+1,m}_{q,1} - \lambda D u^{aux}_{q,1} \;, \quad q = 1$$

other equations can be derived on the other boundaries, e.g $p_{1,r}$ i.e the line $x_1 = 0$ etc, by similar expresions, $Du^{aux}_{q,1} = \frac{2}{\Delta x_2}\left[\left(u^{aux}_{2(q,2)} - u^{aux}_{2(q,1)}\right) - \frac{1}{4}\left(u^{aux}_{2(q,3)} - u^{aux}_{2(q,1)}\right)\right]$

$$+ \frac{1}{2\Delta x_1}\left(u^{aux}_{1(q+1,1)} - u^{aux}_{1(q-1,1)}\right)$$

it is clear that we consider, u^{aux}_i at the boundary is interpreted as u^{n+1}_i.

The whole iteration system i.e. equations (26 a), (26 b), (26 c), converges for all $\lambda > 0$ and converges fastest when $\lambda \sim \lambda_{opt}$.

Because of our representation of $Du = 0$, which expresses the balance of mass in a rectangle of sides $2\Delta x_i$, $i = 1, 2$. The pressure iterations split into to calculations on intertwined meshes , coupled at the boundary. The most efficient ordering for performing the iterations are such that resulting over-all scheme is a DuFort-Frankel scheme for each one of the intertwined meshes. The iterations are to be done until for some k,

$$\max_{q,r} \left| p_{q,r}^{n+1,k+1} - p_{q,r}^{n+1,k} \right| \leqq \varepsilon$$

where ε is a given small number.
The new velocities u_i^{n+1}, $i = 1, 2$ are to be evaluated using (25 b), (25 c), (25 d), (25 e). This must be done only after $p^{n+1,m}$ are converged. It is also better to evaluating Du^{aux} in at the beging of each iteration. There are two advantages for this iteration procedure (1) Du^{n+1} can be made as small as one desire independently of the error in Du^n (2) we could then use latest iterate $p^{n+1,k+1}$ to evaluate u_i^{n+1} through formula such as,

$$u_i^{n+1} = u_i^{aux} - G_i p^{n+1,k+1}$$

where $G_i p$ approximates $\frac{\partial p}{\partial x_i}$. if $\Delta t = O(\Delta x^2)$, when $p^{n+1,k+1}$ and $p^{n+1,k}$ differ by less than ε, $Du^{n+1} = O(\varepsilon/\lambda)$. Also a gain in accuracy appears, which can use to relax the convergence messure for iterations. This gain in accuracy is due to the fact that u_i^{n+1} are evaluated by using an appropriate combination of $p^{n+1,k}$ and $p^{n+1,k+1}$, rather than $p^{n+1,k+1}$.
The problem of stability and convergence will be supported by numerical evidence.

Solution of a Test Problem.

Our method can be applied to a simple-two-dimensional test problem. $\mathcal{D}$ is the square $0 \leqq x_i \leqq \pi$, i = 1, 2. The external forces E_1, E_2 assumed to be zero. i.e.

$E_1 = E_2 = 0$

The boundary data are,

$$u_1\Big|_{x_1=0} = -\sin x_2 e^{-2t}, \qquad u_1\Big|_{x_2=0} = 0$$

$$u_1\Big|_{x_1=\pi} = \sin x_2\, e^{-2t}, \qquad u_1\Big|_{x_2=\pi} = 0$$

$$u_2\Big|_{x_1=0} = 0, \qquad u_2\Big|_{x_2=0} = \sin x_1\, e^{-2t}$$

$$u_2\Big|_{x_1=\pi} = 0, \qquad u_2\Big|_{x_2=\pi} = -\sin x_1 e^{-2t}$$

also the initial data are,

$$u_1\Big|_{t=0} = -\cos x_1 \sin x_2, \qquad u_2\Big|_{t=0} = \sin x_1 \cos x_2$$

The exact solution of the problem is.

$u_1 = -\cos x_1 \sin x_2\, e^{-2t}$, $u_2 = \sin x_1 \cos x_2\, e^{-2t}$

$$p = -R\frac{1}{4}(\cos 2x_1 + \cos 2x_2)\, e^{-4t}$$

where R is the Reynolds number,

We first evaluate λ_{opt} for the equation,

$$-Lu = f$$

in $\mathcal{D}$ with a grid of mesh widths $2\Delta x_1$, $2\Delta x_2$, and u k known on the boundary then,

$$\omega_{opt} = \frac{2}{1 + (1-\alpha^2)^{1/2}}$$

where $\alpha = \frac{1}{2}(\cos 2\Delta x_1 + \cos 2\Delta x_2)$ is the largest eigenvalue of the associated Jacobi matrix [5] , [7] .

weput $\quad q = \frac{\lambda_{opt}}{2}\left(\frac{\Delta t}{\Delta x_1^2} + \frac{\Delta t}{\Delta x_2^2}\right)$

equation (18) can be written in the form

$$\omega_{opt} = \frac{8q}{1+4q}$$

then

$$q = \frac{1}{(1-\alpha^2)^{1/2}}$$

and

$$\lambda_{opt} = \frac{4}{(\Delta t/\Delta x_1^2 + \Delta t/\Delta x_2^2)}\ \frac{1}{(1-\alpha^2)^{1/2}}$$

if we assume $\Delta x_1 = \Delta x_2 = \Delta x$, then,

$$\lambda_{opt} = \frac{2\Delta x^2}{\Delta t \sin(2\Delta x)}$$

CHAPTER III

We record here the list of symbols, Algol 60 listing, and logic flow chart for the two-dimensional viscous flow test problem given at the end of chapter II.

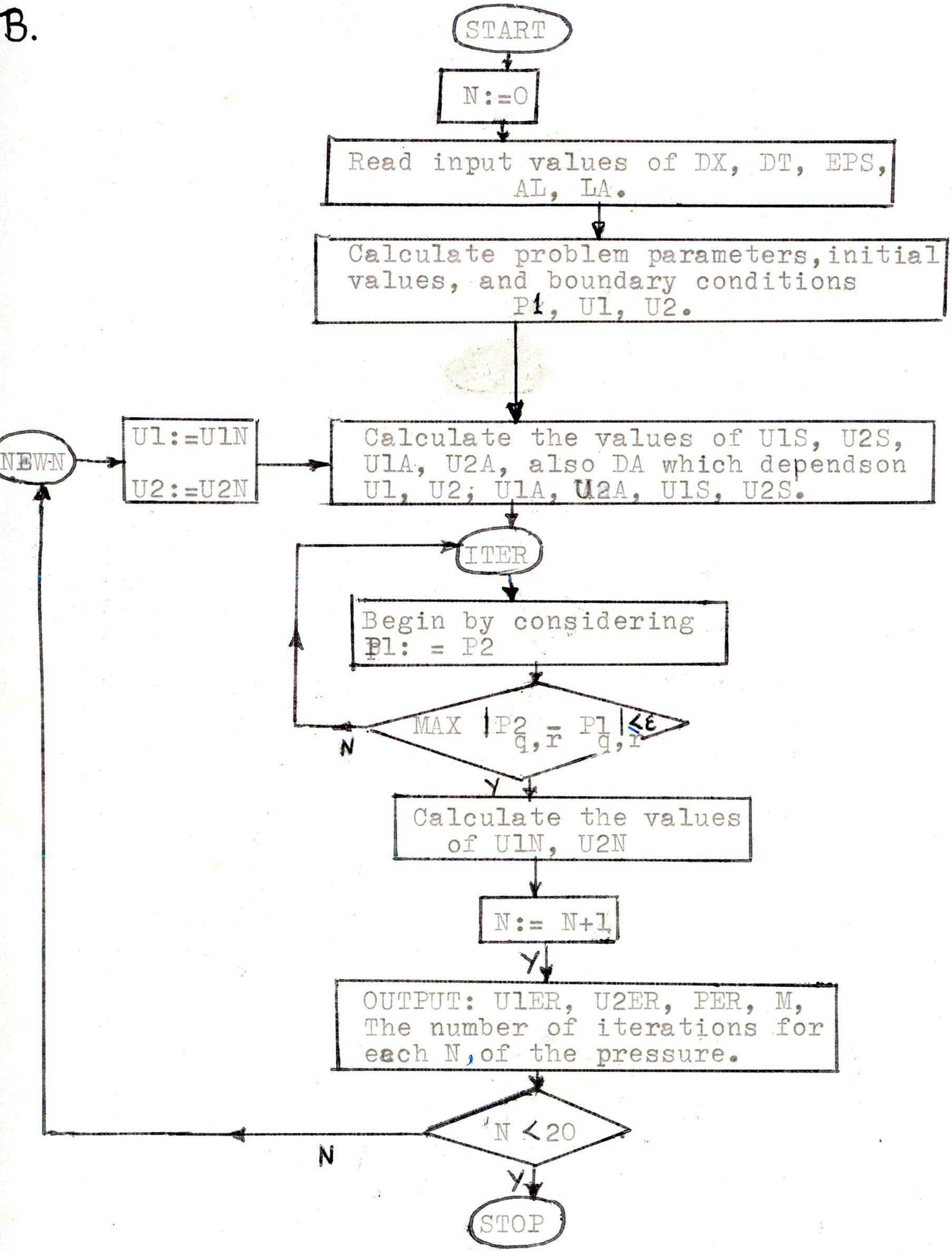
A. List of Symbols

U1A[Q,R]	$u^{aux}_{1(q,r)}$,	x-component of auxiliary velocity.
U2A[Q,R]	$u^{aux}_{2(q,r)}$,	y-component of auxiliary velocity.
U1S[Q,R]	$u^{*}_{1(q,r)}$,	x-component of intermidiate velocity.
U2S[Q,R]	$u^{*}_{2(q,r)}$,	y-component of intermidiate velocity.
P1[Q,R]	$p^{n+1,m}_{(q,r)}$,	the pressure before iteration.
P2[Q,R]	$p^{n+1,m+1}_{(q,r)}$,	the pressure after iteration.
U1[Q,R]	$u^{n}_{1(q,r)}$,	x-component of velocity at time step n.
U2[Q,R]	$u^{n}_{2(q,r)}$,	y-component of velocity at time step n.
U1N[Q,R]	$u^{n+1,m+1}_{1(q,r)}$	the computed x-component of velocity.
U2N[Q,R]	$u^{n+1,m+1}_{2(q,r)}$,	the computed y-component of velocity.
DA[Q,R]	$Du^{aux}\vert_{(q,r)}$	
LA	$\lambda_{opt.}$	

AL	α
DT	time step Δt.
Dx	mesh interval Δx.
EPS	ε
MAX	$\max_{q,r} \left\| p_{q,r}^{n+1,m+1} - p_{q,r}^{n+1,m} \right\|$ at each time step.
U1ER	absolute difference between the exact and computed value of $u_{1(q,r)}$.
U2ER	absolute difference between the exact and computed value of $u_{2(q,r)}$.
PER	absolute difference between the exact and computed value of p.
N	number of time steps.
M	number of iterations.
Q	x-coordinate
R	y-coordinate
T	time

The Algol listing (Section C) and flow chart (Section B) follow.

B.


START
N:=0
Read input values of DX, DT, EPS, AL, LA.
Calculate problem parameters, initial values, and boundary conditions P1, U1, U2.
NEWN
U1:=U1N
U2:=U2N
Calculate the values of U1S, U2S, U1A, U2A, also DA which dependson U1, U2; U1A, U2A, U1S, U2S.
ITER
Begin by considering P1: = P2
MAX |P2 - P1| < ε
q,r q,r
N
Y
Calculate the values of U1N, U2N
N:= N+1
Y
OUTPUT: U1ER, U2ER, PER, M, The number of iterations for each N, of the pressure.
N < 20
N
Y
STOP

Flow chart

Schemes A and B, were used for the solution of the test problem, i.e. formulae (6 a), (6 b), (7 a), (7 b), (7 c) were used to evaluate u_i^{aux} . ε is the convergence criterion. In tables I, II, n is the number of time steps; $e(u_i)$, i = = 1, 2, are the maxima over $\mathfrak{D}$ of the difference between the exact and the computed solution u_i, e(p) in the tables represents the maximum over the grid of the differences between the exact pressure at time $n\Delta t$ and the computed p^n divided by R. The accuracy of the scheme is to be judged by the smallness of $e(u_i)$. m is the number of iterations.

Table I

scheme A ; $\Delta x = \pi/39$; $\Delta t = 2\Delta x^2$; $\varepsilon = \Delta x^2$; $R = 1$

n	$e(u_i)$	$e(u_2)$	$e(p)$	m
1	$2 \cdot 8 \times 10^{-4}$	$2 \cdot 6 \times 10^{-4}$	0•0243	1
2	$2 \cdot 7 \times 10^{-4}$	$2 \cdot \ \times 10^{-4}$	0•0136	7
3	$1 \cdot 5 \times 10^{-4}$	$1 \cdot 3 \times 10^{-4}$	0•0069	4
4	$1 \cdot 8 \times 10^{-4}$	$1 \cdot 9 \times 10^{-4}$	0•0145	4
5	$1 \cdot 3 \times 10^{-4}$	$1 \cdot 7 \times 10^{-4}$	0•0089	5
6	$1 \cdot 3 \times 10^{-4}$	$1 \cdot 8 \times 10^{-4}$	0•0116	4
7	$1 \cdot 6 \times 10^{-4}$	$1 \cdot 9 \times 10^{-4}$	0•0144	4
9	$1 \cdot 4 \times 10^{-4}$	$1 \cdot 7 \times 10^{-4}$	0•0147	4
10	$1 \cdot 3 \times 10^{-4}$	$1 \cdot 6 \times 10^{-4}$	0•0156	4
20	$1 \cdot 8 \times 10^{-4}$	$2 \cdot 3 \times 10^{-4}$	0•0241	4

Table II

scheme B; $\Delta x = \pi/39$; $\Delta t = \Delta x^2$; $\varepsilon = \Delta x^2$; $R = 20$

n	$e(u_1)$	$e(u_2)$	$e(p)$	m
1	$3 \cdot 9 \times 10^{-3}$	$4 \cdot 4 \times 10^{-3}$	0·0404	16
3	$5 \cdot 9 \times 10^{-3}$	$6 \cdot 0 \times 10^{-3}$	0·0466	11
5	$8 \cdot 5 \times 10^{-3}$	$6 \cdot 7 \times 10^{-3}$	0·0505	10
7	$1 \cdot 0 \times 10^{-2}$	$7 \cdot 4 \times 10^{-3}$	0·0551	10
9	$1 \cdot 1 \times 10^{-2}$	$7 \cdot 9 \times 10^{-3}$	0·599	10
20	$1 \cdot 0 \times 10^{-2}$	$7 \cdot 8 \times 10^{-3}$	0·0839	10

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