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Investigating the Limiting Distribution of Character Sums

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Investigating the Limiting Distribution of Character Sums

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Abstract

Dirichlet characters, and their partial sums, play a fundamental role in analytic number theory. In this thesis we study various distributions of these character sums, and find the limiting distribution as the conductor tends to infinity. We consider the limit of two main distributions: the continuous paths of character sums modulo a prime q on the complex plane, and partial sums of quadratic characters with prime conductors in the dyadic range $[Q, 2Q]$ for some $Q > 0$. The limiting distributions are formulated as Fourier series with Steinhaus and Rademacher random multiplicative functions as the respective Fourier coefficients.

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I'd like to thank my supervisor, Jonathan Bober, for all of his help throughout my PhD. Additionally, conversations with Oleksiy Klurman and Andrew Granville have been very insightful, as well as the anonymous referee for my paper. I'm not sure who they were, but I really appreciate all of their comments.

This PhD probably would've happened without Emmy, my little kitten, but would've been significantly less enjoyable. The same could be said to all the members of Office 1.23, affectionately known as the 'Pit' (I still am adamant I'm the one who came up with the moniker). We never did do the 'Pit Olympics', but we definitely did procrastinate enough during our time there.

Thank you also to my friends who stayed with me and the friends I made along the way; you definitely made the past 4 years incredibly enjoyable. Thank you to the Joneses and Albests for making the South West a home; I can't see myself leaving anytime soon. And to my number 1 cheerleaders: Mum, Dad, and Zara - thank you for supporting me during the highs and lows! A special mention should go to my dad, who told everyone back in 2018 that number theory involves me picking a random number and studying it for 4 years. While multiple people asked me how I was enjoying 'the number 2', I'd like to inform them I actually looked at all prime numbers (although '2' was significantly harder than the rest).

Finally, thank you to my partner David. I would promise that you don't have to read long mathematical drafts of my work again, but that would be a lie! In all honesty, I couldn't have done it without you.

Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: .... DATE: ....

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Notation

Symbol	Meaning
$f(x) \ll_k g(x)$	There exists a constant $C := C_k > 0$ such that $\|f(x)\| \leq Cg(x)$.
$f(x) \gg_k g(x)$	There exists a constant $C := C_k > 0$ such that $\|f(x)\| \geq Cg(x)$.
$f(x) = O(g(x))$	$f(x) \ll g(x)$.
χ_0	The principal character ($\chi_0(n) = 1$ for all n that is co-prime to some modulus).
$e(\theta)$	$e^{2\pi i\theta}$ (the complex exponential with period 1).
$p^k \\| n$	$p^j \mid n$ and $p^{k+1} \nmid n$.
$d_N(x)$	$\sum_{x_1 \cdots x_n = x} 1$ (the Nth divisor function for $x \in \mathbb{N}$).
γ	The Euler-Mascheroni constant (roughly 0.57721) [80].
$L(\chi, s)$	Dirichlet L-function $\sum_{n \geq 1} \chi(n) n^{-s}$, for $Re(s) > 1$.
$\omega(n)$	The distinct prime factors of $n \in \mathbb{N}$.
$\Omega(n)$	The prime factors of $n \in \mathbb{N}$, counted with multiplicity.
$\mu(n)$	The Möbius function (± 1 for square free n with an even or odd number of prime factors respectively, and 0 otherwise).
$\phi(q)$	The counting function for all the integers $\leq q$ which are coprime to q, known as the Euler Totient Function.
$\pi(n)$	The prime counting function, counting the number of prime numbers $\leq n$.
$\pi^*(n)$	$(\pi(2n) - \pi(n))$, or counting all prime numbers in-between $2n$ and n.
$P^+(n)$	The largest prime divisor of n (we take $P^+(1) = 1$ as convention).
$P^-(n)$	The smallest prime divisor of n (we take $P^-(1) = \infty$ as convention).
$n = \square$	Shorthand for when n is a square number.
$\mathbb{T}$	$\mathbb{R}/\mathbb{Z}$, or the real numbers modulo 1.
$[x]$	The largest integer not exceeding $x \in \mathbb{R}$.
$\{x\}$	The fractional part of $x \in \mathbb{R}$, or equivalently $x - [x]$.
$C([0, 1])$	Continuous functions taking values $\in [0, 1]$.

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Chapter 1

Introduction

People often attribute the start of analytic number theory to Peter Gustav Lejeune Dirichlet who, in 1837, introduced the world to studying primes in arithmetic progressions [22]. Analytic number theory looks into the multiplicative and additive structure of integers, referred to as multiplicative and additive number theory respectively. In this thesis we will firmly focus on the former, specifically one of Dirichlet's main contributions to multiplicative number theory: Dirichlet characters.

A function $f : \mathbb{N} \to \mathbb{C}$ is multiplicative if $f(a \cdot b) = f(a) \cdot f(b)$ when a, b are coprime, and is called completely multiplicative if $f(a \cdot b) = f(a) \cdot f(b)$ for all integers a, b. As a result, due to the fundamental theorem of arithmetic [51], multiplicative functions are completely determined by their values at the prime powers, and completely multiplicative functions depend only on the prime numbers. Since they interact so well with the factorisation of integers, multiplicative functions are a good tool for studying multiplicative structures.

Multiplicative functions have been rigorously studied by mathematicians over the past couple of hundred years, notably Dirichlet, who defined Dirichlet characters in the early 1800s [26]. Dirichlet characters are group characters on $(\mathbb{Z}/q\mathbb{Z})^*$, and are often extended to the following definition.

Definition 1.0.1. A **Dirichlet Character modulo** q is a function $\chi : \mathbb{Z} \to \mathbb{C}^\times$ where

- $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$,

- $\chi(n+q) = \chi(n)$ for all $n \in \mathbb{Z}$,

- $\chi(n) = 0$ for $\gcd(n, q) > 1$.

They always have size 1, unless at a factor of the period, and there are $\phi(q)$ Dirichlet characters for every modulus q. These characters were first introduced as a way to study primes in arithmetic progressions [27]. This is due to orthogonality relations that Dirichlet characters satisfy:

$$\sum_{n \bmod q} \chi(n) = \begin{cases} \phi(q) & ; \chi = \chi_0, \\ 0 & ; \text{otherwise.} \end{cases}$$
$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \phi(q) & ; n \equiv 1\ (q), \\ 0 & ; \text{otherwise.} \end{cases} \tag{1.1}$$

Here χ_0 is the principal character modulo q[1]. This is especially useful for detecting congruence conditions; as a result, Dirichlet characters appear in many analytic number theory proofs.

It is also convenient to call characters odd or even, depending on whether $\chi(-1)$ equals -1 or 1 respectively. A Dirichlet character is primitive if it generates every value $e(j/q)(= e^{2\pi i j/q})$ for $j \in [1, q]$. For prime moduli, which is the only case we are considering in the thesis, every non-principal character is primitive.

There exists a class of real Dirichlet characters that only take values $\{0, \pm 1\}$, with at least one -1 value. These are also known as quadratic characters as they have order 2, and for prime modulus such a character is given by the Legendre symbol $\left(\frac{\cdot}{q}\right)$ [62].

[1]The principal character modulo q is $\chi_0(n) = 1$ for all $(n, q) = 1$, and vanishes otherwise.

Let us consider all non principal Dirichlet characters modulo q, that is $\chi(n)$ is not 1 for all $(n, q) = 1$. All values are multiplicative and distributed on the unit circle, meaning the total sum $\sum_{n=1}^{q} \chi(n)$ vanishes. To show $\sum_{n \leq q} \chi(n) = 0$, and subsequently proving Equation (1.1), take $a \in \mathbb{Z}$ such that $\gcd(a, q) = 1$ and $\chi(a) \neq 1$. Using the periodicity of characters, we can rearrange the order of the sum. Therefore, noting that $n \mapsto a \cdot n$ is a bijection on $\mathbb{Z}/q\mathbb{Z}$,

$$\sum_{n=1}^{q} \chi(n) = \sum_{n=1}^{q} \chi(a \cdot n).$$

Dirichlet characters are completely multiplicative, meaning we can separate $\chi(a \cdot n)$ to $\chi(a)$ multiplied by $\chi(n)$. As a result,

$$\sum_{n=1}^{q} \chi(n) = \chi(a) \sum_{n=1}^{q} \chi(n).$$

Since $\chi(a) \neq 1$, the total sum $\sum_{n=1}^{q} \chi(n)$ must equal 0.

However, what happens to the partial sums, and how large can the partial sum get before returning to 0? My research has centred on these partial sums of Dirichlet characters with prime moduli, analysing how they relate to other multiplicative functions. Character sums play a fundamental role in analytic number theory, and I have investigated the distribution of the sums, as well as properties of the distribution as the modulus tends to infinity.

For this thesis, we treat partial character sums as functions in t:

Definition 1.0.2. Let χ be a Dirichlet character with modulus q. For $t \in [0, 1]$, let

$S_\chi(t)$ be the **normalised partial character sum**

$$S_\chi(t) := \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n).$$

Partial character sums are 1-periodic in t, so we can also view $S_\chi(t)$ as a Fourier series. We have

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{k \neq 0} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)), \tag{1.2}$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(a/q).$$

Note this is only valid where t is not a discontinuity of the function, so when qt is not an integer value. This is a standard result, also known as Pólya's Fourier expansion [22, Chapter 23]. The proof of this is shown in Appendix A. Additionally, for primitive characters the absolute value of $\tau(\chi)$ is $\sqrt{q}$, hence the normalising factor of $1/\sqrt{q}$ in our definition of $S_\chi(t)$.

There has been a lot of interest in the maxima of partial character sums. Since Dirichlet characters $\chi(n)$ vanish when the modulus q shares a prime factor with n, and are of size 1 otherwise[2], we have the trivial bound [54, Chaper VIII, Section 1]

$$\max_{t \in [0,1]} |S_\chi(t)| \leq \frac{\phi(q)}{\sqrt{q}}.$$

For prime q, $\phi(q) = q - 1$, giving an upper bound of approximately $\sqrt{q}$. This has

[2] For this thesis we are only considering prime moduli. As a result, $\chi(n)$ vanishes only when the modulus divides n. As a result, for very large conductors, characters are of size 1 almost everywhere.

been significantly improved, and notably the estimates hold as q gets large. In 1918, Pólya and Vinogradov independently found the Pólya-Vinogradov bound [73, 85]

$$\max_{t\in[0,1]} |S_\chi(t)| \ll \log q.$$

The implicit constant has been improved since [13,35,45], with the sharpest constant [60, 74]:

$$|S_\chi(t)| \leq \begin{cases} \left(\dfrac{2}{\pi^2} + o(1)\right)\log q & ;\ \chi(-1) = 1, \\ \left(\dfrac{1}{\pi^2} + o(1)\right)\log q & ;\ \chi(-1) = -1. \end{cases}$$

When q belongs to certain structured subsequences of integers, the Pólya-Vinogradov inequality can be improved. For instance, Goldmakher [31, 32] showed that if q is smooth[3] then $|S_\chi(t)| = o(\log q)$. In the late 1970s, assuming the Generalised Riemann Hypothesis (GRH)[4], Montgomery and Vaughan [65] discovered

$$|S_\chi(t)| \ll \log\log q. \tag{1.3}$$

The implied constant has since been improved by various mathematicians, but most notably by Granville and Soundararajan [35] in 2007. This is also the best result possible[5], as proven in 1932 by Paley [69].

Theorem 1.0.1. *[67, Theorem 9.24] Let χ_d be a real Dirichlet character modulo d.*

[3] More specifically, $\prod_{p|q} p \leq \exp\left((\log q)^{3/4}\right)$.

[4] The Generalised Riemann Hypothesis states that for $s \in \mathbb{C}$ and $s \notin \mathbb{R}_{<0}$, $L(\chi, s) = 0$ only if $Re(s) = 1/2$. For $\chi = 1$, this is the Riemann hypothesis.

[5] This is the best result uniformly over all characters. For fixed odd order characters, the upper bound can be improved on GRH, with a nearly matching unconditional lower bound [35].

There exists a positive constant c such that

$$\max_{M,N}\left|\frac{1}{\sqrt{d}}\sum_{n=M+1}^{M+N}\chi_d(n)\right| > c\log\log d$$

for infinitely many d[6].

A stronger form of Paley's result was proven by Bateman and Chowla [4]. This result was then extended to odd complex characters by Granville and Soundararajan [35].

Theorem 1.0.2. *[35, Theorem 3] Let q be a large prime and $C \geq 0$. There exists an absolute constant C_0 such that for at least $q^{1-C_0/(\log\log q)^2}$ characters χ modulo q, with $\chi(-1) = -1$, we have*

$$\frac{1}{\sqrt{q}}\sum_{n\leq x}\chi(n) = C\log\log q + O(\sqrt{\log\log q})$$

for all but $o(q)$ natural numbers $x \leq q$.

Combining the two theorems, the conditional bound in Equation (1.3) is optimal[7], as

$$\max_{t\in[0,1]}|S_\chi(t)| \gg \log\log q.$$

Additionally, we can consider short character sums with prime conductor and

[6] Here d is a positive quadratic discriminant, which is defined in Definition 4.2.1 and used in Chapter 4

[7] Additionally, Lamzouri [58] found the optimal implicit constant for even order characters. For more information, see [59].

how it exhibits cancellation as a function of t. Burgess [16] proved

$$|S_\chi(t)| \ll t^{1/2} q^{3/16} \log q,$$

for small values of t, specifically $t < q^{-3/8}$. Note the bound is trivial if you take $t \leq q^{-3/4}$ [43]. The result has been subsequently slightly improved by Hildebrand [44]. Assuming the Generalised Riemann Hypothesis, Granville and Soundararajan [34] proved that, if $x \ll q^{1/2+o(1)}$ and $\log x / \log\log q \to \infty$ as $q \to \infty$,

$$\left| \sum_{n \leq x} \chi(n) \right| = o(x).$$

This is the best possible result, as for any given $A > 0$ and prime q, there exists a non principal character such that

$$\left| \sum_{n \leq x} \chi(n) \right| \gg_A x,$$

for $x = \log^A q$ [34, Corollary A].

These results are about the maximum for all Dirichlet characters of a certain modulus. If we instead considered the average maximum of character sums, the result is significantly smaller.

Theorem 1.0.3. *[66, Theorem 1] For any real $k > 0$,*

$$\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \max_N \left| \frac{1}{\sqrt{q}} \sum_{n=1}^{N} \chi(n) \right|^{2k} \ll_k 1,$$

where the summation is over all non-principal characters modulo q.

In other words, for all $\varepsilon > 0$, $(100-\varepsilon)\%$ of Dirichlet characters satisfy $|S_\chi(t)| \ll_\varepsilon$

1. This leads naturally to the following questions: how often is the maximum of $S_\chi(t)$ large? What are the properties of the characters where the maximum of $S_\chi(t)$ is large? What is the distribution of the character sums, not just their absolute value? What happens as the modulus goes to infinity? Additionally, Paley showed there is an infinite class of quadratic characters which are large, so what happens if we restrict our distribution to real character sums? The aim of this thesis is to answer these questions.

1.1 Summary of the Chapters

In this thesis, we investigate the distribution of character sums $\sum \chi(n)$. In Chapter 2, we consider the sequence of distributions of all character sums over prime moduli, and find the limiting distribution as the modulus tends to infinity. Then, in Chapter 3, we investigate properties of the limiting distribution, such as the support of the law[8] of the random process used. Next, in Chapter 4, we focus on quadratic characters, of which there is only one for every odd prime conductor q. We investigate the sequence of real character sums with moduli $q \in [Q, 2Q]$ for some large Q, and find the limiting distribution as $Q \to \infty$. In both cases, the limiting distribution is an almost surely continuous random process. These random processes are defined in Sections 4.1 and 4.1.

However the character sum $S_\chi(t)$ is a step function, with jump discontinuities at every $t \in \frac{1}{q}\mathbb{Z}$. In order to circumvent the difficulties posed by these discontinuities, it is natural to consider a continuous modification, where the steps are replaced by straight line interpolations.

[8] A set $C \subset C([0,1])$ is the support of the law of F if C is the intersection of all closed sets $K \subset C([0,1])$ such that $\mathbb{P}(F \in K) = 1$.

Definition 1.1.1. Character paths are paths in the complex plane formed by drawing a straight line between the successive partial sums

$$S_\chi(x) = \tfrac{1}{\sqrt{q}} \textstyle\sum_{n \le qx} \chi(n), \quad S_\chi(x + 1/q) = \tfrac{1}{\sqrt{q}} \textstyle\sum_{n \le qx+1} \chi(n),$$

for $x \in [0, 1)$ and $qx \in \mathbb{Z}$. We parameterise character paths by the function

$$f_\chi(t) \coloneqq S_\chi(t) + \frac{\{qt\}}{\sqrt{q}} \chi\left(\lceil qt \rceil\right),$$

where $\{x\}$ is the fractional part of the number x.

Note the difference between any character sum $S_\chi(t)$ and character path $f_\chi(t)$ is bounded by $\frac{1}{\sqrt{q}}$. Character paths inherit many properties of character sums, most notably periodicity. As such, we can approximate $f_\chi(t)$ as a truncated Fourier series, also shown in Appendix A. Using character paths, we will define a distribution taking values in the Banach space[9] $C([0, 1])$.

The limiting distributions in Chapters 2 and 4 are formulated on Fourier series with random multiplicative functions[10] as the Fourier coefficients. Paley and Zygmund introduced random Fourier series in the 1930's [70–72]. Random multiplicative functions were first introduced by Wintner [87] in 1944 as a model for the Möbius function $\mu(n)$. Since then, they have been used as a model for Dirichlet characters. The sums of random multiplicative functions have a long history. See [12, 37–39, 41, 42] for examples of recent work.

[9] A **Banach space** is a complete normed space. This is a vector space over a scalar field with a distinguished norm.

[10] The two random multiplicative functions used are defined in Definitions 2.1.2 and 4.1.2 respectively.

1.1.1 Summary of Chapter 2 and 3

When investigating the maximum of character sums, Bober, Goldmakher, Granville and Koukoulopoulos [9] studied the distribution function for $\tau > 0$,

$$\Phi_q(\tau) := \frac{1}{\phi(q)}\#\left\{\chi \bmod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi}\tau,\ q \text{ prime}\right\},$$

where γ is the Euler-Mascheroni constant. The limiting distribution of Φ_q is

$$\Phi(\tau) := \mathbb{P}\left(\max_t |F(t)| > 2e^\gamma\tau\right),$$

where $F(t)$ is a random Fourier series[11] properly defined later in the thesis (Equation (2.3)). In Chapter 2, we investigate the distribution of character sums, not just their maxima, and find the limiting distribution as the modulus goes to infinity through the primes. We split the characters sums depending on if they're odd or even. We find the limiting distribution is, unsurprisingly, the same Fourier series $F(t)$, albeit also split by parity.

Theorem 1.1.1 (see Theorem 2.1.1)**.** *Let $(\mathcal{F}_{q,\pm}(t))_q$ be the sequence of the distributions of character paths*[12]

$$\mathcal{F}_{q,\pm}(t) := \{f_\chi(t) : \chi \mod q, \chi(-1) = \pm 1\}.$$

The sequence weakly converges to the random process[13] *$F_\pm(t)$ in the Banach space $C([0,1])$ as q tends to infinity through the primes.*

[11] The random series uses Steinhaus random multiplicative functions.

[12] The distribution $\mathcal{F}_{q,\pm}(t)$ will be further explained in Chapter 2.

[13] This is the same random process as in Bober, Goldmakher, Granville, and Koukoulopoulos' work [9], and is further explained in Chapter 2.

In other words, for any continuous and bounded map

$$\psi : C([0,1]) \to \mathbb{C},$$

we have, for prime q,

$$\lim_{q\to\infty} \mathbb{E}\left(\psi\left(\mathcal{F}_{q,\pm}\right)\right) = \mathbb{E}\left(\psi(F_{\pm})\right).$$

This answers an open problem set by Kowalski and Sawin [56, Section 5.2], where they consider the limiting distribution of 'Kloosterman Paths'. This chapter is taken from the author's paper 'The Limiting Distribution of Character Sums' [48].

In Chapter 3, we investigate properties of the random process $F(t)$. In finding the support of the law of $F(t)$, we uncover interesting connections on the behaviour of character sums.

Proposition 1.1.2. *Let g be in the support of the law of the random process F over $C([0,1])$. For any $\varepsilon > 0$,*

$$\liminf_{q\to\infty} \frac{1}{\phi(q)} \left| \left\{ \chi \bmod q : \sup_{t\in[0,1]} \left| \frac{1}{\sqrt{q}} \sum_{n\le qt} \chi(n) - g(t) \right| < \varepsilon \right\} \right| > 0.$$

Section 3.3 looks at some examples of functions in the support, and by the proposition we know there are ε-close character sums.

1.1.2 Summary of Chapter 4

In Chapter 2, we investigate the behaviour of complex character sums. We have a sequence of distributions of all character sums over prime moduli, and find the limiting distribution of the corresponding character paths as the modulus tends to infinity.

For every prime modulus q, there is only one (non-trivial) real character, otherwise known as the Legendre symbol $\left(\frac{\cdot}{q}\right)$. Legendre symbols are used a lot in Number Theory, and results about their sums are connected with various open problems, such as Vinogradov's quadratic non residue conjecture (see e.g. [33]).

We can apply the Pólya-Vinogradov inequality to sums of Legendre symbols, and more generally Jacobi symbols[14] [20]. Let $X \in \mathbb{N}$. Then, representing square numbers as $\square$,

$$\sum_{\substack{m \leq X \\ m \text{ odd}}} \left(\frac{m}{n}\right) = \begin{cases} \frac{X}{2}\frac{\phi(n)}{n} + O(X^{\varepsilon}) & ; \text{if } n = \square, \\ O(\sqrt{n}\log n) & ; \text{if } n \neq \square, \end{cases}$$

and

$$\sum_{\substack{n \leq X \\ n \text{ odd}}} \left(\frac{m}{n}\right) = \begin{cases} \frac{X}{2}\frac{\phi(m)}{m} + O(X^{\varepsilon}) & ; \text{if } m = \square, \\ O(\sqrt{m}\log m) & ; \text{if } m \neq \square. \end{cases}$$

For complex character sums modulo q, we know the majority of character sums $\frac{1}{\sqrt{q}}\sum \chi(n)$ are bounded by 1, proved by Montgomery and Vaughan [66] and shown in Theorem 1.0.3. Montgomery and Vaughan also showed an analogous theorem for quadratic character sums:

Theorem 1.1.3. *[66, Theorem 2] For any $k > 0$,*

$$\frac{1}{\pi(Q)} \sum_{\substack{2 < q \leq Q \\ q \text{ prime}}} \max_{N} \left| \frac{1}{\sqrt{q}} \sum_{n=1}^{N} \left(\frac{n}{q}\right) \right|^{2k} \ll_k 1.$$

[14] Jacobi symbols $\left(\frac{m}{n}\right)$ are the product of Legendre symbols $\left(\frac{m}{p_i}\right)^{a_i}$, where n is odd and $n = \prod p_i^{a_i}$.

Therefore, we would naturally guess that an analogous distribution of quadratic character sums, as was in [48], has a similiar limiting distribution.

In Chapter 4, we investigate the distribution of real quadratic character sums for the modulus q in a dyadic interval. The results of this chapter rely on the Generalised Riemann Hypothesis, the conjecture that all non-trivial zeroes of $L(s,\chi)$ lie on the critical line, where $Re(s) = 1/2$. The main result follows the same format as Theorem 1.1.1.

Theorem 1.1.4 (see Theorem 4.1.1)**.** *Let $(\mathcal{G}_{Q,\pm}(t))_q$ be the sequence of the distributions of character paths*[15]

$$\mathcal{G}_{Q,\pm}(t) := \{f_\chi(t) : q \in [Q, 2Q], \chi \bmod q \in \mathbb{R}, \chi(-1) = \pm 1\}.$$

Assuming the Generalised Riemann Hypothesis, the sequence weakly converges to the random process[16] *$G_\pm(t)$ in the Banach space $C([0,1])$ as Q tends to infinity through the primes.*
In other words, for any continuous and bounded map

$$\psi : C([0,1]) \to \mathbb{C},$$

we have

$$\lim_{Q\to\infty} \mathbb{E}\left(\psi\left(\mathcal{G}_{Q,\pm}\right)\right) = \mathbb{E}\left(\psi(G_\pm)\right).$$

For future work, the GRH assumption will hopefully be removed.

[15] The distribution $\mathcal{G}_{q,\pm}(t)$ will be further explained in Chapter 4.
[16] The random process uses Rademacher random multiplicative functions and is further explained in Chapter 4.

Chapter 2

The Distribution of Character Sums

2.1 Introduction

Given a primitive Dirichlet character χ modulo q, we define the normalised partial character sum

$$S_\chi(t) \coloneqq \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n),$$

for $t \in [0,1]$. Such character sums play a fundamental role in analytic number theory. Our goal is to study the distribution of character sums for prime modulus q, and find the limiting distribution as $q \to \infty$, answering the open problem set by Kowalski and Sawin [56, Section 5.2].

Bober, Goldmakher, Granville and Koukoulopoulos [9] investigated the maximum of these character sums. Taking $\tau > 0$, they studied the distribution function,

$$\Phi_q(\tau) \coloneqq \frac{1}{\phi(q)} \# \left\{ \chi \mod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi} \tau \right\}. \tag{2.1}$$

The limiting distribution of Φ_q is

$$\Phi(\tau) \coloneqq \mathbb{P}\left(\max_t |F(t)| > 2e^\gamma \tau \right),$$

where $F(t)$ is a random Fourier series defined later in this chapter [9, Theorem 1.4]. We find, through different methods, that the limiting distribution of character sums, not just their maxima, uses the same random series. Our main theorem can also be used to recover their result.

Recall Definition 1.1.1, continuous character paths modulo q.

Definition 2.1.1. Character paths are paths in the complex plane formed by drawing a straight line between the successive partial sums. We parameterise character paths by the function

$$f_\chi(t) \coloneqq S_\chi(t) + \frac{\{qt\}}{\sqrt{q}}\chi\left(\lceil qt \rceil\right),$$

where $\{x\}$ is the fractional part of the number x.

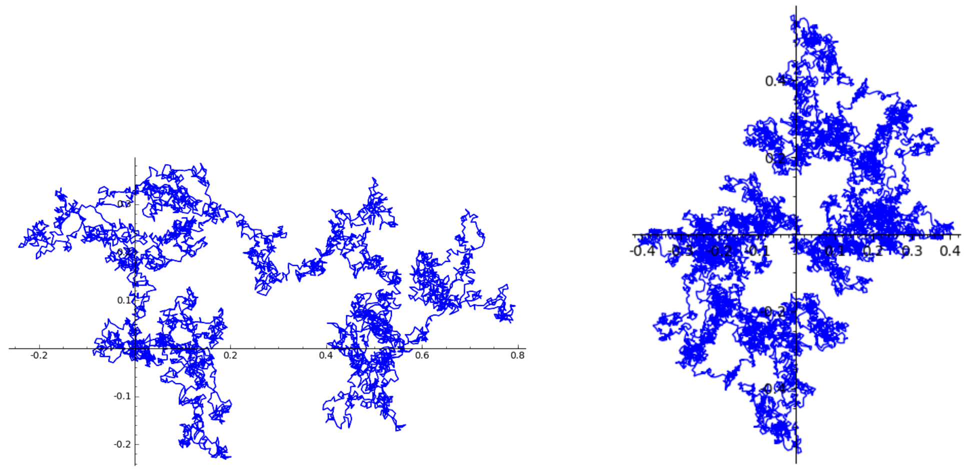
Character paths, like character sums, are periodic and have the truncated Fourier series[1]

$$\frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{0<|k|<q} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + O\left(\frac{\log q}{\sqrt{q}}\right). \tag{2.2}$$

See Appendix A for further details. Character paths are polygonal, continuous and closed. Examples of character paths on the complex plane can be seen in Figure 2.1.

We define the distribution of character paths by mapping a character χ mod q to $f_\chi(t)$ as a random process, choosing χ uniformly at random. Let us write $\mathcal{F}_q$ for

[1] Note the difference between character sums and character paths is bounded by $\frac{1}{\sqrt{q}}$, so error from the truncated Fourier series of $S_\chi(t)$ encapsulates the difference.



Odd path defined by $\chi(5) = e\left(\frac{1}{10007}\right)$. Even path defined by $\chi(5) = e\left(\frac{2}{10007}\right)$.

Figure 2.1: Character paths of an odd and even character modulo 10007.

this random process,

$$\mathcal{F}_q(t) \coloneqq \{f_\chi(t) : \chi \bmod q\}.$$

We also further define $\mathcal{F}_{q,\pm}(t)$ by fixing $\chi(-1)$ as $+1$ or -1 respectively, so distributions of character paths are dependent on the character's parity.

One of the main goals of this chapter is to find the limiting distribution of the sequence $(\mathcal{F}_q)_q$ as $q \to \infty$ through the primes. For this, we need to define Steinhaus random multiplicative functions.

Definition 2.1.2. We define,

1. **Steinhaus random variables** X_p to be random variables uniformly distributed on the unit circle $\{|z| = 1\}$.

2. **Steinhaus random multiplicative functions** X_n, $n \in \mathbb{N}$, to be

$$X_n = \prod_{p^a \| n} X_p^a,$$

where X_p are Steinhaus random variables for prime p. We extend this definition to $n \in \mathbb{Z}$ by taking $X_{-1} = \pm 1$ with probability $1/2$ each, so $X_{-|n|} = X_{-1}X_{|n|}$. (Here $p^a \| n$ means p^a strictly divides n, so $p^a \mid n$ but $p^{a+1} \nmid n$). Note these random variables live on the same ambient probability space.

Steinhaus random multiplicative functions are completely multiplicative[2], with all values distributed on the unit circle. This leads to a natural question: can we compare partial sums of characters with partial sums of Steinhaus random multiplicative functions, assuming similar behaviour? Sums of Steinhaus random multiplicative functions might be a good model for short character sums, but the periodicity of the characters means that for long character sums the model fails. This can be shown by considering moments of $\sum X_n$ and $\sum \chi(n)$ [9]. For sums of Steinhaus random multiplicative functions,

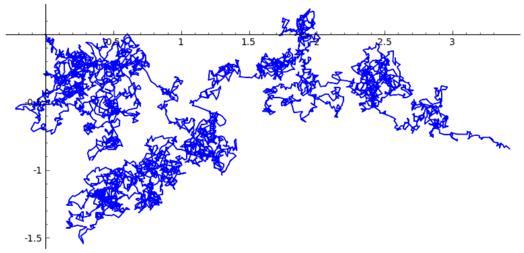
$$\mathbb{E}\left(\left|\sum_{n \leq qt} X_n\right|^2\right) \sim qt.$$

However, for character sums the periodicity means

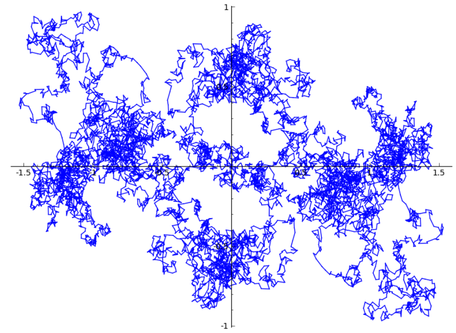
$$\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \left|\sum_{n \leq qt} \chi(n)\right|^2 \sim qt(1-t).$$

Consequently, we must find a way to incorporate the periodicity from the char-

[2]Strictly speaking, for each $x \in X$, the ambient probability space, $n \mapsto X_n(x)$ is completely multiplicative.



A sample of $F_-(t)$.



A sample of $F_+(t)$.

Figure 2.2: Samples of $F_\pm$ with 10007 terms.

acter sums. Let $F(t)$ be the random Fourier series

$$F(t) := \frac{\eta}{2\pi} \sum_{|k|>0} \frac{X_k}{k}(1 - e(kt)), \tag{2.3}$$

where X_k are Steinhaus random multiplicative functions for $k \neq 0$ and η is a random variable uniformly distributed on the unit circle. Additionally, take $F_\pm(t)$ where we fix X_{-1} as $+1$ or -1. The infinite series is almost surely well defined, and we show in Section 2.2 that this is almost surely the Fourier series of a continuous function. Therefore we can think of F as a random process on $C([0,1])$. Examples of the paths formed by $F_\pm$ are shown in Figure 2.2.

Using the random Fourier series F, we state the main theorem of the chapter:

Theorem 2.1.1. *Let $F_\pm$ be defined as above for $t \in [0,1]$. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ weakly converges to the process $F_\pm(t)$ in the Banach space $C([0,1])$ as $q \to \infty$ through the primes. In other words, for any continuous and bounded map*

$$\psi : C([0,1]) \to \mathbb{C},$$

we have, for prime q,

$$\lim_{q\to\infty} \mathbb{E}\left(\psi\left(\mathcal{F}_{q,\pm}\right)\right) = \mathbb{E}\left(\psi(F_{\pm})\right).$$

2.1.1 Proof Outline

Theorem 2.1.1 shows that Steinhaus random multiplicative functions can be used as Fourier coefficients of a random process F to find the limiting distribution of character paths. In Section 2.2 we properly define the random process F and prove some of its properties. Theorem 2.1.1 only makes sense if $F(t)$ is a function almost surely in $C([0,1])$, which is proven in Theorem 2.2.3.

The proof of Theorem 2.1.1 can be split into two parts: proving that the sequence $(\mathcal{F}_{q,\pm}(t))_q$ converges in finite distributions to the random process $F_{\pm}(t)$ and that the sequence of distributions is relatively compact. Convergence of finite distributions is proved in Section 2.3, using the method of moments. To prove relative compactness, it is sufficient to use Prohorov's Theorem [7, Theorem 5.1], which states that if a family of probability measures is tight, then it must be relatively compact. Section 2.4 proves the sequence of distributions $(\mathcal{F}_{q,\pm}(t))_q$ satisfies the tightness property, therefore proving Theorem 2.1.1.

Remark 2.1.1. As referred to earlier, Bober, Goldmakher, Granville and Koukoulopoulos [9] investigated the distribution function $\Phi_q(\tau)$, defined in Equation (2.1). They proved $(\Phi_{q,\pm}(\tau))$ converges weakly to

$$\textbf{Prob}\left(\max_t |F_{\pm}(t)| > 2e^{\gamma}\tau\right),$$

as q tends to infinity through the primes. Theorem 2.1.1 can be used to obtain the same result.

Remark 2.1.2. The definition of character paths is motivated by similar research by Kowalski and Sawin [56, 57]. In their papers they define *'Kloosterman paths'* $K_p(t)$, view the paths as random variables, and find their limiting distribution as $p \to \infty$. This chapter, based on my paper [48], continues in this vein to investigate the analogous limiting distribution of character paths. Due to the multiplicativity of Dirichlet characters, our random multiplicative coefficients X_n are not independent. This leads to interesting properties shown in Section 2.2.

Remark 2.1.3. Theorem 2.1.1 is restricted to q being prime, so a natural question is to consider composite q as well. Steinhaus random multiplicative functions are non-zero so we need a high percentage of primitive characters modulo q. If we take q as not being 'too smooth' we might be able to relax this condition, as the contribution from imprimitive characters could potentially be included in the error terms already produced from the method. Future work could explore the generalised case when the modulus of the characters is not prime.

2.2 Properties of $F(t)$

Recall the random process F, defined by the infinite sum

$$F(t) = \frac{\eta}{2\pi} \sum_{n \neq 0} \frac{1 - e(nt)}{n} X_n,$$

where X_n are Steinhaus random multiplicative functions, defined in Definition 2.1.2, and η is a random variable uniformly distributed on the unit circle. We define the infinite sum as a limit of the smooth sum[3]

$$\frac{\eta}{2\pi}\sum_{\substack{n\neq 0\\ P^+(|n|)\leq y}}\frac{1-e(nt)}{n}X_n,$$

as $y\to\infty$. In this section we deal with samples of the random process and prove some of their properties. We will be using sample functions of the random process, which we define here for ease of notation.

Definition 2.2.1. Let G be defined as the function

$$G(t):=\frac{c}{2\pi}\sum_{n\neq 0}\frac{1-e(nt)}{n}\alpha_n,$$

where c is a sample of a Steinhaus random variable and α_n is a sample of a Steinhaus random multiplicative function. We say G is a **sample function**[4] of the random process F.

The Fourier coefficients are bounded by $O(1/n)$. This will be useful later in the section, where we show the infinite series F can also be defined as the limit of partial symmetric sums.

Our main result of Section 2.2 proves that any sample function of F is almost surely continuous. This is non trivial and involves considering the y-smooth and 'y-rough' parts of the infinite sum $G(t)$.

[3]Here $P^+(|n|)$ denotes the largest prime factor of $|n|$.

[4]A **sample space** is a collection of all possible experimental outcomes. Therefore, a **sample point/function** corresponds to all possible outcomes of the experiment [55].

Let

$$S_y \coloneqq \sum_{\substack{n\neq 0\\ P^+(|n|)\leq y}} \frac{1-e(nt)}{n}\alpha_n \qquad \text{and} \qquad R_y \coloneqq \sum_{\substack{n\neq 0\\ P^+(|n|)> y}} \frac{1-e(nt)}{n}\alpha_n.$$

Note that $S_y + R_y = \frac{\pi}{c}G(t)$ and the two functions are not independent. We will show that the rough part R_y is almost surely 0 as y tends to infinity.

Lemma 2.2.1. *For all $\varepsilon > 0$ and sufficiently large $y > 1$,*

$$\mathbb{P}\left(\|R_y\|_\infty > \varepsilon\right) \ll \exp\left\{\frac{-\varepsilon^2 y^{1/3}}{\log y(\log y + O(1))^2}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + O\left(|\log\varepsilon|\right)\right)\right\}$$

independently of S_y, where $\|\cdot\|_\infty \coloneqq \max_{t\in[0,1]}|\cdot|$. Notably for all $\varepsilon > 0$, we have $\mathbb{P}\left(\|R_y\|_\infty > \varepsilon\right) \to 0$ as $y \to \infty$.

Proof. For all $y \geq 1$,

$$\sum_{\substack{n\geq 1\\ P^+(n)>y}} \frac{1-e(nt)}{n}\alpha_n = \sum_{\substack{n\geq 1\\ P^+(n)\leq y}} \frac{\alpha_n}{n} \sum_{\substack{m>y\\ P^-(m)>y}} \frac{1-e(mnt)}{m}\alpha_m,$$

where $P^-(m)$ is the smallest prime factor of m. By setting

$$T_y(\alpha) \coloneqq \max_{t\in[0,1]}\left|\sum_{\substack{m>y\\ P^-(m)>y}} \frac{1-e(mt)}{m}\alpha_m\right|,$$

we have

$$\|R_y\|_\infty \coloneqq \max_{t\in[0,1]}\left|\sum_{\substack{n\neq 0\\ P^+(n)>y}} \frac{1-e(nt)}{n}\alpha_n\right| \leq 2\sum_{\substack{n\geq 1\\ P^+(n)\leq y}} \frac{T_y(\alpha)}{n}.$$

We then use the following result for smooth sums (see e.g. [46], [67, Theorem 2.7]),

$$\sum_{P+(n)\leq y} \frac{1}{n} = e^{\gamma} \log y + O(1).$$

Consequently, $\|R_y\|_\infty$ is bounded above by

$$\|R_y\|_\infty \leq 2T_y(\alpha) \left(e^{\gamma} \log y + O(1)\right).$$

Adapting [9, Proposition 5.2] for Steinhaus random multiplicative functions[5], for $k \geq 3$ and $y \geq k^3$, we obtain

$$\mathbb{E}\left[\left(\sum_{\substack{m>y \\ P^-(m)>y}} \frac{1-e(mt)}{m}\alpha_m\right)^{2k}\right] \ll \frac{1}{(\log y)^{40k}}.$$

We set ρ_y as the probability $T_y(\alpha) > \varepsilon(y) > 1/(\log y)^{20})$ and

$$k = \left\lfloor \frac{\varepsilon(y)^2 y^{1/3}}{\log y} \right\rfloor.$$

Therefore,

$$\rho_y \leq \frac{\mathbb{E}(T_y(\alpha)^{2k})}{\varepsilon(y)^{2k}} \ll \frac{\varepsilon(y)^{-2k}}{(\log y)^{40k}} \leq \left(\frac{\varepsilon(y)^{-1}}{(\log y)^{20}}\right)^{\frac{2\varepsilon(y)^2 y^{1/3}}{\log y}}.$$

[5]We take $q \to \infty$ and apply a sample of a Steinhaus random multiplicative function α_n instead of $\chi(n)$.

Taking $\varepsilon(y) = \frac{\varepsilon}{2e^\gamma \log y + O(1)}$ for $\varepsilon > 0$,

$$\mathbb{P}\left(\|R_y\|_\infty > \varepsilon\right) \leq \mathbb{P}\left(T_y(\alpha) > \frac{\varepsilon}{2e^\gamma \log y + O(1)}\right) \ll \left(\frac{2e^\gamma \log y + O(1)}{\varepsilon(\log y)^{20}}\right)^{\frac{2\varepsilon^2 y^{1/3}}{\log y(2e^\gamma \log y + O(1))^2}}$$
$$\ll \exp\left(\frac{-\varepsilon^2 y^{1/3}}{\log y(\log y + O(1))^2}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + \log\left(O(\varepsilon)\right)\right)\right).$$

To prove the final part of the lemma, we take $y \to \infty$ to show the probability tends to 0,

$$0 \leq \mathbb{P}\left(\|R_y\|_\infty > \varepsilon\right) \ll \lim_{y\to\infty} \exp\left(\frac{-\varepsilon^2 y^{1/3}}{\log y(\log y + O(1))^2}\left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)}\right) + O\left(\log \varepsilon\right)\right)\right) = 0.$$

□

Lemma 2.2.1 can be appreciated more by taking $\varepsilon = 1/\log y$, leading to the following example.

Example 2.2.2. For sufficiently large $y > 1$, there exists a constant $C > 0$ such that

$$\mathbb{P}\left(\|R_y\|_\infty > \frac{1}{\log y}\right) \ll \exp\left\{\frac{-y^{1/3}}{(\log y)^5}\left(C \log\log y\right)\right\}.$$

This is independent of S_y.

Subsequently, defining F as the limit of its smooth parts, we get the following theorem.

Theorem 2.2.3. *Let F be the random Fourier series*

$$F(t) := \lim_{y\to\infty} \frac{\eta}{2\pi} \sum_{\substack{n\neq 0 \\ P^+(|n|)\leq y}} \frac{1 - e(nt)}{n} X_n,$$

where X_n are Steinhaus random multiplicative functions and η is a random variable uniformly distributed on $\{|z| = 1\}$. With probability 1 this is the Fourier series of a continuous function.

Proof. We will prove this theorem by showing any sample of the random process F is almost surely continuous.

Consider the sequence of functions $(S_y)_y$ and $(R_y)_y$ defined by

$$S_y(t) := \frac{c}{2\pi} \sum_{\substack{n\neq 0 \\ P^+(|n|)\leq y}} \frac{1-e(nt)}{n}\alpha_n,$$
$$R_y(t) := \frac{c}{2\pi} \sum_{\substack{n\neq 0 \\ P^+(|n|)> y}} \frac{1-e(nt)}{n}\alpha_n,$$

where c, α_n are on the unit circle and the sequence $\{\alpha_n\}$ is completely multiplicative. Note that samples of the random process F can be written as $S_y(t) + R_y(t)$ for appropriate choices of α_n and c.

The function S_y is the $y-$smooth part of a sample of the random process F, which we call $G(t)$. The sum S_y converges absolutely, so $S_y(t)$ is a continuous function (see e.g. [82]).

To prove continuity with probability 1, we use the first Borel-Cantelli Lemma [14, 17] to show the $y-$rough part of G vanishes as $y \to \infty$. Consider the sequence $\{R_y : \|R_y\|_\infty > 1/\log y\}_y$, where $\|R_y\|_\infty$ is the maximum of the $y-$rough part of G. Using Example 2.2.2,

$$\sum_{y=1}^{\infty} \mathbb{P}(\|R_y\|_\infty > 1/\log y)$$
$$\ll \sum_{y=1}^{\infty} \exp\left\{ \frac{-y^{1/3}}{(\log y)^3(\log y + O(1))^2} \left(\log\left(\frac{(\log y)^{20}}{\log y + O(1)} \right) + O\left(\log\log y\right) \right) \right\} < \infty.$$

As a result, the probability of $\|R_y\|_\infty > \varepsilon$ occuring infinitely often is zero. Consequently, R_y, the rough part of the sample of F, vanishes almost surely as $y \to \infty$.

As a result, the sequence of continuous functions (S_y) uniformly converges to its limit, which by the Uniform Limit Theorem [68, Chapter 2] must be continuous. By uniform convergence we can compute the Fourier expansion, which recovers exactly what we expect. We defined F as the limit of its smooth parts, so therefore any samples of F are almost surely continuous. $\square$

At the start of this section, we defined $F(t)$ as the limit as $y \to \infty$ of the smooth sum

$$\frac{\eta}{2\pi} \sum_{\substack{n\neq 0 \\ P^+(|n|)\leq y}} \frac{1-e(nt)}{n} X_n.$$

Since the Fourier coefficients are bounded by $O(1/n)$ and F is almost surely a Fourier series of a continuous function, all finite Fourier sums converge to F uniformly [47]. Consequently, we can also define the process F as the limit as $N \to \infty$ of the partial symmetrical sums

$$\frac{\eta}{2\pi} \sum_{|n|\leq N} \frac{1-e(nt)}{n} X_n.$$

Therefore for the rest of the chapter we can interchangeably use either definition for the infinite series $F(t)$.

2.3 Convergence of Finite-Dimensional Distributions of $\mathcal{F}_q$

In order to prove Theorem 2.1.1, we will first show convergence of finite-dimensional distributions. We take $(\mathcal{F}_{q,\pm}(t))_{q \text{ prime}}$ as the sequence of distributions of character paths modulo q dependent on the parity of the characters. We split the distribution as odd and even character paths have different behaviour, due to the constant term vanishing when χ is even. As such, we also want two limiting distributions, for $\mathcal{F}_{q,+}$ and $\mathcal{F}_{q,-}$ respectively. Let $F_\pm(t)$ be random processes defined by

$$F_+(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} \sin(2\pi k t), \qquad \text{and} \qquad F_-(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k}(1 - \cos(2\pi k t)),$$

where X_n are Steinhaus random multiplicative functions, defined in Definition 2.1.2, and η is uniformly distributed on the unit circle.

Theorem 2.3.1. *Let q be an odd prime. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ converges to the process $F_\pm(t)$ in the sense of convergence of finite distributions. In other words, for every $n \geq 1$ and for every n-tuple $0 \leq t_1 < \cdots < t_n \leq 1$, the vectors*

$$(\mathcal{F}_{q,\pm}(t_1), \ldots, \mathcal{F}_{q,\pm}(t_n))$$

converge in law as q tends to infinity through the primes to

$$(F_\pm(t_1), \ldots, F_\pm(t_n)) \,.$$

We prove this by the method of moments. We will define a moment M_q, of our distribution $\mathcal{F}_q$ and a moment M for the random process F. In Section 2.3.3, we prove M is determinate[6]. Subsequently, in Section 2.3.4, we prove this sequence of moments M_q converges to M, the moment of F. This is sufficient to prove Theorem 2.3.1.

We are considering odd and even character paths separately. In this proof we will focus on results for odd character paths as the proof is analogous for the even character case. Where this is not the case, any changes will be addressed throughout the section.

2.3.1 Definitions of the Moments

The Fourier series of the character path is

$$f_\chi(t) = \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{0<|k|<q} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

This results from truncating the Fourier series of the character sum $S_\chi(t)$ and the trivial inequality $|f_\chi(t) - S_\chi(t)| \leq \frac{1}{\sqrt{q}}$. The paths of odd and even characters are shown to differ greatly, exemplified in Figure 2.1, due to the constant term $\frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum \overline{\chi}(k)/k$ vanishing when χ is even. As such, this chapter will assess distributions of these character paths modulo odd prime q separately, dependent on parity. As a Fourier series we split this into

$$f_\chi(t) = \begin{cases} \dfrac{-\tau(\chi)}{\pi\sqrt{q}} \displaystyle\sum_{k=1}^{q} \frac{\overline{\chi}(k)}{k} \sin(2\pi kt) + O\left(\frac{\log q}{\sqrt{q}}\right) & ;\ \text{if } \chi \text{ even,} \\ \dfrac{\tau(\chi)}{\pi i\sqrt{q}} \displaystyle\sum_{k=1}^{q} \frac{\overline{\chi}(k)}{k}(1 - \cos(2\pi kt)) + O\left(\frac{\log q}{\sqrt{q}}\right) & ;\ \text{if } \chi \text{ odd.} \end{cases}$$

[6] Determinacy is defined in Section 2.3.3

We define our moments M_q and M. In this section we will assume χ is odd as the proof is analogous to the even case. Therefore, taking a function from the odd distribution $\mathcal{F}_{q,-}$, we will take the character path modulo q as

$$f_\chi(t) = \frac{\tau(\chi)}{\pi i\sqrt{q}} \sum_{1\leq n\leq q} \frac{\overline{\chi}(n)}{n}(1-\cos(2\pi nt)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

We will also be considering the odd random series

$$F_-(t) = \frac{\eta}{\pi}\sum_{n\geq 1}\frac{X_n}{n}(1-\cos(2\pi nt)),$$

which for ease of notation will be referred to as $F(t)$ for the rest of this section.

Definition 2.3.1. Let $k \geq 1$ be given and $\underline{t} = (t_1, \ldots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Additionally fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$. We define the moment sequence $M_q(\underline{n}, \underline{m})$ as

$$M_q(\underline{n}, \underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \text{ odd}} \prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i},$$

and the moment $M(\underline{n}, \underline{m})$ as

$$M(\underline{n}, \underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right).$$

The moment $M(\underline{n}, \underline{m})$ is well defined. To show this we prove the equivalent result that $\mathbb{E}(|F(t)|^n)$ is bounded for all n. By Fubini's theorem (e.g. [25, III, Theorem 14.1]),

$$\mathbb{E}\left(|F(t)|^n\right) = \int_0^\infty nx^{n-1}\mathbb{P}\left(|F(t)| > x\right)dx.$$

We then use a result by Bober, Goldmakher, Granville and Koukoulopoulos [9]: Let $c = e^{-\gamma}\log 2$. For any $\tau \geq 1$,

$$\mathbb{P}\left(\max_{0\leq t\leq 1}|F(t)| > 2e^{\gamma}\tau\right) \leq \exp\left\{-\frac{e^{\tau-c-2}}{\tau}\left(1+O\left(\frac{\log\tau}{\tau}\right)\right)\right\}.$$

Therefore, combining both equations, the moment is finite and well defined. $F(t)$ is a random process, defined by the almost surely converging sum

$$F(t) = \frac{\eta}{\pi}\sum_{a\geq 1}\frac{X_a}{a}(1-\cos(2\pi a t)).$$

As shown in Section 2.2 we can define F as the limit of the symmetric partial sums. The infinite series F is not absolutely convergent, so justification is needed to manipulate the product $\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}$.

We write the expansion of $F(t_i)^{n_i}$ as

$$\frac{\eta^{n_i}}{\pi^{n_i}}\sum_{a_{i,1},\ldots,a_{i,n_i}\geq 1}\prod_{j=1}^{n_i}\frac{X_{a_{i,j}}}{a_{i,j}}(1-\cos(2\pi a_{i,j}t_i)),$$

and $\overline{F(t_i)}^{m_i}$ in a similar manner. Without changing the order of summation, the product $\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}$ is therefore

$$\frac{\eta^n\overline{\eta}^m}{\pi^{n+m}}\sum\cdots\sum\prod_{i=1}^{k}\prod_{j=1}^{n_i}\prod_{j'=1}^{m_i}\frac{X_{a_{i,j}}\overline{X_{b_{i,j'}}}}{a_{i,j}b_{i,j'}}(1-\cos(2\pi a_{i,j}t_i))(1-\cos(2\pi b_{i,j'}t_i)),$$

where $n = |\underline{n}|$ and $m = |\underline{m}|$ as above. The sums are over $a_{i,j} \geq 1$ and $b_{i,j'} \geq 1$, where $j \in [1, n_i]$ and $j' \in [1, m_i]$ for $i \in [1, k]$.

The moment $M(\underline{n}, \underline{m})$ is the expectation of this multivariate sum. To simplify the equation, we want to swap the order of expectation with the order of summation.

Since the moment is finite, we use Lebesgue's dominated convergence theorem [3, Section 5.6] to bring the expectation inside the sum. Using the multiplicativity of Steinhaus random multiplicative functions, the moment M therefore equals[7]

$$\mathbb{E}\left(\frac{\eta^n\overline{\eta}^m}{\pi^{n+m}}\right)\sum\cdots\sum\mathbb{E}\left(X_a\overline{X_b}\right)\prod_{i=1}^{k}\prod_{j=1}^{n_i}\prod_{j'=1}^{m_i}\frac{(1-\cos(2\pi a_{i,j}t_i))(1-\cos(2\pi b_{i,j'}t_i))}{a_{i,j}b_{i,j'}},$$

where

$$a \coloneqq \prod_{i=1}^{k}\prod_{j=1}^{n_i} a_{i,j} \qquad \text{and} \qquad b \coloneqq \prod_{i=1}^{k}\prod_{j'=1}^{m_i} b_{i,j'}.$$

Steinhaus random multiplicative functions X_n are orthogonal as n can always be written as a unique prime factorisation and $\mathbb{E}(X_p)=0$ for all primes p. In other words,

$$\mathbb{E}\left(X_a\overline{X_b}\right) = \mathbb{1}_{a=b} \coloneqq \begin{cases} 1 & ;\text{ if } a=b, \\ 0, & ;\text{ otherwise.}\end{cases}$$

Therefore we can rewrite the moment $M = M(\underline{n},\underline{m})$ as follows,

$$M = \mathbb{E}\left(\frac{\eta^n\overline{\eta}^m}{\pi^{n+m}}\right)\sum_{l=1}^{\infty}\sum_{\substack{a_{i,j},b_{i,j}>0\\ a=b=l}}\frac{1}{ab}\prod_{i=1}^{k}\prod_{j=1}^{n_i}\prod_{j'=1}^{m_i}(1-\cos(2\pi a_{i,j}t_i))(1-\cos(2\pi b_{i,j'}t_i)),$$

where a and b are the product of $a_{i,j}$ and $b_{i,j'}$ respectively. Taking $\frac{1}{ab}=\frac{1}{l^2}$ and bounding $(1-\cos(x))\le 2$, M is clearly bounded as a function of n_i and m_i. These variables are fixed and finite, and the number of such tuples is $(n+m)\ll l^{\varepsilon}$ in each case, so the moment is absolutely convergent. Therefore, we can swap the order of

[7] Note the moment will be different for F_+, where we have $\sin(2\pi at)$ instead of $(1-\cos(2\pi at))$.

summation. As a result,

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\frac{\eta^n\overline{\eta}^m}{\pi^{n+m}}\right)\sum_{a\geq 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a), \tag{2.4}$$

where

$$\mathcal{B}_{\underline{N},\underline{t}}(x) \coloneqq \sum_{x_1\cdots x_k = x}\prod_{i=1}^{k}\beta_{N_i,t_i}(x_i) \tag{2.5}$$

and

$$\beta_{N_i,t_i}(x_i) = \sum_{y_1\cdots y_{N_i}=x_i}\frac{1}{x_i}\prod_{j=1}^{N_i}(1-\cos(2\pi y_j t_i)). \tag{2.6}$$

The moment $M_q(\underline{n},\underline{m})$ can be also be rewritten using methods from Bober and Goldmakher [8]. First, we use the Fourier expansion of $f_\chi(t)$, so

$$f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i} = \frac{\tau(\chi)^{n_i}\overline{\tau(\chi)}^{m_i}}{(\pi\sqrt{q})^{n_i+m_i}i^{n_i-m_i}}\sum_{\substack{1\leq a\leq q^{n_i}\\ 1\leq b\leq q^{m_i}}}\overline{\chi}(a)\chi(b)\beta_{n_i,q,t_i}(a)\beta_{m_i,q,t_i}(b) + O\left(\frac{(\log q+1)^{n_i+m_i}}{\sqrt{q}}\right),$$

where $\beta_{N,q,t}$ is defined as

$$\beta_{N,q,t}(x) \coloneqq \frac{1}{x}\sum_{\substack{x_1\cdots x_N = x\\ x_i\leq q}}\prod_{k=1}^{N}(1-\cos(2\pi x_k t)), \tag{2.7}$$

for $(x,q)=1$ and 0 otherwise[8].

Continuing to expand $M_q(\underline{n},\underline{m})$, we take a product of all $f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i}$ for

[8]For even characters, $\beta_{N,q,t}$ would instead sum over the product of $\sin(2\pi x_k t)$.

$i \in [1,k]$.Therefore,

$$\prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i} = \frac{\tau(\chi)^n\overline{\tau(\chi)}^m}{(\pi\sqrt{q})^{n+m}i^{n-m}} \sum_{\substack{1\le a\le q^n \\ 1\le b\le q^m \\ \text{for } i\in[1,k]}} \overline{\chi}(a)\chi(b)\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b) + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right),$$

where

$$n \coloneqq n_1 + \cdots + n_k \text{ and } m \coloneqq m_1 + \cdots + m_k$$

and

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \coloneqq \sum_{\substack{x_1\cdots x_k = x \\ x_i \le q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i), \tag{2.8}$$

for $(x,q) = 1$ and 0 otherwise. Note that $\mathcal{B}_{\underline{N},\underline{t}}$ and β_{N_i,t_i} from Equations (2.5) and (2.6) are the limits as $q \to \infty$ of $\mathcal{B}_{\underline{N},q,\underline{t}}$ and β_{N_i,q,t_i} respectively. Furthermore, we take the average of this product over all odd Dirichlet characters χ to find

$$\begin{aligned} M_q(\underline{n},\underline{m}) =& \frac{1}{(\pi\sqrt{q})^{n+m}i^{n-m}} \sum_{\substack{1\le a\le q^n \\ 1\le b\le q^m}} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi \text{ odd}}} \overline{\chi}(a)\chi(b)\tau(\chi)^n\overline{\tau(\chi)}^m \\ & + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right). \end{aligned} \tag{2.9}$$

This form is more useful for future calculations and will be used to prove M_q tends to M as $q \to \infty$ through the primes.

2.3.2 Bounding the Moments

Later in the chapter we will be interested in bounding $\mathcal{B}_{\underline{N},q,\underline{t}}$ and $\mathcal{B}_{\underline{N},\underline{t}}$. The inequality we find is independent of q, so we can consider both bounds at the same time. Therefore for this subsection we will work with $\mathcal{B}_{\underline{N},q,\underline{t}}$.

Recall,

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) = \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i),$$

where

$$\beta_{N,q,t}(x_i) = \sum_{\substack{y_1 \cdots y_N = x_i \\ y_j \leq q}} \frac{1}{x_i} \prod_{j=1}^{N} (1 - \cos(2\pi y_j t)),$$

for $(x_i, q) = 1$ and 0 otherwise. Since $|1 - \cos(2\pi y_j t)| \leq 2$, we always have the bound

$$|\beta_{N,q,t}(x)| \leq \frac{2^N d_N(x)}{x},$$

where $d_N(x)$ is the Nth divisor function[9] $\sum_{x_1 \cdots x_N = x} 1$. As a result,

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N}{x} \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} \prod_{i=1}^{k} d_{N_i}(x_i),$$

where $N = \sum N_i = |\underline{N}|$. To further bound $\mathcal{B}$ we next use the following lemma.

Lemma 2.3.2. *Let $d_{N_1}(x_1), d_{N_2}(x_2)$ be the N_1th and N_2th divisor function of $x_1, x_2 \in$*

[9]For F_+ and even character paths, the 2^N vanishes in the bound of β as $|\sin(2\pi y_j t)| \leq 1$. However, since this bound is only included in error terms and the n_i and m_i terms are fixed, the difference of the constant is irrelevant.

$\mathbb{N}$ *respectively. We have the relation*

$$d_{N_1}(x_1)d_{N_2}(x_2) \leq d_{N_1+N_2}(x_1 \cdot x_2).$$

Proof. We apply a combinatorial argument, where we view $d_N(x)$ as the number of ways of choosing N positive integers that multiply to x. Therefore $d_{N_1+N_2}(x_1 \cdot x_2)$ is at least the number of ways of choosing N_1 integers multiplying to x_1 times the number of ways of choosing N_2 integers multiplying to x_2. □

Using Lemma 2.3.2, we bound $\mathcal{B}_{\underline{N},q,\underline{t}}(x)$ by

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x)}{x} \sum_{\substack{x_1 \cdots x_k = x \\ x_i \leq q^{N_i}}} 1 \leq \frac{2^N d_N(x) d_k(x)}{x} \leq \frac{2^N d^{N+k}(x)}{x}. \tag{2.10}$$

In parts of the proof, it is sufficient to use the looser bound $\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq 2^N x^{\varepsilon}/x$, however we will mainly apply the bound from Equation (2.10). This will be useful in future equations. Note that this is independent of q and $\underline{t}$, so the bounds hold for $\mathcal{B}_{\underline{N},\underline{t}} = \lim_{q\to\infty} \mathcal{B}_{\underline{N},q,\underline{t}}$.

2.3.3 Proving Determinacy

Our aim is to use the method of moments to prove the distribution of character paths modulo q converges to $F(t)$ in the sense of finite distributions. For this we need to show the moment $M(\underline{n}, \underline{m})$ is determinate, or in other words show the moment only has one representing measure. To show that M is a determinate complex moment sequence, it is sufficient to show that it satisfies

$$\sum_{n=1}^{\infty} M(\underline{n}, \underline{n})^{-1/2n} = \infty, \tag{2.11}$$

where $\underline{n} = (n_1, \ldots, n_k)$ and $n = |\underline{n}| = \sum_i n_i$. This is also known as the Carleman condition [78, Theorem 15.11].

Lemma 2.3.3. *The moment $M(\underline{n}, \underline{m})$ satisfies Equation* (2.11)*.*

Proof. This is shown using Equation (2.4) and taking $\underline{n} = \underline{m}$. Setting $n = |\underline{n}| = |\underline{m}|$, we have

$$M(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n}, \underline{t}}(a)^2.$$

We use the bound of $\mathcal{B}$ from Equation (2.10), taking $d_k(a) \leq a^{\varepsilon_k}$ for small $\varepsilon_k > 0$, so

$$M(\underline{n}, \underline{n}) \leq \frac{2^{2n}}{\pi^{2n}} \sum_{a \geq 1} \frac{d_n(a)^2}{a^{2-2\varepsilon_k}} =: \frac{2^{2n}}{\pi^{2n}} \sum_{a \geq 1} \frac{d_n(a)^2}{a^{2\sigma}},$$

taking $\sigma \coloneqq 1 - \varepsilon_k$. We can use Proposition 3.2 from Bober and Goldmakher [8], which states for $1/2 < \sigma \leq 1$ that,

$$\sum_{a=1}^{\infty} \frac{d_n(a)^2}{a^{2\sigma}} \leq \exp\left(2n\sigma \log\log(2n)^{1/\sigma} + \frac{(2n)^{1/\sigma}}{2\sigma - 1} + O\left(\frac{n}{2\sigma - 1} + \frac{(2n)^{1/\sigma}}{\log(3(2n)^{1/\sigma - 1})}\right)\right). \tag{2.12}$$

Here we have shown the sum $\sum_{n=1}^{\infty} M(\underline{n}, \underline{n})^{-1/2n}$ has the lower bound

$$\frac{\pi}{2} \sum_{n=1}^{\infty} \exp\left(-\sigma \log\log\left((2n)^{1/\sigma}\right) - \frac{(2n)^{1/\sigma - 1}}{2\sigma - 1} + O\left(\frac{1}{2\sigma - 1} + \frac{(2n)^{1/\sigma - 1}}{\log(3(2n)^{1/\sigma - 1})}\right)\right).$$

The lower bound can be rewritten as

$$\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sigma^{\sigma}}{(\log 2n)^{\sigma}} \exp\left(-\frac{(2n)^{\frac{1-\sigma}{\sigma}}}{2\sigma - 1}\right) \exp\left(O\left(\frac{1}{2\sigma - 1} + \frac{(2n)^{1/\sigma - 1}}{\log(3(2n)^{1/\sigma - 1})}\right)\right).$$

Tending $\sigma = 1 - \varepsilon_k$ to 1, this sum diverges[10]. Therefore

$$\sum_{n=1}^{\infty} M(\underline{n}, \underline{n})^{-1/2n} = \infty,$$

and the Carleman condition holds. As a result, the claim is proved. $\square$

2.3.4 Convergence of Moments

In this section we show the moment sequence M_q converges to the multivariate moment of F, therefore proving Theorem 2.3.1. Separating the distribution by parity, we have two lemmas.

Lemma 2.3.4. *Let $k \geq 1$ be given and $0 \leq t_1 < \cdots < t_k \leq 1$ be fixed. Fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$. Let*

$$M_{q,-}(\underline{n}, \underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \ \textbf{odd}} \prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i}.$$

Then for all $\varepsilon > 0$,

$$M_{q,-}(\underline{n}, \underline{m}) = M_-(\underline{n}, \underline{m}) + O_{\underline{n},\underline{m},k}\left(q^{-1/2+\varepsilon}\right),$$

where

$$M_-(\underline{n}, \underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F_-(t_i)^{n_i} \overline{F_-(t_i)}^{m_i}\right).$$

Importantly, $M_{q,-}(\underline{n}, \underline{m}) \to M_-(\underline{n}, \underline{m})$ as $q \to \infty$ through the primes.

[10] This is clear by comparison test (see e.g. [28]).

Lemma 2.3.5. *Let $k \geq 1$ be given and fix $\underline{t}, \underline{n}, \underline{m}$ as in Proposition 2.3.4. Let*

$$M_{q,+}(\underline{n}, \underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \textbf{ even}} \prod_{i=1}^{k} f_\chi(t_i)^{n_i} \overline{f_\chi(t_i)}^{m_i}.$$

Then for all $\varepsilon > 0$,

$$M_{q,+}(\underline{n}, \underline{m}) = M_{-}(\underline{n}, \underline{m}) + O_{\underline{n},\underline{m},k}\left(q^{-1/2+\varepsilon}\right),$$

where

$$M_{+}(\underline{n}, \underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F_{+}(t_i)^{n_i} \overline{F_{+}(t_i)}^{m_i}\right).$$

Importantly, $M_{q,+}(\underline{n}, \underline{m}) \to M_{+}(\underline{n}, \underline{m})$ as $q \to \infty$ through the primes.

In this section we only look at the $\mathcal{F}_{q,-}$ case, where χ is odd. There are equivalent propositions and lemmas for the even case, where the proofs are analogous to the proofs shown in the section. In places where the proof differs, we will state the results for $\mathcal{F}_{q,+}$ and how it does not largely affect the proof.

These lemmas are sufficient to prove Theorem 2.3.1, showing $(\mathcal{F}_q(t))_{q \text{ prime}}$ converges in finite distributions to $F(t)$. We prove Lemma 2.3.4 using a combination of the following two propositions.

Proposition 2.3.6. *Let $k \geq 1$ be given and $\underline{t} = (t_1, \ldots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Fix $\underline{n} = (n_1, \ldots, n_k)$ and $\underline{m} = (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$ and*

$$n \coloneqq n_1 + n_2 + \cdots + n_k = m_1 + \cdots + m_k.$$

The moment sequence defined in Proposition 2.3.4 can be expressed as

$$M_{q,-}(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

where $\mathcal{B}_{\underline{N},\underline{t}}$ is defined as

$$\mathcal{B}_{\underline{N},\underline{t}}(a) = \frac{1}{a} \sum_{x_1\cdots x_k=a} \prod_{i=1}^{k} \left(\sum_{y_1\cdots y_{N_i}=x_i} \prod_{j=1}^{N_i} (1-\cos(2\pi y_j t)) \right).$$

Proposition 2.3.7. *Let $k\geq 1$ be given and $\underline{t}=(t_1,\ldots,t_k)$, where $0\leq t_1<\cdots<t_k\leq 1$, be fixed. Fix $\underline{n}=(n_1,\ldots,n_k)$ and $\underline{m}=(m_1,\ldots,m_k)$, where $n_i,m_i\in\mathbb{Z}_{\geq 0}$ and*

$$n \coloneqq n_1+n_2+\cdots+n_k = m_1+\cdots+m_k.$$

Then

$$M_{-}(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a),$$

where $\mathcal{B}_{\underline{N},\underline{t}}$ is defined as in Proposition 2.3.6.

Before the proof of the propositions, we will use them to prove Lemma 2.3.4.

Proof of Lemma 2.3.4. Take $n=n_1+n_2+\cdots+n_k$ and $m=m_1+\cdots m_k$. We split the proof into two cases: $n=m$ and $n\neq m$. The first case has already been shown by Propositions 2.3.6 and 2.3.7:

$$\begin{aligned} M_{q,-}(\underline{n},\underline{m}) &= \frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right) \\ &= M_{-}(\underline{n},\underline{m}) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right). \end{aligned}$$

Therefore, the only case left to show is when $n \neq m$. We recall Equation (2.4):

$$M_-(\underline{n}, \underline{m}) = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum_{a \geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a).$$

Since η is uniformly distributed on the unit circle, $\mathbb{E}(\eta^n \overline{\eta}^m) = 0$ and the moment M_- vanishes. Therefore, to conclude the proof, we need to show the moment $M_{q,-} \ll q^{-1/2+\varepsilon}$, and therefore vanishes as $q \to \infty$. As shown in Equation (2.9), we can write $M_{q,-}(\underline{n}, \underline{m})$ as

$$\begin{aligned} M_{q,-}(\underline{n}, \underline{m}) = & \frac{1}{(\pi\sqrt{q})^{n+m}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m}} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\chi \text{ odd}} \overline{\chi}(a) \chi(b) \tau(\chi)^n \overline{\tau(\chi)}^m \\ & + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right). \end{aligned}$$

Assuming $n > m$, we rewrite $\tau(\chi)^n \overline{\tau(\chi)}^m$ as $q^m \tau(\chi)^{n-m}$. Therefore, taking $\chi(\overline{a}) \coloneqq \overline{\chi}(a)$,

$$\frac{2}{\phi(q)} \sum_{\chi} \overline{\chi}(a) \chi(b) \tau(\chi)^n \overline{\tau(\chi)}^m = \frac{2q^m}{\phi(q)} \sum_{\chi \text{ odd}} \chi(\overline{a} \cdot b) \tau(\chi)^{n-m}.$$

Lemma 2.3.8. *For $N \in \mathbb{N}$,*

$$\frac{2}{\phi(q)} \left| \sum_{\substack{\chi \mod q \\ \chi(-1) = \sigma}} \chi(a) \tau(\chi)^N \right| \leq 2N q^{(N-1)/2},$$

where $\sigma = \{-1, 1\}$.

This lemma is a slight generalisation of a result by Granville and Soundararajan [35, Lemma 8.3] and uses Deligne's bound on hyper-Kloosterman sums. Below follows Granville and Soundararajan's proof, with a modification to include when χ

is even.

Proof. Firstly, we rewrite the sum as exponential sums, using orthogonality of characters and the definition of the Gauss sum $\tau(\chi)$. Taking $1_{\chi(-1)=\sigma} = (1+\sigma\chi(-1))/2$,

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=\sigma}} \chi(a)\tau(\chi)^N = \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1\cdots x_N \equiv \overline{a} \mod q}} e\left(\frac{x_1+\cdots+x_N}{q}\right) + \sigma \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1\cdots x_N \equiv -\overline{a} \mod q}} e\left(\frac{x_1+\cdots+x_N}{q}\right).$$

Then, using Deligne's bound [24]

$$\max_{b\in(\mathbb{Z}/q\mathbb{Z})^\times} \left| \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1\cdots x_N \equiv b \mod q}} e\left(\frac{x_1+\cdots+x_N}{q}\right) \right| \leq Nq^{(N-1)/2},$$

we have proved the lemma. □

As a result, we have the inequality

$$|M_{q,-}(\underline{n},\underline{m})| \leq \frac{2(n-m)}{\pi^{(n+m)}\sqrt{q}} \sum_{\substack{1\leq a\leq q^n \\ 1\leq b\leq q^m}} |\mathcal{B}_{\underline{n},q,\underline{t}}(a)||\mathcal{B}_{\underline{m},q,\underline{t}}(b)| + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right).$$

We also have the bound on $\mathcal{B}$, as shown in Equation (2.10),

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x) d_k(x)}{x}.$$

Therefore, trivially bounding both divisor functions by q^ε for $\varepsilon > 0$,

$$\sum_{1\leq a\leq q^n} |\mathcal{B}_{\underline{n},q,\underline{t}}(a)| \ll 2^n q^\varepsilon \sum_{1\leq a\leq q^n} \frac{1}{a} \leq 2^n q^\varepsilon \log(q^n).$$

We get an analogous result for $\sum_{1\le b\le q^m} |\mathcal{B}_{\underline{m},q,\underline{t}}(b)|$. As a result,

$$M_{q,-}(\underline{n},\underline{m}) \ll \frac{2^{2+n+m}(n-m)q^{\varepsilon}\log(q^n)\log(q^m)}{\pi^{n+m}\sqrt{q}} + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right) \ll_{n,m} q^{-1/2+\varepsilon},$$

which tends to zero as $q\to\infty$. By a similar method we can show this is also the case when $n<m$. Therefore Lemma 2.3.4 holds. □

Having proven Lemma 2.3.4 assuming Propositions 2.3.6 and 2.3.7, we will now prove both results, showing when $|\underline{n}| = |\underline{m}|$ both $\lim_{q\to\infty} M_q$ and M equal

$$\frac{1}{\pi^{2n}}\sum_{a\ge 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a).$$

Recall that we are only proving the odd character case, as the proof is analogous for $\mathcal{F}_{q,+}$. For odd characters, $\mathcal{B}_{N,q,t}(x)$ is defined in Section 2.3.1 as

$$\sum_{\substack{x_1\cdots x_k=x\\ x_i\le q^{N_i}}}\prod_{i=1}^{k}\beta_{N_i,q,t_i}(x_i) = \sum_{\substack{x_1\cdots x_k=x\\ x_i\le q^{N_i}}}\frac{1}{x}\prod_{i=1}^{k}\sum_{\substack{y_1\cdots y_{N_i}=x_i\\ y_i\le q}}\prod_{j=1}^{N_i}(1-\cos(2\pi y_j t)).$$

For ease of notation, in the proofs we refer to $M_{q,-}$ and M_- as M_q and M respectively.

Proof of Proposition 2.3.6. Taking $n=m$, where

$$n := |\underline{n}| = n_1+\cdots+n_k, \qquad\qquad m := |\underline{m}| = m_1+\cdots+m_k,$$

we rewrite Equation (2.9) as

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}}\sum_{1\le a,b\le q^n}(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b))\,\frac{2}{\phi(q)}\sum_{\chi\text{ odd}}\overline{\chi}(a)\chi(b) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

where $\mathcal{B}_{\underline{n},q,\underline{t}}$ is defined as in Equation (2.8).

Using the orthogonality of χ, and noting we are only summing over odd characters[11] χ modulo q, the moment sequence becomes

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}}\Sigma_+(\underline{n},\underline{m}) - \frac{1}{\pi^{2n}}\Sigma_-(\underline{n},\underline{m}) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right), \tag{2.13}$$

where

$$\Sigma_+(\underline{n},\underline{m}) := \sum_{\substack{1\le a,b\le q^n\\ a\equiv b \mod q}} \mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b), \quad \Sigma_-(\underline{n},\underline{m}) := \sum_{\substack{1\le a,b\le q^n\\ a\equiv -b \mod q}} \mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b).$$

The aim is to get the main sum independent of q. Using ideas from Bober and Goldmakher [8, Proof of Lemma 4.1], we consider Σ_+ and Σ_- simultaneously. First we split the sums into arithmetic progressions mod q,

$$\Sigma_\pm(\underline{n},\underline{m}) = \sum_{\substack{1\le a,b<q\\ a\equiv \pm b \mod q}} \sum_{0\le\gamma_1,\gamma_2<q^{n-1}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q).$$

We simplify $\Sigma_\pm$ by splitting the inner sum into $\gamma_1=\gamma_2=0$, $\gamma_1\neq 0$, and $\gamma_2\neq 0$:

$$\begin{aligned}\sum_{0\le\gamma_1,\gamma_2<q^{n-1}} &\mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q)\\ &= (\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)) + \sum_{j=1}^{2}\sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q).\end{aligned}$$

[11] The method for the even character case would differ here. Firstly recall $\mathcal{B}$ involves sin instead of $(1-\cos)$ in the even case. Additionally, since these are even characters, we would have $M_q = \frac{1}{\pi^{2n}}\Sigma_+ + \frac{1}{\pi^{2n}}\Sigma_- + O\left((\log q)^{2n}q^{-1/2}\right)$, where we are adding the Σ_- term instead of subtracting it. However, the Σ_- term is eventually swallowed by the error term, so this does not affect the end result.

For ease of notation, we define the above latter sum as

$$\Omega = \sum_{j=1}^{2} \sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q).$$

We can bound Ω by using the bound of $\mathcal{B}$ shown in Equation (2.10), so

$$\Omega \le 2^{2n} \sum_{j=1}^{2} \sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \frac{d_n(a+\gamma_1 q)d_k(a+\gamma_1 q)}{a+\gamma_1 q}\frac{d_n(b+\gamma_2 q)d_k(b+\gamma_2 q)}{b+\gamma_2 q}.$$

By bounding the divisor functions by $O_{n,k}(q^{\varepsilon})$, we can further bound the sum to

$$\Omega \ll_{n,k} q^{\varepsilon} \sum_{j=1}^{2} \sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \frac{1}{a+\gamma_1 q}\frac{1}{b+\gamma_2 q}.$$

We can use the bound on partial harmonic series (see e.g. [1, Theorem 3.2]),

$$\omega_x \coloneqq \sum_{\gamma=1}^{q^{n-1}} \frac{1}{x+\gamma q} \le \frac{\log(q^{n-1})}{q},$$

to further bound Ω. As a result,

$$\sum_{j=1}^{2} \sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \frac{1}{a+\gamma_1 q}\frac{1}{b+\gamma_2 q} = \left(\frac{1}{a}+\omega_a\right)\omega_b + \omega_a\left(\frac{1}{b}+\omega_b\right) \le \frac{\log(q^{n-1})}{q}\left(\frac{1}{a}+\frac{1}{b}+\frac{2\log(q^{n-1})}{q}\right).$$

Therefore $\Sigma_{\pm}$ can be written as

$$\Sigma_{\pm} = \sum_{\substack{1\le a,b<q\\ a\equiv\pm b \mod q}} (\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)) + O_{n,k}\left(\frac{q^{\varepsilon}\log(q^{n-1})}{q} \sum_{\substack{1\le a,b<q\\ a\equiv\pm b \mod q}} \left(\frac{1}{a}+\frac{1}{b}+\frac{2\log(q^{n-1})}{q}\right)\right).$$

For Σ_+ we have $a \equiv +b \mod q$ and $1 \leq a, b \leq q$. Therefore $a = b$ and we have

$$\Sigma_+ = \sum_{1 \leq a < q} (\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)) + O_{n,k}\left(\frac{q^\varepsilon \log(q^{n-1})}{q} \sum_{1 \leq a < q} \left(\frac{2}{a} + \frac{2\log(q^{n-1})}{q}\right)\right).$$

For Σ_- we have $1 \leq a, b \leq q$ and $a \equiv -b \mod q$. Therefore $b = q - a$ and

$$\Sigma_- = \sum_{1 \leq a < q} (\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(q-a)) + O_{n,k}\left(\frac{q^\varepsilon \log(q^{n-1})}{q} \sum_{1 \leq a < q} \left(\frac{1}{a} + \frac{1}{q-a} + \frac{2\log(q^{n-1})}{q}\right)\right).$$

We bound the partial harmonic series again by $\log q$ to simplify both errors for Σ_+ and Σ_-. Consequently both error terms above can be bounded by

$$O_{n,k}\left(\frac{\log(q^{n-1})}{q^{1-\varepsilon}} \left(2\log q + 2\log(q^{n-1})\right)\right).$$

By combining the error terms, the moment sequence from Equation (2.13) is[12]

$$\begin{aligned} M_q(\underline{n}, \underline{m}) =& \frac{1}{\pi^{2n}}\Sigma_+(\underline{n}, \underline{m}) - \frac{1}{\pi^{2n}}\Sigma_-(\underline{n}, \underline{m}) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right) \\ =& \frac{1}{\pi^{2n}} \sum_{1 \leq a \leq q} (\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)) - \frac{1}{\pi^{2n}} \sum_{1 \leq a \leq q} (\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(q-a)) \\ &+ O_{n,k}\left(\frac{2^{2n+2}(\log q^{n-1})^2}{q^{1-\varepsilon}}\right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right). \end{aligned}$$

Our aim is to only have one main term,

$$\frac{1}{\pi^{2n}} \sum_{a \geq 1} (\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)).$$

[12] Recall we are adding the Σ_- term instead of subtracting it in the even character case.

Therefore, we want to first bound the term

$$\frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(q-a) \right), \tag{2.14}$$

and then extend the sum

$$\sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(a) \right)$$

over all positive integers.

To bound Expression (2.14) we again use the bound of $\mathcal{B}$ from Equation (2.10) to show

$$\begin{aligned} \sum_{1 \le a < q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(q-a) \right) &\le 2^{2n} \sum_{1 \le a < q} \frac{d_n(a) d_k(a)}{a} \frac{d_n(q-a) d_k(q-a)}{q-a} \\ &\ll_{n,\varepsilon} q^{\varepsilon} \sum_{1 \le a < q} \frac{1}{a(q-a)} \le q^{\varepsilon} \frac{2 \log q}{q}. \end{aligned}$$

As a result,

$$M_q(\underline{n}) := M_q(\underline{n}, \underline{n}) = \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a) \mathcal{B}_{\underline{m},q,\underline{t}}(a) \right) + O_{n,k}\left(\frac{\log q}{q^{1-\varepsilon}} \right) + O_{n,k}\left(\frac{(\log q^{n-1})^2}{q^{1-\varepsilon}} \right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}} \right.$$

This can be simplified, as $\mathcal{B}_{\underline{n},q,\underline{t}}$ is equivalent to $\mathcal{B}_{\underline{n},\underline{t}}$ when $1 \le a \le q$, and we can combine the errors. Since $k, \underline{n}, \underline{m}$ are all fixed, we omit the dependencies on the error for ease of notation. Therefore

$$M_q(\underline{n}) = \frac{1}{\pi^{2n}} \sum_{1 \le a \le q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a) \mathcal{B}_{\underline{m},\underline{t}}(a) \right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}} \right). \tag{2.15}$$

The final step is to extend the main sum to infinity. We rewrite the sum

$$\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)=\sum_{a\ge 1}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)-\sum_{a>q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right).$$

By bounding $\mathcal{B}$ as before, the second sum has the upper bound

$$\sum_{a>q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)\le 2^{2n}\sum_{a>q}\left(\frac{d_n^2(a)d_k^2(a)}{a^2}\right).$$

For all ε, we take $d_n(a)^2d_k(a)^2=O(a^{2\varepsilon_{k,n}})=: O_k(a^{\varepsilon})$, so

$$\sum_{a>q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)\ll\sum_{a>q}\frac{a^{\varepsilon}}{a^2}\ll q^{-1+\varepsilon}. \tag{2.16}$$

This bound is clearly smaller than the error term in Equation (2.15), so as a result,

$$M_q(\underline{n})=\frac{1}{\pi^{2n}}\sum_{a\ge 1}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)+O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right).$$

□

To finish proving Lemma 2.3.4 we prove Proposition 2.3.7, showing how the expectation also equals the sum

$$\frac{1}{\pi^{2n}}\sum_{a\ge 1}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right).$$

Proof of Proposition 2.3.7. We are interested in the expectation

$$M(\underline{n},\underline{m})=\mathbb{E}\left(\prod_{i=1}^{k}F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right).$$

Using Equation (2.4) from Section 2.3.1, and $n\coloneqq n_1+\cdots n_k=m_1+\cdots+m_k$, the moment is equivalent to

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\frac{\eta^n\overline{\eta}^n}{\pi^{2n}}\right)\sum_{a\geq 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a).$$

Therefore we have

$$\mathbb{E}\left(\prod_{i=1}^{k} F_-(t_i)^{n_i}\overline{F_-(t_i)}^{m_i}\right) = \frac{1}{\pi^{2n}}\sum_{a\geq 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a),$$

proving Proposition 2.3.7. $\square$

Therefore we have shown the multivariate moment sequence

$$M_q(\underline{n},\underline{m}) = \frac{2}{\phi(q)}\sum_{\chi \text{ odd}}\prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i}$$

converges, as $q \to \infty$ through the primes, to

$$\mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right),$$

for all k-tuples $\underline{n},\underline{m}$ and $0 \leq t_1 < \cdots < t_k \leq 1$. This section only addressed the odd character case, but the proof is similar for even characters and leads to the same results. Therefore $(\mathcal{F}_{q,\pm})_{q \text{ prime}}$, the distribution of odd/even character paths f_χ modulo q, converges to $F_\pm$ as $q \to \infty$ in the sense of convergence of finite distributions.

2.4 Relative Compactness of the Sequence of Distributions

In the previous section we showed $(\mathcal{F}_q)$ converges in finite distributions to the process F as $q \to \infty$ through the primes. If we can prove the sequence of distributions is relatively compact, then it follows that $(\mathcal{F}_q)$ converges in distribution to F [7, Example 5.1]. This is much stronger than convergence of finite-dimensional distributions and concludes the proof of Theorem 2.1.1.

Prohorov's Theorem [75] states that if a sequence of probability measures is tight, then it must be relatively compact [7, Theorem 5.1]. For this we use Kolmorogorov's tightness criterion, quoted from Revuz and Yor:

Proposition 2.4.1. *[76, Th. XIII.1.8] Let $(L_p(t))_{t\in[0,1]}$ be a sequence of $C([0,1])-$valued processes such that $L_p(0) = 0$ for all p. If there exist constants $\alpha > 0$, $\delta > 0$ and $C \geq 0$ such that for any p and any $s < t$ in $[0,1]$, and we have*

$$\mathbb{E}(|L_p(t) - L_p(s)|^{\alpha}) \leq C|t-s|^{1+\delta},$$

then the sequence $(L_p(t))$ is tight.

For our sequence of processes $(\mathcal{F}_q(t))_{t\in[0,1]}$ we have $f_\chi(0) = 0$ for all q. We also have the trivial bound

$$|f_\chi(t) - f_\chi(s)| \leq \sqrt{q}|t-s|,$$

leading to

$$\mathbb{E}\big|f_\chi(t) - f_\chi(s)\big|^{2k} \leq q^k |t-s|^{2k}.$$

As a result, for $k > 1$ if we take $|t-s| < \frac{1}{q^{1-\varepsilon}}$ for $\varepsilon \in (0, \frac{k-1}{2k-1})$, we have

$$\mathbb{E}\big|f_\chi(t) - f_\chi(s)\big|^{2k} \leq |t-s|^{2k - \frac{k}{1-\varepsilon}} =: |t-s|^{1+\delta_1}, \tag{2.17}$$

where $\delta_1 := \frac{k-1+\varepsilon(1-2k)}{1-\varepsilon}$. Therefore if we show a similar bound for $\mathbb{E}\big|f_\chi(t) - f_\chi(s)\big|^{2k}$ for $|t-s| > \frac{1}{q^{1-\varepsilon}}$ then the tightness condition holds for our sequence of processes.

Various authors have found results bounding the average of the difference of character sums [2, 40, 49]. For example, Cochrane and Zheng [19] prove for positive integers k and Dirichlet characters modulo prime q,

$$\frac{1}{q-1} \sum_{\chi \neq \chi_0} \left| \sum_{n=s+1}^{s+t} \chi(n) \right|^{2k} \ll_{\varepsilon,k} q^{k-1+\varepsilon} + |t-s|^k q^{\varepsilon}.$$

To prove tightness however we need the $|t-s|$ term independent of q.

Lemma 2.4.2. *Let q be an odd prime. For all $\varepsilon \in (0,1)$, there exists constants $C_1(\varepsilon), C_2$ independent of q such that for all $0 \leq s < t \leq 1$,*

$$\mathbb{E}\big|f_\chi(t) - f_\chi(s)\big|^{4} \leq C_1(\varepsilon)|t-s|^{1+\delta_2} + C_2 \frac{q^{\varepsilon}}{q},$$

where $\delta_2 := 1 - \varepsilon$.

This lemma can be applied to characters of all moduli, not just primes, but for our work it is sufficient to look only at primitive characters. Clearly if $|t-s|^{1+\delta_2} \geq \frac{q^{\varepsilon}}{q}$, then the equation becomes

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^4 \leq C|t-s|^{1+\delta},$$

which, combined with Equation (2.17) above, proves the sequence $(\mathcal{F}_q)$ is tight for all $s, t \in [0,1]$.

Lemma 2.4.2 is similar to a result of Bober and Goldmakher [8, Lemma 4.1] and we use parts of their work in the proof. Unlike Section 2.3.3, we will consider the odd and even case at the same time.

Proof of Lemma 2.4.2. Using the Fourier expansion of f_χ, the difference $(f_\chi(t) - f_\chi(s))$ can be written as

$$\frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{1\leq|n|\leq q} \frac{\overline{\chi}(n)}{n} e(-sn)\left(1 - e(-(t-s)n)\right) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

Consequently,

$$\left|f_\chi(t) - f_\chi(s)\right|^4 \leq \frac{2^4}{\pi^4}\left|\sum_{1\leq n\leq q} \frac{\overline{\chi}(n)}{n} e(-sn)\left(1 - e(-(t-s)n)\right)\right|^4 + O\left(\frac{(3+\log q)^4}{q^2}\right).$$

Similar to Section 2.3.1 and [8, Lemma 4.1], we define

$$b(n) = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{\substack{n_1 n_2 = n \\ n_i \leq q}} \prod_{j=1}^{2} \left(e(-sn_j)\left(1 - e\left(-(t-s)n_j\right)\right)\right) & ;\ (n,q) = 1, \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Therefore,

$$\Bigg|\sum_{1\le n\le q}\frac{\overline{\chi}(n)}{n}e(-sn)\left(1-e(-(t-s)n)\right)\Bigg|^4=\Bigg|\sum_{1\le n\le q^2}\overline{\chi}(n)b(n)\Bigg|^2.$$

The sum $b(n)$ can be bounded using $(1-e(x))\le\min\{2,2\pi|x|\}$, so

$$|b(n)|\le d(n)\min\left\{\frac{2^2}{n},(2\pi(t-s))^2\right\}. \tag{2.18}$$

As a result, taking $n=a+mq$,

$$\mathbb{E}\Bigg|\sum_{1\le n\le q^2}\overline{\chi}(n)b(n)\Bigg|^2=\sum_{a=1}^{q}\Bigg|\sum_{m=0}^{q}b(a+mq)\Bigg|^2\le 2\sum_{a=1}^{q}|b(a)|^2+2\sum_{a=1}^{q}\Bigg|2^2\sum_{m=1}^{q}\frac{d(a+mq)}{a+mq}\Bigg|^2. \tag{2.19}$$

We are interested in bounding the latter inner sum,

$$\mathcal{D}_a:=\sum_{m=1}^{q}\frac{d(a+mq)}{a+mq}=\sum_{\substack{q<m\le(a+q^2)\\ m\equiv a(q)}}\frac{d(m)}{m}.$$

By Abel summation (see e.g. [83, Theorem 1]) this is

$$\frac{1}{a+q^2}\sum_{\substack{q<m\le(a+q^2)\\ m\equiv a(q)}}d(m)+\int_q^{(a+q^2)}\frac{1}{t^2}\sum_{\substack{m\le t\\ m\equiv a(q)}}d(m)dt.$$

In order to further bound the sum, we use the Shiu's upper bound [79, Theorem 1],

$$\sum_{\substack{n\le x\\ n\equiv a\,(q)}}d(n)\ll_\delta\frac{x\phi(q)\log x}{q^2}<x\cdot\frac{\log x}{q},$$

which is valid for all $x \geq q^{1+\delta}$ for any $\delta > 0$. Therefore,

$$\begin{aligned}\mathcal{D}_a &= \frac{1}{a+q^2} \sum_{\substack{q \leq m \leq (a+q^2) \\ m \equiv a(q)}} d(m) + \int_q^{(a+q^2)} \frac{1}{t^2} \sum_{\substack{m \leq t \\ m \equiv a(q)}} d(m) dt \\ &= O\left(\frac{\log(a+q^2)}{q}\right) + O\left(\int_{q^{1+\delta}}^{(a+q^2)} \frac{\log t}{qt} dt\right) + \int_q^{q^{1+\delta}} \frac{1}{t^2} \sum_{\substack{m \leq t \\ m \equiv a(q)}} d(m) dt.\end{aligned}$$

We bound $d(a) \ll a^{\varepsilon}$, so the sum $\sum_{m \leq t,\, m \equiv a\,(q)} d(a)$ is bounded by $O(a^{\varepsilon}(t/q+1))$. As a result,

$$\mathcal{D}_a = O\left(\frac{\log q}{q}\right) + O_{\delta}\left(\frac{(\log q)^2}{q}\right) + O_{\delta}\left(a^{\varepsilon} \frac{\log q}{q}\right).$$

As a result, fixing $\delta > 0$, Equation (2.19) becomes

$$\sum_{a=1}^{q} \left| \sum_{m=0}^{q} b(a+mq) \right|^2 \leq 4 \sum_{a=1}^{q} |b(a)|^2 + O\left(\sum_{a=1}^{q} \left| a^{\varepsilon} \frac{(\log q)}{q} \right|^2\right) = 4 \sum_{a=1}^{q} |b(a)|^2 + O\left(\frac{q^{\varepsilon}}{q}\right).$$

Therefore the only sum left to evaluate is $\sum_{a \leq q} |b(a)|^2$. Using the bound from Equation (2.18) and splitting the cases $\frac{1}{a} > \pi^2(t-s)^2$ and $\frac{1}{a} < \pi^2(t-s)^2$, we have

$$\sum_{a=1}^{q} |b(a)|^2 \leq 2^4 \left(\pi^4(t-s)^4 \sum_{a \leq \pi^{-2}(t-s)^{-2}} d(a)^2 + \sum_{\pi^{-2}(t-s)^{-2} < a \leq q} \frac{d(a)^2}{a^2}\right).$$

We combine the two sums by Rankin's trick. Taking $x = \pi^{-2}(t-s)^{-2}$,

$$\begin{aligned}\frac{1}{x^2} \sum_{a \leq x} d(a)^2 &\leq \frac{x^{\sigma_1}}{x^2} \sum_{a=1}^{\infty} \frac{d(a)^2}{a^{\sigma_1}}, && 1 < \sigma_1 < 2, \\ \sum_{a \geq x} \frac{d(a)^2}{a^2} &\leq \frac{1}{x^{\sigma_2}} \sum_{a=1}^{\infty} \frac{d(a)^2}{a^{2-\sigma_2}}, && 0 < \sigma_2 < 1.\end{aligned}$$

Note σ_1, σ_2 are bounded so that the sums converge and tend to zero as $x \to \infty$. These sums are addressed by one of Ramanujan's identities (see e.g. [21]). For $\mathrm{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}.$$

Therefore

$$\sum_{a=1}^{q} |b(a)|^2 \leq 2^4 \left(\frac{1}{x^{2-\sigma_1}} \frac{\zeta(\sigma_1)^4}{\zeta(2\sigma_1)} + \frac{1}{x^{\sigma_2}} \frac{\zeta(2-\sigma_2)^4}{\zeta(2(2-\sigma_2))} \right).$$

Taking $\sigma \coloneqq \min(2-\sigma_1, \sigma_2) \in (0,1)$ and substituting back $\pi^{-2}(t-s)^{-2} = x$,

$$\frac{1}{x^{2-\sigma_1}} \frac{\zeta(\sigma_1)^4}{\zeta(2\sigma_1)} + \frac{1}{x^{\sigma_2}} \frac{\zeta(2-\sigma_2)^4}{\zeta(2(2-\sigma_2))} \leq \frac{C}{x^{\sigma}} = C\pi^{2\sigma}(t-s)^{2\sigma},$$

for some $C = C(\sigma) > 0$. As a result,

$$\mathbb{E}\big|f_\chi(t) - f_\chi(s)\big|^4 \leq C(t-s)^{2\sigma} + O\left(\frac{q^{\varepsilon}}{q}\right).$$

Taking $\sigma = 1 - \varepsilon$ and therefore $2\sigma = 2 - \varepsilon$, we have completed the proof. □

Lemma 2.4.2 shows that the Kolmogorov's tightness criterion argument holds for

$$|t-s|^{1+\delta_2} \gg \frac{q^{\varepsilon}}{q},$$

where we take $\alpha = 4$ from Proposition 2.4.1. Therefore, combining with Equation

(2.17), we have shown for constant K,

$$\mathbb{E}\big|f_\chi(t) - f_\chi(s)\big|^4 \leq \begin{cases} |t-s|^{1+\delta_1} & ;\ \text{if } |t-s| \leq q^{-(1-\varepsilon_1)}, \\ K|t-s|^{1+\delta_2} & ;\ \text{if } |t-s| \geq q^{-(1-\varepsilon_2)}. \end{cases}$$

Here we can choose δ_1 and δ_2 in such a way that $\delta_1 = \frac{1-3\varepsilon_1}{1-\varepsilon_1}$ for $\varepsilon_1 \in (0, \frac{1}{3})$ and and $\delta_2 = 1 - \varepsilon_2$ where $\varepsilon_2 \in (0, 1)$. This is possible as our initial parameter choices are flexible enough to allow this.

For the right choice of ε_1 and ε_2 we have

$$\frac{q^{\varepsilon_2}}{q} < \frac{q^{\varepsilon_1}}{q}.$$

Therefore taking $\delta \coloneqq \min(\delta_1, \delta_2)$, Kolmorogorov's tightness criterion holds for all t, s and $(\mathcal{F}_q)$ is tight. As a result, $(\mathcal{F}_q)_{q \text{ prime}}$ converges in distribution to the random process F as $q \to \infty$, proving Theorem 2.1.1. This concurs with the result from Bober, Goldmakher, Granville and Koukoulopoulos for their distribution function,

$$\Phi_q(\tau) \coloneqq \frac{1}{\phi(q)} \#\left\{\chi \mod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi}\tau\right\},$$

weakly converging to their limiting function [9, Theorem 1.4]

$$\Phi(\tau) \coloneqq \mathbb{P}\left(\max_t |F(t)| > 2e^\gamma\tau\right).$$

Chapter 3

Further Properties of the Random Process

3.1 Introduction

In Chapter 2, we considered random multiplicative functions and how they are used in the limiting distribution of complex character sums. Recall the definition of Steinhaus random multiplicative functions X_k from Definition 2.1.2. We therefore define the corresponding random series

$$F_+(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} \sin(2\pi kt) \quad \text{and} \quad F_-(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} (1 - \cos(2\pi kt)). \tag{3.1}$$

Here X_k are Steinhaus random multiplicative functions and η is a random variable uniformly distributed on the unit circle. The main result of Chapter 2 uses the random series to show the limiting distribution of character sums.

Theorem 3.1.1 (Theorem 2.1.1). *Let q be an odd prime and $F_\pm$ be defined as above for $t \in [0, 1]$. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ weakly converges to the process $F_\pm(t)$ as $q \to \infty$ through the primes.*

The infinite sum defining the random process converges with probability 1 [9]. Theorem 2.2.3 also shows F is almost surely continuous.

In this chapter, we investigate the support of $F(t)$ and more connections between character sums and this random process. In Section 3.2 we find the support of the law[1] of $F(t)$. These are the functions of the form

$$g(t) = \frac{c}{\pi} \sum_{n \geq 1} \frac{a_n}{n} (1 - e(-nt)),$$

where $c, \{a_p\}$ are on the unit circle and the sequence $\{a_n\}$ is completely multiplicative. This then leads to the following proposition.

Proposition 3.1.2. *Let g be in the support of the law of the random process F over $C([0,1])$. For any $\varepsilon > 0$,*

$$\liminf_{q \to \infty} \frac{1}{\phi(q)} \left| \left\{ \chi \mod q \middle| \sup_{t \in [0,1]} : \left| \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n) - g(t) \right| < \varepsilon \right\} \right| > 0.$$

In other words, there is a non zero probability that for every character sum there exists an ε-close function in the support of $F(t)$, and vice versa.

Remark 3.1.1. This chapter is motivated by Kowalski and Sawin's unpublished work 'On the support of Kloosterman paths' [57]. They find statistical results for Kloosterman paths by considering the support of their random Fourier series. In this chapter, we find the support of the law of $F(t)$, and use this to prove interesting properties of character sums.

3.2 Support of the Random Process

Let $F(t)$ be the random process defined in Chapter 2 and Equation 3.1. In this section we will consider the support of the law of $F(t)$.

[1] The support of the law of a random process is properly defined in Definition 3.2.1.

Definition 3.2.1. [6, Chapter VII] A set $C \subset C([0,1])$ is the **support of the law** of F if

$$\begin{aligned} C &= \bigcap \{K \subset C([0,1]) \text{ closed } : \mathbb{P}(F \in K) = 1\} \\ &= \{x \in C([0,1]) \, : \mathbb{P}(F \in G) > 0 \text{ for any neighbourhood } G \text{ of } x\}\,. \end{aligned}$$

Using the definition above, we have the following theorem.

Theorem 3.2.1. *The support of the law of the random process $F(t)$, denoted $\mathcal{S}$, over $C([0,1])$ is the set of* ***continuous*** *functions $g : [0,1] \to \mathbb{C}$ with the following properties:*

- $g(0) = g(1) = 0$*,*
- *there exists $c \in \mathbb{U}$ such that $c\hat{g}(nm) = \hat{g}(n)\hat{g}(m)$ for all $n, m \in \mathbb{Z}_{\neq 0}$,*
- $|\hat{g}(\xi)| = 1/2\pi|\xi|$ *for all* $\xi \neq 0$*,*
- $\hat{g}(0) = -\sum_{\xi \neq 0} \hat{g}(\xi)$*.*

Here $\hat{g}(\xi) = \int_0^1 g(t)e(-\xi t)dt$ is the Fourier transform of the function $g(t)$. Note that these properties imply that the Fourier series of g converge uniformly to g [30]. Also note that if we take $n = m = 1$ then the second condition implies that unless $\hat{g}(1) = 0$, which then implies that $g = 0$ identically, $c = \hat{g}(1)$.

Proof. For this proof, we will define a set $\mathcal{S}$, whose elements satisfy all the properties stated in the theorem. We then show the set $\mathcal{S}$ is closed, contained in the support of F, and $\operatorname{supp}(F) \subset \mathcal{S}$.

As in Section 2.2, for $y \geq 1$ we define the smooth and rough parts of the random process F respectively by,

$$S_y = \frac{\eta}{2\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) \leq y}} \frac{1 - e(nt)}{n} X_n \quad \text{ and } R_y = F - S_y = \frac{\eta}{2\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) > y}} \frac{1 - e(nt)}{n} X_n.$$

By absolute convergence, we know that the support of S_y is the closure of the set of elements

$$\tilde{g}_y := \frac{c}{2\pi} \sum_{\substack{n \neq 0 \\ P^+(|n|) \leq y}} \frac{1 - e(nt)}{n} \alpha_n$$

with α_n completely multiplicative and $|c| = 1$.

Let $\tilde{g} = \lim_{y \to \infty} \tilde{g}_y$, which is a convergent series, and $\varepsilon > 0$ be fixed. Then for all sufficiently large y we have

$$\left\| \frac{c}{2\pi} \sum_{P^+(|n|) > y} \frac{1 - e(nt)}{n} \alpha_n \right\|_\infty := \max_{t \in [0,1]} \left| \frac{c}{2\pi} \sum_{P^+(|n|) > y} \frac{1 - e(nt)}{n} \alpha_n \right| < \varepsilon.$$

Therefore $\tilde{g}_y$ belongs to the intersection of the support of S_y and the open ball U_ε of radius ε around $\tilde{g}$. As a result, for all large enough y we have

$$\mathbb{P}(S_y \in U_\varepsilon) > 0.$$

We are interested in the probability

$$\begin{aligned} \mathbb{P}(\|F - \tilde{g}\| < 2\varepsilon) &\geq \mathbb{P}(\|S_y - \tilde{g}\|_\infty < \varepsilon \text{ and } \|R_y\|_\infty < \varepsilon) \\ &= \mathbb{P}(\|S_y - \tilde{g}\|_\infty < \varepsilon)\mathbb{P}(\|R_y\|_\infty < \varepsilon : \|S_y - \tilde{g}\| < \varepsilon). \end{aligned}$$

Note that S_y and R_y are not independent, which is why the probability $\|R_y\| < \varepsilon$

is dependent on $\|S_y - \tilde{g}\|$. However, by Lemma 2.2.1, we know $\mathbb{P}(\|R_y\|_\infty < \varepsilon)$ approaches 1 as $y \to \infty$ independent of the value of $\|S_y\|_\infty$ and is positive for all large enough y. Therefore

$$\mathbb{P}(\|F - \tilde{g}\| < 2\varepsilon) > 0.$$

Since $\varepsilon > 0$ is arbitrary, we have $\tilde{g} \in \operatorname{supp}(F)$. Therefore the closure of the set of the **convergent** series

$$\frac{c}{2\pi} \sum_{n \neq 0} \frac{1 - e(nt)}{n} \alpha_n$$

is contained in the support of F.

As such, let $\mathcal{S}$ be the set

$$\mathcal{S} = \left\{ g \in C([0,1]) \;\middle|\; \begin{array}{c} g \text{ periodic with period 1,} \\ \exists c \in \mathbb{U} : c\hat{g}(nm) = \hat{g}(n)\hat{g}(m) \quad \forall n, m \in \mathbb{Z}_{\neq 0}, \\ |\hat{g}(\xi)| = 1/2\pi|\xi| \quad \forall \xi \neq 0, \\ \hat{g}(0) = -\sum_{\xi \neq 0} \hat{g}(\xi). \end{array} \right\}.$$

Let g be an arbitrary function in the set $\mathcal{S}$. Since $\hat{g}(n) = O(1/n)$ and g is continuous, then the Fourier series of g converges uniformly to g [81, Chapter 2, Question 3.b)(iii)]. Therefore,

$$g(t) = \sum_{\xi \in \mathbb{Z}} \hat{g}(\xi) e(\xi t) = \sum_{\xi \neq 0} (e(\xi t) - 1)\hat{g}(\xi).$$

We claim $\mathcal{S}$ is closed and therefore is exactly the set of convergent functions. To prove this, we let h be a limit point of the set $\mathcal{S}$. That is, suppose there exists a sequence (h_n) such that all functions $h_n(t)$ are in $\mathcal{S}$ and h_n converges uniformly to h on $C([0,1])$.

For all $n \in \mathbb{N}$ we can write $h_n \in \mathcal{S}$ as

$$h_n(t) = \sum_{\xi \in \mathbb{Z}} e(\xi t) \hat{h}_n(\xi),$$

where $\hat{h}_n$ has all the necessary properties. The sequence (h_n) is a sequence of continuous functions and uniformly converges, so by the Uniform Limit Theorem [68, Chapter 2] the limit $h(t)$ is also continuous.

We claim $h \in \mathcal{S}$. The limit h is periodic with period 1 and $\hat{h} = \lim_{n\to\infty} \hat{h}_n$. Clearly the second and third properties of $\mathcal{S}$ are preserved, so it is left to show that

$$\hat{h}(0) = -\lim_{n\to\infty} \sum_{\xi \neq 0} \hat{h}_n(\xi) = -\sum_{\xi \neq 0} \lim_{n\to\infty} \hat{h}_n(\xi) = -\sum_{\xi \neq 0} \hat{h}(\xi).$$

By the Dominated Convergence Theorem (see e.g. [36, Section 5.6]) we can swap the limit with the infinite sum, therefore proving the final defining property of $\mathcal{S}$ is satisfied and $h \in \mathcal{S}$. Since h is an arbitrary limit point, $\mathcal{S}$ is closed and the set of convergent series

$$\sum_{\xi \neq 0} (e(\xi t) - 1) \hat{g}(\xi)$$

is contained in the support of F.

Finally, we show all functions in the support of the law of F are contained in $\mathcal{S}$, therefore showing $\mathcal{S} = \operatorname{supp}(F)$. Let

$$\tilde{f}(t) := \frac{c}{2\pi} \sum_{k \neq 0} \frac{1 - e(kt)}{k} \alpha_n$$

be an arbitrary function in $\operatorname{supp}(F)$, with c on the unit circle and α_k completely multiplicative. Clearly $\tilde{f}$ is periodic with period 1, so has the Fourier transform

$$\begin{aligned}\hat{\tilde{f}}(\xi) &= \frac{c}{2\pi} \int_0^1 e(\xi t) \left(\sum_{k \neq 0} \frac{1 - e(kt)}{k} \alpha_k \right) dt = \frac{c}{\pi} \sum_{k \neq 0} \frac{\alpha_k}{k} \int_0^1 e(\xi t) \left(1 - e(kt)\right) dt \\ &= \begin{cases} \dfrac{c}{2\pi} \displaystyle\sum_{k \neq 0} \frac{\alpha_k}{k} & ;\ \xi = 0, \\ -\dfrac{c}{\pi} \dfrac{\alpha_{-\xi}}{\xi} & ;\ \xi \neq 0. \end{cases}\end{aligned}$$

As a result, $\tilde{f}$ satisfies all the properties of $\mathcal{S}$ and therefore $\operatorname{supp}(F) \subset \mathcal{S}$. □

Once we know the support, we have the following proposition.

Proposition 3.2.2 (See Proposition 3.1.2)**.** *Let g be in the support of the law of the random process F over $C([0,1])$. For any $\varepsilon > 0$,*

$$\liminf_{q \to \infty} \frac{1}{\phi(q)} \left| \left\{ \chi \bmod q \ :\ \sup_{t \in [0,1]} \left| \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n) - g(t) \right| < \varepsilon \right\} \right| > 0.$$

Additionally, if we assume g is not in the support of F, then there exists a $\delta > 0$ such that

$$\lim_{q \to \infty} \frac{1}{\phi(q)} \left| \left\{ \chi \bmod q \ :\ \sup_{t \in [0,1]} \left| \frac{1}{\sqrt{q}} \sum_{1 \leq n \leq qt} \chi(n) - g\left(t\right) \right| < \delta \right\} \right| = 0.$$

Proof. Recall the sequence of distributions of character paths $(\mathcal{F}_q)$ for prime q. From Theorem 2.1.1 we know the sequence $(\mathcal{F}_q)$ converges in distribution to F as $q \to \infty$

through the primes. Therefore using the standard result [7, Theorem 2.1], it is equivalent to say that for any open set $U \subset C([0,1])$, we have

$$\liminf_{q\to\infty} \frac{1}{\phi(q)} \big| \{\chi \bmod q \,:\, f_\chi(t) \in U\} \big| > \mathbb{P}(F \in U).$$

Let g be a continuous sample function of F. If U is an open neighbourhood of g in $C([0,1])$, then by definition we have $\mathbb{P}(F \in U) > 0$. Consequently,

$$\liminf_{q\to\infty} \frac{1}{\phi(q)} \big| \{\chi \mod q \,:\, f_\chi(t) \in U\} \big| > 0.$$

Now take U as the open ball of radius $\varepsilon > 0$ around g. Thus, $f_\chi(t)$ is a member of U if and only if

$$\sup_{t\in[0,1]} |f_\chi(t) - g(t)| < \varepsilon,$$

for all χ modulo q. Therefore, noting the difference between the character paths f_χ and character sums tends to 0 as $q \to \infty$, we have

$$\liminf_{q\to\infty} \frac{1}{\phi(q)} \left| \left\{ \chi \mod q \,:\, \sup_{t\in[0,1]} \left| \frac{1}{\sqrt{q}} \sum_{n\leq qt} \chi(n) - g(t) \right| < \varepsilon \right\} \right| > 0.$$

For the latter part of Proposition 3.2.2, we assume g is not in the support of law of F, for example the non continuous saw-tooth function

$$\frac{1}{\pi} \sum_{n\geq 1} \frac{\sin(2\pi k t)}{k}.$$

There exists a neighbourhood $U \subset C([0,1])$ of g such that $\mathbb{P}(F \in U) = 0$. Therefore for some $\delta > 0$, U contains the closed ball C of radius δ around g. We use the standard result [7, Theorem 2.1] again to show

$$0 \leq \limsup_{q\to\infty} \frac{1}{\phi(q)} \big| \{\chi \bmod q \,:\, f_\chi(t) \in C\} \big| \leq \mathbb{P}(F \in C) = 0.$$

Therefore by the same reasoning, we have

$$\liminf_{q\to\infty} \frac{1}{\phi(q)} \left| \left\{ \chi \bmod q \,:\, \sup_{t\in[0,1]} \left| \frac{1}{\sqrt{q}} \sum_{n\leq qt} \chi(n) - g(t) \right| < \delta \right\} \right| = 0.$$

$\square$

Finally, by choosing specific functions from the support of the law of F, we find some interesting arithmetic statements. This is explored in the following section.

3.3 Examples of Functions in the Support of the Law of $F(t)$

Proposition 3.2.2 states that the Fourier series is a good approximation for the character sum. If we find examples of $g(t)$, a function in the support of the law of the random process F, then we know there exists a character sum which exhibits the same behaviour. In this section, we will consider some interesting examples of $g(t)$.

3.3.1 Example 1

Let P be large and $g_{1,P}(t)$ be defined as

$$g_{1,P}(t) := \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \sin(2\pi n t).$$

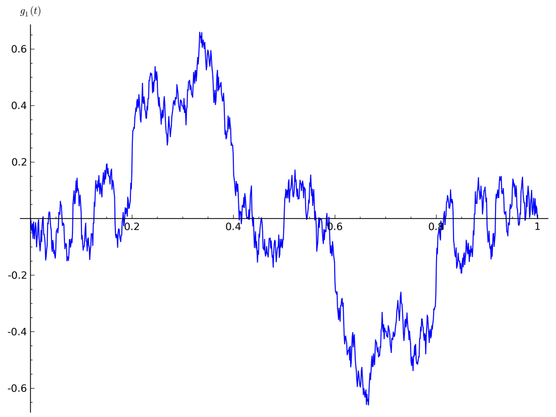


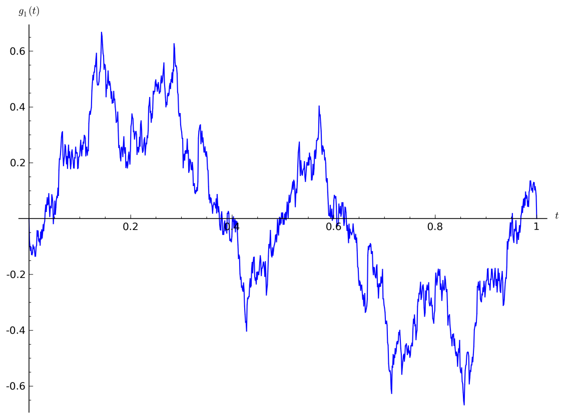
Figure 3.1: The function $g_{1,10007}(t)$, where the sum is truncated at $10,000$ terms.

Here $\lambda(n)$ is completely multiplicative and defined by

$$\lambda(p) = \begin{cases} -1 & ; p \le P, \\ X_p & ; \text{otherwise}, \end{cases}$$

where X_p are random variables uniformly distributed on the unit circle. In other words, $\lambda(n)$ is the completely multiplicative Liouville function[2] when $P^+(n) \le P$ and is randomly distributed on the unit circle otherwise. The function, by Theorem 2.2.3, is almost surely continuous. Therefore $g_{1,P}(t)$, for any $P > 0$, is almost surely in the support of the law of $F(t)$ by Theorem 3.2.1. See Figure 3.1 for an example of $g_{1,P}$ when $P = 10,007$.

Using Proposition 3.2.2, there is a positive proportion of character sums which are ε-close to $g_{1,P}(t)$.



The function $g_2(t)$, where $a_2 = 1$ and $a_p = -1$ for all $p \neq 2$.

The function $g_3(t)$, where $a_2, a_3 = 1$ and $a_p = -1$ for all $p \neq 2, 3$.

Figure 3.2: The function $g_m(t)$, truncated with 10,000 terms, for $m = 2$ and 3.

3.3.2 Example 2

In Example 1, we took $\lambda(p) = -1$ for all primes $\leq P$. Here we define the function

$$g_{m,P}(t) := \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(2\pi n t),$$

where $a_p = 1$ for the first $m-1$ primes, -1 for all other primes $\leq P$, and randomly uniformly distributed on the unit circle otherwise. For example, $g_{2,P}(t)$ has $a_2 = 1$ and $a_p = -1$ for primes $p \in (2, P]$, and randomly distributed on the unit circle for all $p > P$. Figure 3.2 shows $g_{2,10{,}007}(t)$ and $g_{3,10007}(t)$, and Figure 3.1 shows the case $a_p = -1$ for all primes $p \leq 10,007$. As in Example 1, the function is almost surely continuous for all finite $m \in \mathbb{Z}$ and almost surely in the support of the law of $F(t)$. Note that as m and P tend to infinity, this tends to the saw tooth function $g_\infty(t) = 1/2 - \{t\}$, which is not continuous and not in the support of the law of

[2] Let $n = \prod_{i=1}^{k} p_i^{a_i}$. The Liouville function is defined by $\lambda(n) = (-1)^{\sum a_i}$. We also define $\lambda(1) = 1$. This is equivalent to defining λ as a completely multiplicative function where $\lambda(p) = -1$. See [63] for more information on the function.

$F(t)$.

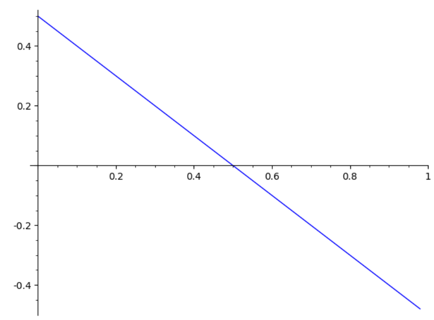


Figure 3.3: The function $g_\infty(x)$ with $x \in [0, 1]$.

It is future work to further investigate these functions in the support of the law of our random process F, using them to discover more about the behaviour of character sums.

Chapter 4

The Distribution of Real Character Sums

4.1 Introduction

Given the real Dirichlet character modulo prime q, otherwise known as the Legendre symbol $\left(\frac{\cdot}{q}\right)$, we define the normalised partial character sum

$$S_q(t) := \frac{1}{\sqrt{q}} \sum_{n \le qt} \left(\frac{n}{q}\right), \tag{4.1}$$

for $t \in [0,1]$. Legendre symbols, and therefore quadratic residues modulo primes, play an important role in many areas of number theory. Davenport and Erdös [23] studied the distribution of a similar function,

$$S_{q,H}(x) := \sum_{x < n \le x+H} \left(\frac{n}{q}\right),$$

as x runs through the positive integers. They found that it tends to a normal distribution with mean zero and variance H, provided $\log H = o(\log q)$ and $H \to \infty$ as $q \to \infty$.

In a follow up to Chapter 2, our goal is to study the distribution of real character sums $S_q(t)$ for $q \in [Q, 2Q]$ with $Q \in \mathbb{Z}$, and the corresponding limiting distribution as $Q \to \infty$. The notation has changed for this chapter, but the same methodology of Chapter 2 can be used in the quadratic case.

The character sum defined in Equation (4.1) is discontinuous, with jumps at every $t \in \frac{1}{q}\mathbb{Z}$. As in Definition 1.1.1, we define a *character path* as the continuous function $f_q(t)$, where we concatenate the points where the function changes. For the rest of the chapter, we will refer to character paths $f_q(t)$ instead of the discontinuous sums $S_q(t)$. Character paths are continuous, periodic in t with period 1, and can be approximated by the truncated Fourier expansion as in Appendix A:

$$f_q(t) = \frac{\tau\left(\left(\frac{\cdot}{q}\right)\right)}{2\pi i\sqrt{q}} \sum_{0<|n|<Q} \left(\frac{n}{q}\right) \frac{1-e(-nt)}{n} + O\left(\frac{\sqrt{q}\log Q}{Q}\right). \tag{4.2}$$

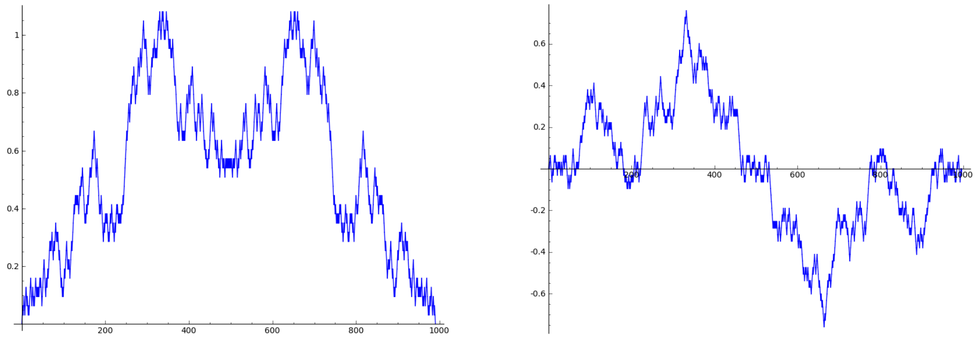
The difference between the character sum and character path is always bounded by $\frac{1}{\sqrt{q}}$, and since we are taking $q \in [Q, 2Q]$, we get that $|S_q(t) - f_q(t)| \leq \frac{1}{\sqrt{Q}}$. This is smaller than the error in the Fourier truncation, making Equation (4.2) an acceptable approximation of the character path. Additionally, $\tau\left(\left(\frac{\cdot}{q}\right)\right)$ is the Gauss sum,

$$\tau\left(\left(\frac{\cdot}{q}\right)\right) := \sum_{a=1}^{q} \left(\frac{a}{q}\right) e(a/q) = \begin{cases} \sqrt{q}, & q \equiv 1\,(4), \\ i\sqrt{q}, & q \equiv -1\,(4). \end{cases}$$

The value of the prime modulo 4 influences the shape of the character path, as shown in Figure 4.1. We refer to character paths as odd if $q \equiv -1 \bmod 4$, as then $\left(\frac{-1}{q}\right) = -1$, and even if $q \equiv 1 \bmod 4$, as $\left(\frac{-1}{q}\right) = 1$.

We now define the distribution of real character paths.

Definition 4.1.1. Let Q be a large integer. We define the **distribution of character paths** by taking $q \mapsto f_q(t)$ as a random process, choosing a prime $q \in [Q, 2Q]$



Character path for $q = 991 \equiv 1 \bmod 4$ Character path for $q = 997 \equiv 3 \bmod 4$.

Figure 4.1: Character paths for $q = 991$ and $q = 997$, where the x-axis is qt.

uniformly at random. Let $F_Q(t)$ denote the distribution, where

$$\mathcal{F}_Q(t) := \{f_q(t) : Q \leq q \leq 2Q,\, q \text{ prime}\}\,.$$

We also define $\mathcal{F}_{Q,\pm}(t)$ by fixing the value of $\left(\frac{-1}{q}\right)$ as either $+1$ or -1 for all $q \in [Q, 2Q]$.

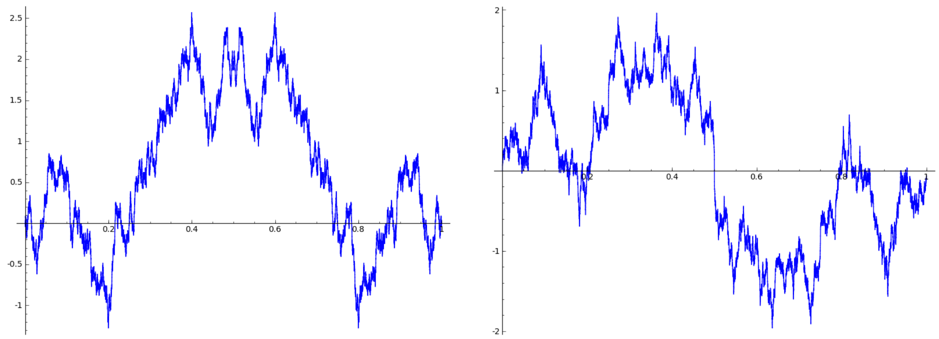
We are investigating the limit of the distribution of real character sums, i.e. when $Q \to \infty$. For this, we need Rademacher random multiplicative functions.

Definition 4.1.2. Let $X_p = \{\pm 1\}$ be **Rademacher random variables**, independently taking values ± 1 with equal probability. We define **Rademacher random multiplicative functions** X_n, for $n \in \mathbb{N}$, as

$$X_n = \prod_{p^a \| n} X_p^a,$$

where X_p are Rademacher random variables. We also let $X_{-1} = \{\pm 1\}$ so the definition extends to negative numbers.

More properties of Rademacher random multiplicative functions can be seen



Sample of $F(t)$ with $X_{-1} = -1$. Sample of $F(t)$ with $X_{-1} = 1$

Figure 4.2: Samples, with 10,000 points, of $F(t)$, where the x axis is t.

in [18, 38, 39, 61].

Our aim is to find the limiting distribution of $\mathcal{F}_Q$. To begin, we define a sum very similar to Equation (4.2). Let $F_{\pm}(t)$ be the random functions

$$F_+(t) := \frac{1}{\pi}\sum_{n\geq 1} X_n \frac{\sin(2\pi n t)}{n} \qquad \text{and} \qquad F_-(t) := \frac{1}{\pi}\sum_{n\geq 1} X_n \frac{1-\cos(2\pi n t)}{n},$$

where X_n are Rademacher random multiplicative functions. By defining $X = \{1, i\}$, depending on if $X_{-1} = 1$ or -1 respectively, we can combine the random functions to obtain

$$F(t) := \frac{X}{2\pi i}\sum_{n\neq 0} X_n \frac{1-e(-nt)}{n}.$$

See Figure 4.2 for examples of the random process.

This infinite sum is defined as the limit of the symmetrical partial sums

$$F_N(t) := \frac{X}{2\pi i}\sum_{0<|n|<N} X_n \frac{1-e(-nt)}{n}.$$

Benatar, Nishry, and Rodgers [5] studied a similar function,

$$P_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n e(nt).$$

Combining partial summation with [5, Theorem 1.3], for any $\varepsilon > 0$,

$$\max_t |F_N(t)| \leq \frac{\exp(3\sqrt{\log N \log\log N})}{\sqrt{N}} + O\left(1 - \frac{N^\varepsilon}{\sqrt{N}}\right).$$

Additionally, the function $F_N(t)$ is clearly in $L^2([0,1])$, since

$$\sum_{0<|n|<N} \left| X_n \frac{1-e(-nt)}{n} \right|^2 \leq 2^3 \sum_{0<n<N} \frac{1}{n^2} \leq 2^3 \left(\frac{\pi^2}{6} - \frac{1}{N}\right).$$

For fixed $t \in [0,1]$, the limit of the partial sums converges almost surely [53, Lemma 1]. In the proof of Theorem 4.2.1, we will show the process is almost surely continuous.

We can now state the main theorem of the chapter:

Theorem 4.1.1. *Let $F_\pm(t)$ be the random processes defined above for $t \in [0,1]$. Assuming the Generalised Riemann Hypothesis (GRH) for all Dirichlet L-functions, the sequence of distributions of real character paths $(\mathcal{F}_{Q,\pm}(t))_Q$ weakly converges to the process $F_\pm(t)$ in the Banach space $C([0,1])$ as $Q \to \infty$. In other words, for any continuous and bounded map*

$$\phi : C([0,1]) \to \mathbb{R},$$

we have

$$\lim_{Q\to\infty} \mathbb{E}(\phi(\mathcal{F}_{Q,\pm})) = \mathbb{E}(\phi(F_{\pm})).$$

The proof of the theorem is split into 2 main steps: showing convergence in finite distributions by proving the multivariate moments of $\mathcal{F}_Q$ converge to multivariate moments of F as $Q \to \infty$, and showing that the sequence $(\mathcal{F}_Q)_Q$ is relatively compact. The combination of both steps proves convergence in distribution, thereby proving Theorem 4.1.1.

Remark 4.1.1. Theorem 4.1.1 requires $F(t)$ to be a function almost surely taking values in $C([0,1])$. In Section 4.2, we prove the distribution $\mathcal{F}_Q(t)$ converges in finite distribution to

$$\frac{X}{2\pi i}\sum_{n\neq 0} X_n \frac{1-e(-nt)}{n}. \tag{4.3}$$

A priori, both the series and $F(t)$ are distributions on $L^2([0,1])$. On L^2, distributions are determined by their finite dimensional distributions [10, Corollary 2.4].

In proving Theorem 4.2.1, we will prove the limiting distribution of $(\mathcal{F}_Q(t))_Q$ is continuous and the finite dimensional distributions match those of $F(t)$. Therefore, as a random process in $L^2([0,1])$, the function in Equation (4.3) is almost surely the Fourier series of a continuous function. Combined with the fact that the Fourier coefficients are bounded by $O(1/n)$, the Fourier series converges uniformly to its defining function [47]. Therefore, by the Uniform Limit Theorem [68, Chapter 2], the Fourier series is almost surely continuous and uniquely defines $F(t)$ [88, Theorem 9.2].

Remark 4.1.2. The assumption of GRH comes from using a strong version of the

Prime Number Theorem in Arithmetic Progressions (see e.g. [11, Chapter 7]). Without the Generalised Riemann Hypothesis, the error term is too large and does not vanish as $Q \to \infty$. GRH is also needed to prove the tightness condition for the sequence $(\mathcal{F}_Q)$. Siegel zeroes are likely to be the worst problem, but for this proof it is not sufficient to only assume there are not real zeroes. It is a future project to remove the dependence on GRH in both parts of the proof. Future ideas involve changing where we truncate the Fourier series in Equation (4.2), using the zero density estimate for quadratic L-functions $L(1/2, \chi_d)$, and using different methods so we are not relying on the error bound on the Prime Number Theorem in Arithmetic Progressions.

Remark 4.1.3. Theorem 4.1.1 follows on from Chapter 2, and the paper [48] which Chapter 2 is based on, where we look at the limiting distribution of character sums modulo q as $q \to \infty$. The sequence of distributions of complex character sums weakly converges to a random Fourier series, with Steinhaus random multiplicative functions as Fourier coefficients, as $q \to \infty$. In this chapter, we find an analogous result for real Dirichlet characters, which includes a similar random process. We cannot use the orthogonality of characters in the same way, so the methods are slightly different. Additionally, since the values are all real in this chapter, the graphs shown are instead time graphs instead of maps of the complex plane.

4.2 Proof of Convergence in Finite Distributions

Recall the definitions of the real character path, which we have split into odd and even sums,

$$f_{q,+}(t) = \frac{1}{\pi}\sum_{a=1}^{Q}\left(\frac{a}{q}\right)\frac{\sin(2\pi at)}{a} + O\left(\frac{\sqrt{q}\log Q}{Q}\right),$$
$$f_{q,-}(t) = \frac{1}{\pi}\sum_{a=1}^{Q}\left(\frac{a}{q}\right)\frac{1-\cos(2\pi at)}{a} + O\left(\frac{\sqrt{q}\log Q}{Q}\right).$$

We do the same with the random Fourier series, where we fix $X_{-1} = +1$ or -1,

$$F_{+}(t) = \frac{1}{\pi}\sum_{a\geq 1} X_a\frac{\sin(2\pi at)}{a}, \qquad F_{-}(t) = \frac{1}{\pi}\sum_{a\geq 1} X_a\frac{1-\cos(2\pi at)}{a}.$$

As stated in the introduction, in this section we will prove the following theorem:

Theorem 4.2.1. *Assume GRH and let $Q \in \mathbb{Z}$ and $(\mathcal{F}_{Q,\pm}(t))_Q$ be the sequence of distributions of character paths, where for $\varrho = \{\pm\}$*

$$\mathcal{F}_{Q,\varrho}(t) := \left\{f_{q,\varrho}(t) : Q \leq q \leq 2Q,\ q \ \textit{prime}\right\}.$$

Then $(\mathcal{F}_{Q,\pm})_Q$ converges to the process $F_{\pm}$ in the sense of convergence of finite distributions[1]*. In other words, for every $n \geq 1$ and for every $n-$tuple $0 \leq t_1 <$*

[1] This is a weaker form of convergence. To prove Theorem 4.1.1, we also need the sequence $(\mathcal{F}_Q)$ to be relatively compact.

$\cdots < t_n \leq 1$, *the vectors*

$$(\mathcal{F}_{Q,\pm}(t_1), \ldots, \mathcal{F}_{Q,\pm}(t_n))$$

converge in law as $Q \to \infty$ *to*

$$(F_{\pm}(t_1), \ldots, F_{\pm}(t_n)).$$

The behaviour of both f_q and F vary greatly depending on the values of $\left(\frac{-1}{q}\right)$ and X_{-1} respectively. For this section we will fix the sums as even, so $\left(\frac{-1}{q}\right) = X_{-1} = 1$, as the odd case is analogous.

Theorem 4.2.1 can be proved by the method of moments. We define the limiting moment and moment sequence as follows: let $k \geq 1$ and $\underline{n} = (n_1, \ldots, n_k)$ for $n_i \in \mathbb{Z}_{\geq 0}$. Then the moments are

$$M_{\pm}(\underline{n}) := \mathbb{E}\left(\prod_{i=1}^{k} F_{\pm}(t_i)^{n_i}\right), \tag{4.4}$$

$$M_{Q,\pm}(\underline{n}) := \frac{2}{\pi^*(Q)} \sum_{\substack{q \in [Q,2Q] \\ q \equiv \pm 1\,(4)}} \prod_{i=1}^{k} f_{q,\pm}(t_i)^{n_i}, \tag{4.5}$$

where $\pi^*(Q) := \pi(2Q) - \pi(Q)$.

Proposition 4.2.2. *Let* $k \geq 1$, $\underline{n} = (n_1, \ldots, n_k)$ *for* $n_i \in \mathbb{Z}_{\geq 0}$, *and* $M_{\pm}(\underline{n})$ *and* $M_{Q,\pm}(\underline{n})$ *be defined as above. Then, assuming GRH,*

$$M_{Q,\pm}(\underline{n}) = M_{\pm}(\underline{n}) + O\left(\frac{(\log Q)^{n+2}}{\sqrt{Q}}\right).$$

Additionally, the moment $M_{\pm}(\underline{n})$ *only has one representing measure, and the mo-*

ment sequence is therefore determinate.

This proposition will be proved in 2 parts, first showing the limiting moment $M_\pm(\underline{n})$ equals a certain sum and is determinate, then proving the moment sequence $M_{Q,\pm}(\underline{n})$ equals the same sum, plus an error which converges to 0 as $Q \to \infty$. Since we are only considering the even case, we will point out when the odd case differs.

4.2.1 The Limiting Moment

Using the definition of $F_+(t)$, we can heavily simplify the moment in Equation (4.4). Taking $n = \sum n_i$,

$$M_+(\underline{n}) = \frac{1}{\pi^n}\mathbb{E}\left(\sum_{a_{1,1},\ldots,a_{k,n_k}>0}\prod_{i=1}^{k}\prod_{j=1}^{n_i} X_{a_{i,j}}\frac{\sin(2\pi a_{i,j}t_i)}{a_{i,j}}\right) = \frac{1}{\pi^n}\mathbb{E}\left(\sum_{a>0}\frac{X_a}{a}\mathcal{B}_{\underline{n},\underline{t}}(a)\right),$$

where[2]

$$\mathcal{B}_{\underline{n},\underline{t}}(a) := \sum_{a_1\cdots a_k=a}\prod_{i=1}^{k}\sum_{b_{i,1},\cdots b_{i,n_i}=a_i}\prod_{j=1}^{n_i}\sin(2\pi b_{i,j}t_i).$$

We are allowed to reorder the sums in the expectation as $F(t)$ converges almost surely [53, Lemma 1].

Lemma 4.2.3. *The sum $\mathcal{B}_{\underline{n},\underline{t}}(a)$ is bounded above by a^ε.*[3]

Proof. Firstly, we introduce absolute value signs and take $|\sin(2\pi b_{i,j}t_i)| \leq 1$. The

[2] For the odd case, $\mathcal{B}_{\underline{n},\underline{t}}(a)$ is defined in the same way, but $(1-\cos(2\pi b_{i,j}t_i))$ instead of $\sin(2\pi b_{i,j}t_i)$.

[3] The equivalent sum for $M_-(\underline{n})$ is also bounded by a^ε. The proof is analogous.

sum is then a product of divisor sums, so

$$\mathcal{B}_{\underline{n},\underline{t}}(a) \leq \sum_{a_1\cdots a_k=a} \prod_{i=1}^{k} d_{n_i}(a_i).$$

We then use Proposition 2.3.2, showing $d_{N_1}(x_1)d_{N_2}(x_2) \leq d_{N_1+N_2}(x_1 \cdot x_2)$. As a result, taking $n = \sum n_i$,

$$\mathcal{B}_{\underline{n},\underline{t}}(a) \leq \sum_{a_1\cdots a_k=a} \prod_{i=1}^{k} d_{n_i}(a_i) \leq d_k(a)d_n(a) \leq d^{k+n}(a).$$

By bounding $d(a)$ by a^{ε}, we have finished the proof. For later parts of the section, we may also use the stronger bound $\mathcal{B}_{\underline{n},\underline{t}}(a) \ll d_n(a)$. □

Taking the expectation inside the sum, which is permitted since $F(t)$ converges almost surely,

$$M_+(\underline{n}) = \frac{1}{\pi^n}\sum_{a>0}\frac{\mathbb{E}(X_a)}{a}\mathcal{B}_{\underline{n},\underline{t}}(a) = \frac{1}{\pi^n}\sum_{a'>0}\frac{1}{a'^2}\mathcal{B}_{\underline{n},\underline{t}}(a'^2). \tag{4.6}$$

This is due to $\mathbb{E}(X_a) = 1$ when a is a square number (i.e. $a = a'^2$) and vanishes otherwise. In the next section, we will show $M_{Q,+}(\underline{n})$ equals the same sum, with an error that vanishes as $Q \to \infty$.

Additionally, we have the following lemma.

Lemma 4.2.4. *The moment $M(\underline{n})$ only has one representing measure.*

Proof. In this proof we will only show the lemma is true for $M_+(\underline{n})$, but the same bounds hold for $M_-(\underline{n})$. It is sufficient to show $M(\underline{n})$ satisfies the Carleman condition [78, Theorem 15.11]:

$$\sum_{n=1}^{\infty} |M(\underline{2n})|^{-\frac{1}{2n}} = +\infty, \tag{4.7}$$

for $\underline{n} = (n_1, \ldots, n_k)$ and $n = |\underline{n}| = \sum n_i$.

Using the definition of $M(\underline{2n})$,

$$M(\underline{2n}) = \frac{1}{\pi^{2n}} \sum_{a>0} \frac{1}{a^2} \mathcal{B}_{\underline{2n},\underline{t}}(a^2) \leq \sum_{a>0} \frac{d_{2(n+k)}(a^2)}{a^2}.$$

We ideally want to apply a result from Bober and Goldmakher [8, Proposition 3.2]:

$$\sum_{a=1}^{\infty} \frac{d_{2(n+k)}(a)^2}{a^2} \leq \exp\left(4(n+k)\log\log(4(n+k)) + O\left(\frac{4(n+k)}{\log 3}\right)\right). \tag{4.8}$$

Our aim is to directly apply Bober and Goldmakher's result. This follows from the claim:

Claim. Let $a, N > 0$ and $d_N(a)$ to be the Nth divisor function of a. Then $d_{2N}(a^2) \leq d_{2N}(a)^2$.

Proof. The divisor function is multiplicative, so for $a = p_1^{j_1} \cdots p_k^{j_k}$, $d_N(x) = \prod_{i \leq k} d_N(p_i^{j_i})$. For prime powers [8],

$$d_N(p^j) = \binom{N+j-1}{j}.$$

Therefore,

$$d_N(a^2) = \prod_{i \leq k} \binom{N+2j_i-1}{2j_i} \qquad \text{and} \qquad (d_N(a))^2 = \prod_{i \leq k} \binom{N+j_i-1}{j_i}^2.$$

Using the definition of binomial coefficients,

$$\binom{N+2j-1}{2j} = \binom{N+j-1}{j} \prod_{k=0}^{j-1} \frac{N+j+k}{j+1+k}.$$

For $j, k, N \geq 1$, $\frac{N+j+k}{j+1+k} \leq \frac{N+k}{k+1}$. Therefore,

$$\binom{N+2j-1}{2j} \leq \binom{N+j-1}{j} \prod_{k=0}^{j-1} \frac{N+k}{k+1} = \binom{N+k-1}{j}^2.$$

As a result, the claim holds. $\square$

Therefore, we use Equation (4.8), the result from Bober and Goldmakher [8], and $|M(\underline{2n})|^{-\frac{1}{2n}}$ is bounded below by

$$\begin{aligned} &\exp\left(-2\log\log(4(n+k)) - \frac{2k}{n}\log\log(4(n+k)) + O\left(\frac{2(n+k)}{n\log 3}\right)\right) \\ &> \exp\left(-2\log\log(4(n+k))\right). \end{aligned}$$

This can be rewritten as $C(\log(4(n+k)))^{-2}$ for some constant C. As a result,

$$\sum_{n=1}^{\infty} |M(\underline{2n})|^{-\frac{1}{2n}} \geq C \sum_{n=1}^{\infty} \frac{1}{(\log(4(n+k)))^2}.$$

The lower bound diverges, so the sum in Equation (4.7) is therefore infinite, finishing the proof. $\square$

4.2.2 The Moment Sequence

The moment sequence $M_{Q,+}(\underline{n})$ sums over all primes $q \in [Q, 2Q]$ which are equivalent to 1 modulo 4. Using the definition of $f_{q,+}(t)$,

$$M_{Q,+}(\underline{n}) = \frac{1}{\pi^n} \frac{1}{2\pi^*(Q)} \sum_{\substack{q \in [Q,2Q] \\ q \equiv 1\,(4)}} \prod_{i=1}^{k} \left(\sum_{a=1}^{Q} \left(\frac{a}{q}\right) \frac{\sin(2\pi a t_i)}{a} + O\left(\frac{\sqrt{q}\log Q}{Q} \right) \right)^{n_i}, \quad (4.9)$$

where $n = \sum n_i$ and we define $\pi^*(Q)$ as $(\pi(2Q) - \pi(Q))$. As in Section 4.2.1, we similarly define $\mathcal{B}_{\underline{n},Q,\underline{t}}$ as

$$\mathcal{B}_{\underline{n},Q,\underline{t}}(a) := \begin{cases} \displaystyle\sum_{a_1 \cdots a_k = a} \prod_{i=1}^{k} \sum_{\substack{b_{i,1}, \cdots b_{i,n_i} = a_i \\ b_{i,j} < Q}} \prod_{j=1}^{n_i} \sin(2\pi b_{i,j} t_i) & ;\ (a,q) = 1 \\ 0 & ;\ (a,q) > 1. \end{cases} \quad (4.10)$$

Note that we included a condition that the sum $\mathcal{B}_{\underline{n},Q,\underline{t}}(a)$ vanishes when $(a, q) > 1$. This helps us later in the proof, and comes from $\left(\frac{a}{q}\right) = 0$ when a and q share a factor. Also note that the major difference to $\mathcal{B}_{\underline{n},\underline{t}}$ is that the $b_{i,j}$ are all less than Q. Consequently, we can also bound $|\mathcal{B}_{\underline{n},Q,\underline{t}}(a)|$ by a^{ε}. As a result, we multiply out the brackets in Equation (4.9), so

$$M_{Q,+}(\underline{n}) = \frac{1}{\pi^n} \frac{1}{2\pi^*(Q)} \sum_{\substack{q \in [Q,2Q] \\ q \equiv 1\,(4)}} \sum_{a=1}^{Q^n} \left(\frac{a}{q}\right) \frac{1}{a} \mathcal{B}_{\underline{n},Q,\underline{t}}(a) + O\left(\frac{(\log Q)^n}{\sqrt{Q}} \right). \quad (4.11)$$

The sum can be split into whether a is a square number or not. If a is a square, then $\left(\frac{a}{q}\right) = 1$ and

$$\frac{1}{\pi^n}\frac{1}{2\pi^*(Q)}\sum_{\substack{q\in[Q,2Q]\\ q\equiv 1\,(4)}}\sum_{\substack{a=1\\ a=\Box}}^{Q^n}\left(\frac{a}{q}\right)\frac{1}{a}\mathcal{B}_{\underline{n},Q,\underline{t}}(a) = \frac{1}{\pi^n}\sum_{a=1}^{Q^n}\frac{1}{a^2}\mathcal{B}_{\underline{n},Q,\underline{t}}(a^2).$$

This is the same as Equation (4.6), the value of $M_+(\underline{n})$. If a is not a square, then the sum is

$$\frac{1}{\pi^n}\frac{1}{2\pi^*(Q)}\sum_{\substack{q\in[Q,2Q]\\ q\equiv 1\,(4)}}\sum_{\substack{a=1\\ a\neq\Box}}^{Q^n}\left(\frac{a}{q}\right)\frac{1}{a}\mathcal{B}_{\underline{n},Q,\underline{t}}(a).$$

Following the ideas of Montgomery and Vaughan [66, Lemma 6], we instead sum over fundamental discriminants.

Definition 4.2.1. A **fundamental discriminant** d satisfies one of the following conditions:

- $d \equiv 1 \bmod 4$ and d is square free,
- $d = 4D$, where $D \equiv 2, 3 \bmod 4$ and D is square free.

For each non zero integer a, we can write $4a$ uniquely in the form dr^2, where d is a fundamental discriminant. As a result, $\left(\frac{a}{q}\right) = \left(\frac{d}{q}\right)$, unless $q|r$. Let

$$f_d = \sum_{\substack{a\leq Q^n\\ 4a=dr^2}}\frac{1}{a}\mathcal{B}_{\underline{n},Q,\underline{t}}(a).$$

Therefore,

$$\sum_{\substack{a=1 \\ a\neq\square}}^{Q^n} \left(\frac{a}{q}\right)\frac{1}{a}\mathcal{B}_{\underline{n},Q,\underline{t}}(a) = \sideset{}{'}\sum_{d=1}^{4Q^n} \left(\frac{d}{q}\right) f_d + O\left(\sum_{\substack{a=1 \\ a\neq\square \\ q^2|a}}^{Q^n} \left(\frac{a}{q}\right)\frac{1}{a}\mathcal{B}_{\underline{n},Q,\underline{t}}(a)\right),$$

where $\sum'$ is a sum over fundamental discriminants. Note that the $a \neq \square$ condition is no longer needed for the main term. The error can be simplified by taking $a = q^2 a'$ and $|B(q^2 a)| \leq Q^\varepsilon$, so

$$\sum_{\substack{a\leq Q^n \\ a\neq\square \\ q^2|n}} \frac{|B(a)|}{a} = \frac{1}{q^2}\sum_{\substack{a'\leq Q^n \\ a'\neq\square}} \frac{|B(q^2 a)|}{a'} \leq \frac{Q^\varepsilon \cdot \log Q}{q^2}.$$

Therefore, taking $\pi^*(Q) := (\pi(2Q) - \pi(Q)) \approx \frac{Q}{\log Q}$, the moment sequence from Equation (4.11) is

$$\begin{aligned} M_{Q,+}(\underline{n}) =& \frac{1}{\pi^n}\sum_{a=1}^{Q^n} \frac{1}{a^2}\mathcal{B}_{\underline{n},Q,\underline{t}}(a^2) + O\left(\frac{\log Q}{Q}\left|\sideset{}{'}\sum_{d=1}^{4Q^n} f_d \sum_{\substack{q\in[Q,2Q] \\ q\equiv 1\ (4)}} \left(\frac{d}{q}\right)\right|\right) \\ &+ O\left(\frac{Q^\varepsilon(\log Q)^2}{Q^2}\right) + O\left(\frac{(\log Q)^n}{\sqrt{Q}}\right). \end{aligned} \tag{4.12}$$

Consider the sum in the first error,

$$\sideset{}{'}\sum_{d=1}^{4Q^n} f_d \sum_{\substack{q\in[Q,2Q] \\ q\equiv 1\ (4)}} \left(\frac{d}{q}\right).$$

The Legendre symbol $\left(\frac{d}{\cdot}\right)$ is the unique primitive character modulo $|d|$ [67, Theorem 9.13], which we will denote as $\chi_d(q)$. Using the orthogonality of Dirichlet characters, the sum $\sum \chi_d(q)$ over the primes can be rewritten as

$$\sum_{q\in[Q,2Q]} \chi_d(q) \left(\frac{1}{\varphi(4)} \sum_{\chi \mod 4} \chi(q) \right) = \frac{1}{\varphi(4)} \sum_{q\in[Q,2Q]} \chi_d(q) \left(\chi_0(q) + \chi_{-4}(q)\right),$$

where χ_0 is the trivial character and χ_{-4} is the Kronecker symbol $\left(\frac{-4}{n}\right)$. In other words,

$$\begin{array}{c|ccc} n & 1 & 2 & 3 \\ \hline \chi_0 & 1 & 0 & 1 \\ \chi_{-4} & 1 & 0 & -1 \end{array}.$$

Therefore[4],

$$\sum_{\substack{q\in[Q,2Q] \\ q\equiv 1 \mod 4}} \chi_d(q) = \frac{1}{2} \left(\sum_{q\in[Q,2Q]} \chi_d(q) + \sum_{q\in[Q,2Q]} \chi_d(q)\chi_{-4}(q) \right).$$

Since d be a fundamental discriminant, then $\chi_d(q) = \left(\frac{d}{q}\right)$ is uniquely given as a primitive quadratic character modulo d [67, Theorem 9.13]. Additionally, $\chi_d(q)\chi_{-4}(q) = \chi_{4d}(q)$[5]. Therefore,

$$\sum_{\substack{q\in[Q,2Q] \\ q\equiv 1 \mod 4}} \left(\frac{d}{q}\right) = \frac{1}{2} \sum_{q\in[Q,2Q]} \chi_d(q) + \frac{1}{2} \sum_{q\in[Q,2Q]} \chi_{4d}(q).$$

[4]For the odd case where $q \equiv 3(4)$, everything uses the same method except we have

$$\sum_{\substack{q\in[Q,2Q] \\ q\equiv 3(4)}} \chi_d(q) = \frac{1}{\phi(4)} \sum_{q\in[Q,2Q]} \chi_d(q) \left(\chi_0(3q) + \chi_{-4}(3q)\right).$$

[5]Note this could be imprimitive if $4|d$.

Assuming GRH[6], the sum $\sum_{q\in[Q,2Q]}\chi_d(q)$ is bounded by [67, Theorem 13.7]

$$O\left(Q^{1/2}\log(dQ)\right).$$

Therefore,

$$\sum_{d=1}^{4Q^n}{}' f_d \sum_{\substack{q\in[Q,2Q]\\ q\equiv 1\,(4)}} \left(\frac{d}{q}\right) \ll \sqrt{Q}\sum_{d=1}^{4Q^n}{}'|f_d|\log(dQ) \ll_n \sqrt{Q}\log(Q)\sum_{d=1}^{4Q^n}{}'|f_d|.$$

To bound $\sum|f_d|$, we rearrange the sum back to non fundamental discriminants:

$$\sum_{0<d\le 4Q^n}{}'|f_d| = \sum_{0\le d\le 4Q^n}\left|\sum_{\substack{a\le Q^n\\ 4a=dr^2}}\frac{\mathcal{B}_{\underline{n},Q,\underline{t}}(a)}{a}\right| \le \sum_{a\le Q^n}\frac{|\mathcal{B}_{\underline{n},Q,\underline{t}}(a)|}{a}.$$

Recall the definition

$$\mathcal{B}_{\underline{n},Q,\underline{t}}(a) := \sum_{a_1\cdots a_k=a}\prod_{i=1}^{k}\sum_{\substack{b_{i,1},\cdots b_{i,n_i}=a_i\\ b_{i,j}<Q}}\prod_{j=1}^{n_i}\sin(2\pi b_{i,j}t_i).$$

Therefore,

$$\sum_{a\le Q^n}\frac{\mathcal{B}_{\underline{n},Q,\underline{t}}(a)}{a} \le \prod_{i=1}^{k}\left(\sum_{a\le Q}\frac{|\sin(2\pi a t_i)|}{a}\right)^{n_i} \le \prod_{i=1}^{k}(\log Q)^{n_i} = (\log Q)^n,$$

where $n=\sum n_i$.

[6]All bounds for $\sum_q \chi_d(q)$ without assuming GRH are not sufficient for this case, as we are summing over large values of d and the potential zeroes near 1 cause the double sum to explode as $Q\to\infty$.

As a result, the moment sequence in Equation (4.12) is

$$\begin{aligned} M_{Q,+}(\underline{n}) &= \frac{1}{\pi^n}\sum_{a=1}^{Q^n}\frac{1}{a^2}\mathcal{B}_{\underline{n},Q,\underline{t}}(a^2) + O\left(\frac{\log Q}{Q}\cdot\sqrt{Q}(\log Q)^{n+1}\right) + O\left(\frac{Q^{\varepsilon}(\log Q)^2}{Q^2}\right) + O\left(\frac{(\log Q)^n}{\sqrt{Q}}\right) \\ &= \frac{1}{\pi^n}\sum_{a=1}^{Q^n}\frac{1}{a^2}\mathcal{B}_{\underline{n},Q,\underline{t}}(a^2) + O\left(\frac{(\log Q)^{n+2}}{\sqrt{Q}}\right). \end{aligned}$$

To finish the proof, the main term has to be independent of Q. This is achieved in 2 steps, removing the dependence from $\mathcal{B}_{\underline{n},Q,\underline{t}}$ then showing the tail of the sum is smaller than the error term in Equation (4.13). By rearranging the sum into arithmetic progressions,

$$\sum_{a=1}^{Q^n}\frac{1}{a^2}\mathcal{B}_{\underline{n},Q,\underline{t}}(a^2) = \sum_{1\le a\le Q}\sum_{0\le m\le Q^{n-1}}\frac{1}{(a+mQ)^2}\mathcal{B}_{\underline{n},Q,\underline{t}}((a+mQ)^2).$$

When $m=0$, all the summands are $\le Q$, so trivially $\mathcal{B}_{\underline{n},Q,\underline{t}} = \mathcal{B}_{\underline{n},\underline{t}}$. Consequently,

$$\sum_{a=1}^{Q^n}\frac{1}{a^2}\mathcal{B}_{\underline{n},Q,\underline{t}}(a^2) = \sum_{1\le a\le Q}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2) + \sum_{1\le a\le Q}\sum_{1\le m\le Q^{n-1}}\frac{1}{(a+mQ)^2}\mathcal{B}_{\underline{n},Q,\underline{t}}((a+mQ)^2).$$

By bounding $\mathcal{B}((a+mQ)^2)$ by Q^{ε}, using the claim stated earlier in the proof,

$$\sum_{1\le a\le Q}\sum_{1\le m\le Q^{n-1}}\frac{1}{(a+mQ)^2}\mathcal{B}_{\underline{n},Q,\underline{t}}((a+mQ)^2) \le \frac{Q^{\varepsilon}}{Q^2}\sum_{1\le a\le Q}\sum_{1\le m\le Q^{n-1}}\frac{1}{(a/Q+m)^2} \ll \frac{Q^{\varepsilon}}{Q}.$$

As a result, the summands are independent of Q, and

$$M_{Q,+}(\underline{n}) = \frac{1}{\pi^n}\sum_{a\le Q}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2) + O\left(\frac{Q^{\varepsilon}}{Q}\right) + O\left(\frac{(\log Q)^{n+2}}{\sqrt{Q}}\right). \tag{4.13}$$

The final step is to extend the sum

$$\sum_{a\leq Q}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2)=\sum_{a\geq 1}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2)+O\left(\sum_{a>Q}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2)\right).$$

By Rankin's trick, for any $\sigma>0$

$$\sum_{a>Q}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2)\leq\frac{1}{Q^\sigma}\sum_{a\geq 1}\frac{1}{a^{2-\sigma}}\mathcal{B}_{\underline{n},\underline{t}}(a^2).$$

By choosing $\sigma\in(1/2,1)$, the error term is less than the error in Equation (4.13) and

$$M_{Q,+}(\underline{n})=\frac{1}{\pi^n}\sum_{a\geq 1}\frac{1}{a^2}\mathcal{B}_{\underline{n},\underline{t}}(a^2)+O\left(\frac{(\log Q)^{n+2}}{\sqrt{Q}}\right).$$

Combined with Section 4.2.1, this concludes the proof of Proposition 4.2.2.

4.3 Relative Compactness of the Sequence of Distributions $\mathcal{F}_Q$

Our aim is to prove Theorem 4.1.1, the sequence of real character paths $(\mathcal{F}_Q)$ converges in distribution to the random process F. If we can prove $(\mathcal{F}_Q)$ is relatively compact, then it follows from Theorem 4.2.1 that the sequence converges in distribution to F [7, Example 5.1]. This finishes the proof of the main theorem of the chapter.

We will prove relative compactness in the same way as in Chapter 2 (and [48, Section 4]). Firstly, Prohorov's Theorem [7, Theorem 5.1] states that if a sequence of probability measures is tight then it is relatively compact. We prove tightness by

Kolmogorov's tightness criterion (see Proposition 2.4.1 or [76, Theorem XIII.1.8]), shown in the following proposition.

Proposition 4.3.1. *The sequence of real character paths,*

$$\mathcal{F}_Q(t) := \{f_q(t) : Q \leq q \leq 2Q,\ q \text{ prime}\}$$

is a sequence of real continuous processes, where $f_q(0) = 0$ for all Q. Furthermore, there exist constants $\alpha > 0$, $\delta > 0$ and $C \geq 0$ such that for any prime q and any $s < t$ in $[0, 1]$ we have

$$\mathbb{E}\left(|f_q(t) - f_q(s)|^{\alpha}\right) \leq C|t - s|^{1+\delta}. \tag{4.14}$$

We prove Proposition 4.3.1 in 2 parts:

Lemma 4.3.2. *Let $s < t$ in $[0, 1]$. There exists $\varepsilon_1 > 0$ and $C > 0$ such that,*

$$\mathbb{E}\left(|f_q(t) - f_q(s)|^4\right) < C|t - s|^{1+\varepsilon_1},$$

whenever[7]

$$|t - s| > \frac{1}{Q^{1/2-\varepsilon_1'}}.$$

Lemma 4.3.3. *Let $s < t$ in $[0, 1]$. Assuming GRH (or the weaker Generalised*

[7] Taking $Q^{\varepsilon_1'} = (\log Q)^{\frac{6}{1+\varepsilon_1}} Q^{\frac{\varepsilon_1}{2(1+\varepsilon_1)}}$.

Lindelof Hypothesis[8]*.), for any* $\varepsilon_2 > 0$,

$$\mathbb{E}\left(|f_q(t) - f_q(s)|^4\right) < 4|t-s|^{1+\varepsilon_2},$$

where[9]

$$|t-s| < \frac{1}{Q^{\varepsilon_2'}}.$$

Combining the 2 Lemmas, Proposition 4.3.1 is true for all $s < t$ in $[0,1]$.

Proof of Lemma 4.3.2. In Section 4.2.2, we consider the moment sequence $M_Q(\underline{n})$. By abusing the definition, we get results for the expectation of $|f_q(t) - f_q(s)|$. Using Equation (4.13) and taking $\underline{n}$ with only one element, so $\underline{n} = (4)$,

$$\mathbb{E}\left(|f_q(t) - f_q(s)|^4\right) = \frac{1}{\pi^4}\sum_{a=1}^{Q}\frac{1}{a^2}B_{4,s,t}(a^2) + O\left(\frac{(\log Q)^6}{\sqrt{Q}}\right),$$

where

$$B_{4,s,t}(a^2) := \sum_{a_1\cdots a_4 = a^2}\prod_{j=1}^{4} e(-a_j s)\left(1 - e(-a_j(t-s))\right).$$

By bounding $(1 - e(x))$ by $\min\{2, 2\pi|x|\}$,

$$B_{4,Q,s,t}(a^2) \le d_4(a^2)\min\{2^4, a^2(2\pi|t-s|)^4\}. \tag{4.15}$$

[8]The Generalised Lindelof Hypothesis states that for any primitive character modulo q and any $\varepsilon > 0$, $L(1/2,\chi) \ll q^{\varepsilon}$ [64]

[9]Taking $\varepsilon_2' = \frac{\varepsilon_2}{1+\varepsilon_2}$.

As a result,

$$\frac{1}{\pi^4}\sum_{a=1}^{Q}\frac{1}{a^2}B_{4,Q,s,t}(a^2) \le \frac{2^4}{\pi^4}\sum_{a\ge\frac{1}{(\pi|t-s|)^2}}\frac{d_4(a^2)}{a^2} + (2|t-s|)^4\sum_{a\le\frac{1}{(\pi|t-s|)^2}} d_4(a^2).$$

The first sum can be bounded using Rankin's trick. For $\sigma \in (0,1)$,

$$\frac{2^4}{\pi^4}\sum_{a\ge\frac{1}{(\pi|t-s|)^2}}\frac{d_4(a^2)}{a^2} \le C|t-s|^{2\sigma}\sum_{a\ge 1}\frac{d_4(a^2)}{a^{2-\sigma}} = C'|t-s|^{2\sigma},$$

for some constant C'. The second sum can be trivially bounded by taking $d_4(a^2) = O(a^\varepsilon)$ for any $\varepsilon > 0$. As a result,

$$(2|t-s|)^4\sum_{a\le\frac{1}{(\pi|t-s|)^2}} d_4(a^2) \le C|t-s|^{4-2(1+\varepsilon)} = C|t-s|^{2-2\varepsilon}.$$

By choosing $\sigma = (1+\varepsilon)/2$ for any ε, then

$$\mathbb{E}\left(|f_q(t)-f_q(s)|^4\right) \le C'|t-s|^{1+\varepsilon} + O\left(\frac{(\log Q)^6}{\sqrt{Q}}\right).$$

Therefore for $|t-s| > \frac{(\log Q)^{\frac{6}{1+\varepsilon}}}{\sqrt{Q}^{\frac{1}{1+\varepsilon}}}$ for any $\varepsilon > 0$,

$$\mathbb{E}\left(|f_q(t)-f_q(s)|^4\right) \le C'|t-s|^{1+\varepsilon}.$$

$\square$

Proof of Lemma 4.3.3. Recall the definition of $f_q(t)$: the concatenation of points in the time graph of $\frac{1}{\sqrt{q}} \sum_{n \le qt} \chi(n)$. We note the trivial bound

$$|f_q(t) - f_q(s)| \le \frac{1}{\sqrt{q}} |qt - qs| = \sqrt{q}|t - s|.$$

Burgess [15] and Wang [86] proved that if $|t - s| > q^{-3/4+\varepsilon}$ for any $\varepsilon > 0$,

$$|f_q(t) - f_q(s)| \ll \frac{q^{1/2}|t - s|}{\log^A q},$$

for any $A > 0$. Using Abel Summation, we can improve this. As a result,

$$\sum_{n \le qt} \chi(n) \le |qt|^{1/2} \sum_{n \le qt} \frac{\chi(n)}{\sqrt{n}} - |qs|^{1/2} \sum_{n \le qs} \frac{\chi(n)}{\sqrt{n}} + \int_{qs}^{qt} u^{-1/2} \sum_{n \le u} \frac{\chi(n)}{\sqrt{n}} du.$$

By Burgess' bound [16, Theorem 3],

$$\sum_{n \le u} \frac{\chi(n)}{\sqrt{n}} = O(q^{3/16+\varepsilon}),$$

for any $\varepsilon > 0$. Consequently,

$$\sum_{qs \le n \le qt} \chi(n) \le q^{1/2} \cdot 3q^{3/16+\varepsilon} (|t|^{1/2} - |s|^{1/2}).$$

Since $s < t$, $(|t|^{1/2} - |s|^{1/2}) \le |t - s|^{1/2}$. As a result,

$$|f_q(t) - f_q(s)| \le q^{3/16+\varepsilon} |t - s|^{1/2}.$$

In this case,

$$\mathbb{E}\left(|f_q(t)-f_q(s)|^4\right) \leq \frac{|t-s|^2}{\pi(2Q)-\pi(Q)} \sum_{q\leq[Q,2Q]} q^{3/4+\varepsilon} \leq Q^{3/4+\varepsilon}|t-s|^2.$$

Therefore for any δ,

$$\mathbb{E}\left(|f_q(t)-f_q(s)|^4\right) \leq |t-s|^{1+\delta}$$

is satisfied if

$$|t-s| < Q^{-\frac{3/4+\varepsilon}{1-\delta}}.$$

However for any $\delta > 0$, this covers only a small percentage of values $t, s \in [0,1]$.

Let us assume the Generalised Lindelöf Hypothesis [50, Corollary 5.20], an implication of GRH [29, Section 1.9]: for any $\varepsilon > 0$,

$$L(1/2,\chi) \ll q^{\varepsilon}.$$

It is a folklore conjecture (see e.g. [32]), using the Generalised Lindelof Hypothesis, that

$$|f_q(t)-f_q(s)| \leq q^{\varepsilon}|t-s|^{1/2}.$$

As a result,

$$\mathbb{E}\left(|f_q(t)-f_q(s)|^4\right) \leq Q^{\varepsilon}|t-s|^2.$$

Consequently for $|t-s| < \frac{1}{Q^{\frac{\varepsilon}{1-\delta}}}$,

$$\mathbb{E}\left(|f_q(t) - f_q(s)|^4\right) \leq |t-s|^{1+\delta}.$$

Combined with Lemma 4.3.2, Equation 4.14 is true for all $s < t$ in $[0, 1]$.

$\square$

Therefore, assuming GRH, the sequence of distributions satisfies Kolmogorov's tightness criterion and is therefore tight. As a result, $(\mathcal{F}_Q)$ is relatively compact and, using Theorem 4.2.1, conditionally converges in distribution to the random process F.

We have therefore proven Theorem 4.1.1, the main theorem of the chapter.

Chapter 5

Concluding Remarks and Future Work

In this final chapter, we conclude the main results of the thesis.

We started by considering character sums

$$S_\chi(t) := \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n),$$

and the continuous modification to character paths

$$f_\chi(t) \coloneqq S_\chi(t) + \frac{\{qt\}}{\sqrt{q}} \chi\left(\lceil qt \rceil\right).$$

In Chapter 2, we considered the distribution of complex character paths with prime modulus,

$$\mathcal{F}_{q,\pm}(t) : = \{f_\chi(t) : \chi \mod q, \chi(-1) = \pm 1\}.$$

The limiting distribution of the complex case was a random process, formulated as Fourier series with Steinhaus random multiplicative functions as the Fourier coefficients. This mirrored the work of Bober, Goldmakher, Granville, and Koukoulopoulos [9], who found the limiting distribution of the maximum of character sums used the same random distribution. In Chapter 3, we identified the support of the law of the aforementioned random process, and found examples of functions in the support. Future work is to further investigate the links between the random process and character sums.

In Chapter 4, we modified our distribution from Chapter 2 to the distribution of real character paths with prime modulus:

$$\mathcal{G}_{Q,\pm}(t) := \{f_\chi(t) : q \in [Q, 2Q], \chi \bmod q \in \mathbb{R}, \chi(-1) = \pm 1\}.$$

Here we picked our prime conductor in a dyadic range $[Q, 2Q]$ for some large Q, and investigated the behaviour as $Q \to \infty$. By assuming GRH, a condition we hope to remove, the limiting distribution of $(\mathcal{G}_Q)$ is formulated as a Fourier series with Rademacher, instead of Steinhaus, random multiplicative functions as Fourier coefficients.

Independent to removing the condition on the Generalised Riemann Hypothesis, further advances in this field could include removing the condition that the conductor of the character paths has to be a prime number. There could then exist a scenario where there are too many characters that vanish due to sharing a factor with the modulus. As a result, the limiting distributions with Steinhaus and Rademacher random multiplicative functions as Fourier coefficients may not provide the best model. However, for suitably smooth conductors, where the number of prime factors is not too large, the same limiting distributions shown in this thesis could suffice.

Appendix A

The Fourier Expansion of $S_\chi(t)$

Recall the definition

$$S_\chi(t) := \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n),$$

for any primitive Dirichlet character $\chi \mod q$. We view this as a function over 'time' t, for $0 \leq t \leq 1$. We extend this domain by using the periodicity of χ and finding the Fourier expansion.

Lemma A.0.1. *The normalised partial character sum has the Fourier series*

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{k \neq 0} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)),$$

where $\tau(\chi)$ denotes the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e(a/q).$$

Note that this is only valid for t which is not a discontinuity of the function.

Proof. First, we find the Fourier coefficients of the periodic function $S_\chi(t)$:

$$\hat{S}_\chi(k) = \int_0^1 S_\chi(t) e(-kt) dt.$$

By splitting the integral into $t \in [a/q, (a+1)/q]$ for $0 \leq a < q$ and using the definition of $S_\chi(t)$, the Fourier transform equals

$$\hat{S}_\chi(k) = \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) \int_{n/q}^{1} e(-kt)dt = \begin{cases} \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) \left(1 - \frac{n}{q}\right) & ; k = 0, \\ \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) \frac{1 - e(-kn/q)}{-2\pi i k} & ; k \neq 0. \end{cases}$$

The total sum $\sum_{n=1}^{q} \chi(n)$ vanishes, so we can simplify $\hat{S}_\chi(k)$. Therefore,

$$\hat{S}_\chi(0) = -\frac{1}{q^{3/2}} \sum_{n=1}^{q} n\chi(n).$$

We use this for $k \neq 0$ as well, so

$$\hat{S}_\chi(k) = \frac{1}{2\pi i k\sqrt{q}} \sum_{n=1}^{q} \chi(n) e(-kn/q) =: \frac{\overline{\chi}(-k)\tau(\chi)}{2\pi i k\sqrt{q}},$$

using the definition of the Gauss sum $\tau(\chi)$ and noting that χ is primitive.

As a result, the Fourier series is

$$S_\chi(t) = \sum_{k\in\mathbb{Z}} \hat{S}_\chi(k) e(kt) = -\frac{1}{q^{3/2}} \sum_{n=1}^{q} n\chi(n) - \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{k\neq 0} \frac{\overline{\chi}(k)}{k} e(-kt).$$

We can simplify this further by noting that $S_\chi(0) = 0$. Therefore,

$$-\frac{1}{q^{3/2}} \sum_{n=1}^{q} n\chi(n) = \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{k\neq 0} \frac{\overline{\chi}(k)}{k},$$

leading to the desired result. □

In Chapters 2 and 4, we use the following truncation of the Fourier series [67, Equation 9.19]:

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{0<|k|\le K} \frac{\overline{\chi}(k)}{k}(1-e(-kt)) + O\left(\frac{\phi(q)}{K\sqrt{q}}\log(K)\right). \tag{A.1}$$

This holds whenever t is not a discontinuity of the function. This equates to whenever $tq \in \mathbb{Z}$. This is particularly useful when considering character paths, a continuous version of character sums, as there are no discontinuities to worry about.

The error is calculated using the following theorem.

Theorem A.0.2. *[67, Theorem D.2] If f has a bounded variation*[1] *on $[0,1]$, then for any t,*

$$\left|\frac{f(t^+)+f(t^-)}{2} - \sum_{|k|\le K} \hat{f}(k)e(kt)\right| \le \int_{0^+}^{1^-} \min\left(\frac{1}{2}, \frac{1}{(2K+1)\pi\sin(\pi x)}\right)|df(t+x)|.$$

Proof. Let $D_K(x)$ be the Dirichlet kernel,

$$D_K(x) = \sum_{k=-K}^{K} e(kx).$$

We have,

$$\sum_{k=-K}^{K} \hat{f}(k)e(kt) = \int_{\mathbb{T}} f(x)D_K(t-x)dx = \int_{\mathbb{T}} f(t-x)D_K(x)dx,$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Dirichlet kernels have certain properties that can help us analyse

[1] A function f has bounded variation if the total variation is bounded [84]. The total variation on an interval $[a,b]$ is the supremum of the sum of $|f(x_{i+1}) - f(x_i)|$ over the set of all partitions of $[a,b]$, $\{P = \{x_0,\ldots,x_{n_P}\}\}$ [52].

the integral. Firstly, D_K is an even function, so the integral is equivalent to

$$\int_{\mathbb{T}} D_K(x) f(t+x)dx. \tag{A.2}$$

If $x \notin \mathbb{Z}$, $D_K(x)$ is a geometric series and

$$D_K(x) = \frac{e((K+1)x) - e(-Kx)}{e(x) - 1} = \frac{\sin((2K+1)\pi x)}{\sin(\pi x)}.$$

Additionally let $E_K(x)$ be the function

$$E_K(x) = s(x) + \sum_{k=1}^{K} \frac{\sin(2\pi k x)}{\pi k},$$

where $s(x)$ is the saw-tooth function

$$s(x) = \begin{cases} \{x\} - \dfrac{1}{2} & ; \text{ if } x \notin \mathbb{Z}, \\ 0 & ; \text{ if } x \in \mathbb{Z}. \end{cases}$$

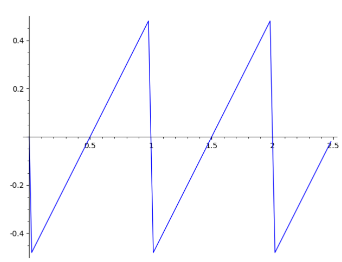


Figure A.1: The saw-tooth function $s(x)$ for $x \in [0, 5/2]$.

The function is also known as the secant coefficient, and has the bound [67, Lemma D.1]

$$|E_K(x)| \leq \min\left(\frac{1}{2}, \frac{1}{(2K+1)\pi|\sin(\pi x)|}\right). \tag{A.3}$$

Since E_K is an odd periodic function, we can take $x \in [0, 1/2]$. For $x \in \mathbb{Z}$, this is now just the $x = 0$ case. For $x \notin \mathbb{Z}$, we can differentiate E_K to obtain

$$E_K^{'}(x) = 1 + 2\sum_{k=1}^{K} \cos(2\pi kx) = D_K(x).$$

Therefore, since f has bounded variation on $\mathbb{T}$, Equation (A.2) equals

$$\int_{0^+}^{1^-} E_K^{'}(x) f(t+x)dx = \int_{0^+}^{1^-} f(t+x)dE_K(x).$$

Using integration by parts (see [67, Theorem A.2] for more),

$$\int_{0^+}^{1^-} f(t+x)dE_K(x) = E_K(1^-)f(t+1^-) - E_K(0^+)f(t+0^+) - \int_{0^+}^{1^-} E_K(x)|d(f(t+x)|.$$

From the definition of E_K, and using the $1-$periodicity of f,

$$\sum_{k=-K}^{K} \hat{f}(k)e(kt) = \int_{0^+}^{1^-} f(t+x)dE_K(x) = \frac{f(t^-)+f(t^+)}{2} - \int_{0^+}^{1^-} E_K(x)df(t+x).$$

Finally, rearranging the equation and applying the bound from Equation (A.3),

$$\left|\frac{f(t^+)+f(t^-)}{2} - \sum_{k=-K}^{K} \hat{f}(k)e(kt)\right| \leq \int_{0^+}^{1^-} \min\left(\frac{1}{2}, \frac{1}{(2K+1)\pi|\sin(\pi x)|}\right)|df(t+x)|.$$

$\square$

Character sums have a bounded variation on $[0, 1]$, notably $\phi(q)$. We can improve this by using the Legendre-Eratosthenes sieve.

Theorem A.0.3. *[67, Theorem 3.1] Let $S(x, y; q)$ denote the number of integers n such that $x < n \leq x + y$ and $(n, q) = 1$. Then for any real x and $y \geq 0$,*

$$S(x, y; q) = \frac{\phi(q)}{q} y + O\left(2^{\omega(q)}\right),$$

where $\omega(q)$ is the number of distinct primes factors[2] of q.

The theorem tells us that the residues are well distributed in the interval $[0, q]$. Additionally, for $y \geq q^{\varepsilon}$ the error term is smaller than the main term, as $\omega(n) \ll \log n / \log \log n$ for $n \geq 3$ [77, Theorem 11].

As a result, for $\alpha, \beta \in [0, 1]$ where $(\beta - \alpha) > q^{-1+\varepsilon}$,

$$\mathrm{Var}_{[\alpha,\beta]} S_{\chi}(t) \ll \frac{\phi(q)}{\sqrt{q}} (\beta - \alpha). \tag{A.4}$$

Here $\mathrm{Var}_{[\alpha,\beta]}$ is the total variation[3]. For $(\beta - \alpha) < q^{-1}$, the variation is bounded by $O(1/\sqrt{q})$ or vanishes, depending on the locations of α and β. Therefore, knowing the variation, we apply Theorem A.0.2 to character sums $S_{\chi}(t)$. When t is not a discontinuity of the function,

$$\begin{aligned} &\left| S_{\chi}(t) - \frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{|k| \leq K} \frac{\overline{\chi}(k)}{k} (1 - e(-kt)) \right| \\ &\quad \leq \int_{0^+}^{1^-} \min\left(\frac{1}{2}, \frac{1}{(2K+1)\pi \sin(\pi x)}\right) |dS_{\chi}(t + x)| + O\left(\frac{1}{\sqrt{q}}\right). \end{aligned}$$

The error term is due to $S_{\chi}(t)$ being a step function, so the Fourier series could be

[2] In our case, q is prime so $\omega(q) = 1$.

[3] The total variation on an interval $[a, b]$ is the supremum of the sum of $|f(x_{i+1}) - f(x_i)|$ over the set of all partitions of $[a, b]$, $\{P = \{x_0, \ldots, x_{n_P}\}\}$ [52].

up to $1/\sqrt{q}$ away from the character sum. We split the integral into K intervals of equal length. For $K \leq q^{1-\varepsilon}$, we can use Equation (A.4) to bound $|dS_\chi(t+x)|$ by $\frac{\phi(q)}{K\sqrt{q}}dx$. As a result, the integral has the upper bound

$$\frac{\phi(q)}{K\sqrt{q}} \sum_{i=1}^{K} \int_{(i-1)/K}^{i/K} \min\left(\frac{1}{2}, \frac{1}{(2K+1)\pi\sin(\pi x)}\right) dx. \tag{A.5}$$

To simplify the proof, we will take the minimum as $1/((2K+1)\pi\sin(\pi x))$, except for the first and final interval. Therefore, the integral is bounded by

$$\frac{\phi(q)}{K^2\sqrt{q}} + \frac{\phi(q)}{\pi K(2K+1)\sqrt{q}} \sum_{i=1}^{K-2} \int_{i/K}^{(i+1)/K} \frac{1}{\sin(\pi x)} dx.$$

Taking $1/\sin(\pi x) \ll \frac{1}{x}$, which holds for $0 < x < 1/2$, we see that Equation (A.5) is bounded by

$$\Phi := \frac{\phi(q)}{K^2\sqrt{q}} + \frac{\phi(q)}{K(2K+1)\sqrt{q}} \left(\sum_{i=1}^{K/2-1} \left| \log\left(\frac{i}{K}\right) \right| + \sum_{i=K/2+1}^{K-2} \frac{1}{K\sin\left(\pi\frac{K-1}{K}\right)} \right).$$

Therefore,

$$\Phi \leq \frac{\phi(q)}{K^2\sqrt{q}} + \frac{\phi(q)}{K(2K+1)\sqrt{q}} \left(K\log K + \frac{1}{\sin\left(\pi\frac{K-1}{K}\right)} \right) \ll \frac{\phi(q)}{K\sqrt{q}} \log K.$$

Consequently, for $K \leq q^{1-\varepsilon}$, as the integral bound is larger than $1/\sqrt{q}$

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{|k|\leq K} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + \left(\frac{\phi(q)}{K\sqrt{q}} \log K \right).$$

When $K > q^{1-\varepsilon}$, we can still split the integral into K intervals. Recalling that character sums are step functions with $\phi(q)$ jumps, for $K > \phi(q)$ there must be intervals where the variation vanishes[4]. Therefore we say the variation is bounded by 1 for $\phi(q)$ intervals and vanishes otherwise. For ease of notation, we will order the $\phi(q)$ intervals by $n_1 < \cdots < n_{\phi(q)}$, where $|n_{j+1} - n_j| < \phi(q)$. Therefore for $K > q$,

$$\left| S_\chi(t) - \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{|k|\leq K} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) \right|$$
$$\ll \frac{1}{\sqrt{q}} \sum_{i=1}^{\phi(q)-1} \int_{n_i/K}^{n_{i+1}/K} \min\left(\frac{1}{2}, \frac{1}{(2K+1)\pi\sin(\pi x)}\right) dx + O\left(\frac{1}{\sqrt{q}}\right).$$

As above, we take $1/((2K+1)\pi\sin(\pi x))$ as the minimum except for the first and last term. Therefore the integral is bounded by

$$\frac{\phi(q)}{K\sqrt{q}} + \frac{1}{\pi(2K+1)\sqrt{q}} \sum_{i=2}^{\phi(q)-1} \int_{n_{i/K}}^{n_{i+1}/K} \frac{1}{\sin(\pi x)} dx.$$

By bounding $1/\sin(\pi x)$ by $1/x$, the above equation is bounded by

$$\frac{\phi(q)}{K\sqrt{q}} + \frac{1}{K\sqrt{q}} \sum_{i=1}^{\phi(q)-1} \left|\log\left(\frac{n_i}{K}\right)\right| dx.$$

To err on the side of caution, we take $|\log(n_i/K)| \leq \log K$. The second term is consequently bigger than the first, and therefore for $K > q$,

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{|k|\leq K} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + O\left(\frac{\phi(q)}{K\sqrt{q}} \log K\right) + O\left(\frac{1}{\sqrt{q}}\right).$$

In Chapters 2 and 4, we take $K = q$ and Q respectively, where $q \in [Q, 2Q]$ in the latter case. For both cases, the first error is larger than $1/\sqrt{q}$, so we use the

[4]This is shown by the pigeonhole principle.

following formula:

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{|k|\leq K} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + O\left(\frac{\phi(q)}{K\sqrt{q}}\log K\right).$$

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