ASPECTS OF ABSTRACT REGULAR POLYTOPES AND THE COMBINATORICS OF COXETER GROUPS

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In this thesis, we examine abstract regular polytopes and some combinatorics of Coxeter groups.

For abstract regular polytopes, we define the notion of when such polytopes are *unravelled*. We then go on to examine and catalogue examples of these abstract regular polytopes. We construct four different non-trivial infinite families and analyse some small interesting examples. Chapter 2 gives an introduction, some concrete examples and a bird's-eye-view of the existence of such polytopes before Chapters 3 and 4 construct the specific non-trivial families.

In Chapter 5 we move on to Coxeter groups. Here we examine a neat combinatorial bijection between classes of reduced words of Coxeter groups and certain tilings of polygons known as Elnitsky's tilings. Chapter 6 examines the Bruhat order in relation to Elnitsky's tilings. In Chapter 7 we define E-embeddings; embeddings of Coxeter groups into the symmetric group that we show also give rise to bijections between tilings and reduced words. Chapter 8 provides an outline for a strategy to create E-embeddings but does not deliver an actual proof that this strategy indeed works. Chapter 9 examines the notions of 'subtilings' of tilings in the context of our E-embeddings. Chapter 10 provides some suggestions for further research.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Introduction

This is a thesis of two parts. The first examines abstract regular polytopes in Chapters 2, 3 and 4 and considers them in a mostly group theoretic light. These chapters summarise joint work with my supervisor Professor Peter Rowley, specifically [39] and [40]. The second half concerns Coxeter groups and explores some combinatorial objects associated to them. This is also all joint work with Professor Rowley and contains the work of [37] and [38].

The definitions of abstract regular polytopes and Coxeter groups are intimately related and Chapter 1 introduces them both together. Abstract regular polytopes are generalisations of the beloved Platonic solids and also admit a group theoretical characterisation as described by a wonderful correspondence theorem in [34]. We will cover this correspondence in some detail and provide a friendly example for clarification. Suffice it to say that the interplay between Abstract Geometry and Group Theory is naturally always present, explicitly or otherwise. We also list a number of fundamental objects associated to Coxeter groups here too. No matter which other chapters one wishes to read, this one should be a prerequisite.

In Chapter 2 we start our examination of abstract regular polytopes in earnest. Our main focus in Chapters 2, 3 and 4 is to provide a new property that an abstract regular polytope might have. We call this property *unravelledness*. These chapters focus mostly of computing examples and providing a total of four different non-trivial families of these so-called unravelled polytopes.

The remaining chapters, Chapters 5, 6, 7, 8, 9 and 10, focus on some combinatorics of Coxeter groups. These chapters can be read independently of Chapters 2, 3 and 4 and contain more exposition. Specifically, we focus on the work of Elnitsky ([11]) that creates three bijections between classes of reduced words of some given families of Coxeter groups and, rather surprisingly, tilings of polygons. Due to the inherent visual nature of this work we try provide many diagrams to illustrate examples. We give this work a detailed introduction in Chapter 5 and generalise this to all finite irreducible Coxeter groups in Chapter 7 by using an insight into the relationship of such tilings and the Bruhat orders. We then show some attempts to construct new tilings in Chapter 8 where we outline a a strategy for making new tilings. However, we are not able to prove this strategy indeed works. In Chapter 9 we make new tilings form old by considering the notion of a subtiling. We then finish on some more ideas for future generalisations and alternative constructions in Chapter 10.

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Chapter 1

Background on abstract regular polytopes and Coxeter groups

We will spend this chapter gently introducing the main objects used in this thesis. Most of the core ideas lie somewhere in the intersection of Group Theory, Abstract Geometry and Combinatorics. The core objects we study are all C-groups. The C here stands for Coxeter but we make the distinction between C-groups and Coxeter groups as names. There are three special cases of C-groups that interest us: Coxeter groups, string C-groups and their intersection, string Coxeter groups. The containments are demonstrated in the following diagram where each arrow denotes that the source contains the target.

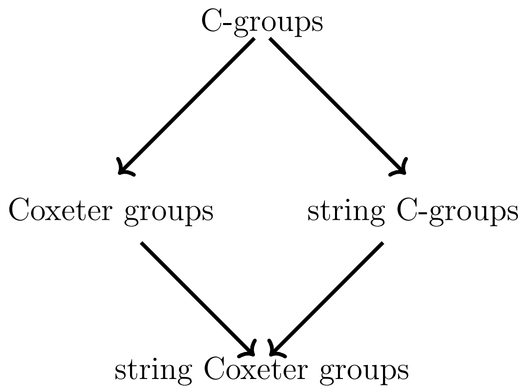


Figure 1.1: The containment of C-groups, Coxeter groups string C-group and string Coxeter groups.

1.1 The main definitions

In this section we define the main objects of this thesis and discuss some of their elementary properties. Three books cover all of the elementary definitions involved here unless otherwise stated: [1], [26] and [34].

Definition 1.1.1 (C-group)**.** *Let G be a group and S some subset of involutions of G. Let $G_I = \langle s \mid s \in I \rangle$ for all $I \subseteq S$. We call G a* C-group with respect to S, *denoted (G, S), if $G = \langle S \rangle$ and*

$$G_I \cap G_J = G_{I \cap J}$$

for all $I, J \subseteq S$.
If G has such a set S then we call G a C-group*. We call $|S|$ the* rank *of (G, S).*

Although S may be infinite, in this thesis, we limit ourselves to the case that S is finite. A group G may have many generating sets S such that (G, S) forms a C-group.

Definition 1.1.2 (Words, Count and Length)**.** *Let (G, S) be a C-group. We call a finite sequence with entries in S, a* word *of (G, S) and denote the set of all such words as S^*. Suppose $s_{i_1}, s_{i_2}, \dots, s_{i_n}$ is a word, then we say it* evaluates *to $g \in G$ exactly when $g = s_{i_1} s_{i_2} \dots s_{i_n}$ (as the product of generators) and we denote the set of such words as $S^*(g)$. In practice, we often write the word $s_{i_1}, s_{i_2}, \dots, s_{i_n}$ in the form $s_{i_1} s_{i_2} \dots s_{i_n}$ where such a lack of distinction causes little ambiguity.*
Let $c : S^* \to \mathbb{Z}_{\geq 0}$ be the function that takes the word $g^* = s_{i_1} s_{i_2} \dots s_{i_n}$ and returns n. We call this the count function *of (G, S). Then we define the* length function *of (G, S) to be $l : G \to \mathbb{Z}_{\geq 0}$ such that $l(g) = \min(\{c(g^*) | g^* \in S^*(g)\})$. So the length function tells us the minimum number of generators needed to construct an element of our group as a word. We describe words of this minimum length as* reduced *and denote the set of all reduced words evaluating to g by $\mathcal{R}(g) = \{\, g^* \in S^*(g) \mid c(g^*) = l(g) \,\}$ along with $\mathcal{R}(G) = \bigsqcup_{g \in G} \mathcal{R}(g)$.*

Next, we define what a Coxeter group is. We will see that these are special cases of C-groups.

Definition 1.1.3 (Coxeter system)**.** *Let S be some set and m an $S \times S$ symmetric matrix whose entries are either positive integers, or (the symbol) ∞ subject to the* *conditions that* $m_{s,s} = 1$ *and* $m_{s,r} = m_{r,s}$ *for all* $r, s \in S$*. Then we say that* m *is a* Coxeter matrix*. We define* W *to be the group presentation whose generating set is* S *subject to the relations of the form* $(sr)^{m_{s,r}} = id$ *for all (not necessarily distinct) generators* $s, r \in S$ *if* $m_{s,r}$ *is an integer (we omit relations corresponding to pairs for which* $m_{s,r} = \infty$*). That is,*

$$W = \langle\, S \,|\, (sr)^{m_{s,r}} = id \;\text{ for all }\; r, s \in S \;\;\text{ such that }\;\; m_{s,r} \neq \infty \rangle.$$

We call W *equipped with* S *a* Coxeter system *and denote it by* (W, S)*. We call* W *a* Coxeter group *if for some* $S \subseteq W$*,* (W, S) *is a Coxeter system.*
We call $|S|$ the rank *of* (W, S)*. For this thesis, we will assume that* S *is finite. It is convention to use the letter* T *to denote the set of conjugates of* S *in* W*,* $T = S^W$*, and call these the* reflections *of the group.*

Since a Coxeter system is determined by its Coxeter matrix m, such a concise definition allows us to capture all of the information determining the system in a labelled graph. In what follows, and the rest of the thesis, for a positive integer n we will use $[n]$ to denote $\{1, \ldots, n\}$ for brevity.

Definition 1.1.4 (Coxeter diagram)**.** *Let* (W, S) *be a Coxeter system. The* Coxeter diagram *of* (W, S) *is the labelled graph* Γ *whose vertex set is* S *along with an edge labelled from* s *to* r *if* $m_{s,r} > 2$*. It is convention to omit labels when* $m_{s,r} = 3$*. We refer to the* primitive Coxeter diagram *to be the underlying, unlabelled graph induced from* Γ*.*

Example 1.1.5. *Let* S *be the set of* $n - 1$ *elements denoted by* $\{s_1, \ldots, s_{n-1}\}$*. Define* m *to be the* $S \times S$ *matrix such that* $m_{s_i,s_j} = 1$ *if* $i = j$*,* $m_{s_i,s_j} = 3$ *if* $|i - j| > 2$ *and* $m_{s_i,s_j} = 2$ *otherwise.*
Let W *be the Coxeter group induced from* m*. One can prove that* $W \cong \mathrm{Sym}(n)$ *by sending* s_i *to the adjacent transposition* $(i, i + 1)$ *and the corresponding Coxeter diagram is given by Figure 1.2.*

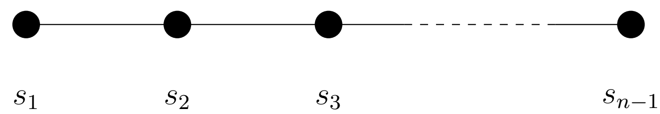


Figure 1.2: The Coxeter diagram of type A with rank $n - 1$.

Such a Coxeter system is of special importance and is said to be of type A*.*

Definition 1.1.6 (Irreducible Coxeter group)**.** *We call a Coxeter system* (W, S) irreducible *if and only if its primitive Coxeter diagram is connected.*

Proposition 1.1.7. *Let* (W, S) *be a Coxeter system. Then* $W \cong W_1 \times \ldots \times W_k$ *and* $S = S_1 \sqcup \ldots \sqcup S_n$ *where* (W_i, S_i) *are each irreducible Coxeter systems.*

Definition 1.1.8 (Standard parabolic subgroups)**.** *Let* (W, S) *be a Coxeter system and let* $I \subseteq S$*. Define* $W_I = \langle s | s \in I \rangle$*. We call such subgroups* parabolic*.*

We state some well-established facts about parabolic subgroups.

Proposition 1.1.9. *Let* (W, S) *be a Coxeter system. Then for all* $I \subset S$*,* (W_I, I) *is a Coxeter system in its own right.*

Proposition 1.1.10. *For all* $w \in W$ *and for all* $s \in S$*,* $l(ws) = l(w) \pm 1$*.*

Define $I^{\pm}(w) = \{ s \in S \,|\, l(ws) = l(w) \pm 1 \}$.

Proposition 1.1.11 (The intersection property)**.** *Let* (W, S) *be a Coxeter system. Then*

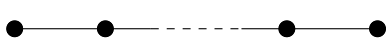
$$W_I \cap W_J = W_{I \cap J}$$

for all $I, J \subseteq S$*.*

C-groups are *smooth quotients* of Coxeter groups; these are quotients of Coxeter groups that preserve the orders of the products of pairs of generators as well at the intersection property. This allows us to consider *the underlying Coxeter group* of a C-group and speak of its properties. We make this precise below.

Definition 1.1.12 (Underlying Coxeter group)**.** *Let* (G, S) *be a C-group. Let* m *be the* $|S| \times |S|$ *matrix such that* $m_{s,r}$ *is the order of* sr *for all* $s, r \in S$*. We call the Coxeter group,* (W_G, S_G)*, whose corresponding Coxeter matrix is* m*,* the underlying Coxeter group of G*.*

All Coxeter groups are C-groups but the converse is not true.
We now discuss the adjective *string*.

Definition 1.1.13 (string C-groups)**.** *We give the adjective* string *to a C-group* (G, S) *with respect to some* $S = \{s_1, \ldots, s_n\}$*, (equiped with some implicit total order on the generators) if for all* $s_i, s_j \in S$*,* $|i - j| \geq 2$ *implies that* $s_i s_j = s_j s_i$*. This is equivalent to requesting the underlying Coxeter group has a primitive Coxeter diagram that is a path graph:*  .

Although it is not standard notation, we call S itself a C-string in this context as it is a useful object to name.
For convenience, given $\{i, j, \ldots, k\} \subseteq [|S|]$, we will write $G_{ij\ldots k} = \langle s_i, s_j, \ldots, s_k \rangle$ along with $G_{\hat{i}} = \langle s_a \,|\, a \neq i \rangle$.
The Schläfli symbol *(or Schläfli type) of a C-string $\{s_1, \ldots, s_n\}$ is the sequence $[\tau_{12}, \tau_{23}, \ldots, \tau_{n-1n}]$ where τ_{jj+1} is the order of $s_j s_{j+1}$. We will often display this information as the labels on the underlying Coxeter diagram.*
The i^{th} Betti number *of a C-string is given by $\beta_i = |G/G_{\hat{i}}|$ and the Betti numbers are given by the sequence $[\beta_1, \ldots, \beta_n]$ (in the standard notation the indices are each decreased by 1 but this will not affect this thesis).*
Reversing the order of the generators of S produces another C-string. We call this the dual *C-string to S.*
Let (G, S) and (H, T) with $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_k\}$. We say (G, S) and (H, T) are isomorphic as string C-groups if and only if $m = k$ and the map sending s_i to t_i for each $i = 1, \ldots, m$ is a group isomorphism between G and H.

C-strings are in correspondence with *abstract regular polytopes* - an abstract geometrical generalisation of the beloved Platonic Solids. Below we give an overview of the geometric definition of an abstract regular polytope. We do not focus on the geometric properties of these polytopes in this thesis; our focus in on their more group theoretic properties as C-strings. The precise definition, and further background, may be found in Sections 2B and 2E of [34].

Definition 1.1.14 (Abstract regular polytopes)**.** *An abstract polytope is a certain kind of poset. Let $\mathcal{P}$ be a set and $\prec$ a (strict) partial order; a transitive, anti-symmetric, anti-reflexive binary operation on $\mathcal{P}$. Let $\preceq$ denote the reflexive closure of $\prec$. Typically, we call the elements of $\mathcal{P}$* faces *in this context and say two faces $F, G \in \mathcal{P}$ are incident if $F \preceq G$ or $G \preceq F$. We start by describing the conditions for $\mathcal{P}$ to be an abstract n-polytope for some non-negative integer, n.*
Our first requirement is that $\mathcal{P}$ has both a minimum and maximum face. That is, there exists a unique pair of faces we call F_{-1} and F_n, such that for all $G \in \mathcal{P}$, $F_{-1} \preceq G$ and $G \preceq F_n$.
An ordered set of faces $H_1, H_2, \ldots, H_k \in \mathcal{P}$ forms a flag if

$$H_1 \prec H_2 \prec \ldots \prec H_k.$$

More concisely, a flag is a totally ordered subset of $\mathcal{P}$. Naturally, a flag is *described as being maximal if it is not a proper subset of another flag. Our second requirement for $\mathcal{P}$ to form an abstract n-polytope is that all maximal flags contain exactly $n+2$ elements.*

In the literature, the term flag is sometimes used to denote what we have called a maximal flag here (and the word chain *for what we have called flag). We will assume a flag is maximal unless otherwise stated.*

Let $\mathcal{F}(\mathcal{P})$ denote the set of all flags of $\mathcal{P}$. An elementary property of each flag is that F_{-1} and F_n always appear as the minimum and maximum faces. Moreover, each face has a fixed position in each flag that contains it: if $F \in \mathcal{P}$ is the i^{th} least face in a flag then we say it has rank $i-2$, which we denote by writing $rank(F) = i-2$.

For our next requirement, we consider a notion of connectedness associated to these flags. For $\mathcal{H}, \mathcal{G} \in \mathcal{F}(\mathcal{P})$ we say $\mathcal{H}$ and $\mathcal{G}$ are adjacent if $\mathcal{H}$ and $\mathcal{G}$ differ in exactly one face respectively. Necessarily, the faces which they differ must be of the same rank, i say. More specifically, in this case, we describe $\mathcal{G}$ and $\mathcal{H}$ as being i-adjacent. We call $\mathcal{P}$ strongly flag connected if for all flags $\mathcal{H}, \mathcal{G} \in \mathcal{F}(\mathcal{P})$ there exists a finite sequence of flags $\mathcal{F}_0, \ldots, \mathcal{F}_k$ such that $\mathcal{F}_0 = \mathcal{H}$, $\mathcal{F}_k = \mathcal{G}$ where $\mathcal{H} \cap \mathcal{G} \subseteq \mathcal{F}_j$ and $\mathcal{F}_{j-1}$ is adjacent to $\mathcal{F}_j$ for all $j = 1, \ldots, k$. To be an abstract n-polytope, $\mathcal{P}$ must be strongly flag connected.

Our final condition is known as the diamond condition: for all faces $F, G \in \mathcal{P}$ such that $rank(G) - rank(F) = 2$, there exist exactly two faces $H_1, H_2 \in \mathcal{P}$ such that $F \prec H_1 \prec G$ and $F \prec H_2 \prec G$.

If $\mathcal{P}$ satisfies these four conditions, it is an abstract n-polytope.

To be an abstract regular n-polytope, one needs to examine the automorphism group of $\mathcal{P}$. Let $\Gamma(\mathcal{P})$ be the subset of permutations of $\mathcal{P}$, $\mathrm{Sym}(\mathcal{P})$, that preserves $\prec$. That is,

$$\Gamma(\mathcal{P}) = \{\delta \in \mathrm{Sym}(\mathcal{P}) \mid \textit{for all } \ F, H \in \mathcal{P}, \ F \prec H \ \textit{if and only if } \delta(F) \prec \delta(H)\}.$$

The induced group action of $\Gamma(\mathcal{P})$ on $\mathcal{P}$ can be extended to $\mathcal{F}(\mathcal{P})$ by defining $\delta(\mathcal{F}) := \{\delta(F_{-1}), \delta(F_0), \ldots, \delta(F_n)\}$ for all flags $\{F_{-1}, F_0, \ldots, F_n\} = \mathcal{F} \in \mathcal{F}(\mathcal{P})$. An abstract n-polytope is called regular, and thus an abstract regular n-polytope, if the action of $\Gamma(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ is a regular group action. That is, for all $\mathcal{G}, \mathcal{H} \in \mathcal{F}(\mathcal{P})$ there exists a unique $\delta \in \Gamma(\mathcal{P})$ such that $\delta(\mathcal{G}) = \mathcal{H}$.

Note that for any (not necessarily regular) abstract polytope there is at most one $\delta \in \Gamma(\mathcal{P})$ such that $\delta(\mathcal{G}) = \mathcal{H}$. So abstract regular polytopes are exactly those with most amount of symmetry available in this sense. In a truly beautiful correspondence theorem, abstract regular polytopes are completely characterised by their automorphism groups which are exactly the string C-groups (see Section 2E of [34] for details). We describe a very brief overview of this correspondence for context omitting the justifications.

Given an abstract regular polytope, we obtain a C-string by choosing some distinguished flag of $\mathcal{P}$, Φ say, let g_i be that automorphism that sends Φ to the unique i-adjacent flag. Then $Aut(\mathcal{P})$ is a string C-group with respect to $\{g_1, \dots g_n\}$.

How do we derive an abstract regular polytope from string C-group? Given $G = \langle g_1, \dots g_n \rangle$, we create the poset whose elements consist of the cosets of $G_{\widehat{i}} = \langle g_j \,|\, j \neq i \,\rangle$ for $i = \{0, \dots, n+1\}$ with $G_{\widehat{0}} := G$ and $G_{\widehat{n+1}} := G$ being considered distinct elements in our poset despite being equal as groups. We define the partial order relation $\prec$ so that for all $i, j \in I$ and $g, h \in G$, $gG_{\widehat{i}} \prec hG_{\widehat{j}}$ if and only if $i < j$ and $gG_{\widehat{i}} \cap hG_{\widehat{j}} \neq \emptyset$. We denote this abstract regular polytope as $\mathcal{P}(G)$.

The i^{th} Betti number as defined in Definition 1.1.13 counts the number of rank-i faces in the corresponding abstract regular polytope for a given C-string.

If (G, S) is a rank n C-string, then we also have a correspondence between $G_{\widehat{i}}$ and the stabilizer of (any) rank i face. In this vein we will sometimes call $G_{\widehat{1}}$ the vertex group of G and $G_{\widehat{n}}$ the facet group of G.

Given any poset, $P, \prec$, we may display its data in the form of a *Hasse diagram*. The covering relations of a poset are those of the form $x \prec y$ such that there is no intermediate $z \in P$ with $x \prec z \prec y$. For each element of P and assign it a node in the plane such that x is vertically higher than y if $x \prec y$ is a covering relation and draw a line between the elements. If P is graded then we choose to draw the elements of the same rank are the same height. We may assign a direction to each line to point from x to y exactly when $x \prec y$ and call this the Hasse graph and consider it as a directed graph. These are standard ways of viewing the information of an abstract polytope and we will provide an example shortly.

1.2 Specific details concerning Coxeter groups

Here we add some specific details to the theory of Coxeter groups. The main definitions and results in this section are essential in the theory of Coxeter groups can all be found between [1] and [26] which introduce the subjects. We will make

clear those results that lie outside these books scope and we may change notation slightly in places to suit our purposes.
In this thesis, we are mostly concerned with finite groups. Of particular importance are the finite irreducible Coxeter groups: those finite Coxeter groups whose primitive Coxeter diagrams are connected.

Theorem 1.2.1 (The classification of finite irreducible Coxeter groups)**.** *The following is a complete classification of the irreducible Coxeter systems whose Coxeter groups are finite.*

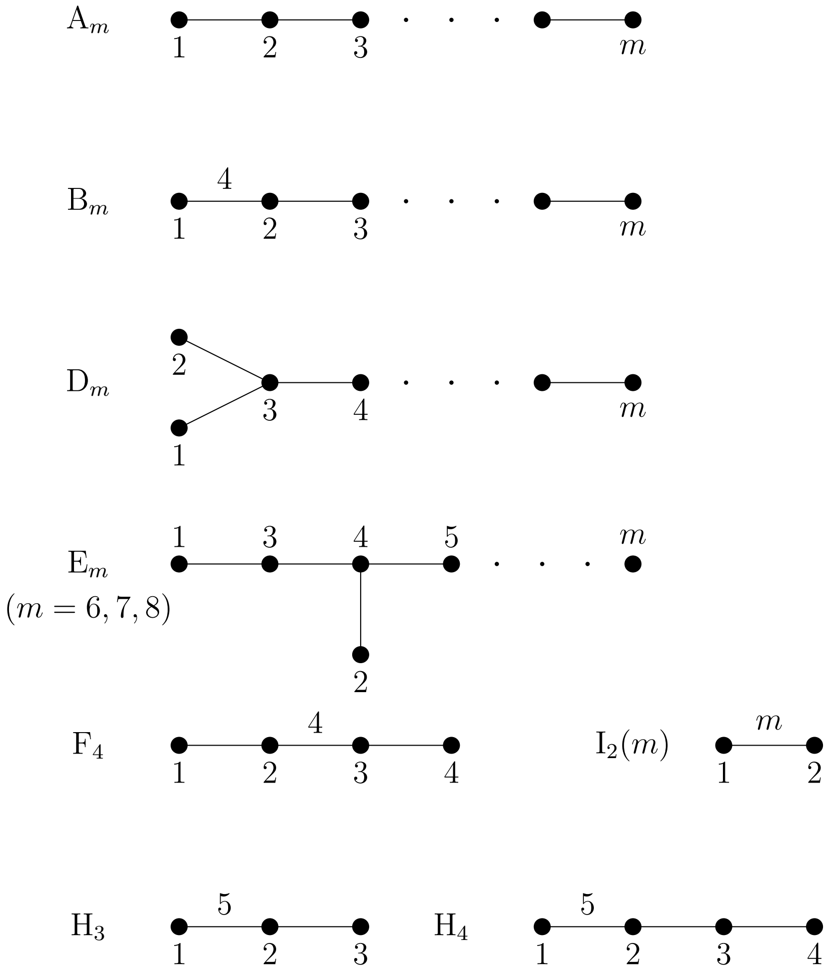


Figure 1.3: The Coxeter diagrams for the finite irreducible Coxeter groups.

Proposition 1.2.2. *W is finite if and only if it has a unique element of longest length, ω_0. Moreover, ω_0 is an involution.*

Now we concern ourselves with the properties of reduced words in Coxeter groups. In particular, the structure of reduced words in Coxeter groups. Let $w^* = s_{i_1}s_{i_2}\dots s_{i_m}$ be a reduced word in $\mathcal{R}(W)$ as in Definition 1.1.2. For all $s, r \in S$, let us use the notation $[sr]^k = \underbrace{srs\dots}_{k}$. We may also extend this to other words also where it is not ambiguous.
Then we can define the (s,r)-braid relation, $\alpha_{s,r}$ to be the *relation* that interchanges the consecutive subsequences in words, $[sr]^{m_{s,r}} \to [rs]^{m_{r,s}}$ for finite $m_{s,r}$. So we have

$$s_{i_1}s_{i_2}\dots s_{i_l}\underbrace{srs\dots}_{m_{s,r}}s_{i_k}\dots s_{i_m} \longrightarrow_{\alpha_{s,r}} s_{i_1}s_{i_2}\dots s_{i_l}\underbrace{rsr\dots}_{m_{s,r}}s_{i_r}\dots s_{i_k}\dots s_{i_m}$$

$$s_{i_1}s_{i_2}\dots s_{i_l}\underbrace{rsr\dots}_{m_{s,r}}s_{i_k}\dots s_{i_m} \longrightarrow_{\alpha_{r,s}} s_{i_1}s_{i_2}\dots s_{i_l}\underbrace{srs\dots}_{m_{s,r}}s_{i_r}\dots s_{i_k}\dots s_{i_m}$$

We also define the *s-nil* relation to be the relation, η_s that exchanges adjacent instances of s as a consecutive subsequence in a word with the empty word;

$$s_{i_1}s_{i_2}\dots s_{i_l}(rr)s_{i_k}\dots s_{i_m} \longrightarrow_{\eta_r} s_{i_1}s_{i_2}\dots s_l s_{i_k}\dots s_{i_m}.$$

The word property in Coxeter groups can the be stated as follows.

Theorem 1.2.3 (The word property)**.** *For all $w^* \in S^*(w)$ there exists a sequence of nil and braid relations that sends w^* to a reduced word $w^{\circledast}$ evaluating to the same element.*
Moreover, for any two reduced words evaluating to the same element, there exists a sequence of braid relations sends one to the other.

We note that the second part of the theorem is often also known as *Matsumoto's Theorem* ([32]) and can provide a useful alternative presentation of Coxeter groups in terms of the braid relations.
We now describe the Bruhat order.

Definition 1.2.4 (The Bruhat order)**.** *The* Bruhat order *is the poset whose underlying set is W with partial relation $<_B$ such that the following are equivalent for all $u, v \in W$:*

(i) $u <_B v$,

(ii) there exists a sequence $t_1, \dots, t_k \in T$ such that $v = ut_1 \dots t_k$ and $l(u) < l(ut_1) < \dots < l(ut_1 \dots t_k)$.

Corollary 2.2.3 of [1] shows us the following characterisation of the Bruhat order in terms of subwords. If u^* and v^* are words in S^* then we say u^* is a subword of v^* if underlying sequence for u^* is a subsequence of that of v^*.

Corollary 1.2.5 (Corollary 2.2.3 of [1])**.** *For all $u, v \in W$, the following are equivalent:*

(*i*) $u <_B v$,

(*ii*) *Every reduced word for v has a subword that is a reduced word for u.*

(*iii*) *Some reduced word for v has a subword that is a reduced word for u.*

Definition 1.2.6 (The weak order)**.** *The* weak (right) order *is the poset whose underlying set is W with binary relation $<_R$ such that the following are equivalent:*

(*i*) $u <_R v$,

(*ii*) *there exists a sequence $s_1, \dots, s_k \in S$ such that $v = us_1 \dots s_k$ and $l(us_1 \dots s_i) = l(u) + i$ for each $i = 1, \dots k$.*

Proposition 3.1.2 of [1] shows us that the weak order can be characterised in terms of 'prefixes' of reduced words. Note that the weak order is a subposet of the strong Bruhat order since $S \subseteq T$. For the finite irreducible Coxeter groups, the strong order satisfies all of the axioms of being an abstract polytope.
We mention one last piece of information we will refer back to for Coxeter groups. It seems this result is much less frequently used in the literature but appears in [36] and is ripe for applications. We edit the notation and presentation to suit our own purposes.

Definition 1.2.7 (Admissible partitions [36])**.** *Let (W, S) be a Coxeter system and $\Sigma = \{\Sigma_i \mid i \in I\}$ a partition of S (indexed by some set I). If for each $i \in I$, $W_{\Sigma_i} = \langle s \in S \mid s \in \Sigma_i \rangle$ is a finite parabolic subgroup of W, then we call Σ spherical. When this is the case, for each $\Sigma_i \in \Sigma$ we set s_{Σ_i} to be the longest element of W_{Σ_i} and choose some fixed reduced word x_i for s_{Σ_i} over S. Note that if Σ_i consists of pairwise commuting generators, then*

$$s_{\Sigma_i} = \Pi_{s \in \Sigma_i} s$$

and so x_i is just some ordering of Σ_i. Set $S_\Sigma = \{s_{\Sigma_i} \mid \ i \in I\}$ and $W_\Sigma = \langle s_{\Sigma_i} \mid \ i \in I \rangle$.

We say $w \in W$ is Σ-consistent if $w \in W_\Sigma$ also.
We call Σ admissible at $w \in W_\Sigma$ if for all $i \in I$, we have either $\Sigma_i \subseteq I^+(w)$ or $\Sigma_i \subseteq S \setminus I^+(w)$. Then we call Σ admissible if it is admissible at all $w \in W_\Sigma$. When this is the case, we obtain an embedding of W_Σ into W and denote this by $W_\Sigma \lessdot_\Sigma W$.

We now coalesce many of the main results of [36] into one statement, relevant to this thesis.

Theorem 1.2.8 (Mühlherr, [36])**.** *Suppose (W, S) is a Coxeter system and Σ an admissible partition of S. Then*

(i) *(W_Σ, S_Σ) is itself a Coxeter System in its own right,*

(ii) *If $s_{\Sigma_{i_1}} \dots s_{\Sigma_{i_k}}$ is a reduced word in W_Σ, then $x_{i_1} x_{i_2} \dots x_{i_k}$ is reduced word in W,*

(iii) *The partitions in Table 1.1 are admissible.*

Type of W	Type of W_Σ	Σ
$\mathrm{A}_{2n-1}\,(n \geq 2)$	B_n	$\{\{i, 2n-i\}, \{n\} \mid i = 1, \dots, n-1\}$
$\mathrm{A}_{2n}\,(n \geq 2)$	B_n	$\{\{i, 2n-i\} \mid i = 1, \dots, n\}$
$\mathrm{D}_{n+1}\,(n \geq 2)$	B_n	$\{\{1, 2\}, \{i\} \mid i = 3, \dots, n+1\}$
E_6	F_4	$\{\{1, 6\}, \{3, 5\}, \{2\}, \{4\}\}$
D_6	H_3	$\{\{1, 4\}, \{2, 6\}, \{3, 5\}\}$
E_8	H_4	$\{\{1, 8\}, \{2, 5\}, \{3, 7\}, \{4, 6\}\}$

Table 1.1: Some admissible partitions for the finite irreducible Coxeter groups.

We note that the list in Table 1.1 is not claimed to be exhaustive (and no exhaustive list seems to exist in the literature). We have deliberately omitted those known admissible partitions involving the dihedral group. Also, we have only taken examples up to automorphisms the Coxeter diagram (to avoid repetitions). A natural consequence of Theorem 1.2.8 *(ii)* is that for all $u, v \in W_\Sigma$, $u <_B v$ implies $u' <_B v'$ where u' and v' are the image of u and v in the embedding of W_Σ in W. This can be seen by applying the subword characterisation of the Bruhat order from Corollary 1.2.5.

1.3 A familiar example

Consider $W = \mathrm{Sym}(3)$. We saw in Example 1.1.5 that this we can think of this as a Coxeter when generated by $s_1 = (1,2)$ and $s_2 = (2,3)$. Then $\mathcal{P}(W)$ represents the regular triangle as Figure 1.4:

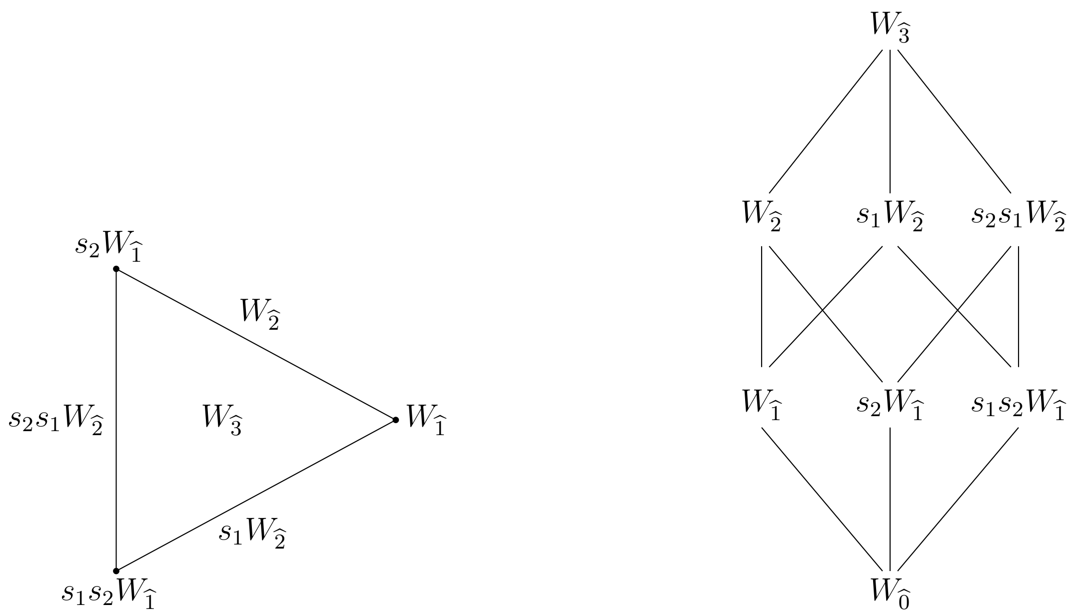


Figure 1.4: The regular triangle (left) and its Hasse diagram (right) as $\mathcal{P}(\mathrm{Sym}(3))$.

We note that the Hasse diagram of $\mathrm{Sym}(n-1)$ is indeed isomorphic to the (directed) n-hypercubic graph.

The elements of $\mathrm{Sym}(3)$ are of course the permutations $\{id, (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$. We multiply our permutations from left to right and when dealing with $w \in \mathrm{Sym}(n)$, write the image of $i \in [n]$ by $(i)w$. Given a permutation w we may write it in *one-line form* where we write the numbers $(1)w, (2)w, \ldots, (n)w$ in the order. So $(1,3,2)$ is written as $3\,1\,2$. The reduced words of each element are given by

w		$\mathcal{R}(w)$
id	1 2 3	
(1,2)	2 1 3	s_1
(2,3)	1 3 2	s_2
(1,3,2)	3 1 2	$s_1 s_2$
(1,2,3)	2 3 1	$s_2 s_1$
(1,3)	3 2 1	$s_1 s_2 s_1, s_2 s_1 s_2$

Table 1.2: The elements of Sym(3) and their reduced words.

Note that $s_1 s_2 s_1$ and $s_2 s_1 s_2$ are indeed connected by a braid relation as Matsumoto's theorem suggests. Using the subword criterion for reduced words, we can now readily compute the weak order and Bruhat order.

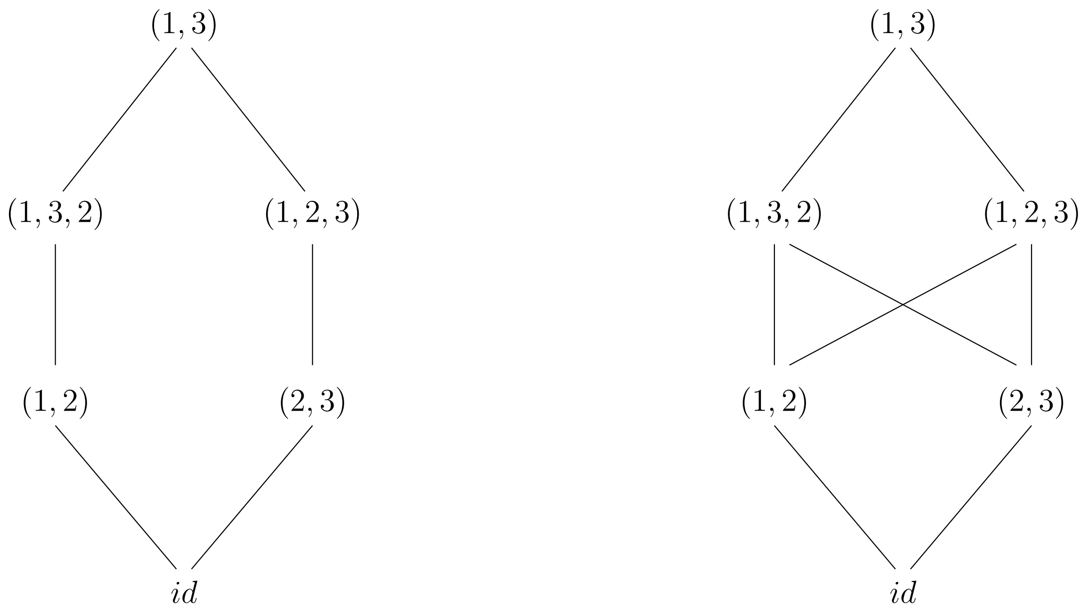


Figure 1.5: The Hasse diagrams of the weak order (left) and strong Bruhat order (right) of $\mathcal{P}(\mathrm{Sym}(3))$.

As observed in [1], the strong Bruhat Order on $\mathrm{Sym}(n)$ reduces to a nice characterisation: given $u \in \mathrm{Sym}(n)$ and transposition $t = (a, b) \in T$ (a reflection), $u <_B ut$ if and only if $a < b$ and $(a)u^{-1} < (b)u^{-1}$.

1.4 Unfamiliar examples

We give an example of two distinct (up to isomorphism) abstract regular polytopes whose automorphism groups are equal as groups. Consider Sym(4). The regular tetrahedron is a familiar abstract regular polytope whose corresponding C-string is given by the adjacent transpositions $\{(1,2),(2,3),(3,4)\}$. However, the following three involutions also produce a C-string: $\{(1,2)(3,4),(2,3),(3,4)\}$. This C-string corresponds to the abstract regular polytope known as *the hemi-cube*. One can check that the Schläfli types of the two strings are $[3,3]$ and $[4,3]$. This shows they are not isomorphic as C-strings.
This serves an example of a Coxeter group having a C-string which is not a Coxeter system. We also note that there exist groups that are not Coxeter groups but are C-groups. M_{12}, the sporadic simple group, is a good example of this. It is shown in [25] that it does indeed contain C-strings. Since it is simple, if it is a Coxeter group, it would be required to be both irreducible and finite. The order of the finite irreducible Coxeter groups is well-known due to the classification. The only finite irreducible Coxeter group of order $|M_{12}| = 95040$ is $\mathrm{I}_2(47520)$. But clearly $\mathrm{I}_2(47520)$ is not isomorphic to M_{12} since it is a dihedral group and therefore not simple.

1.5 An overview of string C-groups

In this section we provide a brief overview of some of the recent literature surrounding string C-groups. This includes cataloguing all the C-strings for certain families of groups such as the symmetric, alternating and sporadic simple groups. We'll also highlight some very useful theoretical results that are of general use when trying to prove that some set of involutions of a group are indeed a C-string. One theorem that is often used in practice for proving the existence of a C-string is the following (edited slightly to be more consistent with our notation).

Theorem 1.5.1 (Theorem 2E16 of [34]). *Suppose that $G = \langle s_1, \ldots, s_n \rangle$ with $\{s_1, \ldots, s_n\}$ a generating set of G with each s_i an involution. Suppose that $G_{[2,n]} := \langle s_2, \ldots, s_n \rangle$ is string C-group.*

(i) If $G_{[n-1]} = \langle s_1, \ldots, s_{n-1} \rangle$ is a string C-group also, then

$$G_{[2,n]} \cap G_{[n-1]} = \langle s_2, \ldots, s_{n-1} \rangle$$

implies G is a string C-group with respect to $\{s_1, \ldots, s_n\}$.

(ii) If

$$G_{[2,n]} \cap \langle s_1, \ldots, s_k \rangle = \langle s_2, \ldots, s_k \rangle$$

for all $k \in \{1, \ldots, n-1\}$ *then G is also a string C-group.*

Theorem 1.5.1 really does serve as a staple in enumerating all C-strings of a given Group. For example, one can use it to exhaustively find all C-strings of a given group. A depth-first algorithm describing this procedure is found in [29] and we have implemented this in MAGMA for our own investigations (see Appendix A).

In [23], Hartley produces an Atlas of C-strings for groups of order at most 2000. This originally excluded those with groups of order 1024 and 1536. There about 10,000 non-degenerate, abstract regular polytopes of order 1536. Those of order 1024 were later classified in [18] using different techniques leveraging some knowledge of Coxeter groups.

In [31], Leemans and Vauthier classified the C-strings for the almost simple groups up to a certain order. In particular, for every C-string of a group G such that $S \leq G \leq Aut(S)$ and S is a simple group of order less than 900,000 has been listed.

Hartley and Hulpke enumerated all of the abstract regular polytopes for the sporadic simple groups of up to the Held Group of order 4030387200 in [25]. This has been extended to include the smallest Conway group Co_3 in [29] and impressive partial results for ON in [7]. This is still an ongoing area of enquiry and soon to be released work concerns itself with more enumeration for large finite simple groups with new, more effective algorithms (see [30]) where the data for ON is completed.

A lot of work on symmetric, alternating and transitive permutation groups has been carried out. Some noticeable examples include [4], [5], [14], [16] and [17]. We present some of the relevant tables enumerating the abstract regular polytopes of the symmetric groups from [15].

Group \ Rank	3	4	5	6	7	8	9	10	11	12	13
Sym(4)	2	0	0	0	0	0	0	0	0	0	0
Sym(5)	4	1	0	0	0	0	0	0	0	0	0
Sym(6)	2	4	1	0	0	0	0	0	0	0	0
Sym(7)	35	7	1	1	0	0	0	0	0	0	0
Sym(8)	68	36	11	1	1	0	0	0	0	0	0
Sym(9)	129	37	7	7	1	1	0	0	0	0	0
Sym(10)	413	203	52	13	7	1	1	0	0	0	0
Sym(11)	1221	189	43	25	9	7	1	1	0	0	0
Sym(12)	3346	940	183	75	40	9	7	1	1	0	0
Sym(13)	7163	863	171	123	41	35	9	7	1	1	0
Sym(14)	23126	3945	978	303	163	54	35	9	7	1	1

Table 1.3: The number of abstract regular polytopes of Sym(n) up to duality.

And information for the alternating groups can be found in [31].

Group \ Rank	3	4	5	6	7	8	9	10	11	12	13
Alt(5)	2	0	0	0	0	0	0	0	0	0	0
Alt(6)	0	0	0	0	0	0	0	0	0	0	0
Alt(7)	0	0	0	0	0	0	0	0	0	0	0
Alt(8)	0	0	0	0	0	0	0	0	0	0	0
Alt(9)	41	6	0	0	0	0	0	0	0	0	0
Alt(10)	94	2	4	0	0	0	0	0	0	0	0
Alt(11)	64	0	0	3	0	0	0	0	0	0	0
Alt(12)	194	90	22	0	0	0	0	0	0	0	0
Alt(13)	1558	102	25	10	0	0	0	0	0	0	0
Alt(14)	4347	128	45	9	0	0	0	0	0	0	0
Alt(15)	5820	158	20	42	6	0	0	0	0	0	0

Table 1.4: The number of abstract regular polytopes of Alt(n) up to duality.

From [25] with the addition of complete data of ON from [30] we have the following table for the sporadic simple groups.

Group	Order	3	4	5	≥ 6
M_{11}	7920	0	0	0	0
M_{12}	95040	23	14	0	0
J_1	175560	148	2	0	0
M_{22}	443520	0	0	0	0
J_2	604800	137	17	0	0
M_{23}	10200960	0	0	0	0
$^2F_4(2)'$	17971200	244	31	0	0
HS	44352000	252	57	2	0
J_3	50232960	303	2	0	0
M_{24}	244823040	490	155	2	0
McL	898128000	0	0	0	0
He	244823040	1188	76	0	0
ON	6536	16	0	0	0

Figure 1.6: The number of abstract regular polytopes for sporadic simple groups up to duality.

Another useful tool gaining more popularity is the CPR graph.

Definition 1.5.2 (CPR graph)**.** *Let (G, S) be a C-group and $\phi : G \hookrightarrow \mathrm{Sym}(n)$ an embedding into the symmetric group on n elements. Then the* CPR graph *of G with respect to ϕ is the labelled multi-graph (more than one edge may exist between the same two vertices) whose vertex set is $[n]$ and there exists a label from i to j labelled k if and only if $\phi(s_k)$ transposes i and j.*

This definition comes from Pellicer in [41] where it is explained that CPR stands for C-group permutation representation. It gives some useful theorems that allow one to deduce whether some group is a (low ranking) C-group from graph theoretic facts and has influenced the methods involved for classifying polytopes as in [17].

Chapter 2

An introduction to unravelled polytopes

Here we focus on the ideas of quotients in abstract regular polytopes. There already exists a vast literature with some surprising results (see [20]) for general abstract polytopes. Of particular relevance for this section is Hartley's work on so-called *semisparse subgroups* and quotients, see [22]. For a detailed exposition of a quotient polytope, we suggest [34].
The work on these unravelled polytopes (Chapters 2, 3 and 4) is a collaboration with my PhD supervisor, Professor Peter Rowley. This work is edited and adapted from our paper in progress, [40].

2.1 Introduction

Definition 2.1.1 (Quotient of a polytope)**.** *Let $\mathcal{P}$ be an abstract polytope of rank n, with partial ordering $\prec$ and automorphism group $Aut(\mathcal{P})$. For a given subgroup $\Sigma \leq Aut(\mathcal{P})$, let $\mathcal{P}/\Sigma$ denote the set of orbits of Σ acting on $\mathcal{P}$ as order-preserving permutations on the faces. We define the new partial ordering, denoted $\prec_\Sigma$, on $\mathcal{P}/\Sigma$ as follows: for all $\widehat{F}, \widehat{G} \in \mathcal{P}/\Sigma$ we say $\widehat{F} \prec_\Sigma \widehat{G}$ exactly when there exists some $F \in \widehat{F}$ and $G \in \widehat{G}$ for which $F \prec G$. So $\mathcal{P}/\Sigma$ equipped with $\prec_\Sigma$ defines a new poset which we call* the quotient of $\mathcal{P}$ with respect to Σ.

Notice that we have not called this new poset a polytope; the reason being, that it may not be one. When this is the case, we call the resulting polytope the *quotient polytope.*

The set of subgroups for which the quotient of an abstract polytope is again an abstract polytope are the so-called *semisparse subgroups*. The group theoretic conditions for being a semisparse subgroup are quite complex and we do not take the diversion to study them in this thesis; Proposition 12 of [33] and the further work of Hartley in [21] and [22] give a thorough account of them.

If we restrict ourselves to examining when an abstract regular polytope quotients to another abstract regular polytope, matters simplify somewhat. Specifically, since we must send C-strings to C-strings, the subgroups we quotient by must be normal. So we need to check that if (G, S) is a string C-group with respect to $S = \{s_1, \ldots, s_n\}$ and $N \trianglelefteq G$, then $\{s_1N, \ldots, s_nN\}$ is a C-string for G/N. We consider adding an extra condition to this already well studied phenomena: what if we require that we preserve the rank of the abstract regular polytopes also. We use this as inspiration to define what we call *unravelled polytopes*.

Definition 2.1.2 (Unravelled polytopes and C-strings)**.** *Let $G = \langle s_1, \ldots, s_n \rangle$ be a rank n string C-group. If for all non-trivial normal subgroups of G, N, it is true that $G/N = \langle s_1N, \ldots, s_nN \rangle$ is **not** a rank n string C-group, we call G, and its corresponding abstract regular polytope,* unravelled*.*

If for any particular N, $G/N = \langle s_1N, \ldots, s_nN \rangle$ is not *a rank n string C-group, then we say G is N-*unravelled*.*

From a geometric perspective this serves as a filter to find those abstract regular polytopes that can never be quotiented to form another abstract regular polytope of the same rank. One could go further, naturally, by considering what else we might strengthen or weaken and this certainly merits further enquiry. From a group theoretic perspective, these unravelled polytopes offer a natural focus on the interaction between the C-strings of a group and the normal subgroups. So, heuristically, we might expect it to be harder to find unravelled polytopes in some groups than others. Our first observation concerns the triviality of unravelled C-strings for simple groups.

Proposition 2.1.3. *If G is a simple group then every C-string for G is unravelled.*

To the best of my knowledge, unravelled polytopes have not been studied in isolation in the literature. The remaining sections will highlight some first steps in understanding the landscape of these polytopes and finding noticeable examples.

2.2 A detailed example

Let us consider the triple cover of Sym(6), $G = 3.\text{Sym}(6)$: that group that when quotiented by its normal cyclic group of order 3, C_3, gives Sym(6). A construction and examination of the group can be found in [47]. Of importance to us is that G has exactly two non-trivial normal subgroups: the normal C_3 of index 120 and the triple cover of Alt(6), 3.Alt(6) of index 2.

Note that since $N = 3.\text{Alt}(6)$ has index 2, if $\{r_1, \dots r_n\}$ is a C-string for G with $n > 1$ then $\{r_1N, r_2N, \dots, r_{k-1}N\}$ is not a rank n C-string for $G/N \cong C_2$ since C_2 only has one involution. Since 3.Alt(6) is not generated by a single involution (otherwise it'd be isomorphic to C_2!), it is always impossible for any C-string to preserve its rank and regularity when quotiented by 3.Alt(6). Hence, for this group, we only need to test if this is the case for when $N = C_3$.

With the help of Magma ([2]) we can exhaustively find all such C-strings of G up to automorphism. We also do the same for $G/C_3 \cong \text{Sym}(6)$. Both groups happen to have exactly 11 C-strings and we list these by their Schläfli types (which happen to be unique to each C-string for these groups respectively). We check if a C-string from 3.Sym(6) is sent to one of Sym(6) by quotienting by C_3 in Table 2.1. Absent from this list are the C-strings with Schläfli type $[4, 5, 4]$ in 3.Sym(6) and $[3, 3, 3, 3]$ in Sym(6) since $[4, 5, 4]$ does not quotient to any C-string in Sym(6), nor does any C-string in 3.Sym(6) quotient to $[3, 3, 3, 3]$.

3.Sym(6)	Covers	Sym(6)
$[4, 12, 4]$	$\rightarrow$	$[4, 4, 4]$
$[3, 6, 4]$	$\rightarrow$	$[3, 6, 4]$
$[4, 6, 3]$	$\rightarrow$	$[4, 6, 3]$
$[3, 12, 4]$	$\rightarrow$	$[3, 4, 4]$
$[4, 12, 3]$	$\rightarrow$	$[4, 3, 3]$
$[4, 6, 4]$	$\rightarrow$	$[4, 6, 4]$
$[4, 6, 4]$	$\rightarrow$	$[4, 6, 4]$
$[6, 5]$	$\rightarrow$	$[6, 5]$
$[5, 6]$	$\rightarrow$	$[5, 6]$
$[6, 6]$	$\rightarrow$	$[6, 6]$

Table 2.1: The C-strings for 3.Sym(6) and Sym(6) respectively denoted by their Schläfli symbols and the coverings between them.

We can see explicitly from Table 2.1 that the only C-string of 3.Sym(6) that is not a C-string of the same rank for Sym(6) is that with symbol $[4, 5, 4]$ and so is the only unravelled C-string for 3.Sym(6). Let $G = \langle s_1, s_2, s_3, s_4 \rangle$ denote this C-string. We mention a charming fact: as groups, $G_{123} \cong G_{234} \cong \mathrm{Sym}(5)$ yet they are not conjugate in G. One is that of usual permutation representation of Sym(5) within Sym(6) and the other is of the exotic transitive permutation representation.
According to Hartley's atlas ([23]) there is only one abstract regular polytope of Sym(5) with Schläfli types $[4, 5]$ and $[5, 4]$ respectively. They are both locally spherical, non-orientable, compact quotients of hyperbolic space.
Since G is the only C-string of its Schläfli type, it is necessarily self-dual.
By computation, one can check that the Betti numbers for G are $[1, 18, 135, 135, 18, 1]$.
Let Δ_i denote the number of elements of length $i = 0, 1 \ldots$ in (G, S). We often refer to these as the disc sizes in the chamber graph due to their connection to buildings. They are given by

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert\Delta_i\rvert$	4	9	18	34	61	108	162	218	303	358	373	276	154	70	9	2

The two elements of maximal distance are also mutually equidistant and so there exists a triangle between the identity and these two elements - which also happen to form our central C_3.

2.3 Some small computed examples

In Table 2.2 we present the existence of unravelled polytopes (computed again via Magma) for a selection of groups. An entry for the group G of the form $x(y)[z]$ is used to denote that there exists exactly x C-strings (up to automorphism) for G, of which, y are self-dual and z are unravelled. We highlight a few more chosen unravelled polytopes from this selection, examining their Schläfli type, Betti numbers and number of elements of a given length.

2.3.1 $G \sim 3^{\cdot}\mathrm{Sym}(7)$

Just as in Subsection 2.2, G has, up to isomorphism, exactly one unravelled C-string. Again, it has rank 4, while its Schläfli symbol is $[4, 6, 4]$ and Betti

Group	Total	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
3.Sym(6)	11(3)[1]	3(1)[0]	8(2)[1]	0	0	0	0
3.Sym(7)	167(5)[1]	142(4)[0]	23(1)[1]	2	0	0	0
$3.\mathrm{PSL}_3(7):2$	3256(48)[1]	3240(44)[0]	16(4)[1]	0	0	0	0
$3.\mathrm{PSL}_3(13):2$	38594(174)[1]	38534(166)[0]	60(8)[1]	0	0	0	0
$3.\mathrm{M}_{22}:2$	727(13)[5]	550(10)[0]	177(3)[5]	0	0	0	0
$3.G_2(3):2$	725(25)[0]	705(25)[0]	20(0)[0]	0	0	0	0
$2^4:\mathrm{Sym}(6)$	22(2)[11]	6(0)[0]	8(0)[4]	8(2)[7]	0	0	0
B_3	8(0)[0]	8(0)[0]	0	0	0	0	0
B_4	14(2)[0]	6(2)[0]	8(0)[0]	0	0	0	0
B_5	165(0)[0]	63(0)[0]	88(0)[0]	14(0)[0]	0	0	0
B_6	130(0)[0]	24(0)[0]	76(0)[0]	20(0)[0]	10(0)[0]	0	0
B_7	2965(21)[14]	1031(21)[0]	1428(0)[10]	400(0)[4]	84(0)[0]	22(0)[0]	0
B_8	3051(33)[38]	1020(32)[0]	1494(0)[32]	304(0)[8]	192(0)[0]	27(1)[0]	14(0)[0]
D_3	3(1)[3]	3(1)[3]	0	0	0	0	0
D_4	0	0	0	0	0	0	0
D_5	39(1)[16]	21(1)[0]	16(0)[14]	2(0)[2]	0	0	0
D_6	132(0)[2]	24(0)[0]	48(0)[2]	60(0)[0]	0	0	0
D_7	628(16)[210]	348(16)[0]	226(0)[166]	42(0)[36]	10(0)[6]	2(0)[2]	0
D_8	3537(27)[24]	887(19)[0]	1598(8)[14]	826(0)[10]	172(0)[0]	54(0)[0]	0

Table 2.2: Number of C-strings and those which unravel.

numbers are $[1, 63, 945, 945, 63, 1]$. The disc sizes of the chamber graph are

i	1	2	3	4	5	6	7	8	9	10	11
$\lvert\Delta_i\rvert$	4	9	18	34	62	113	204	366	601	963	1454

12	13	14	15	16	17	18	19	20	21	22
2036	2562	2696	2005	1219	514	188	57	10	4	1

So we have a unique element of maximum length 22. This element is an involution. Both G_{123} and G_{234} are isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(5)$ and are named as $\{4,6\} * 240a$ in Hartley's atlas ([23]).

2.3.2 $G \sim 3^{\cdot}\mathrm{M}_{22}:2$

In this case, there are five unravelled C-strings, all of rank 4, with details given in Table 2.3.

Schläfli symbol	Betti numbers
$[4,5,4]$	$[1,2016,166320,166320,8316,1]$
$[4,5,4]$	$[1,8316,166320,166320,2016,1]$
$[4,6,4]$	$[1,693,166320,166320,693,1]$
$[4,6,4]$	$[1,693,166320,166320,6930,1]$
$[4,6,4]$	$[1,6930,166320,166320,693,1]$

Table 2.3: Unravelled C-strings for $3^{\cdot}M_{22}:2$.

We note that the five abstract regular polytopes in Table 2.3 consists of a dual pair of $[4,5,4]$ abstract regular polytopes and a dual pair of $[4,6,4]$ abstract regular polytopes and one self-dual $[4,6,4]$ abstract regular polytope.

2.3.3 $G \sim 2^4 : \mathrm{Sym}(6)$

Here in Table 2.4 we find eleven unravelled C-strings, four of which have rank 4 and the remainder rank 5.

Schläfli symbol	Betti numbers
$[6,6,4]$	$[1,60,720,480,16,1]$
$[4,6,6]$	$[1,16,480,720,60,1]$
$[6,5,4]$	$[1,72,720,480,16,1]$
$[4,5,6]$	$[1,16,480,720,72,1]$
$[4,4,6,3]$	$[1,16,120,240,90,6,1]$
$[3,6,4,4]$	$[1,6,90,240,120,16,1]$
$[4,4,4,3]$	$[1,16,120,240,90,10,1]$
$[3,4,4,4]$	$[1,10,90,240,120,16,1]$
$[3,6,4,3]$	$[1,6,120,320,120,16,1]$
$[3,4,6,3]$	$[1,16,120,320,120,6,1]$
$[3,4,4,3]$	$[1,16,120,320,120,16,1]$

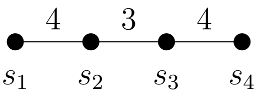
Table 2.4: Unravelled C-strings for $2^4 : \mathrm{Sym}(6)$.

Only two of the eleven, namely those with symbols $[4,5,6]$ and $[6,5,4]$, decrease in rank when quotienting, whereas the others have at least one case of the intersection property failing. We also note that the only self-dual C-string in Table 2.4 is the one with symbol $[3,4,4,3]$.

2.3.4 G of order $1296 = 2^4.3^4$.

The more normal subgroups a group has, the more stringent the unravelled condition becomes. We close this section including an example of a soluble group which possess an unravelled C-string. Let s_1, s_2, s_3, s_4 be the elements of Sym(27) as follows.

$$\begin{aligned}
s_1 =&(4,10)(7,15)(9,17)(12,20)(14,22)(16,23)(19,25)(21,26)(24,27),\\
s_2 =&(2,4)(5,10)(6,9)(11,17)(12,15)(13,16)(18,23)(19,22)(24,26),\\
s_3 =&(2,3)(5,8)(7,9)(11,13)(12,16)(15,17)(19,21)(20,23)(25,26) \text{ and}\\
s_4 =&(1,3)(2,6)(4,9)(5,11)(7,14)(10,17)(12,19)(15,22)(20,25).
\end{aligned}$$

Set $G = \langle s_1, s_2, s_3, s_4 \rangle$. Then $\{s_1, s_2, s_3, s_4\}$ is an unravelled C-string for G with diagram 

and Betti numbers $[1, 27, 81, 81, 27, 1]$. Both the vertex group and facet groups of this abstract regular polytope are isomorphic to the Coxeter group B_3 as their automorphism groups. For the lengths of the elements we have the following.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\lvert\Delta_i\rvert$	4	9	17	28	42	60	81	105	129	147	157	155	138	109	71	33	9	1

Chapter 3

Two families of unravelled polytopes

In this chapter and Chapter 4, we aim to find some non-trivial families of unravelled C-strings. This follows some of the work contained in [40] and so is joint work with Professor Peter Rowley. Here we concentrate on the matrix groups of the form $\mathrm{SL}_3(q) \rtimes \langle t \rangle \sim 3^{\cdot}L_3(q):2$, where t acts upon $\mathrm{SL}_3(q)$ as the transpose inverse automorphism and q a prime power. These examples demonstrate that being unravelled is not too restrictive a property as to force the C-strings to be uninteresting in nature and that infinite non-trivial examples exist. For the small groups of this form, it seems that those C-strings that unravel are often the exception: $\mathrm{SL}_3(7) \rtimes \langle t \rangle$ has 3256 abstract regular polytopes but only one of which is unravelled, for example.
We will prove two main theorems in this chapter. Both concern the existence of infinite families of polytopes. Our first theorem shows a method to construct rank 4 unravelled C-strings.

Theorem 3.0.1. *Suppose that q is a prime power and $G = \mathrm{SL}_3(q) \rtimes \langle t \rangle$ where t acts upon $\mathrm{SL}_3(q)$ as the transpose inverse automorphism. Assume that*

(*i*) $q \equiv 7 \pmod{24}$;

(*ii*) *at least one of $-3^{-1} + (3^{-2} - 1)^{1/2}$ and $-3^{-1} - (3^{-2} - 1)^{1/2}$ has order $q+1$ in $\mathrm{GF}(q^2)^*$.*

Then G possesses an unravelled rank 4 C-string with Schläfli symbol $[4, q+1, 4]$.

We comment now on the conditions imposed: there are infinitely many q satisfying (*i*) of Theorem 3.0.1 (for example, taking $q = p$, a prime with $p \equiv 1 \pmod 3$ and

$p \equiv 7 \pmod 8$ gives infinitely many q by Dirichlet's Theorem). However we do not know if there are infinitely many q satisfying both conditions in the theorem. Of the 157 primes p less than or equal to 10000 with $p \equiv 1 \pmod 3$ and $p \equiv 7 \pmod 8$, 20 of them do not satisfy (ii) (and they are $199, 343, 919, 1039, 1063, 2239, 3079, 3919, 4423, 4759, 4783, 5167, 6967, 7039, 7759, 7879, 8287, 8887, 9511, 9679$).
We also note that (i) is equivalent to having $q \equiv 1 \pmod 6$ and the existence of some $\lambda, \mu \in \mathrm{GF}(q)$ such that $2\lambda^2 = 1$ and $2\mu^2 - \lambda^2 = 0$. This fact is an indirect consequence of the second supplementary law of Gauss's quadratic reciprocity. Condition (ii), we will see, determines the Schläfli symbol $q+1$ for the desired C-strings. This fact is essential in our proof that that C-strings are unravelled. Our second theorem is targeted at only on those $G = \mathrm{SL}_3(q) \rtimes < t >$ with q being prime (with some additional congruence conditions imposed) but aims to prove a similar statement in Section 3.3.

Theorem 3.0.2. *Let p be a prime with $p \equiv 1 \pmod 3$ and $p \equiv 5 \pmod 8$. Then $G = \mathrm{SL}_3(p) \rtimes \langle t \rangle$, where t is the transpose inverse automorphism of $\mathrm{SL}_3(p)$, has an unravelled rank 4 C-string with Schläfli symbol $[4, p, 4]$.*

3.1 Some notation

Recall that we will be investigating C-strings in the group $G = \mathrm{SL}_3(q) \rtimes \langle t \rangle$ where q is some prime power with $3 \mid q-1$ and t is the transpose inverse automorphism of $\mathrm{SL}_3(q)$. We establish some relevant notation. Put $H = \mathrm{SL}_3(q)$ and let U be the natural 3-dimensional $\mathrm{GF}(q)H$-module. Set $V = U \oplus U^*$, where U^* is the dual of U. Choosing a basis for U and a dual basis for U^* (viewing U and U^* as subspaces of V) we may take t to be $t = \left(\begin{array}{c|c} & I_3 \\ \hline I_3 & \end{array}\right)$. We note that G has two conjugacy classes of involutions, namely t^G and s^G where $s \in G' = H$. These classes may be easily distinguished as $\dim C_V(t) = 3$ whereas $\dim C_V(s) = 2$. Also, since $3 \mid q-1$, G has shape $3^{\cdot}L_3(q) : 2$, where the multiplicative group is cyclic.

3.2 C-strings with Schläfli symbol $[4, q+1, 4]$

In this section we prove Theorem 3.0.1 in a series of steps. We use the set up given at the end of Section 3.1. Since $6 \mid q-1$, we may select $\rho \in \mathrm{GF}(q)^*$ such that ρ has multiplicative order 6. Further, we have $\lambda, \mu \in \mathrm{GF}(q)$ for which $2\lambda^2 = 1$ and $2\mu^2 - \lambda^2 = 0$. We now introduce five other elements of $\mathrm{GF}(q)$.

Definition 3.2.1.

$$\begin{aligned}
\alpha &= 3^{-1} \\
\beta &= 2\alpha\lambda\mu^{-1} \\
\xi &= \rho^2 + (1-\rho^2)2^{-1} \\
\eta &= (1-\rho^2)2^{-1} \\
\tau &= \rho^4
\end{aligned}$$

Note that $\alpha = (\mu^{-2} - 1)^{-1}$. From $2\mu^2 = \lambda^2 = 2^{-1}$ we get $2^{-1}\mu^{-2} = 2$, and so $\mu^{-2} = 4$. Therefore $\alpha = (\mu^{-2} - 1)^{-1} = 3^{-1}$. Also, since $\beta = 2\alpha\lambda\mu^{-1}$, $\beta^2 = 4\alpha^2\lambda^2\mu^{-2} = 8\alpha^2$.

Hence $\alpha^2 + \beta^2 = 9\alpha^2 = 9(3^{-1})^2 = 1$. Thus $\alpha^2 + \beta^2 = 1$.

Using these elements we now define our C-string, $\{t_1, t_2, t_3, t_4\}$. We shall show that $\{t_1, t_2, t_3, t_4\}$ is an unravelled C-string for G where the t_i are specified as follows.

Definition 3.2.2.

$$t_1 = \left(\begin{array}{ccc|ccc}
 & & & \mu & \lambda & \mu \\
 & 0 & & \lambda & 0 & -\lambda \\
 & & & \mu & -\lambda & \mu \\
\hline
\mu & \lambda & \mu & & & \\
\lambda & 0 & -\lambda & & 0 & \\
\mu & -\lambda & \mu & & &
\end{array}\right)$$

$$t_2 = \left(\begin{array}{ccc|ccc}
-1 & & & & & \\
 & 1 & & & 0 & \\
 & & -1 & & & \\
\hline
 & & & -1 & & \\
 & 0 & & & 1 & \\
 & & & & & -1
\end{array}\right) = \mathrm{diag}(-1, 1, -1, -1, 1, -1)$$

$$t_3 = \left(\begin{array}{ccc|ccc} \alpha & \beta & 0 & & & \\ \beta & -\alpha & 0 & & \mathbf{0} & \\ 0 & 0 & -1 & & & \\ \hline & & & \alpha & \beta & 0 \\ & \mathbf{0} & & \beta & -\alpha & 0 \\ & & & 0 & 0 & -1 \end{array}\right)$$

$$t_4 = \left(\begin{array}{ccc|ccc} & & & \xi & 0 & \eta \\ & \mathbf{0} & & 0 & \tau & 0 \\ & & & \eta & 0 & \xi \\ \hline \xi\rho^{-2} & 0 & \eta\rho & & & \\ 0 & \tau\rho^{-2} & 0 & & \mathbf{0} & \\ \eta\rho & 0 & \xi\rho^{-2} & & & \end{array}\right)$$

Lemma 3.2.3. *For $i = 1, 2, 3, 4$, t_i are involutions with $t_1, t_4 \in t^G$ and $t_2, t_3 \in s^G$.*

Proof. The diagonal blocks of t_2 and t_3 are easily seen to be involutions, and so t_2 and t_3 are involutions. Since

$$\begin{pmatrix} \mu & \lambda & \mu \\ \lambda & 0 & -\lambda \\ \mu & -\lambda & \mu \end{pmatrix}^2 = \begin{pmatrix} 2\mu^2 + \lambda^2 & 0 & 2\mu^2 - \lambda^2 \\ 0 & 2\lambda^2 & 0 \\ 2\mu^2 - \lambda^2 & 0 & 2\mu^2 + \lambda^2 \end{pmatrix}$$

the conditions on μ and λ imply that t_1 is an involution.

Moving on to t_4, we look at the product

$$\begin{pmatrix} \xi & 0 & \eta \\ 0 & \tau & 0 \\ \eta & 0 & \xi \end{pmatrix} \begin{pmatrix} \xi\rho^{-2} & 0 & \eta\rho \\ 0 & \tau\rho^{-2} & 0 \\ \eta\rho & 0 & \xi\rho^{-2} \end{pmatrix} = \begin{pmatrix} \xi^2\rho^{-2} + \eta^2\rho & 0 & \xi\eta\rho + \eta\xi\rho^{-2} \\ 0 & \tau^2\rho^{-2} & 0 \\ \eta\xi\rho^{-2} + \xi\eta\rho & 0 & \eta^2\rho + \xi^2\rho^{-2} \end{pmatrix} = A.$$

Note that ρ^3 has multiplicative order 2, and so $\rho^3 = -1$. Now

$$\begin{aligned} \eta\xi\rho^{-2} + \xi\eta\rho &= \eta\xi\rho^{-2}(1 + \rho^3) \\ &= \eta\xi\rho^{-2}(1 + -1) = 0, \end{aligned}$$

and using Definition 3.2.1 we have

$$\tau^2\rho^{-2} = \rho^8\rho^{-2} = \rho^6 = 1$$

Again, from Definition 3.2.1

$$\begin{aligned}
\xi &= \rho^2 + \eta \\
\xi^2 &= \rho^4 + 2\rho^2\eta + \eta^2 \\
\xi^2\rho^{-2} &= \rho^2 + 2\eta + \eta^2\rho^{-2} \\
\xi^2\rho^{-2} + \eta^2\rho &= \rho^2 + 2\eta + \eta^2\rho^{-2} + \eta^2\rho \\
&= \rho^2 + (1-\rho^2) + \eta^2\rho^{-2} + \eta^2\rho
\end{aligned}$$

as $2\eta = 1 - \rho^2$. Then, as $\eta^2\rho^{-2} + \eta^2\rho = \eta^2\rho^{-2}(1+\rho^3) = 0$, we get

$$\xi^2\rho^{-2} + \eta^2\rho = 1.$$

Hence $A = I_3$, whence t_4 is also an involution. Since $\dim C_V(t_i) = 3$ for $i = 1, 4$ and $\dim C_V(t_i) = 2$ for $i = 2, 3$, Lemma 3.2.3 is proved. □

Lemma 3.2.4.

$$C_G(t) = \langle t \rangle \times C_H(t) \cong 2 \times \mathrm{SO}_3(q) \cong 2 \times \mathrm{PGL}_2(q).$$

Proof. Because t acts by inverse conjugation on H, $C_H(t)$ consists of all orthogonal matrices of determinant 1. The well-known isomorphism $\mathrm{SO}_3(q) \cong \mathrm{PGL}_2(q)$ (see [44]) now gives Lemma 3.2.4. □

We define

$$r = \left(\begin{array}{ccc|ccc}
 & & & \rho & 0 & 0 \\
 & 0 & & 0 & \rho & 0 \\
 & & & 0 & 0 & \rho^{-2} \\
\hline
\rho^{-1} & 0 & 0 & & & \\
0 & \rho^{-1} & 0 & & 0 & \\
0 & 0 & \rho^2 & & &
\end{array}\right).$$

Observe that $r \in t^G$ and so $C_G(r) \cong 2 \times \mathrm{PGL}_2(q)$.

Lemma 3.2.5. *$tr = \mathrm{diag}(\rho^{-1}, \rho^{-1}, \rho^2, \rho, \rho, \rho^{-2}) \in H$ has order 6 and $(tr)^2 \in Z(H)$. Further, $C_G(t) \cap C_G(r) \leq C_G(tr) = C_H(tr) \cong \mathrm{GL}_2(q)$.*

Proof. Since $[G : H] = 2$, we have $tr \in H$ and, as ρ has multiplicative order 6, tr has order 6 with $(tr)^2 \in Z(H)$. Thus $C_G(tr) = C_H(tr) = C_H((tr)^3) \cong \mathrm{GL}_2(q)$. $\square$

Lemma 3.2.6. *We have $t_1, t_2, t_3 \in C_G(t)$ and $t_2, t_3, t_4 \in C_G(r)$.*

Proof. It is straightforward to check Lemma 3.2.6, though for $t_4 r = r t_4$ we use the fact that $\rho^2 = \rho^{-4}$. $\square$

Lemma 3.2.7. *$C_G(t) \cap C_G(r) \cong \mathrm{Dih}(2(q + \epsilon))$ where $\epsilon = \pm 1$.*

Proof. First we observe that $C_G(t) \cap C_G(r) = C_{C_G(tr)}(t)$. Since $C_G(tr) = C_H(tr) \cong \mathrm{GL}_2(q)$ by Lemma 3.2.5 and t acts by transpose inverse upon $C_H(tr)$, $C_{C_G(tr)}(t) \cong \mathrm{O}_2^\epsilon(q)$ (the 2-dimensional orthogonal group of type ϵ). Since $\mathrm{O}_2^\epsilon(q) \cong \mathrm{Dih}(2(q - \epsilon))$, (see [44]), we have Lemma 3.2.7. $\square$

Lemma 3.2.8. *The order of $t_1 t_2$ is 4.*

Proof. We have $t_1 t_2 = \left(\begin{array}{c|c} & A \\ \hline A & \end{array}\right)$ where $A = \begin{pmatrix} -\mu & \lambda & -\mu \\ -\lambda & 0 & \lambda \\ -\mu & -\lambda & -\mu \end{pmatrix}$. Now

$$A^2 = \begin{pmatrix} 2\mu^2 - \lambda^2 & 0 & 2\mu^2 + \lambda^2 \\ 0 & -2\lambda^2 & 0 \\ 2\mu^2 + \lambda^2 & 0 & 2\mu^2 - \lambda^2 \end{pmatrix} \text{ and hence } A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore $t_1 t_4$ has order 4. $\square$

Lemma 3.2.9. *$t_1 t_3 = t_3 t_1$.*

Proof. Let $A = \begin{pmatrix} \mu & \lambda & \mu \\ \lambda & 0 & -\lambda \\ \mu & -\lambda & \mu \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then $t_1 t_3 = t_3 t_1$ provided $AB = BA$. Now

$$AB = \begin{pmatrix} \mu\alpha + \lambda\beta & \mu\beta - \alpha\lambda & -\mu \\ \lambda\alpha & \lambda\beta & \lambda \\ \mu\alpha - \lambda\beta & \mu\beta + \alpha\lambda & -\mu \end{pmatrix} \quad \text{and}$$

$$BA = \begin{pmatrix} \alpha\mu + \beta\lambda & \alpha\lambda & \alpha\mu - \beta\lambda \\ \beta\mu - \alpha\lambda & \beta\lambda & \beta\mu + \alpha\lambda \\ -\mu & \lambda & -\mu \end{pmatrix}.$$

So we need to know that

$$\begin{aligned} \alpha\lambda &= \mu\beta - \alpha\lambda, \\ -\mu &= \alpha\mu - \beta\lambda \quad \text{and} \\ \lambda &= \beta\mu + \alpha\lambda. \end{aligned}$$

Since $\mu\beta = \mu 2(\mu^{-2}-1)^{-1}\lambda\mu^{-1} = 2(\mu^{-2}-1)^{-1}\lambda = 2\alpha\lambda$, we have $\alpha\lambda = \mu\beta - \alpha\lambda$. From $\lambda\beta = \lambda 2(\mu^{-2}-1)^{-1}\lambda\mu^{-1} = (\mu^{-2}-1)^{-1}\mu^{-1} = \alpha\mu^{-1}$, we get

$$\begin{aligned} \mu\alpha - \lambda\beta &= \mu\alpha - \alpha\mu^{-1} \\ &= \mu\alpha(1-\mu^{-2}) \\ &= \mu(\mu^{-2}-1)^{-1}(1-\mu^{-2}) \\ &= -\mu. \end{aligned}$$

Finally we show $\lambda = \beta\mu + \alpha\lambda$. Using $\beta = 2\alpha\mu^{-1}$, we have

$$\begin{aligned} \mu\beta + \alpha\lambda &= 2\alpha\lambda + \alpha\lambda \\ &= 3\alpha\lambda \\ &= 3(\mu^{-2}-1)^{-1}\lambda \\ &= 3.3^{-1}\lambda = \lambda, \end{aligned}$$

as $4\mu^2 = 1$ implies $\mu^{-2} - 1 = 3$. Hence Lemma 3.2.9 holds. □

Lemma 3.2.10. *The order of* t_2t_3 *is* $q+1$ *and* $C_G(t) \cap C_G(r) = \langle t_2, t_3\rangle$

Proof. We use that

$$t_2t_3 = \left(\begin{array}{c|c} X & \\ \hline & X \end{array}\right) \text{ where } X = \begin{pmatrix} -\alpha & -\beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the order of t_2t_3 is the same as the order of Y where $Y = \begin{pmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{pmatrix}$.

Recalling that $\alpha^2 + \beta^2 = 1$, the characteristic polynomial of Y is

$$x^2 + 2\alpha x + 1.$$

Therefore the eigenvalues of Y are $-\alpha \pm (\alpha^2 - 1)^{1/2} = -3^{-1} \pm (3^{-2} - 1)^{1/2}$. If these two eigenvalues are equal, then $2(\alpha^2 - 1)^{1/2} = 0$ which implies the impossible $\alpha^2 = 1$. So the two eigenvalues of Y are different. Consequently Y is diagonalizable in $\mathrm{GL}_2(q^2)$ and hence, by assumption (ii) of Theorem 3.0.1, Y has order $q + 1$. Hence, using Lemmas 3.2.6 and Lemma 3.2.7, we obtain $C_G(t) \cap C_G(r) = \langle t_2, t_3 \rangle$. □

Lemma 3.2.11. $[t_2, t_4] = 1$

Proof. Since t_2 is a diagonal matrix with 1 and -1 as its only diagonal entries, a matrix commutes with t_2 if and only if it is of the form

$$\left(\begin{array}{ccc|ccc} * & 0 & * & * & 0 & * \\ 0 & * & 0 & 0 & * & 0 \\ * & 0 & * & * & 0 & * \\ \hline * & 0 & * & * & 0 & * \\ 0 & * & 0 & 0 & * & 0 \\ * & 0 & * & * & 0 & * \end{array}\right),$$

and t_4 is of this form. □

Lemma 3.2.12. $[t_1, t_4] = 1$

Proof. Writing $t_1 = \left(\begin{array}{c|c} & A \\ \hline A & \end{array}\right)$ and $t_4 = \left(\begin{array}{c|c} & C \\ \hline D & \end{array}\right)$, Lemma 3.2.12 will hold if we show that $AD = CA$ and $AC = DA$. Calculating gives

$$AD = \begin{pmatrix} \mu\xi\rho^{-2} + \mu\eta\rho & \lambda\tau\rho^{-2} & \mu\eta\rho + \mu\xi\rho^{-2} \\ \lambda\xi\rho^{-2} - \lambda\eta\rho & 0 & \lambda\eta\rho - \lambda\xi\rho^{-2} \\ \mu\xi\rho^{-2} + \mu\eta\rho & -\lambda\tau\rho^{-2} & \mu\eta\rho + \mu\xi\rho^{-2} \end{pmatrix} \quad \text{and}$$

$$CA = \begin{pmatrix} \xi\mu + \eta\mu & \xi\lambda - \eta\lambda & \xi\mu + \mu\eta \\ \tau\lambda & 0 & -\tau\lambda \\ \eta\mu + \xi\mu & \eta\lambda - \xi\lambda & \eta\mu + \xi\mu \end{pmatrix}.$$

Therefore $AD = CA$ holds provided

$$\begin{aligned}\mu\xi\rho^{-2} + \mu\eta\rho &= \xi\mu + \eta\mu,\\ \lambda\xi\rho^{-2} - \lambda\eta\rho &= \tau\lambda \quad \text{and}\\ \lambda\tau\rho^{-2} &= \xi\lambda - \eta\lambda.\end{aligned}$$

Since $\lambda \neq 0$ and $\mu \neq 0$ this is equivalent to showing that

$$\begin{aligned}\xi\rho^{-2} + \eta\rho &= \xi + \eta,\\ \xi\rho^{-2} - \eta\rho &= \tau \quad \text{and}\\ \tau\rho^{-2} &= \xi - \eta.\end{aligned}$$

First we observe that $\xi = \rho^2 + \eta$, and recall that $\rho^3 = -1$. Hence

$$\begin{aligned}\xi + \eta &= \rho^2 + 2\eta\\ &= \rho^2 + 2(1 - \rho^2)2^{-1}\\ &= \rho^2 + 1 - \rho^2 = 1.\end{aligned}$$

While

$$\begin{aligned}\xi\rho^{-2} + \eta\rho &= (\rho^2 + \eta)\rho^{-2} + \eta\rho\\ &= 1 + \eta\rho^{-2} + \eta\rho\\ &= 1 + \eta\rho^{-2}(1 + \rho^3)\\ &= 1 + \eta\rho^{-2}(1 - 1) = 1.\end{aligned}$$

Next,

$$\begin{aligned}\xi\rho^{-2} - \eta\rho &= (\rho^2 + \eta)\rho^{-2} - \eta\rho\\ &= 1 + \eta\rho^{-2} - \eta\rho\\ &= \rho^4(\rho^2 + \eta - \eta\rho^{-3})\\ &= \rho^4(\rho^2 + 2\eta),\end{aligned}$$

and substituting for η yields

$$\begin{aligned}\xi\rho^{-2} - \eta\rho &= \rho^4(\rho^2 + 2(1-\rho^2)2^{-1}) \\ &= \rho^4 = \tau.\end{aligned}$$

Since $\xi - \eta = \rho^2 + \eta - \eta = \rho^2 = \rho^4\rho^{-2} = \tau\rho^{-2}$,we have shown that $AD = CA$. Similar considerations verify that $AC = DA$, whence Lemma 3.2.12 holds. □

Lemma 3.2.13. *t_3t_4 has order 4.*

Proof. Let

$$X = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$A = \begin{pmatrix} \xi & 0 & \eta \\ 0 & \tau & 0 \\ \eta & 0 & \xi \end{pmatrix} \textit{ and}$$

$$B = \begin{pmatrix} \xi\rho^{-2} & 0 & \eta\rho \\ 0 & \tau\rho^{-2} & 0 \\ \eta\rho & 0 & \xi\rho^{-2} \end{pmatrix}.$$

To show that t_3t_4 has order 4 we verify that $(t_3t_4)^2$ is an involution. Now

$$(t_3t_4)^2 = \left(\begin{array}{c|c} XAXB & \\ \hline & XBXA \end{array}\right).$$

We will see in a moment that the $(3,2)^{th}$-entry of XBXA is non-zero, so $(t_3t_4)^2 \neq 1$. Thus recalling that $X = X^{-1}$ and $A^{-1} = B$, we must show

$$\begin{aligned}XAXB &= (XAXB)^{-1} = AXBX \text{ and} \\ XBXA &= (XBXA)^{-1} = BXAX.\end{aligned}$$

Observe that $XBXA = BXAX$ implies

$$A(XBXA)B = A(BXAX)B,$$

giving $AXBX = XAXB$. Hence it suffices to show that $XBXA = BXAX$.

We calculate that

$$BXAX = \begin{pmatrix} \alpha^2\xi^2\rho^{-2} - \alpha\eta^2\rho + \beta^2\xi\tau\rho^{-2} & \alpha\beta\xi^2\rho^{-2} - \beta\eta^2\rho - \alpha\beta\xi\tau\rho^{-2} & \eta\xi\rho - \alpha\xi\eta\rho^{-2} \\ \alpha\beta\xi\tau\rho^{-2} - \alpha\beta\tau^2\rho^{-2} & \beta^2\xi\tau\rho^{-2} + \alpha^2\tau^2\rho^{-2} & -\beta\eta\tau\rho^{-2} \\ \alpha^2\xi\eta\rho - \alpha\xi\eta\rho^{-2} + \beta^2\tau\eta\rho & \alpha\beta\xi\eta\rho - \beta\xi\eta\rho^{-2} - \alpha\beta\tau\eta\rho & \eta^2\rho^{-2} - \alpha\xi^2\rho \end{pmatrix} \text{ and}$$

$$XBXA = \begin{pmatrix} \alpha\xi^2\alpha\rho^{-2} - \alpha\eta^2\rho + \beta^2\tau\rho^{-2}\xi & \alpha\xi\rho^{-2}\beta\tau - \beta\tau^2\rho^{-2}\alpha & \alpha^2\xi\rho^{-2}\eta - \alpha\eta\rho\xi + \beta^2\tau\rho^{-2}\eta \\ \beta\xi^2\alpha\rho^{-2} - \beta\eta^2\rho - \alpha\tau\rho^{-2}\beta\xi & \beta^2\xi\rho^{-2}\tau + \alpha^2\tau^2\rho^{-2} & \beta\xi\rho^{-2}\alpha\eta - \beta\eta\rho\xi - \alpha\tau\rho^{-2}\beta\eta \\ -\eta\rho\alpha\xi + \xi\rho^{-2}\eta & \eta\rho\beta\tau & -\eta^2\rho\alpha + \xi^2\rho^{-2} \end{pmatrix}.$$

First we note that $XBXA$ and $BXAX$ have the same diagonal entries. For the $(2,1)^{th}$-co-ordinate of $XBXA$ and $BXAX$ we require

$$\alpha\beta\xi\tau\rho^{-2} - \alpha\beta\tau^2\rho^{-2} = \beta\xi^2\alpha\rho^{-2} - \beta\eta^2\rho - \alpha\tau\rho^{-2}\beta\xi.$$

Multiplying through by $\beta\rho^2$ this is equivalent to

$$\tau\xi\alpha - \alpha\tau^2 = \xi^2\alpha + \eta^2 - \alpha\tau\xi,$$

using $\beta^3 = -1$. Since $-2\xi = \tau$, this is equivalent to

$$-2\alpha\tau^2 = \xi^2\alpha + \eta^2.$$

Substituting for ξ, η and $\alpha = 3^{-1}$ reduces this to

$$0 = 1 + \rho^2 + \rho^4,$$

which holds. Therefore $XBXA$ and $BXAX$ have the same $(2,1)^{th}$-co-ordinate. Similarly we may check all the off-diagonal entries of $XBXA$ and $BXAX$ are equal. Therefore $XBXA = BXAX$ and hence Lemma 3.2.13 holds. □

Lemma 3.2.14. *$\langle t_1, t_2, t_3\rangle = C_G(t)$ and $\langle t_2, t_3, t_4\rangle = C_G(r)$.*

Proof. Since $t_1, t_2 \in C_G(t)$ with $t_2 \in C_H(t) \trianglelefteq C_G(t)$, $[t_1, t_2] \in C_H(t) \cong \mathrm{PGL}_2(q)$.

Now, from Lemma 3.2.8,

$$[t_1, t_2] = (t_1 t_2)^2 = \left(\begin{array}{ccc|ccc} & & 1 & & & \\ & -1 & & & & \\ 1 & & & & & \\ \hline & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \end{array}\right).$$

A quick calculation reveals that $[t_1, t_2] \notin C_G(r)$, and so $[t_1, t_2] \notin C_G(t) \cap C_G(r)$. By Lemma 3.2.7 and [43], $C_G(t) \cap C_G(r)$ is a maximal subgroup of $C_H(t)$, whence, as $t_1 \notin C_H(t)$, we infer that $\langle t_1, t_2, t_3 \rangle = C_G(t)$. Similar considerations show that $\langle t_2, t_3, t_4 \rangle = C_G(r)$. □

Proposition 3.2.15. $\{t_1, t_2, t_3, t_4\}$ *is a C-string for* G *with Schläfli symbol* $[4, q+1, 4]$.

Proof. This comes from combining the fact that $C_H(t)$ is a maximal subgroup of H (see [35]) with Lemmas 3.2.8, 3.2.9, 3.2.10, 3.2.11, 3.2.12 and 3.2.13. □

Proposition 3.2.16. $\{t_1, t_2, t_3, t_4\}$ *is an unravelled C-string of* G.

Proof. The only non-trivial proper normal subgroups of G are H,$Z(H)$. Since $[G : H] = 2$, we only need show $\{t_1, t_2, t_3, t_4\}$ is $Z(H)$-unravelled. Put $G_{123} = \langle t_1, t_2, t_3 \rangle$, $G_{234} = \langle t_2, t_3, t_4 \rangle$ and $\overline{G} = G/Z(H)$. Since $Z(H) = \langle \mathrm{diag}(\rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^2, \rho^2, \rho^2) \rangle$, we see that $|\{\overline{t_1}, \overline{t_2}, \overline{t_3}, \overline{t_4}\}| = 4$. Also $\langle \overline{t_2}, \overline{t_3} \rangle \cong C_G(t) \cap C_G(r) \cong \mathrm{Dih}(2(q+1))$ as $\langle t_2, t_3 \rangle \cap Z(H) = 1$. Now $\overline{G}_{123} = C_{\overline{G}}(\overline{t})$ and $\overline{G}_{234} = C_{\overline{G}}(\overline{r})$, as the orders of t and r are coprime to $|Z(H)|$. From Lemma 3.2.5 $\overline{t}\overline{r} = \overline{r}\overline{t}$ has order 2. That is $\overline{t}$ and $\overline{r}$ commute. So $\overline{t}, \overline{r} \in \overline{G}_{123} \cap \overline{G}_{234}$ and therefore $\overline{G}_{123} \cap \overline{G}_{234} \gneqq \langle \overline{t_2}, \overline{t_3} \rangle$. Consequently, the intersection property fails for $\{\overline{t_1}, \overline{t_2}, \overline{t_3}, \overline{t_4}\}$. □

Together, Propositions 3.2.15 and 3.2.16 prove Theorem 3.0.1.

3.3 C-strings with Schläfli symbol $[4, p, 4]$

In this section we are concerned with proving Theorem 3.0.2. We re-use the notation used in the previous section. Here, $G = \mathrm{SL}_3(p) \rtimes \langle t \rangle$ where p is a prime such that $p \equiv 1 \pmod 3$ and $p \equiv 5 \pmod 8$. Because $p \equiv 1 \pmod 3$ we may choose, and keep fixed, $\rho \in \mathrm{GF}(p)$ of multiplicative order 3. Further, $p \equiv 5 \pmod 8$ means we may choose $\iota \in \mathrm{GF}(p)$, also now to be fixed, such that $\iota^2 = -1$. Set $\alpha = \sqrt{(1+\rho^2)^{-1}}$, again making a choice from the (at most) two possibilities. Now we define a slew of elements in $\mathrm{GF}(p)$.

Definition 3.3.1.

$$\begin{aligned}
\lambda &= \alpha(\iota+1)(-1+\rho-\iota\rho^2) \\
\epsilon &= -\iota\lambda \\
\beta &= -2^{-1}\lambda^2\iota \\
\gamma &= 2^{-1}\lambda^2 - 1 \\
\delta &= -1 - 2^{-1}\lambda^2 \\
\mu &= 1 - \rho.
\end{aligned}$$

Note that $\lambda \neq 0$ and $\lambda^2 = -\epsilon^2$. Also recall that $1+\rho+\rho^2 = 0$ and so $\alpha^2 = -\rho^2$. Hence $\alpha \neq 0$.

The elements in Definition 3.3.1 appear as entries in $\{t_1, t_2, t_3, t_4\}$, elements of G, which we now define.

Definition 3.3.2.

$$t_1 = \left(\begin{array}{ccc|ccc}
 & & & 0 & \alpha & -\alpha\rho \\
 & 0 & & \alpha & \rho & 1 \\
 & & & -\alpha\rho & 1 & \rho^2 \\
\hline
0 & \alpha & -\alpha\rho & & & \\
\alpha & \rho & 1 & & 0 & \\
-\alpha\rho & 1 & \rho^2 & & &
\end{array}\right)$$

$$t_2 = \left(\begin{array}{ccc|ccc}
1 & & & & & \\
 & -1 & & & 0 & \\
 & & -1 & & & \\
\hline
 & & & 1 & & \\
 & 0 & & & -1 & \\
 & & & & & -1
\end{array}\right) = \mathrm{diag}(1,-1,-1,1,-1,-1)$$

$$t_3 = \left(\begin{array}{ccc|ccc} 1 & \lambda & \epsilon & & & \\ \lambda & \gamma & \beta & & 0 & \\ \epsilon & \beta & \delta & & & \\ \hline & & & 1 & \lambda & \epsilon \\ & 0 & & \lambda & \gamma & \beta \\ & & & \epsilon & \beta & \delta \end{array}\right)$$

$$t_4 = \left(\begin{array}{ccc|ccc} & & & -\rho & 0 & 0 \\ & 0 & & 0 & 0 & \rho \\ & & & 0 & \rho & -\mu\rho^2 \\ \hline -\rho^2 & 0 & 0 & & & \\ 0 & \mu & \rho^2 & & 0 & \\ 0 & \rho^2 & 0 & & & \end{array}\right).$$

In order to define a further element in t^G, we introduce more elements in $\mathrm{GF}(p)$.

Definition 3.3.3.

$$\begin{aligned} a &= 2(2\rho^2 + (1-\rho)\iota)^{-1} \\ x &= -2^{-1}a\rho(1-\rho) \\ y &= -x\rho \\ b &= a^{-1}(\rho^2 + x^2) \\ c &= a^{-1}(1 + x^2\rho) \\ d &= a\rho \end{aligned}$$

Observe that $a \neq 0$, so b and c are well-defined. Now set

$$r = \left(\begin{array}{ccc|ccc} & & & \rho & 0 & 0 \\ & 0 & & 0 & a & x \\ & & & 0 & x & b \\ \hline \rho^2 & 0 & 0 & & & \\ 0 & c & y & & 0 & \\ 0 & y & d & & & \end{array}\right).$$

Lemma 3.3.4.

(*i*) t_1, t_2, t_3, t_4 *and* r *are involutions.*

(*ii*) $t_1, t_4, r \in t^G$ *and* $t_2, t_3 \in s^G$.

Proof. To show that t_1 is an involution, we must verify that $X^2 = I_3$ where $X = \begin{pmatrix} 0 & \alpha & -\alpha\rho \\ \alpha & \rho & 1 \\ \alpha\rho & 1 & \rho^2 \end{pmatrix}$.

Now $X^2 = \begin{pmatrix} \alpha^2 + \alpha^2\rho & 0 & 0 \\ 0 & \alpha^2 + \rho^2 + 1 & -\alpha^2\rho + \rho + \rho^2 \\ 0 & -\alpha^2\rho + \rho + \rho^2 & \alpha^2\rho + 1 + \rho^4 \end{pmatrix}$ and using $\alpha^2 = -\rho^2$, we see $X^2 = I_3$. Similarly, using Definition 3.3.1, we may show t_3 is an involution. While it is straightforward to check that t_2 and t_4 are involutions, for r it suffices, using Definition 3.3.3, to show that

$$\begin{pmatrix} a & x \\ x & b \end{pmatrix}^{-1} = \begin{pmatrix} c & y \\ y & d \end{pmatrix},$$

so proving (*i*). Since, by calculation, $\dim C_V(t_1) = \dim C_V(t_4) = \dim C_V(r) = 3$ and $\dim C_V(t_2) = \dim C_V(t_3) = 2$, we have part (*ii*). □

Lemma 3.3.5. $t_1t_3 = t_3t_1$, $t_1t_4 = t_4t_1$ *and* $t_2t_4 = t_4t_2$.

Proof. Checking $t_1t_4 = t_4t_1$ uses $\mu = 1 - \rho$ whereas $t_1t_3 = t_3t_1$ requires the definitions of $\lambda, \epsilon, \beta, \gamma$ and δ. That $t_2t_4 = t_4t_2$ is easily seen. □

Lemma 3.3.6.

(*i*) t_1t_2 *and* t_3t_4 *both have order 4.*

(*ii*) t_2t_3 *has order* p.

Proof. Part (*i*) can be checked following the same strategy as in 3.2.13.

Now $t_2t_3 = \left(\begin{array}{c|c} X & \\ \hline & X \end{array}\right)$ where $X = \begin{pmatrix} 1 & \lambda & \epsilon \\ -\lambda & -\gamma & -\beta \\ -\epsilon & -\beta & -\delta \end{pmatrix}$.

We demonstrate that X has order p, from which (*ii*) will follow. Consider X acting on the 3-dimensional vector space U, setting $U_1 = C_U(X)$ and letting U_2 be the inverse image of $C_{U/U_1}(X)$ in U. For $(u,v,w) \in U, (u,v,w) \in U_1$ if and only if

$$\begin{aligned} u - \lambda v - \epsilon w &= u \\ \lambda u - \gamma v - \beta w &= v \\ \epsilon u - \beta v - \delta w &= w \end{aligned}$$

The first equation gives $v = -\lambda^{-1}\epsilon w = (-\lambda^{-1})(-\iota\lambda w) = \iota w$, and then the second yields

$$\lambda u = (\gamma\iota + \beta + \iota)w = 0,$$

using the definitions of γ and β. Since $\lambda \neq 0, u = 0$. Thus
$U_1 = \{(0, \iota w, w) | w \in \mathrm{GF}(p)\}$. Similar calculations show that
$U_2 = \{(u, \iota w, w) | u, w \in \mathrm{GF}(p)\}$. Now $(0,0,1)X - (0,0,1) = (-\epsilon, -\beta, -\delta - 1) \in U_2$, as $\iota(-\delta - 1) = -\beta$. Hence as $(0,0,1) \notin U_2$, X acts nilpotently on U, whence X has p-power order. Since Sylow p-subgroups of $\mathrm{SL}_3(p)$ have exponent p and $X \neq I_3$, X has order p. This completes the proof of Lemma 3.3.6.

□

Let $g_0 = \left(\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & 0 & & 0 & 0 & -1 \\ & & & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & & & \\ 0 & 0 & -1 & & 0 & \\ 0 & 1 & 0 & & & \end{array}\right)$ and $z = \mathrm{diag}(\rho, \rho, \rho, \rho^2, \rho^2, \rho^2)$. Note that $z \in Z(H)$, and straightforward calculation gives

Lemma 3.3.7.

(i) $g_0 \in C_G(t)$ *and* $zg_0 \in C_G(r)$,

(ii) $g_0^2 = t_2 = (zg_0)^2$.

Proof. Set $L_{123} = G_{123} \cap C_H(t)'$ and $L_{234} = G_{234} \cap C_H(t)'$. Note that
$L_{123} \cong \mathrm{PSL}_2(p) \cong L_{234}$. □

Lemma 3.3.8.

(i) $C_G(t) \geq G_{123}$ *and* $C_G(r) \geq G_{234}$,

(ii) $G_{123} = \langle t_1 \rangle L_{123}$ *and* $G_{234} = \langle t_4 \rangle L_{234}$,

(iii) $G_{123} \cong \mathrm{PGL}(2,p) \cong G_{234}.$

Proof. First, calculation reveals that t_1, t_2 and t_3 commute with t and t_2, t_3 and t_4 commute with r, so part (i) holds.

Observe that, as $C_G(t) = \langle t \rangle \times C_H(t)$ with $C_H(t) \cong \mathrm{PGL}(2,p)$, $C_G(t) = \langle t \rangle \times C_H(t)$ with $C_H(t) \cong \mathrm{PGL}(2,p), C_G(t)/L_{123}$ is elementary abelian of order 4, Lemma 3.3.7 implies that $t_2 \in L_{123}$. Clearly we also have $t_2 t_3 \in L_{123}$, so $G_{23} = \langle t_2, t_3 \rangle \leq L_{123}$. Since by Lemma 3.3.6 (ii), $\mathrm{Dih}(2p) \cong G_{23}$ is a maximal subgroup of $L_{123} \cong \mathrm{PSL}(2,p)$ and t_1 does not normalize G_{23}, $G_{123} = \langle t_1 \rangle L_{123}$. A similar argument establishes $G_{234} = \langle t_4 \rangle L_{234}$.

Since $p \equiv 5 \pmod 8$, Lemma 3.3.6 (i) implies that $\langle t_1, t_2 \rangle \in \mathrm{Syl}_2(G_{123})$. Hence $t \notin G_{123}$ and so, by (ii), $G_{123} \cong \mathrm{PGL}(2,p)$. Likewise we have $G_{234} \cong \mathrm{PGL}(2,p)$, so proving Lemma 3.3.8. □

Proposition 3.3.9. $G = \langle t_1, t_2, t_3, t_4 \rangle$.

Proof. Put $\overline{G} = G/Z(H)$. Then $\overline{H} \cong \mathrm{PSL}_3(p)$ and $\overline{G}_{123}$ contains a subgroup isomorphic to $\mathrm{PSL}_2(p)$ by Lemma 3.3.7. Since $\overline{C_G(t)}$ is the only maximal subgroup of $\overline{G}$ containing $\overline{G}_{123}$ and $\overline{t}_4 \notin \overline{C_G(t)}$, $\overline{G} = \langle \overline{G}_{123}, \overline{t}_4 \rangle$. Now H being a non-split central extension this then implies Proposition 3.3.9.

Lemma 3.3.10. $G_{23} = G_{123} \cap G_{234} = C_G(t) \cap C_G(r)$.

From Lemma 3.3.7 $G_{23} \leq G_{123} \cap G_{234} \leq C_G(t) \cap C_G(r)$. Now

$$tr = \left(\begin{array}{ccc|ccc} \rho^2 & 0 & 0 & & & \\ 0 & c & y & & 0 & \\ 0 & y & d & & & \\ \hline & & & \rho & 0 & 0 \\ & 0 & & 0 & a & x \\ & & & 0 & x & b \end{array}\right).$$

Let $g = ztr$ (recall that $z = \mathrm{diag}(\rho, \rho, \rho, \rho^2, \rho^2, \rho^2)$. Then

$$g = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & \rho c & \rho y & & 0 & \\ 0 & \rho y & \rho d & & & \\ \hline & & & 1 & 0 & 0 \\ & 0 & & 0 & \rho^2 a & \rho^2 x \\ & & & 0 & \rho^2 x & \rho^2 b \end{array}\right).$$

Investigating the action of g on V we discover that g acts nilpotently on V, and therefore g has order p. Hence $tr = z^{-1}g$ has order $3p$ with $\langle z \rangle \leq \langle tr \rangle$. Consequently $C_G(tr) \leq C_G(z) = H$. So $C_G(tr) = C_H(g)$. Since $G_{23} \leq C_G(t) \cap C_G(r) \leq C_G(tr), C_G(tr)$ has even order by Lemma 3.3.6 (i). Thus from centralizers of p-elements in $\mathrm{SL}_3(p)$ we have $C_G(tr) = C_H(g) \sim p^3 : (p-1)$. Let $P \in \mathrm{Syl}_p C_H(g)$. Then $P \trianglelefteq C_H(g)$. Also t acts upon $C_H(g)/P \cong p - 1$. If t centralizes $C_H(g)/P$, then $C_H(g) = C_{C_H(g)}(t)P$. Now $\langle t_2t_3 \rangle \leq C_H(t)$ and from $C_H(t) \cong \mathrm{PGL}_2(p)$ we have $N_{C_H(t)}(\langle t_2t_3 \rangle) \sim p : p-1$, so $C_{C_H(g)}(t)$ normalizes $\langle t_2t_3 \rangle$ which contradicts the structure of $C_H(g)$. Therefore t does not centralize $C_H(g)/P$. Since $C_H(g)/P$ is a cyclic group, t must act by inverting which implies $C_{C_G(tr)}(t)$ has order dividing $2p^3$. But the largest power of p dividing $|\mathrm{PGL}_2(p)|$ is p and so $|C_{C_G(tr)}(t)| = 2p$. Now we infer that $C_G(t) \cap C_G(r) = C_{C_G(tr)}(t) = G_{23}$. □

Proposition 3.3.11. $\{t_1, t_2, t_3, t_3\}$ *is an unravelled C-string for G with Schläfli symbol* $[4, p, 4]$.

Proof. Combining Lemma 3.3.4(i), 3.3.6, Proposition 3.3.9 and Lemma 3.3.10 gives that $\{t_1, t_2, t_3, t_3\}$ is a C-string with Schläfli symbol $[4, p, 4]$. We now show it is unravelled.

Since $L_{123} \cong \mathrm{PSL}(2, p)$ and, by assumption $p \equiv 5 \pmod 8$, the Sylow 2-subgroup of L_{123} are elementary abelian. In particular, L_{123} contains no elements of order 4. Hence, if h is an element of G_{123} of order 4, $G_{123} = \langle h \rangle L_{123}$. As a consequence any G_{123}-conjugate of h is L_{123}-conjugate. By Lemma 3.3.8(iii) $G_{123} \cong \mathrm{PGL}(2, p)$ and so, as its Sylow 2-subgroups are isomorphic to $\mathrm{Dih}(8)$, has only one $G_{123}-$conjugacy class of elements of order 4. Now

$$t_1t_2 = \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right).$$

Thus we conclude, as $L_{123} \leq H$, that all order 4 elements of G_{123} must have this shape. From 3.3.7(i) $g_0 \in C_G(t)$ and, since $C_G(t) = \langle t \rangle G_{123}$, either g_0 or tg_0 are in G_{123}. But tg_0 has shape $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$, whence we deduce that $g_0 \in G_{123}$. Because of Lemma 3.3.7, a similar argument yields that $zg_0 \in G_{234}$. Let $\overline{G} = G/Z(H)$. Then, as $z \in Z(H)$, we have

$$\overline{g_0} = \overline{zg_0} \in \overline{G}_{123} \cap \overline{G}_{234},$$

but $\overline{g_0} \notin \overline{G}_{23} = \langle \bar{t}_2, \bar{t}_3 \rangle$ as $\overline{g_0}$ has order 4. Thus $\{t_1, t_2, t_3, t_4\}$ is an unravelled C-string, so proving Proposition 3.3.10. □

Chapter 4

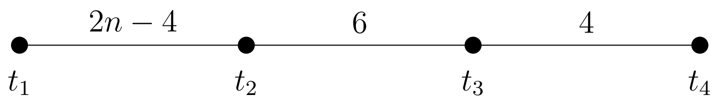
Two families of unravelled polytopes for type B Coxeter groups

This chapter consists of the work of joint work with Professor Rowley in [39]. We have cut certain parts out that were repeated within the previous chapters and lightly edited some of the results for consistency and added some exposition for motivation.

Our main concern in this chapter is to find unravelled C-strings for the type B Coxeter groups. Table 2.2 showed that B_7 and B_8 do contain unravelled C-strings whereas for $n < 7$ they did not. We will produce to infinite families of C-strings: one rank 4 family within B_n with n odd, and one rank $n-4$ family within B_n with n even. This shows that non-trivial unravelled C-strings exist for unbounded ranks also.

Our theorems are as follows.

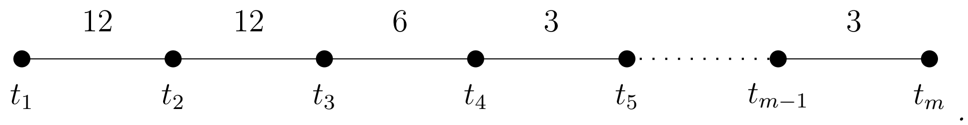
Theorem 4.0.1. *Suppose that $G = B_n$ where n is odd and $n \geq 5$. Then G has a rank 4 C-string $\{t_1, t_2, t_3, t_4\}$ with Schläfli symbol*



,

which is unravelled when $n > 5$. Further $G_{123} \cong \mathrm{Sym}(n)$ and $G_{234} \cong \mathbb{Z}_2 \times \mathrm{Sym}(5)$.

Theorem 4.0.2. *Suppose that $G = B_n$ where $n \geq 8$, and set $m = n-4$. Then G has a rank m C-string $\{t_1, t_2, \ldots, t_m\}$ with Schlafli symbol*



Further, when n is even, this C-string is unravelled.

4.1 Preliminary results

This short section contains the results we need in the two following sections. The first of these results is one which identifies the Coxeter groups of type B_n. For a set $\Omega = \{1, \ldots, n\}$, $\mathrm{Sym}(n) = \mathrm{Sym}(\Omega)$ denotes the symmetric group of degree n defined on Ω.

Lemma 4.1.1. *Suppose that $\Omega = \{1, 2 \ldots, n, n+1, \ldots, 2n\}$. Let $\beta_0 = (1, n+1)$ and $\beta_i = (i, i+1)(n+i, n+i+1)$ for $1 \leq i < n$. Then $\langle \beta_0, \beta_1, \ldots, \beta_{n-1} \rangle$ is isomorphic to B_n.*

Proof. See (2.10) of [26]. □

In a similar vein to Lemma 4.1.1, we have the well-known characterization of $\mathrm{Sym}(n)$.

Lemma 4.1.2. *Suppose that H is a group with presentation $\langle r_1, \ldots, r_{n-1} | (r_i r_j)^{m_{ij}}) \rangle$. If $m_{ii} = 1$ for $i = 1, \ldots, n-1, m_{ij} = 3$ if $|i - j| = 1$ and $m_{ij} = 2$ if $|i - j| > 1$, then $H \cong \mathrm{Sym}(n)$.*

Proof. See (6.4) of [26]. □

4.2 Rank 4 unravelled C-strings

Here we establish Theorem 4.0.1. So we are assuming that n is odd and $n \geq 5$. We shall construct the C-string for B_n working in $\mathrm{Sym}(2n)$. First we define the involutions t_i, $i = 1, 2, 3, 4$ in $\mathrm{Sym}(2n) = \mathrm{Sym}(\Omega)$, where $\Omega = \{1, \ldots, 2n\}$.

Definition 4.2.1.

$$t_1 = \prod_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (1+2i, 2+2i)(n+1+2i, n+2+2i)$$

$$t_2 = \prod_{i=1}^{\lfloor \frac{n-3}{2} \rfloor} (2+2i, 3+2i)(n+2+2i, n+3+2i)$$

$$t_3 = (1,3)(2,4)(n+1, n+3)(n+2, n+4)$$

$$t_4 = (1,2)(n+1, n+2) \prod_{i=1}^{n-2} (2+i, n+2+i)$$

Observe that t_1t_2, when written as a product of pairwise disjoint cycles, has two of length 2 and two of length $n-2$. Hence, as $n-2$ is odd, t_1t_2 has order $2n-4$. It is easy to check that t_2t_3 has order 6 and t_3t_4 has order 4. Also we see that $t_1t_3 = t_3t_1$ and $t_2t_4 = t_4t_2$.

Put $G = \langle t_1, t_2, t_3, t_4 \rangle$. We will show in Proposition 4.2.10 that $G \cong \mathrm{B}_n$, after we have first investigated the subgroups $G_{123} = \langle t_1, t_2, t_3 \rangle$ and $G_{234} = \langle t_2, t_3, t_4 \rangle$. Beginning with G_{234} and setting

$$\begin{aligned}
\Delta_1 &= \{1,2,3,4,5,n+1,n+2,n+3,n+4,n+5\},\\
\Delta_6 &= \{6,7,n+6,n+7\},\\
\Delta_8 &= \{8,9,n+8,n+9\},\\
&\vdots\\
\Delta_{n-1} &= \{n-1, n, 2n-1, 2n\}.
\end{aligned}$$

Lemma 4.2.2. *The G_{234} orbits of the Ω are $\Delta_1, \Delta_6, \Delta_8, \ldots, \Delta_{n-1}$.*

Lemma 4.2.3. *The induced action of $\langle t_2, t_4 \rangle$ on each of $\Delta_6, \Delta_8, \ldots, \Delta_{n-1}$ is identical to its action on $\Delta_4 = \{4, 5, n+4, n+5\}$.*

Set $s_2 = (4,5)(9,10), s_3 = (1,3)(2,4)(6,8)(7,9)$ and $s_4 = (1,2)(3,8)(4,9)(5,10)(6,7)$ (these are t_2, t_3, t_4 for the case $n = 5$), and $H = \langle s_2, s_3, s_4 \rangle$.

Lemma 4.2.4. *$G_{234} \cong H \cong \mathbb{Z}_2 \times \mathrm{Sym}(5)$ with $\{t_2, t_3, t_4\}$ a C-string for G_{234}. Further $\langle (t_2t_3t_4)^5 \rangle = Z(G_{234})$.*

Proof. Restricting G_{234} to Δ_1 yields a homomorphism from G_{234} to H, and then Lemma 4.2.3 implies $G_{234} \cong H$. Employing MAGMA[2] quickly reveals the structure of H and that $\{s_2, s_3, s_4\}$ is a C-string for H. This proves Lemma 4.2.4. □

We now turn our attention to G_{123}.

Lemma 4.2.5. *(i). The G_{123}- orbits of Ω are $\Lambda_1 = \{1, .., n\}$ and $\Lambda_{n+1} = \{n+1, \dots, 2n\}$.*

(ii). For $j \in \Lambda_1$ and $g \in G_{123}$, $(j)g = k$ if and only if $(j+n)g = k+n$.

Lemma 4.2.6. *For $1 \leq i < n$, we have $(i, i+1)(n+i, n+i+1) \in G_{123}$.*

Proof. In view of Lemma 4.2.5(ii) it will be sufficient to look at the action of elements of G_{123} on Λ_1. So, for $i = 1, 2, 3$, let $\hat{t}_i$ denote the induced action of t_i on Λ_1. Hence

$$\begin{aligned} \hat{t}_1 &= \prod_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (1+2i, 2+2i), \\ \hat{t}_2 &= \prod_{i=1}^{\lfloor \frac{n-3}{2} \rfloor} (2+2i, 3+2i) \text{ and} \\ \hat{t}_3 &= (1,3)(2,4). \end{aligned}$$

Consequently

$$\hat{t}_1\hat{t}_2 = (1,2)(3,5,7,\dots,n,n-1,n-3,\dots,6,4).$$

Since $n-2$ is odd, $(\hat{t}_1\hat{t}_2)^{n-2} = (1,2)$. Therefore, $(1,2)(n+1,n+2) \in G_{123}$. Also

$$\begin{aligned} \hat{t}_3\hat{t}_1\hat{t}_2 &= (1,3)(2,4)(1,2)(3,5,7,\dots,n,n-1,n-3,\dots,6,4) \\ &= (1,5,7,\dots,n,n-1,n-3,\dots,6,4)(2,3) \end{aligned}$$

which is in disjoint cycle form. Again, since $n-2$ is odd, we have $(\hat{t}_3\hat{t}_1\hat{t}_2)^{n-2} = (2,3)$. Hence $(2,3)(n+2,n+3) \in G_{123}$. Now $(3,4) = (1,2)^{\hat{t}_3}$, so $(3,4)(n+3,n+4) \in G_{123}$.
Now we recursively construct the remaining $(i,i+1)(n+i,n+i+1)$ for all i, with $3 < i < n$. Supposing we have $(i,i+1)(n+i,n+i+1) \in G_{123}$ for all $3 \leq i \leq k$. We show that $(k+1,k+2)(n+k+1,n+k+2) \in G_{123}$. If k is even, then

$$(k,k+1)(n+k,n+k+1)^{t_1} = (k-1,k+2)(n+k-1,n+k+2).$$

Since

$$(k-1,k+2)(n+k-1,n+k+2)^{(k-1,k+1)(n+k-1,n+k+1)} = (k+1,k+2)(n+k+1,n+k+2)$$

and

$$(k-1,k+1)(n+k-1,n+k+1) = (k-1,k)(n+k-1,n+k)^{(k,k+1)(n+k,n+k+1)} \in G_{123},$$

we deduce that $(k+1,k+2)(n+k+1,n+k+2) \in G_{123}$. When k is odd, a similar calculation using t_2 in place of t_1, also yields the same conclusion, so proving Lemma 4.2.6. □

Proposition 4.2.7. *$G_{123} \cong \mathrm{Sym}(n)$.*

Proof. From Lemma 4.2.6

$$K = \langle\ (i,i+1)(n+i,n+i+1) \mid 1 \leq i < n\ \rangle \leq G_{123},$$

with the generators of K satisfying the Coxeter relations for $\mathrm{Sym}(n)$. Thus, by Lemma 4.1.2, K is isomorphic to a quotient of $\mathrm{Sym}(n)$ and hence $K \cong \mathrm{Sym}(n)$. The action of G_{123} on $\{\{i,n+i\} | i \in \Lambda_1\}$ forces $K = G_{123}$, so giving Proposition 4.2.7. □

Proposition 4.2.8. *$\{t_1,t_2,t_3\}$ is a C-string for G_{123}.*

Proof. We only need check $\langle t_1,t_2\rangle \cap \langle t_2,t_3\rangle = \langle t_2\rangle$, the other intersections being clear. Now $\langle t_2t_3\rangle$ has $\{1,3\}$ and $\{2,4,5\}$ as orbits on Ω and so $\langle (t_2t_3)^2\rangle$ has $\{2,4,5\}$

as an orbit and $\langle (t_2t_3)^3 \rangle$ has $\{1,3\}$. Since $\langle t_1t_2 \rangle$ has $\{1,2\}$ as an orbit and t_2t_3 has order 6, we conclude that $\langle t_1, t_2 \rangle \cap \langle t_2, t_3 \rangle = \langle t_2 \rangle$. So Proposition 4.2.8 holds. $\square$

Set $\omega_0 = \prod_{i=1}^{n} (i, n+i)$.

Proposition 4.2.9. $\omega_0 = (t_1t_2t_3t_4)^n$.

Proof. We calculate that

$$\begin{aligned}
t_1t_2 &= (1,2)(3,5,7,\ldots,n,n-1,\ldots,6,4) \\
&\quad (n+1,n+2)(n+3,\ldots,2n,2n-1,\ldots n+4), \\
t_1t_2t_3 &= t_1t_2(1,3)(2,4)(n+1,n+3)(n+2,n+4) \\
&= (1,4)(3,5,\ldots,n,n-1,\ldots,8,6,2) \\
&\quad (n+1,n+4)(n+3,\ldots,2n,2n-1,\ldots,n+4), \\
t_1t_2t_3t_4 &= t_1t_2t_3(1,2)(n+1,n+2)\prod_{i=1}^{n-2}(2+i,(n+2)+i) \\
&= (\underbrace{1,n+4,n+2,3,n+5,7,\ldots,n,2n-1,n-3,2n-5,\ldots,n+6,n+1}_{n+1,} \\
&\quad \underbrace{4,2,n+3,\ldots,6}_{n-1}) \\
&\quad \text{when } n \equiv 1 \mod 3 \text{ and} \\
&= (\underbrace{1,n+4,n+2,3,n+5,7,\ldots,2n,n-1,2n-3,n-5,\ldots,n+6,n+1}_{n+1}, \\
&\quad \underbrace{4,2,n+3,\ldots,6}_{n-1}) \\
&\quad \text{when } n \equiv 3 \mod 3.
\end{aligned}$$

Therefore $t_1t_2t_3t_4 = \prod_{i=1}^{n} (i, n+i)$. $\square$

Proposition 4.2.10. $G \cong \mathrm{B}_n$.

Proof. Set $\beta_0 = (1, n+1)$ and, for $1 \leq i < n$, $\beta_i = (i, i+1)(n+i, n+i+1)$. Put $L = \langle \beta_0, \beta_1, \ldots, \beta_{n-1} \rangle$. By Lemma 4.1.2 $L \cong \mathrm{B}_n$.
Directly from their definitions, we have

$$t_1 = \prod_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \beta_{1+2i} \text{ and}$$
$$t_2 = \prod_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \beta_{2+2i}.$$

An easy check shows $t_3 = {\beta_1}^{\beta_2}{\beta_2}^{\beta_3}$. Since

$$\begin{aligned} t_4 &= (1,2)(n+1,n+2)\prod_{i=1}^{n-2}(2+i,n+2+i) \\ &= \beta_1 \prod_{i=2}^{n} \beta_0^{\prod_{j=i}^{i}\beta_j}, \end{aligned}$$

we infer that $G \leq L$.

By Lemma 4.2.6 $\beta_i \in G$ for $1 \leq i < n$. Thus to complete the proof of Proposition 4.2.10 we need to demonstrate that $\beta_0 \in G$. Now

$$\begin{aligned} {t_2}^{t_3} &= (\prod_{i=1}^{\lfloor \frac{n-3}{2} \rfloor}(2+2i,3+2i)(n+2+2i,n+3+2i))^{(1,3)(2,4)(n+1,n+3)(n+2,n+4)} \\ &= (2,5)(n+2,n+5)t_2(4,5)(n+4,n+5) \end{aligned}$$

and

$${t_2}^{t_3t_4} = (1,n+5)(n+1,5)t_2(4,5)(n+4,n+5).$$

Hence

$$\begin{aligned} {t_2}^{t_3}{t_2}^{t_3t_4} &= (2,5)(n+2,n+5)t_2(4,5)(n+4,n+5) \\ &\quad (1,n+5)(n+1,5)t_2(4,5)(n+4,n+5) \\ &= (2,5)(n+2,n+5)(1,n+5)(n+1,5) \\ &= (2,n+1,5)(1,n+5,n+2). \end{aligned}$$

Therefore

$$\begin{aligned} t_1{t_2}^{t_3}{t_2}^{t_3t_4} &= t_1t_2(2,n+1,5)(1,n+5,n+2) \\ &= (1,n+1)(2,n+5,n+7,\ldots,2n,2n-1,\ldots,n+4,n+3, \end{aligned}$$

$$n+2,5,7,9,\dots,n,n-1,\dots,6,4,3),$$

and so

$$(t_1 {t_2}^{t_3} {t_2}^{t_3 t_4})^{n-1} = \prod_{i=2}^{n} (i, n+i),$$

Hence, using Proposition 4.2.9,

$$\beta_0 = \prod_{i=1}^{n} (i, n+i)(t_1 {t_2}^{t_3} {t_2}^{t_3 t_4})^{-n+1} \in G,$$

which proves Proposition 4.2.10. □

Proposition 4.2.11. *$\{t_1, t_2, t_3, t_4\}$ is a C-string for G.*

Proof. From Lemma 4.2.4 $G_{234} \cong \mathbb{Z}_2 \times \mathrm{Sym}(5)$ and $\langle (t_2 t_3 t_4)^5 \rangle = Z(G_{234})$. If $G_{123} \cap G_{234} > G_{23}$, then, as $G_{23} \cong \mathrm{Dih}(12)$, we must have either $(t_2 t_3 t_4)^5 \in G_{123} \cap G_{234}$ or $G_{123} \cap G_{234}$ has index at most 2 in G_{234} (and so $t_3 t_4 \in G_{123}$). Either of these possibilities would contradict Lemma 4.2.5(i) as $(t_2 t_3 t_4)^5 : 1 \to n+1$ and $t_3 t_4 : 5 \to n+5$. Thus $G_{123} \cap G_{234} = G_{23}$. Using Lemma 4.2.4 and Proposition 4.2.8 we now obtain Proposition 4.2.11. □

Proposition 4.2.12. *When $n \geq 7$, $\{t_1, t_2, t_3, t_4\}$ is an unravelled C-string for G.*

Proof. Let $M_1 = \langle\ (i, n+i) \mid 1 \leq i \leq n\ \rangle (= \mathrm{O}_2(G))$ and $M_2 = \langle\ (1, n+1)(i, n+i) \mid 1 < i \leq n\ \rangle$. From Proposition 4.2.10 $G \cong \mathrm{B}_n$ and hence for $N \trianglelefteq G$, $1 \neq N \neq G$, we either have $[G : N] \leq 4$ or $N = \langle \omega_0 \rangle$, M_1 or M_2. Set $\overline{G} = G/N$. Since $\{t_1, t_2, t_3, t_4\}$ has rank 4, we are only required to check that $\{\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4\}$ is not a C-string for $N = \langle \omega_0 \rangle, M_1$ and M_2.
Suppose $N = M_1$ or M_2. Then

$$g = (t_3 t_4)^2 = (1, n+2)(2, n+1)(3, n+4)(4, n+3) \in G_{234}.$$

Also, using Lemma 4.2.6,

$$h = (1,2)(3,4)(n+1, n+2)(n+3, n+4) \in G_{123}.$$

Since

$$gh^{-1} = (1, n+1)(2, n+2)(3, n+3)(4, n+4) \in M_2 \leq M_1,$$

we get $\overline{g} = \overline{h} \in \overline{G}_{123} \cap \overline{G}_{234}$. The non-trivial elements of M_1 either fix an element of Λ_1 or maps it to an element of Λ_2. Hence, as $\{1,3\}$ is a G_{23}-orbit, $\overline{h} \notin \overline{G}_{23}$. Thus $\overline{G}_{123} \cap \overline{G}_{234} > \overline{G}_{23}$ when $N = M_1$ or M_2. Now suppose $N = \langle \omega_0 \rangle$. This time we take

$$g = \prod_{i=1}^{5}(i, n+i) \prod_{j=1}^{\lfloor \frac{n-5}{2} \rfloor} (4+2j, n+5+2j)(5+2j, n+4+2j) \text{ and}$$

$$h = \prod_{j=1}^{\lfloor \frac{n-5}{2} \rfloor} (4+2j, 5+2j)(n+4+2j, n+5+2j).$$

Then $g \in G_{234}$, $h \in G_{123}$ and $gh^{-1} = \omega_0$. Therefore $\overline{g} = \overline{h} \in \overline{G}_{123} \cap \overline{G}_{234}$. It is straightforward to also see that $\overline{g} \notin \overline{G}_{23}$, and consequently Proposition 4.2.12 is proven. □

Combining Propositions 4.2.10, 4.2.11 and 4.2.12 completes the proof of Theorem 4.0.1.

4.3 Rank $n-4$ unravelled C-strings

This final section is devoted to the proof of Theorem 4.0.2. Thus we assume $n \geq 8$ and we set $m = n - 4$.
Just as in the proof of Theorem 4.0.1 we construct $\{t_1, t_2, \ldots, t_m\}$ as a subset of $\mathrm{Sym}(2n)$ and then show that it is a C-string for B_n. Finally, when n is even, we prove that it is an unravelled C-string. So again, let $\Omega = \{1, \ldots, 2n\}$ and define the t_i as follows.

Definition 4.3.1.

$$t_1 = (2,3)(n+2, n+3)(4,5)(n+4, n+5) \prod_{i=6}^{n}(i, n+i)$$

$$t_2 = (1,2)(n+1, n+2)(3,4)(n+3, n+4)(5,6)(n+5, n+6) \prod_{i=7}^{n}(i, n+i)$$

$$t_3 = (2,3)(n+2, n+3)(6,7)(n+6, n+7)$$

and for $k = 4, \ldots, m$,

$$t_k = (k+3, k+4)(n+k+3, n+k+4).$$

Set $I = \{1, 2, \ldots, m\}$.

Lemma 4.3.2. $t_3t_4 = (2, 3)(n + 2, n + 3)(6, 8, 7)(n + 6, n + 8, n + 7)$.

Next we show that

Lemma 4.3.3. $(t_1t_2 \ldots t_mt_3t_4)^n = \prod_{i=1}^{n}(i, n + i)$.

Proof.

We calculate that

$$t_4t_5 \ldots t_m = (7, n, n-1, n-2, \ldots, 9, 8)(n + 7, 2n, 2n - 1, \ldots, n + 8)$$

and

$$t_1t_2t_3 = (1, 3)(n + 1, n + 3)(2, 4, 7, 6, n + 5, n + 2, n + 4, n + 7, n + 6, 5).$$

Hence

$$\begin{aligned} t_1t_2 \ldots t_m =& (1, 3)(n + 1, n + 3)(2, 4, n, n - 1, \ldots, 9, 8, 7, 6, n + 5, \\ & n + 2, n + 4, 2n, 2n - 1, \ldots, n + 8, n + 7, n + 6, 5). \end{aligned}$$

Using Lemma 4.3.2 we now get

$$\begin{aligned} t_1t_2 \ldots t_mt_3t_4 =& (1, 2, 4, n, n - 1, \ldots, 9, 7, 8, 6, n + 5, \\ & n + 3, n + 1, n + 2, n + 4, 2n, \ldots, n + 9, n + 7, n + 8, n + 6, 5, 3), \end{aligned}$$

which yields Lemma 4.3.3.

□

Lemma 4.3.4. $(t_1t_2 \ldots t_{m-1}t_3t_4)^{n-1} = \prod_{i=1}^{n-1}(i, n + i)$.

Proof.

First we have

$$t_1t_2\dots t_{m-1} = (1,3)(n+1,n+3)(2,4,n-1,n-2,\dots,6,\\ n+5,n+2,n+4,2n-1,\dots,n+7,n+6,5),$$

then, using Lemma 4.3.2,

$$t_1t_2\dots t_{m-1}t_3t_4 = (1,2,4,n-1,\dots,9,7,8,6,n+5,n+3,\\ n+1,n+2,n+4,2n-1,\dots,n+9,n+7,n+8,n+6,5,3).$$

This gives the desired expression for $(t_1t_2\dots t_{m-1}t_3t_4)^{n-1}$.

□

Combining Lemmas 4.3.3 and 4.3.4, we observe that

Lemma 4.3.5. $(t_1t_2\dots t_{m-1}t_3t_4)^{n-1}(t_1t_2\dots t_mt_3t_4)^n = (n,2n)$.

Lemma 4.3.6. *For $i,j \in \{1,2,\dots,m\}$, the order of*

$$t_it_j \text{ is } \begin{cases} 1 & \text{if } i=j \\ 2 & \text{if } |i-j| \geq 2 \\ 12 & \text{if } i=1, j=2 \text{ or } i=2, j=3 \\ 6 & \text{if } i=3, j=4 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. It is evident that each t_i is an involution as they are defined as the products of pairwise disjoint transpositions. Since

$$t_1t_2 = (1,2,4,6,n+5,n+3,n+1,n+2,n+4,n+6,5,3),$$

t_1t_2 has order 12. Similarly we have

$$t_2t_3 = (1,3,4,2)(n+1,n+3,n+4,n+2)(5,7,n+6,n+5,n+7,6)\prod_{i=8}^{n}(i,n+i),$$

and so t_2t_3 also has the order 12. From Lemma 4.3.2 we see that order of t_3t_4 is 6. If, $|i-j|=1$ and $4 \leq i < j \leq m$, then

$$t_it_j = (3+i, 5+i, 4+i)(n+3+i, n+5+i, n+4+i)$$

has order 3. That t_i and t_j commute when $|i-j| \geq 2$ is readily checked, so verifying Lemma 4.3.6. □

Put $G = \langle t_1, t_2, \ldots, t_m \rangle$.

Proposition 4.3.7. $G \cong \mathrm{B}_n$.

Proof. We again employ Lemma 4.1.1 to identify G. So set $\beta_0 = (1, n+1)$, $\beta_i = (i, i+1)(n+i, n+i+1)$, for $1 \leq i < n$, and $L = \langle \beta_0, \beta_1, \ldots, \beta_{n-1} \rangle \leq \mathrm{Sym}(2n)$. Also set $\eta_i = (i, n+i)$ for $1 \leq i \leq n$. Note that $\eta_1 = \beta_0$ and $\eta_i = \beta_0^{\prod_{j=1}^{i-1} \beta_j}$ for $i = 2, \ldots, n$. Therefore $\eta_i \in L$ for $i = 1, \ldots, n$. Because

$$\begin{aligned} t_1 &= \beta_2\beta_4 \prod_{i=6}^{n} \eta_i, \\ t_2 &= \beta_1\beta_3\beta_5 \prod_{i=7}^{n} \eta_i, \\ t_3 &= \beta_2\beta_6 \quad \text{and, for } 4 \leq i \leq m, \\ t_i &= \beta_{i+3} \end{aligned}$$

we conclude that $G \leq L$.

From Lemma 4.3.5, $\eta_n = (n, 2n) \in G$. Now let $g = t_mt_{m-1} \ldots t_4t_3t_2t_1t_2t_1t_2 \in G$. Then we see that $\eta_n{}^g = \eta_1 = \beta_0$, whence $\beta_0 \in G$. Since $\beta_i = t_{i-3}$ for $i = 7, \ldots, n-1$, it remains to show that $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ and β_6 are in G. Employing Lemma 4.3.2 again we have

$$\begin{aligned} \beta_2 &= (t_3t_4)^3, \\ \beta_6 &= t_3(t_3t_4)^3, \\ \beta_1 &= \beta_6{}^{t_2t_3t_1t_2t_1t_2t_3} \\ \beta_3 &= \beta_6{}^{t_2t_3t_1t_2t_1t_3} \\ \beta_4 &= \beta_6{}^{t_2t_1t_2t_1t_3t_2} \text{ and} \\ \beta_5 &= \beta_6{}^{t_2t_3\eta_6}. \end{aligned}$$

Since $\eta_6 = {\eta_n}^h$ where $h = t_m t_{m-1} \dots t_4 t_3$, we have now shown that $\beta_i \in G$ for $i = 0, \dots, n-1$. Thus $G = L$, and Proposition 4.3.7 is proven. □

We now turn our attention to showing that $\{t_1, \dots, t_m\}$ is a C-string.
For $t_1, \dots, t_m$ if we wish to highlight that they are permutations in $\mathrm{Sym}(2n)$ we shall write $t_1^{(n)}, \dots, {t_m}^{(n)}$. Set $G_{1234}^{(n)} = \langle t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)} \rangle$, $G_{123}^{(n)} = \langle t_1^{(n)}, t_2^{(n)}, t_3^{(n)} \rangle$, $G_{234}^{(n)} = \langle t_2^{(n)}, t_3^{(n)}, t_4^{(n)} \rangle$ and $G_{23}^{(n)} = \langle t_2^{(n)}, t_3^{(n)} \rangle$.

Proposition 4.3.8. *For $n \geq 8$, $\{t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}\}$ is a C-string for $G_{1234}^{(n)}$.*

Proof. First we may verify Proposition 4.3.8 for $n = 8$ using Magma. Then we may define $\mu \in \mathrm{Sym}(2n)$ by

$$\begin{aligned}\mu &: \omega \to \omega - n + 8 \text{ for } \omega \in \Lambda = \{n+1, n+2, n+3, n+4, n+5, n+6, n+7, n+8\}\\ &: \omega \to \omega \text{ for } \omega \in \Omega \setminus \Lambda.\end{aligned}$$

For $i = 1, 2, 3, 4$ define

$$\phi(t_i^{(n)}) = \widehat{(t_i^{(n)})^{\mu}}$$

where $\widehat{\quad}$ denotes the induced action upon the set $\Phi = \{1, \dots, 16\}$. When we write equalities in this context, it is as a permutation of Φ. Observe that ϕ extends to a homomorphism from $G_{1234}^{(n)}$ to $\mathrm{Sym}(\Phi)$ with $\phi\left(G_{1234}^{(n)}\right) = G_{1234}^{(8)}$.
Because

$$\phi(G_{23}^{(n)}) \leq \phi(G_{123}^{(n)} \cap G_{234}^{(n)}) \leq \phi(G_{123}^{(n)}) \cap \phi(G_{234}^{(n)}) = G_{123}^{(n)} \cap G_{234}^{(n)}$$

and Proposition 4.3.8 holds for $n = 8$, we have

$$\phi(G_{123}^{(n)} \cap G_{234}^{(n)}) = G_{23}^{(8)}.$$

We now investigate the structure of $H = G_{234}^{(n)}$. Set $s_2 = t_2^{(n)}, s_3 = t_3^{(n)}, s_4 = t_4^{(n)}$ and $R = \langle (s_2 s_3)^3, (s_3 s_4)^3 \rangle$. Now $(s_3 s_4)^3$ inverts $(s_2 s_3)^3$ which has order 4. Therefore $R \cong \mathrm{Dih}(8)$. Calculation shows that s_2, s_3 and s_4 normalize R and hence $R \trianglelefteq H$. Set $C = C_H(R)$. Further calculation shows that $s_2, s_3, s_2 s_3 \notin C$ but $s_4, (s_2 s_3)^2 \in C$. Therefore, as $H = \langle s_2, s_3, s_4 \rangle$, $H = \langle s_2, s_3 \rangle C$ with $H/C \cong 2^2$. Also we have $(s_2 s_3)^6 \in Z(H)$ with $s_4 (s_2 s_3)^4$ of order 4. Thus, as $(s_2 s_3)^4$ has order 3 and s_4 has order 2, $\langle (s_2 s_3)^4, s_4 \rangle \cong \mathrm{Sym}(4)$. Thus $S = \langle (s_2 s_3)^2, s_4 \rangle \cong 2 \times \mathrm{Sym}(4)$ with $S \leq C$. Since s_2 and s_3 normalize S, we infer $C = S$. In particular, we have shown $G_{234}^{(n)} = H$ has order $2^6.3$ for all $n \geq 8$.

Consequently ϕ restricted to $G_{234}^{(n)} \to G_{234}^{(8)}$ is an isomorphism. So calculation in $G_{234}^{(n)}$ may be performed in $G_{234}^{(8)}$ and, using ϕ, we may keep track of the action on Ω. Now $G_{234}^{(n)}$ has orbit $\{5, 6, 7, 8, n+5, n+6, n+7, n+8\}$ on Ω and $G_{123}^{(n)}$ has $\{8, n+8\}$ as an orbit. Thus

$$G_{23}^{(n)} \leq G_{123}^{(n)} \cap G_{234}^{(n)} \leq Stab(5, 6, 7, n+5, n+6, n+7\}) = T^{(n)}.$$

Calculation shows that $T^{(n)} = \langle G_{123}^{(n)}, (6,7)(n+6, n+7) \rangle$ with $[T^{(n)} : G_{23}^{(n)}] = 2$. If $G_{23}^{(n)} < G_{123}^{(n)} \cap G_{234}^{(n)}$, then $G_{123}^{(n)} \cap G_{234}^{(n)} = T^{(n)}$ must contain a normal subgroup of order 2 intersecting $G_{23}^{(n)}$ trivially (the kernel of ϕ, restricted to $G_{123}^{(n)} \cap G_{234}^{(n)}$), but it does not. Thus $G_{23}^{(n)} = G_{123}^{(n)} \cap G_{234}^{(n)}$. That $\{t_2^{(n)}, t_3^{(n)}, t_4^{(n)}\}$ is a C-string for $G_{234}^{(n)}$ follows from $G_{234}^{(n)}$ and $G_{234}^{(8)}$ being isomorphic and the fact that ϕ maps generator to generator. Observe that the G_{23}−orbits of Ω are
$\{1, 2, 3, 4\}, \{5, 6, 7, n+5, n+6, n+7\}, \{n+1, n+2, n+3, n+4\}, \{i, n+i\} (i = 8, \dots, n)$. If $G_{12} \cap G_{23} > G_2$ then one of

$$(t_1t_2)^4 = (1, n+5, n+4)(2, n+3, n+6)(4, n+1, 5)(6, n+2, 3) \text{ and}$$
$$(t_2t_3)^6 = (1, n+1)(2, n+2)(3, n+3)(4, n+4)(5, n+5)(6, n+6)$$

would be in G_{23}. But then 1 and $n+1$ would be in the same G_{23}−orbit, a contradiction. Therefore $G_{12} \cap G_{23} = G_2$. Appealing to Theorem 1.5.1 this now proves Proposition 4.3.8.

□

Lemma 4.3.9. *Let* $4 \leq k \leq m$ *and set* $J_k = I_k \setminus \{1, 2\}$*. Then*

(*i*) $G_{J_k} \cong \mathrm{Sym}(k-2)$ *and*

(*ii*) $\{t_4, \dots, t_k\}$ *is a C-string for* G_{J_k}*.*

Proof. Since $G \cong \mathrm{B}_n$ by Proposition 4.3.7, $\langle t_4, \dots, t_k \rangle = \langle \beta_1, \dots, \beta_{k+3} \rangle$ is a standard parabolic subgroup of G. Hence Lemma 4.3.9 (*i*) and (*ii*) follow. □

Lemma 4.3.10. *Let* $4 \leq k \leq m$ *and set* $J_k = I_k \setminus \{1, 2\}$*. Then*

(*i*) $G_{J_k} \cong \mathbb{Z}_2 \times \mathrm{Sym}(k-1)$; *and*

(*ii*) $\{t_3, \dots, t_k\}$ *is a C-string for* G_{J_k}*.*

Proof. Recall that $t_3 = \beta_2\beta_6$, and that $t_i = \beta_{i+3}$ for $4 \le i \le k$. Since $(\beta_2\beta_6\beta_7)^3 = \beta_2$, we have

$$\begin{aligned}\langle t_3, \ldots, t_k\rangle =&\langle \beta_2\beta_6, \beta_7 \ldots, \beta_{k+3}\rangle\\ =&\langle \beta_2, \beta_6, \beta_7 \ldots, \beta_{k+3}\rangle\\ \cong&\mathbb{Z}_2 \times \mathrm{Sym}(k-1),\end{aligned}$$

so giving part (i). Further, as $\langle t_3, \ldots, t_k\rangle$ is a Coxeter group,

$$\begin{aligned}\langle t_3, t_4, \ldots, t_{k-1}\rangle \cap \langle t_4, \ldots t_k\rangle &= \langle \beta_2, \beta_6, \beta_7, \ldots, \beta_{k+2}\rangle \cap \langle \beta_7, \ldots \beta_{k+3}\rangle\\ &= \langle \beta_7, \ldots, \beta_{k+2}\rangle\\ &= \langle t_4, \ldots, t_{k-1}\rangle.\end{aligned}$$

To prove (ii) we may argue by induction on k, $k = 4$ being covered by Proposition 4.3.8. Thus $\{t_3, t_4, \ldots, t_{k-1}\}$ is a C-string for $G_{\{3,\ldots,k-1\}}$ and, as $\{t_4, \ldots, t_k\}$ is a C-string for $G_{\{4,\ldots,k\}}$, Theorem 1.5.1 yields (ii). □

Set $t_0 = t_4{}^{(t_3t_4t_2t_3)}t_2$.

Lemma 4.3.11. *For $4 \le k \le m$, set $J_k = I_k \setminus \{1\}$. Then*

$$\begin{aligned}G_{J_k} =&\langle t_0, \beta_2\rangle \times \langle \beta_5, \beta_6, \ldots, \beta_{k+3}\rangle\\ \cong&\mathrm{Dih}(8) \times \mathrm{Sym}(k).\end{aligned}$$

Proof. Clearly $t_0 \in G_J$ and calculation reveals that

$$t_0 = (1,2)(n+1, n+2)(3,4)(n+3, n+4)\prod_{i=7}^{n}(i, n+i).$$

Hence $t_0\beta_5 = t_2$.
Set $H = \langle t_0, \beta_2, \beta_5, \beta_6, \ldots, \beta_{k+3}\rangle$. Observing that t_0 and β_2 commute with each of $\beta_5, \beta_6, \ldots, \beta_{k+3}$, we have

$$H = \langle t_0, \beta_2\rangle \times \langle \beta_5, \beta_6, \ldots, \beta_{k+3}\rangle.$$

We show that $G_{J_k} = H$. Recalling that $\beta_{i+3} = t_i$, $4 \le i \le m$ and $t_3 = \beta_2\beta_6$, $t_2 = t_0\beta_5$ implies $G_{J_k} \le H$. Since $t_0 = t_2\beta_5$, $\beta_5 = t_0t_2$, $\beta_6 = t_3(t_3t_4)^3$ and $\beta_2 = t_3\beta_6$, we also have $H \le G_{J_k}$. Because $t_0\beta_2$ has order 4 and $\langle \beta_2, \beta_6, \ldots, \beta_{k+3}\rangle$ is a standard parabolic subgroup of B_n, we deduce that $G_{J_k} \cong \text{Dih}(8) \times \text{Sym}(k)$. $\square$

Lemma 4.3.12. *For $J_k = I_k \setminus \{1\}$ where $4 \leq k \leq m$, $\{t_2, \ldots, t_k\}$ is a C-string for G_{J_k}.*

Proof. We argue by induction on k. By our induction hypothesis we have that $\{t_2, t_3, \ldots, t_{k-1}\}$ is a C-string for $G_{J_k \setminus \{k\}}$. From Lemma 4.3.10 $\{t_3, \ldots, t_k\}$ is a C-string for $G_{J_k \setminus \{2\}}$. Also, from Lemma 4.3.10,

$$G_{J_k \setminus \{2,k\}} \cong \mathbb{Z}_2 \times \text{Sym}(n-6) \quad \text{and}$$
$$G_{J_k \setminus \{k\}} \cong \mathbb{Z}_2 \times \text{Sym}(n-5).$$

Hence $G_{J_k \setminus \{2,k\}}$ is a maximal subgroup of $G_{J_k \setminus \{k\}}$. So, if $G_{J_k \setminus \{k\}} \cap G_{J_k \setminus \{2\}} > G_{J_k \setminus \{2,k\}}$, then $G_{J_k \setminus \{k\}} \leq G_{J_k \setminus \{2\}}$ which means $t_2 \in G_{J_k \setminus \{2\}}$. But $G_{J_k \setminus \{2\}}$ fixes 1 whereas t_2 does not, a contradiction. Therefore $G_{J_k \setminus \{k\}} \cap G_{J_k \setminus \{2\}} = G_{J_k \setminus \{2,k\}}$. Thus, using Theorem 1.5.1, we get Lemma 4.3.12. $\square$

Proposition 4.3.13. *For $4 \leq k \leq m$, $\{t_1, t_2, \ldots, t_k\}$ is a C-string for G_{I_k}.*

Proof. We again argue by induction on k, with Proposition 4.3.8 starting the induction. So $k > 4$ and $\{t_1, t_2, \ldots, t_{k-1}\}$ is a C-string for $G_{I_{k-1}}$. By Lemma 4.3.12 we have that $\{t_2, \ldots, t_{k-1}\}$ is a C-string for $G_{I_{k-1} \setminus \{1\}}$. Then, using Lemma 4.3.11 we have that

$$G_{I_k \setminus \{1\}} \cong \text{Dih}(8) \times \text{Sym}(k) \quad \text{and}$$
$$G_{I_{k-1} \setminus \{1\}} \cong \text{Dih}(8) \times \text{Sym}(k-1).$$

Hence $G_{I_{k-1} \setminus \{1\}}$ is a maximal subgroup of $G_{I_k \setminus \{1\}}$. So, if $G_{I_k \setminus \{1\}} \cap G_{I_{k-1}} > G_{I_{k-1} \setminus \{1\}}$, then $G_{I_k \setminus \{1\}} \leq G_{I_{k-1}}$. But then $t_k \in G_{I_{k-1}}$, which is not the case as t_k moves $k+3$ to $k+4$ which are in different $G_{I_{k-1}}$-orbits. Therefore $G_{I_k \setminus \{1\}} \cap G_{I_{k-1}} = G_{I_{k-1} \setminus \{1\}}$. Thus, using Theorem 1.5.1, we get Proposition 4.3.13. $\square$

Proposition 4.3.14. *$\{t_1, t_2, \ldots, t_k\}$ is a C-string for G.*

Proof. Taking $m = k$ in Proposition 4.3.13 gives Proposition 4.3.14. $\square$

Proposition 4.3.15. *If n is even, then $\{t_1, \ldots, t_m\}$ is an unravelled C-string.*

Proof. Set $w_0 = \prod_{i=1}^{n} (i, n+i)$, $g = (t_3(t_3t_4))^3 (= \beta_6)$ and

$$h = (t_1t_2)^2 t_1 (t_3t_2t_1)^3 t_3^{t_2} (t_1t_3t_2)^4.$$

Then

$$h = \prod_{i=1}^{5}(i, n+i)(6, 7, n+6, n+7)\prod_{i=8}^{n}(i, n+i).$$

Hence $gh = w_0$, and $g \in G_{\{3,4\}}, h \in G_{\{1,2,3\}}$.
Put $M_1 = \langle (i, n+i) | 1 \leq i \leq n \rangle$ and $M_2 = \langle (1, n+1)(i, n+i) | 1 < i \leq n \rangle$. Since n is even, $\langle w_0 \rangle \leq M_2 \leq M_1$. Let $\overline{G} = G/N$ where N is one of $\langle w_0 \rangle, M_2, M_1$. Then $\overline{g}\overline{h^{-1}} \in \overline{G}_{\{1,2,3\}} \cap \overline{G}_{\{3,4\}}$ with $\overline{g} \neq \overline{t_3}$ whence $\overline{G}_{\{1,2,3\}} \cap \overline{G}_{\{3,4\}} > \overline{G_{\{3\}}}$, which proves Proposition 4.3.14. $\square$

Chapter 5

An introduction to Elnitsky's tilings

In his PhD dissertation ([12]), Elnitsky gives an elegant bijection between rhombic tilings of $2n$-gons and commutation classes of reduced words in the Coxeter group of type A_{n-1}. An accessible and streamlined description of this is found in [11]. Moreover, similar systems of tilings were given for type B and D Coxeter groups. In consequent chapters we will describe efforts to generalise this work to further finite irreducible Coxeter groups.

5.1 The type A tiling

We follow Elnitsky's original construction in [11]. We have added some minor changes for the sake of consistency and brevity.
Let (W, S) denote our type A Coxeter system of rank $n-1$ (so $W = \mathrm{Sym}(n)$ and $S = \{s_1, \ldots, s_{n-1}\}$ with each $s_i = (i, i+1)$). Let $w \in W$ be some permutation, then we construct $\mathrm{Y}(w)$, a (possibly degenerate) $2n$-gon. We describe the so-called *regular* construction of $\mathrm{Y}(w)$ before mentioning the degrees of freedom we have at our disposal that preserve the relevant mathematical properties.

(*i*) To construct $Y(w)$, we first declare M to be its lowermost vertex.

(*ii*) Construct the first n edges clockwise from M so that they have form a set of n consecutive, unit length edges of a regular $2n$-gon. We label these edges consecutively from M in clockwise order with the labels $i = 1, \ldots, n$.

(*iii*) Construct and label the first n edges *anti-clockwise* from M consecutively, so that the j^{th} edge from M is again unit length whilst also being parallel to, and labelled as, the edge labelled $(j)w^{-1}$ in (*ii*).

See Example 5.1.1 for reference. Later, in Chapter 6 we will provide an equivalent, more rigorous construction of $Y(w)$.

As Elnitsky remarks, the choice of angles and side lengths for the edges constructed in (*ii*) do not actually matter, so long as the vertices formed produce a convex set. When examining these polygons in this chapter, we will keep to this regular construction. We denote those edges constructed in (*iii*), $\mathrm{B}(w)$, and call this the *border of* w. We denote its i^{th} edge from M (and thus labelled $(i)w^{-1}$) by $B(w)_i$. This naturally gives us a bijection between $\mathrm{Sym}(n)$ and $\mathrm{B}(W) = \{\, \mathrm{B}(w) \mid w \in W \,\}$ and $\mathrm{Y}(W) = \{\, \mathrm{Y}(w) \mid w \in W \,\}$.

Example 5.1.1. *We show an example here for the permutation* $(1,6)(2,4,5)$ *in* $\mathrm{Sym}(6)$.

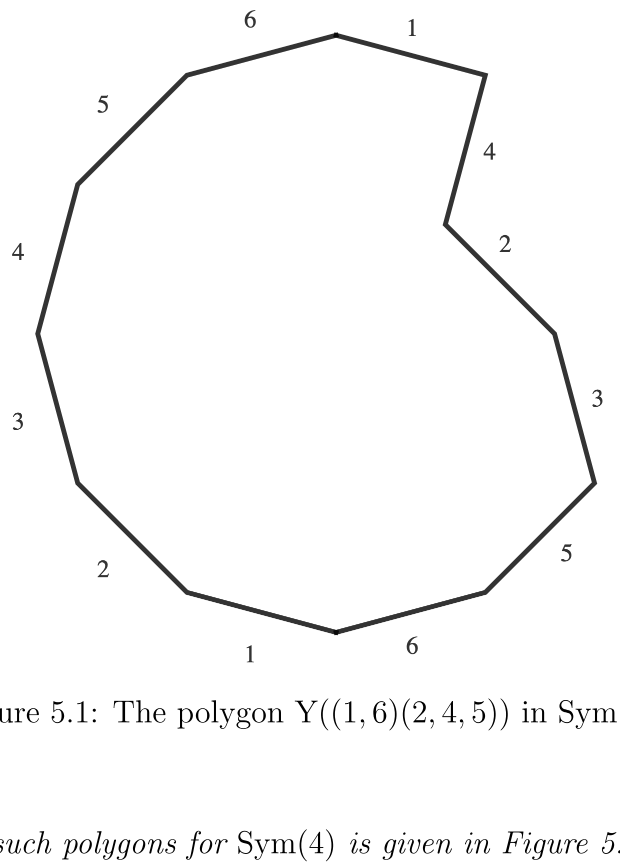


Figure 5.1: The polygon $\mathrm{Y}((1,6)(2,4,5))$ in $\mathrm{Sym}(6)$.

And the set of all such polygons for $\mathrm{Sym}(4)$ *is given in Figure 5.2.*

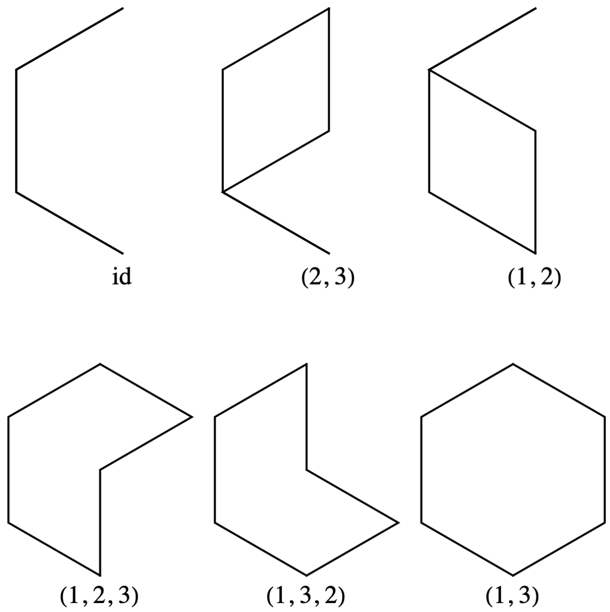


Figure 5.2: $\mathrm{Y}(w)$ for all $w \in \mathrm{Sym}(3)$.

We now consider *tilings* of $\mathrm{Y}(w)$ by rhombi. Implicitly, by a rhombic tiling we mean a covering of $\mathrm{Y}(w)$ by regions of rhombi with unit edge lengths that intersect only on their boundaries. Call the set of all such rhombic tilings of $\mathrm{Y}(w)$ by $T(w)$. We will see that such tilings are associated to (equivalence classes of) reduced words in (W, S).

Example 5.1.2. *Consider the permutation* $(1,4)(2,3)$ *in* $\mathrm{Sym}(4)$. *We can exhaustively find all such tilings:*

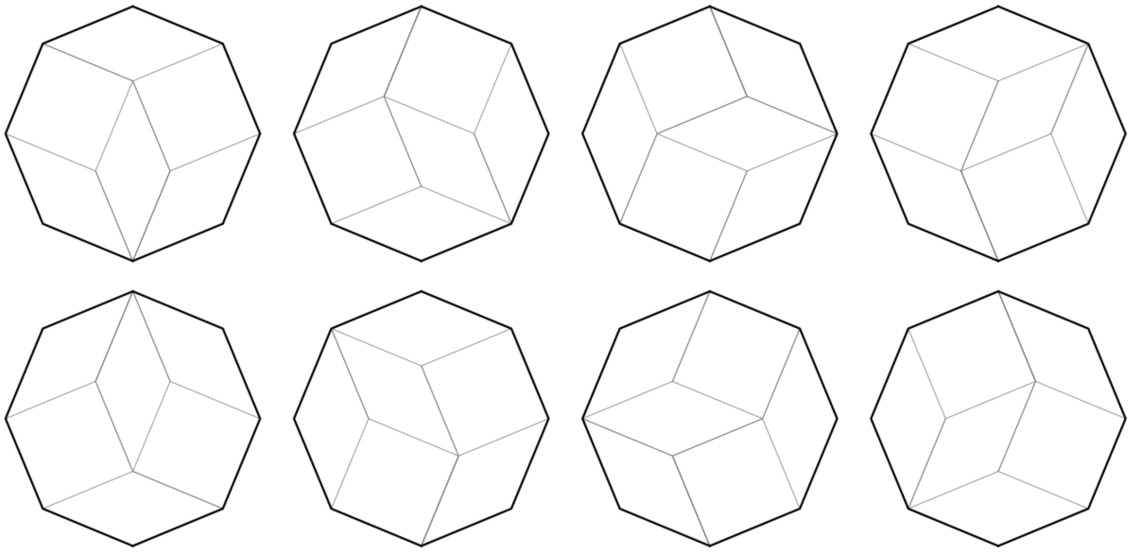


Figure 5.3: All rhombic tilings of Y((1, 4)(2, 3)).

Let J $= \{\{s_i, s_j\} \subseteq S \,|\, |i-j| \geq 2\}$. Elnitsky proves that if two reduced words are connected by braid relations between pairs of generators in J, then the rhombic tilings associated to these words are identical. Let $\mathcal{R}_J(w)$ denotes the set of reduced words of w over our generators S, up to commuting generators. Quite directly, Elnitsky is able to prove the following elegant fact.

Theorem 5.1.3 (Theorem 2.2. of [11])**.** *For all $w \in W$, there exists a bijection between $T(w)$ and $\mathcal{R}_J(w)$.*

We give an outline of Elnitsky's proof of Theorem 5.1.3 to give a flavour of the methods involved. We edit this to be consistent with our notation. Elnitsky's proof is more detailed and contains helpful diagrams and examples.

Proof. Suppose $w \in W$ and $s_i \in S$. Observe that $B(ws_i)$ is identical to B(w) except that their i^{th} and $(i+1)^{th}$ edges (from the lowermost vertex, M) have been transposed. Therefore, the region enclosed by these borders is necessarily a rhombus. Either this rhombus is contained in $Y(w)$ or it is not. It is not contained in $Y(w)$ if and only if the exterior angle of vertex common to $B(w)_i$ and $B(w)_{i+1}$ (when viewed as part of $Y(w)$) have an angle less than π. Due to our construction of $B(w)$, this is equivalent to requiring that $(i)w^{-1} < (i+1)w^{-1}$. But this is exactly the condition for s_i to belong to $I^+(w)$ (see page 67 of [1], for example). Given a reduced word, $w = s_{j_1} \dots s_{j_k}$, we show how one can form a rhombic tiling of $Y(w)$. Starting with $B(id)$, place a rhombus that shares its edges with $B(id)_{j_1}$

and $B(id)_{(j_1+1)}$. Next, place a rhombus that shares its edges with $B(s_{j_1})_{j_2}$ and $B(s_{j_1})_{(j_2+1)}$. Continue in this fashion for each $i = 1, \dots k$. Since $w = s_{j_1} \dots s_{j_k}$ is reduced, $s_{j_i} \in I^+(s_{j_1} \dots s_{j_{i-1}})$ for each $i = 1, \dots k$. Thus, by our previous discussion, none of these rhombi overlap and the final set of rhombi is a tiling of $Y(w)$.
Note that if $|i - j| \geq 2$ then the rhombi associated to s_{j_i} and s_{j_i+1} can never share any edges. Thus, the in which we place these rhombi on a given border produces the same ultimate tiling of $Y(w)$.
Conversely, given a tiling of $Y(w)$, we can extract a reduced word for w like so:

(*i*) Choose some rhombus that shares two edges with $B(id)$.

(*ii*) If these edges are the ${j_1}^{th}$ and $(j_1 + 1)^{th}$ (from the lowermost vertex, M) then let the first generator in our reduced word for w be s_{j_1}.

(*iii*) Choose some rhombus that shares two edges with $B(s_{j_1})$.

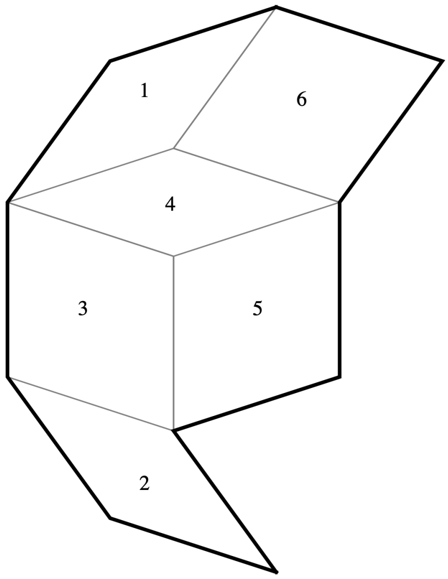
(*iv*) If these edges are the ${j_2}^{th}$ and $(j_2 + 1)^{th}$ (from the lowermost vertex, M), then let the second generator in our reduced word for w be s_{j_2}.

(*v*) Continue in this fashion until all rhombi in the tiling have been chosen.

At the i^{th} step, say, the rhombus we choose shares two edges with, but is not contained in $Y(s_{j_1} \dots s_{j_{i-1}})$. By our previous discussion, this is equivalent to requiring that $s_{j_i} \in I^+(s_{j_1} \dots s_{j_{i-1}})$ for each $i = 1, \dots k$. Thus we indeed have a reduced word. There may be more than one reduced word we may obtain in this manner given a fixed tiling of $Y(w)$. However, given a choice of two rhombi at some step, our procedure ensures that these rhombi share no common edges and thus correspond to commuting generators again.
Therefore we have a bijection between rhombic tilings and reduced words up to commutations. □

We provide a detailed example of how to transform some reduced word into a rhombic tiling.

Example 5.1.4. *We consider the reduced word* $(1, 5, 2) = s_4 s_1 s_2 s_3 s_2 s_4$ *in* Sym(5). *We imagine placing our rhombic tiles, tile-by-tile in sequence, starting with border* $B(id)$*. Since the first generator in our word is* s_4*, the first rhombus we place is that sharing the* 4^{th} *and* 5^{th} *edges of* $B(id)$ *from the lowermost vertex,* M*. The* *resulting border after swapping the edges is $B(s_4)$. Our next generator is s_1. So we place the tile that shares edges of $B(s_4)$ that are 1^{st} and 2^{nd} from the M. The new border formed is $B(s_4s_1)$. Carrying on in this manner produces the tiling displayed in Figure 5.4.*



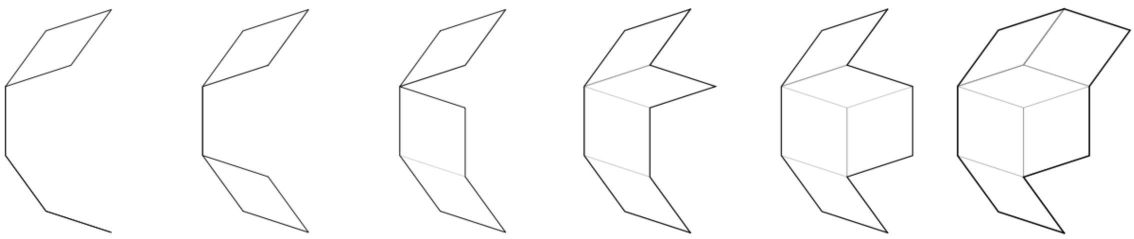


Figure 5.4: The tiling of Y((1, 5, 2)) associated to the reduced word $s_4s_1s_2s_3s_2s_4$ and the sequence in which the tiles were placed.

In Figure 5.4, the tiles labelled 1 and 2 both share no edges with that of label 3. This means we could place the third tile before the first two and produce the same final tiling. This ordering of tiling would correspond to the word $s_1s_2s_4s_3s_2s_4$ and has the analogous sequences in Figure 5.5. This reflects the fact that we have a bijection between the commutation classes of words as opposed to the set of all reduced words: indeed, by applying commutation relations we see $s_1s_2s_4s_3s_2s_4 = s_1s_4s_2s_3s_2s_4 = s_4s_1s_2s_3s_2s_4$.

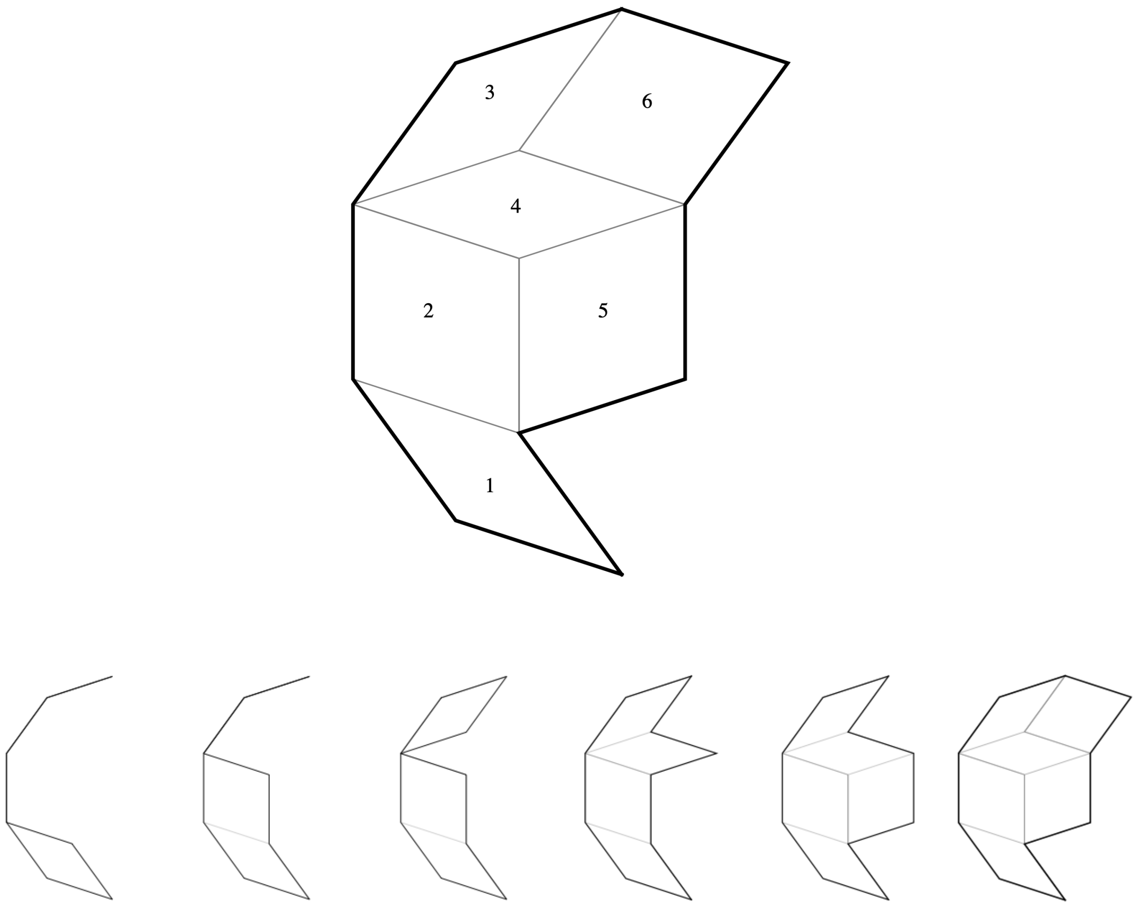


Figure 5.5: The tiling of Y((1, 5, 2)) associated to the reduced word $s_1s_2s_4s_3s_2s_4$ and the sequence in which the tiles were placed. Ultimately, this creates the same tiling as in Figure 5.4.

Through this guise, one is able to translate facts about these rhombic tilings to those of the commutation classes of reduced words in (W, S). This has lead to some fruitful cross-pollination of ideas. See [45] and [46] for recent examples. In [11], Elnitsky goes on to describe such ideas for these tilings giving some wonderful combinatorial insights and some enumerative identities.

It is worth noting that the study of commutation classes of reduced words of Sym(n) has a rich and vibrant literature in its own right. But these classes of words have given rise to many other strange and charming bijections too. Some examples of these bijections are listed in [10].

5.2 The type B tilings

Elnitsky's tilings of type B are exactly his tilings of type A that are *horizontally symmetric* - they can be flipped about the horizontal line that is equidistant from the uppermost and lowermost vertex of the polygon. Here, rhombi which are reflections of one another in this horizontal line are considered belonging to the same 'tile'. Elnitsky gives an analogous bijection between commutation classes of reduced words of type B Coxeter groups and his type B tilings in Theorem 6.1. of [11]. We do not explore this in any detail here but acknowledge its existence; it will inspire the work in Chapter 9 where we will re-examine it in from the perspective of Mühlherr's admissible partitions [36].

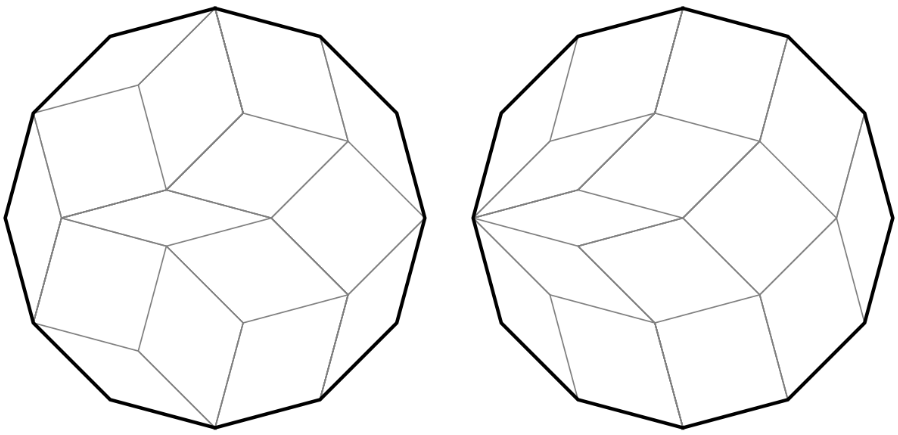


Figure 5.6: Two of Elnitsky's type B tilings associated to reduced words of the longest element in B_3.

5.3 The type D tilings

We now remark on Elnitsky's construction for type D Coxeter groups. As is typical in the literature, it seems type D tilings have received considerably less attention than their type A counterparts.

Let (W, S) be $(\mathrm{D}_n, \{s_1, s_2, \ldots, s_n\})$, a standard embedding of D_n into $\mathrm{Sym}(\{-n, \ldots, -1, 1, \ldots, n\})$ with $s_1 = (1, -2)(2, -1)$ and $s_i = (i-1, i)(-(i-1), -i)$ for each $i \in \{2, \ldots, n\}$. In this setting, Elnitsky describes a similar construction for tilings of a $4n$-gon that again are in a correspondence with classes of reduced words.

Here is the *regular construction* of the type D polygon $\mathrm{Y}(w)$ for all $w \in W$. We assume that $\mathrm{Y}(w)$ is a $4n$-gon with unit length edges.

(i) Let U be the uppermost vertex of our $4n$-gon, L the lowermost vertex and M the vertex that is an equal distance from both.

(ii) Construct the first n edges clockwise from L so that they have form a set of $2n$ consecutive, unit length edges of a regular $4n$-gon. We label these edges consecutively from L in clockwise order with the labels $i = -n, \dots, -1, 1, \dots, n$.

(iii) For each $i = 1, \dots, n$, we now construct and label the first n edges anti-clockwise from L consecutively so that the i^{th} edge from L is parallel to, and labelled as, that edge with label $(-(n+1-i))w^{-1}$ constructed in (ii). Similarly, for $i = n+1, \dots, 2n$, we construct the next n edges anti-clockwise from L consecutively so that the i^{th} edge from L is parallel to, and labelled as, that edge with label $(i-n)w^{-1}$ constructed in (ii).

We are allowed certain modifications in angles and edge length for the edges constructed in step (ii) although we do not explore here.

Example 5.3.1. *We present an example of such a polygon for* D_4.

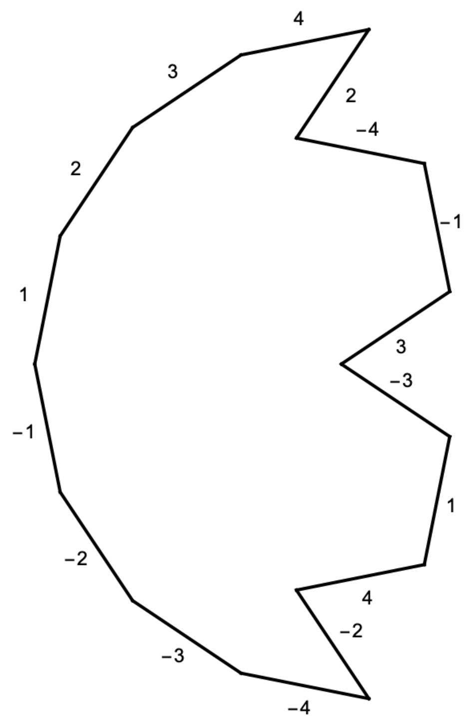


Figure 5.7: $\mathrm{Y}(w)$ for $w = \pm(1, -2, -4, 3)$.

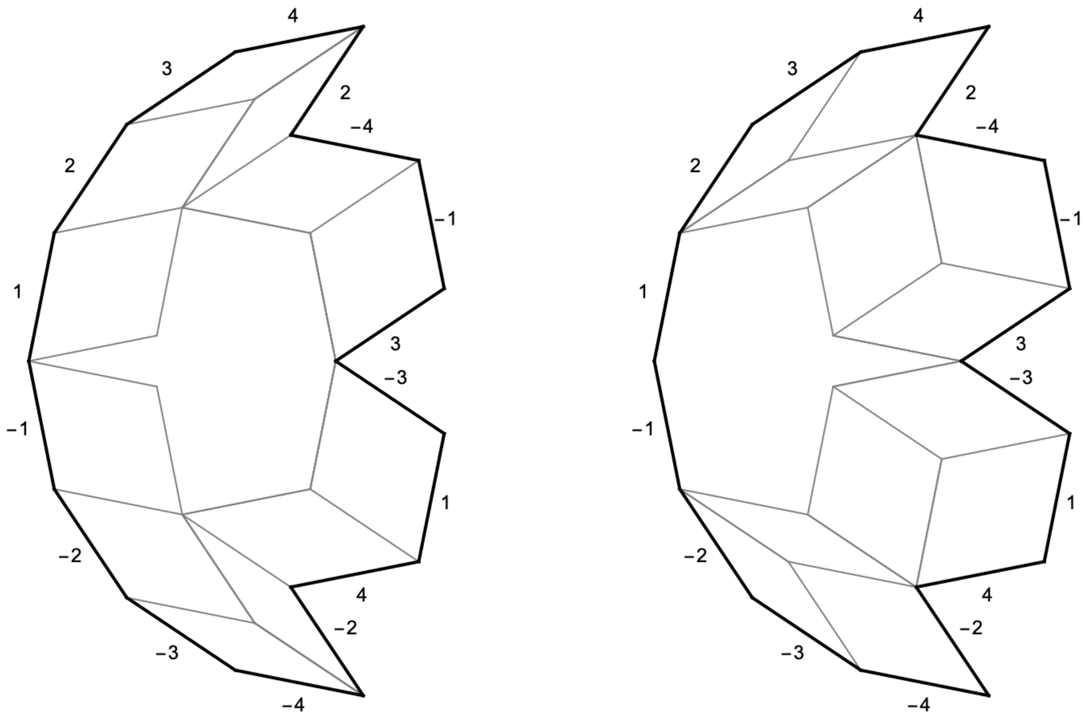
Let us reuse the language of B(w) to denote the edges of Y(w) constructed in step (iii). This time we consider a new set of tiles which are more complex than that of type A. In particular, we now have a set of *megatiles* at our disposal. The megatiles are a subset of octagons with unit edge lengths which can be constructed. Its uppermost vertex, U′, and lowermost vertex, L′, must lie on a vertical line. Its first four edges anti-clockwise from U′ must be symmetric through the horizontal line passing through the middle vertex. Call these edges E. Then to make the remaining edges perform the following on E:

(i) Reflect E through the vertical line passing through U′ and L′.

(ii) Next, transpose the first and second resulting edges below U' and the third and fourth resulting edges below U' respectively.

In order to obtain a tiling bijection, to each generator, we associate the action of placing certain tiles. Given $w \in W$, for each s_i with $i = 2, \dots, n$, if $s_i \in I^+(w)$ then B(ws_i) is obtained by transposing the i^{th} and $(i-1)^{th}$ edges of B(w) above M as well as transposing the i^{th} and $(i-1)^{th}$ edges of B(w) below M too. Effectively, this results in appending two rhombi. If $s_1 \in I^+(w)$, then B(ws_i) is obtained from B(w) by placing the megatile.

Example 5.3.2. *We present two different tilings for the polygon of Example 5.3.1.*



(a) $\delta = s_4 s_3 s_2 s_1 s_4 s_3 s_2$ (b) $\delta = s_3 s_4 s_3 s_1 s_3 s_2 s_3$

Figure 5.8: Two Elnitsky tilings for D_4 with $\delta = \pm(1, -2, -4, 3)$.

We call the set of all such tilings for $\mathrm{Y}(w)$, $T(w)$. The relation set, J, is determined by whether or not the tiles associated to each generators may share an edge. In this case, it is $\{s_i, s_j\}$ for $i, j \in \{1, \ldots, n\}$ with $|i - j| \geq 2$ but excluding $\{s_1, s_3\}$.

Theorem 5.3.3 (Theorem 7.1. of [11])**.** *For all $w \in W$ with W of type* D*, there is a bijection between $T(w)$ and $\mathcal{R}_J(w)$.*

Elnitsky's proof of Theorem 5.3.3 is analogous to that of Theorem 5.1.3.
Elnitsky notices that, given this regular construction, certain tilings may have self-intersections. Fortunately, he also provides a remedy for these intersections: if the angles from the horizontal of each edge in the border created in step (*ii*) of the construction are at least $\pi/3$, then these self-intersections are removed. We present an example in D_5.

Example 5.3.4. *We consider the reduced word $s_3 s_4 s_5 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_1$ in* D_5*. We present one tiling using the usual round construction and one using edges with angles at least $\pi/3$ in Figure 5.9.*

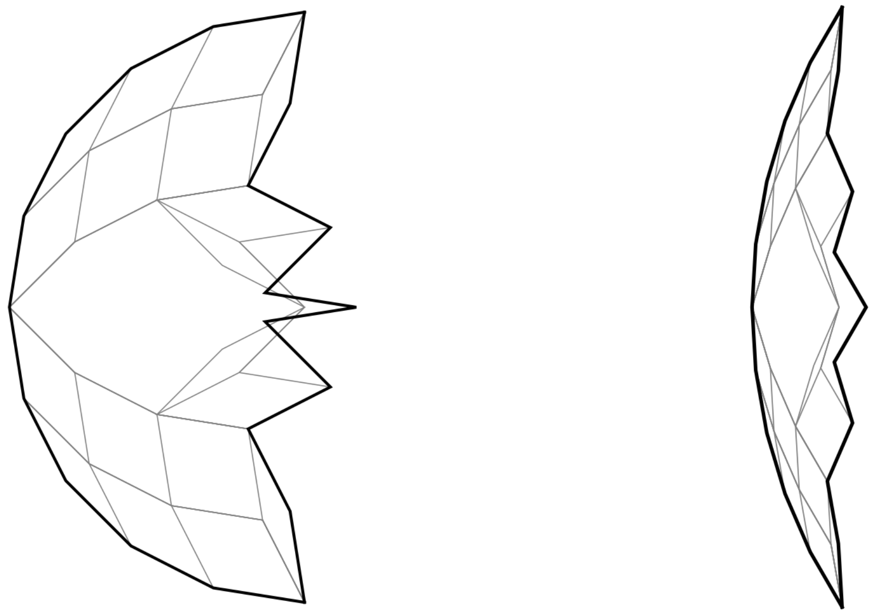


Figure 5.9: Elnitsky's type D tilings for the reduced word $s_3s_4s_5s_2s_3s_4s_5s_1s_2s_3s_1$ using the usual round representation (left) and ensuring that the angles of each edge constructed in the starting border are at least $\pi/3$ from the horizontal (right).

We can see that the left-hand-side contains a self-intersection whereas the right does not.

Elnitsky's proof of this fact is very direct using geometric observations. We will show later that one should be able to relax this bound slightly, by requiring edges of at least $\pi/4$ from the horizontal instead.

Chapter 6

The Bruhat order on Elnitsky's tilings

Theorem 5.1.3 shows that for all $u \in W$, $s \in S$ we have $u <_R us$ if and only if $\mathrm{Y}(us)$ is obtained by appending a rhombus to $\mathrm{Y}(u)$. This gives the weak order a succinct interpretation in the context of Elnitsky's type A tiling. What about the Bruhat order? Is this also reflected visually in Elnitsky's tilings? In this chapter, we will develop some notation for constructing these tilings before answering this question.

Theorem 6.0.1. *For all* $u, v \in W(= \mathrm{Sym}(n))$, $u <_B v$ *implies* $\mathrm{B}^{\pi/4}(u) \prec \mathrm{B}^{\pi/4}(v)$.

We will see later that $\mathrm{B}^{\pi/4}(u)$ denotes some modified border of u. The relation $\prec$ is used to capture the notion of one border not 'crossing' another; this will be defined precisely later. To the best of my knowledge, the observations in this chapter do not appear in the literature. The work in this chapter is a lightly edited version of joint work with Professor Peter Rowley from the paper a preparation [37].

6.1 E-polygons

Let $(W, S) = (\mathrm{Sym}(n), \{s_1, \ldots, s_{n-1}\})$ and fix some $\alpha \in (0, \pi/2)$.

Definition 6.1.1. *Let* $\beta_n^k(\alpha)$ *denote the 2-dimensional, real, unit vector*

$$\beta_n^k(\alpha) = \begin{pmatrix} -\cos\left(\frac{(k-1)(\pi-2\alpha)}{n-1} + \alpha\right) \\ \sin\left(\frac{(k-1)(\pi-2\alpha)}{n-1} + \alpha\right) \end{pmatrix}$$

for $k = 1, \ldots, n$. We will sometimes refer to upper and lower entries of the vectors as the x and y coordinates respectively. We call $\mathcal{B}_n^\alpha = \{\beta_n^1(\alpha), \ldots, \beta_n^n(\alpha)\}$ the set of underlying vectors.

When α is clearly fixed from context, we will write $\beta_n^k(\alpha)$ more simply as β_n^k. Visually, these vectors are distributed evenly on the upper half of the unit circle whose absolute angles from the horizontal axis is at least α, see Figure 6.1. In practice, the angles do not need to be evenly distributed - we just need the angle of β_n^i measured anti-clockwise from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to be greater than that of β_n^j whenever $i > j$.

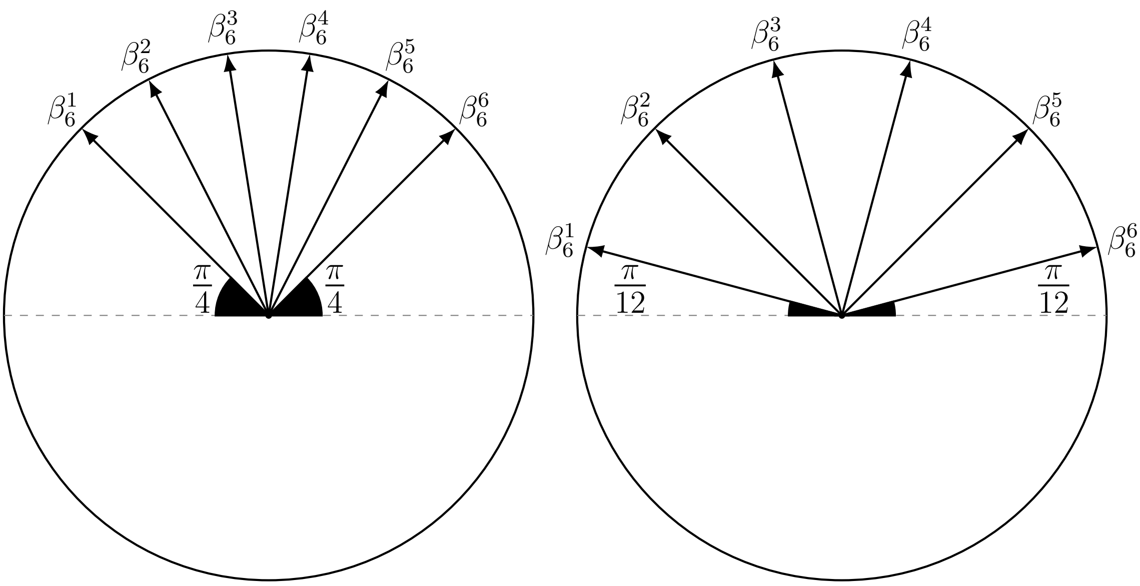


Figure 6.1: The set $\mathcal{B}_6^\alpha$ for $\alpha = \pi/4$ and $\pi/12$ respectively.

Definition 6.1.2. *For all $w \in W$ we define the ordered set*

$$\mathcal{B}_n^\alpha(w) := \{\beta_n^{(1)w^{-1}}, \ldots \beta_n^{(n)w^{-1}}\}$$

to be the w-image of $\mathcal{B}_n^\alpha$.

Definition 6.1.3. *Given $w \in W$, for $i = 1, \ldots, n$, we define $\mathrm{B}_n^\alpha(w)_i$ to be the unit length line segment whose end points are $\Sigma_{j=1}^{i-1}\beta_n^{(j)w^{-1}}$ and $\Sigma_{j=1}^{i}\beta_n^{(j)w^{-1}}$. Here it is understood that $\Sigma_{j=1}^{0}\beta_n^{(j)w^{-1}}$ is the zero-vector. We call*

$$\mathrm{B}_n^\alpha(w) = \bigcup_{i=1}^{n} \mathrm{B}_n^\alpha(w)_i$$

the border of w *and* $\mathrm{B}_n^\alpha(w)_i$ *its edge in* i^{th} *position.*

The borders $\mathrm{B}_n^\alpha(id)$ and $\mathrm{B}_n^\alpha((1,2)(4,5,6))$ in $\mathrm{Sym}(6)$ are displayed in Figure 6.2:

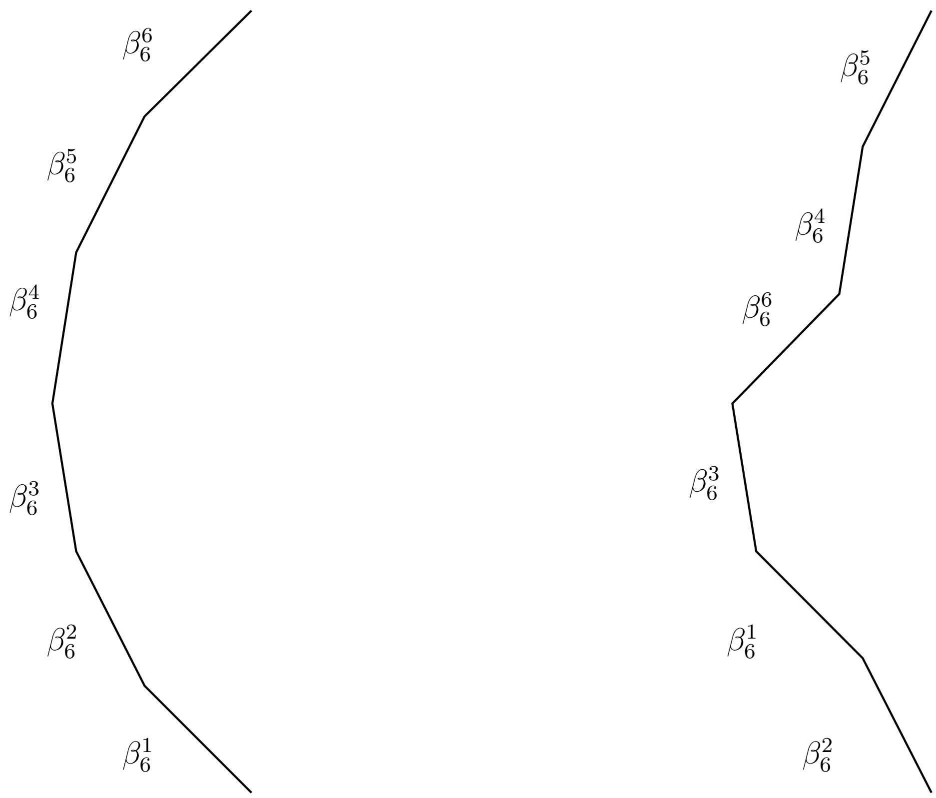


Figure 6.2: The borders $\mathrm{B}_n^\alpha(id)$ and $\mathrm{B}_n^\alpha((1,2)(4,5,6))$ in $\mathrm{Sym}(6)$ with $\alpha = \pi/4$.

Definition 6.1.4. *For all* $u, v \in \mathrm{Sym}(n)$*, we define the* E-polygon *of* (u, v) *(with respect to* n *and* α*), denoted* $\mathrm{P}_n^\alpha(u, v)$*, to be the* $2n$*-gon formed from the union of* $\mathrm{B}_n^\alpha(u)$ *and* $\mathrm{B}_n^\alpha(v)$*:*

$$\mathrm{P}_n^\alpha(u, v) = \mathrm{B}_n^\alpha(u) \bigcup \mathrm{B}_n^\alpha(v).$$

Consequently, $\mathrm{P}_n^\alpha(u, v) = \mathrm{P}_n^\alpha(v, u)$. If $u = id$, we simplify $\mathrm{P}_n^\alpha(u, v)$ to $\mathrm{P}_n^\alpha(v)$.

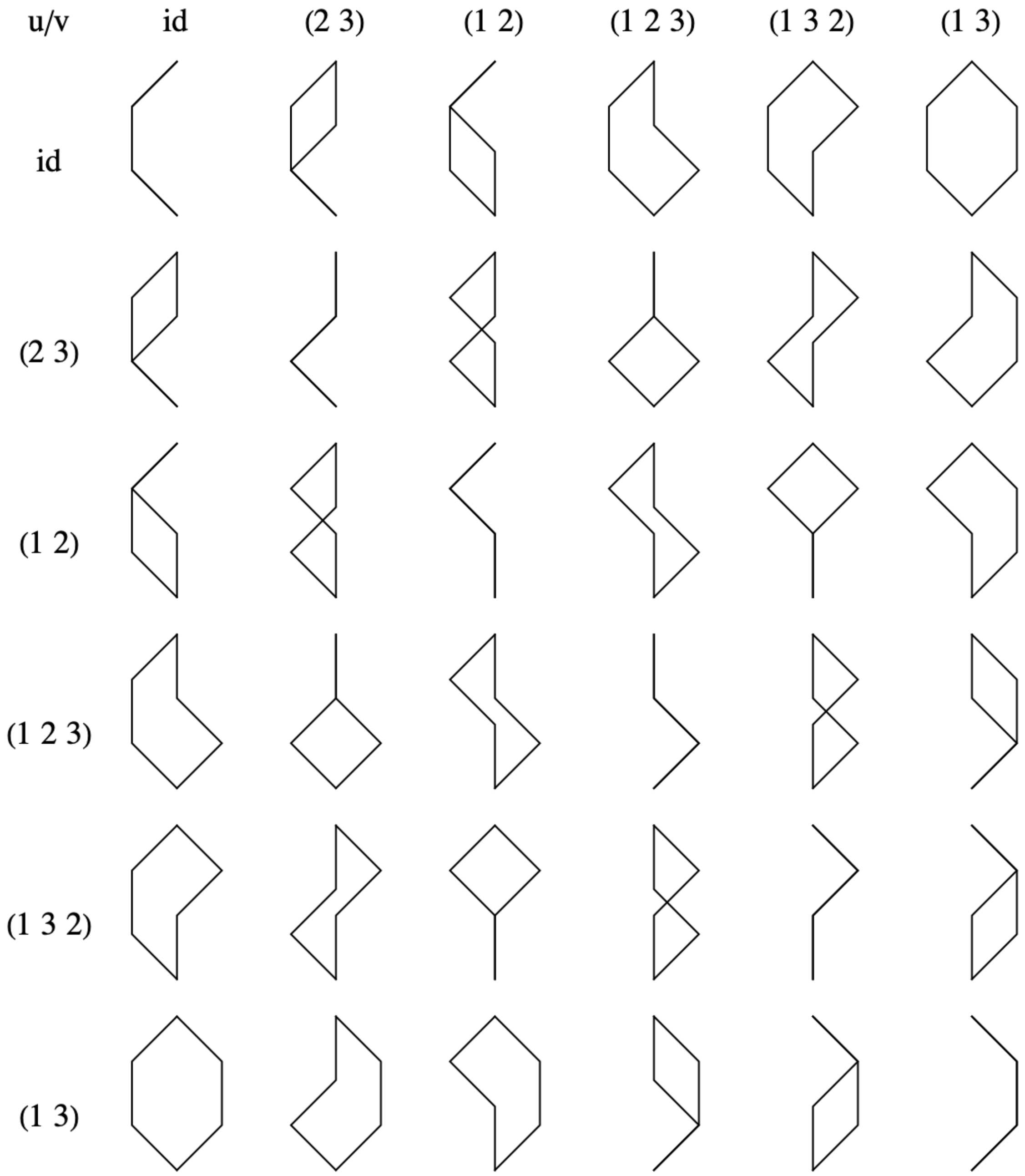


Figure 6.3: $\mathrm{P}_3^\alpha(u, v)$ for all $u, v \in \mathrm{Sym}(3)$ with $\alpha = \pi/4$.

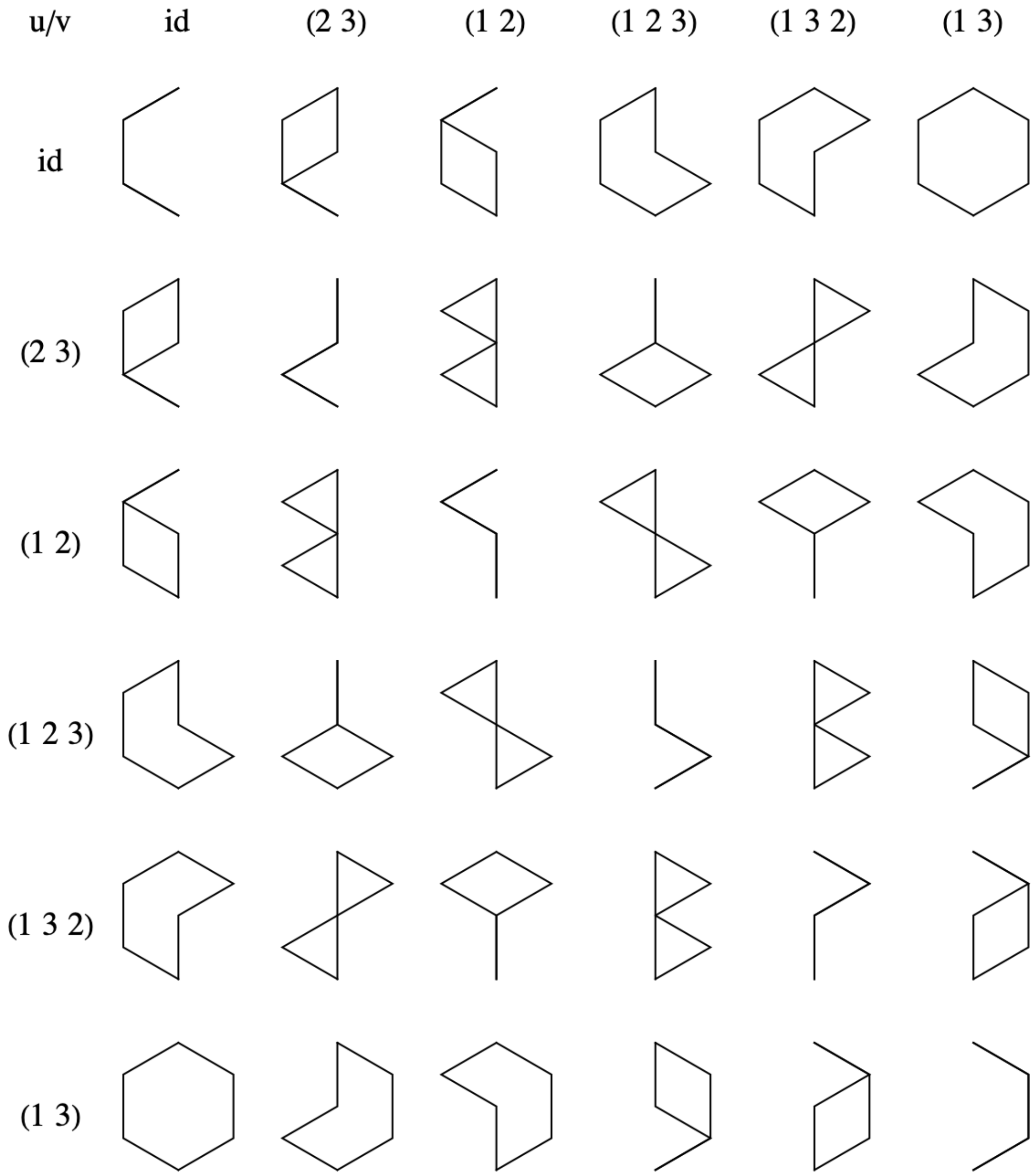


Figure 6.4: $\mathrm{P}_3^\alpha(u,v)$ for all $u,v \in \mathrm{Sym}(3)$ with $\alpha = \pi/6$.

Note that, by construction, for all $u,v \in \mathrm{Sym}(n)$, $\mathrm{B}_n^\alpha(u) = \mathrm{B}_n^\alpha(v)$ if and only if $u = v$.

It would be desirable to be able to define a sensible notion of when a pair of borders produce a tile - when does it make sense to do so? We give a crude but general notion of this. Given fixed n and α, all borders have the same maximal y-coordinate any point may achieve, namely,

$h_n^\alpha := \sum_{k=1}^n \sin\left(\dfrac{(k-1)(\pi-2\alpha)}{n-1}+\alpha\right)$. For each $0 \le y \le h_n^\alpha$, there is a unique point for each border with that y-coordinate. Denote the x-coordinate of this point by $\mathrm{H}(\mathrm{B}_n^\alpha(w), y)$ for $w \in W$ and $0 \le y \le h_n^\alpha$.

Definition 6.1.5. *For all $u, v \in W$ we say $\mathrm{B}_n^\alpha(u)$ precedes $\mathrm{B}_n^\alpha(v)$, denoted $\mathrm{B}_n^\alpha(u) \prec \mathrm{B}_n^\alpha(v)$, if for all $0 \le y \le h_n^\alpha$,*

$$\mathrm{H}(\mathrm{B}_n^\alpha(u), y) \le \mathrm{H}(\mathrm{B}_n^\alpha(v), y).$$

One can define the interior of any $\mathrm{P}_n^\alpha(u, v)$ to be the union of the set of all line segments whose endpoints are $\mathrm{H}(\mathrm{B}_n^\alpha(u), y)$ and $\mathrm{H}(\mathrm{B}_n^\alpha(v), y)$ for all $0 \le y \le h_n^\alpha$. We use the notion of precedence to determine when we assign the word *tile* to some E-polygon for reasons that will become apparent in Chapter 7.

Definition 6.1.6. *For all $u, v \in W$ we call $\mathrm{P}_n^\alpha(u, v)$ a tile if either $\mathrm{B}_n^\alpha(u) \prec \mathrm{B}_n^\alpha(v)$ or $\mathrm{B}_n^\alpha(v) \prec \mathrm{B}_n^\alpha(u)$.*

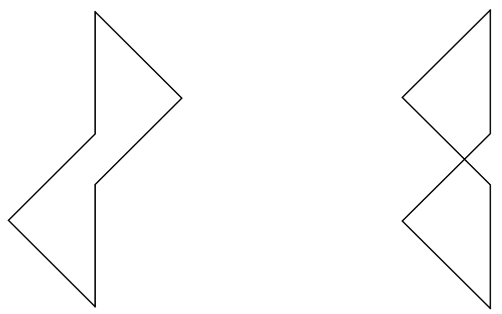


Figure 6.5: For $W = \mathrm{Sym}(3)$, (left) $\mathrm{P}_3^{\pi/4}((2,3),(1,3,2))$ is a tile and (right) $\mathrm{P}_3^{\pi/4}((2,3),(1,2))$ is not.

We also note here that being a tile is dependent on the choice of α as Figure 6.6 shows.

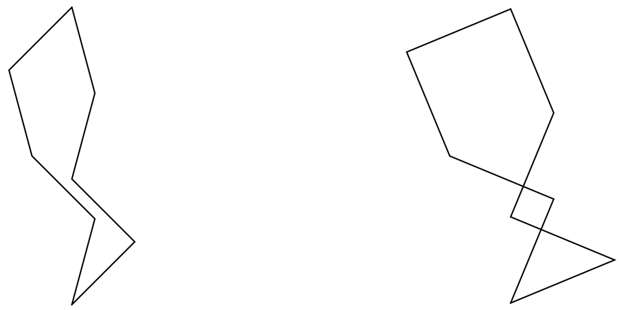


Figure 6.6: For $W = \mathrm{Sym}(4)$, (left) $\mathrm{P}_4^{\pi/4}((1,2,3),(1,2,4))$ is a tile and (right) $\mathrm{P}_4^{\pi/8}((1,2,3),(1,2,4))$ is not.

6.2 Proof of Theorem 6.0.1

The examples presented in the previous section show that self-intersections are dependent on the choice of the minimum angle of edges from the horizontal. Moreover, when this minimum is at least $\pi/4$ for Sym(3), these self-intersections are in bijection with incomparable elements in the strong Bruhat order. However, in general this is not the case as we observe in Figure 6.7 where we have $W = \mathrm{Sym}(4)$ and elements $(1,2,3)$ and $(1,4,2)$ which are not comparable in the Bruhat order. This is the only such pair (up to inverses) amongst the 87 non-comparable elements of Sym(4) exhibiting this behaviour.
For what follows, when $\alpha = \pi/4$ we omit α from our notation.

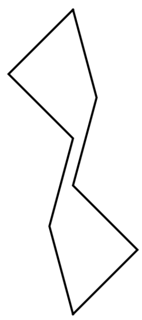


Figure 6.7: For $W = \mathrm{Sym}(4)$, $\mathrm{P}_4((1,3,2),(1,2,4))$.

We recall the following characterisation of the covering relations of the Bruhat order for the symmetric group.

Theorem 6.2.1. *Let $w \in W = \mathrm{Sym}(n)$, and let $t = (a, b) \in T$ with $a < b$. Then $w <_B wt$ if and only if $(a)w^{-1} < (b)w^{-1}$.*

Proof. See Lemma 2.1.4 of [1] (note that we write permutations on the right here). □

We are now ready to prove Theorem 6.0.1 which by way of contrast demonstrates that any two Bruhat comparable elements of $\mathrm{Sym}(n)$ forms an E-polygon which is a tile. We emphasise that the proof is not dependent on consecutive edges of $\mathcal{B}^\alpha$ being spaced apart by angles of equal measure: they all need only to have absolute angle $\pi/4$ from the horizontal.

Proof. It is enough to observe this statement for a covering set of relations of the Bruhat order. That is, for all $w \in \mathrm{Sym}(n)$ and $t \in T$, if $w <_B wt$ and $l(w) < l(wt)$, then $\mathrm{B}_n(w) \prec \mathrm{B}_n(wt)$. But, by Theorem 6.2.1, this is equivalent to the condition that $t = (a, b)$ with $a < b$ and $(a)w^{-1} < (b)w^{-1}$. Let us suppose this is the case. Then the images of $\mathcal{B}_n(w)$ and $\mathcal{B}_n(wt)$ must be identical apart from the transpositions of the vectors, $\beta_n^{(a)w^{-1}}$ and $\beta_n^{(b)w^{-1}}$. Since $a < b$, $\beta_n^{(a)w^{-1}}$ appears in a lower position to $\beta_n^{(b)w^{-1}}$ as line segment in $\mathrm{B}_n^\alpha(w)$. From $(a)w^{-1} < (b)w^{-1}$, $\beta_n^{(a)w^{-1}}$'s x coordinate is more negative than that of $\beta_n^{(b)w^{-1}}$ - intuitively meaning that $\beta_n^{(a)w^{-1}}$ points further left. Hence, $w <_B wt$ implies Figure 6.8 is a sufficiently accurate representation of the situation.

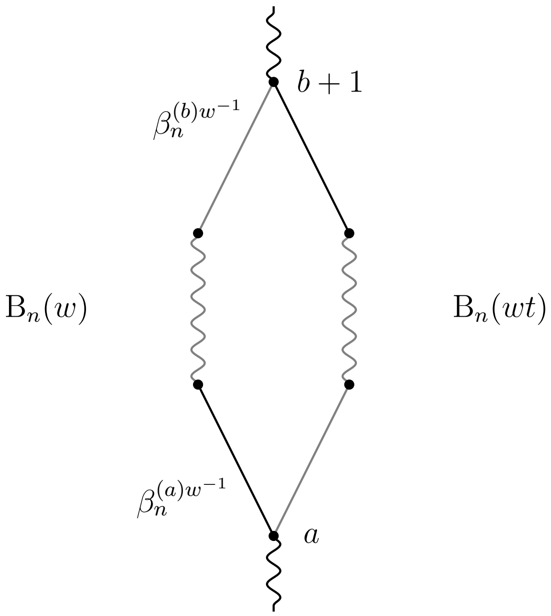


Figure 6.8: The borders $\mathrm{B}_n(w)$ and $\mathrm{B}_n(wt)$.

We define *the critical region* to be the union of the edges $\mathrm{B}_n(w)_i$ and $\mathrm{B}_n(wt)_i$ for $a \leq i \leq b$. We will show that $\mathrm{B}_n(w)_i \cap \mathrm{B}_n(wt)_j = \emptyset$ for all $i, j \in \{a, \ldots, b\}$, excluding the common points of $\mathrm{B}_n(w)_a$ and $\mathrm{B}_n(wt)_a$ (the unique point of least y-coordinate) and $\mathrm{B}_n(w)_b$ and $\mathrm{B}_n(wt)_b$ (the unique point of largest y-coordinate). This is sufficient to prove $\mathrm{B}_n(w) \prec \mathrm{B}_n(wt)$.

Given two distinct vectors $\beta_n^i, \beta_n^j \in \mathcal{B}_n$ with $i < j$, we call the difference between them, $\beta_n^j - \beta_n^i$, their *difference vector*. We extend this notion to $\mathrm{B}_n(w)$ and $\mathrm{B}_n(wt)$ by defining the difference vector of these borders to be the difference vector of $\beta_n^{(a)w^{-1}}$ and $\beta_n^{(b)w^{-1}}$. Note that this difference vector is equal to the difference of $\Sigma_{j=1}^c \mathcal{B}_n^\alpha(w)_j$ and $\Sigma_{j=1}^c \mathcal{B}_n^\alpha(wt)_j$ for all $a \leq c < b$ respectively.

We examine some of the properties our difference vectors may possess. Consider two distinct vectors in this region and let γ and θ denote their angles from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ measured in an anticlockwise rotation, with $\gamma < \theta$ say. By construction, the angles for each β_n^i can possibly take is within the range $(\frac{\pi}{4}, \frac{3\pi}{4})$. The gradient of the chord is the same as the tangent to the circle at the point that intersects the bisector of the chord. The bisector is that vector with angle $\frac{\gamma+\theta}{2}$ and hence the tangent has angle $\frac{\gamma+\theta}{2} - \frac{\pi}{2}$. So the range of gradients a difference vector can take is contained in the open interval $(-\frac{\pi}{4}, \frac{\pi}{4})$.

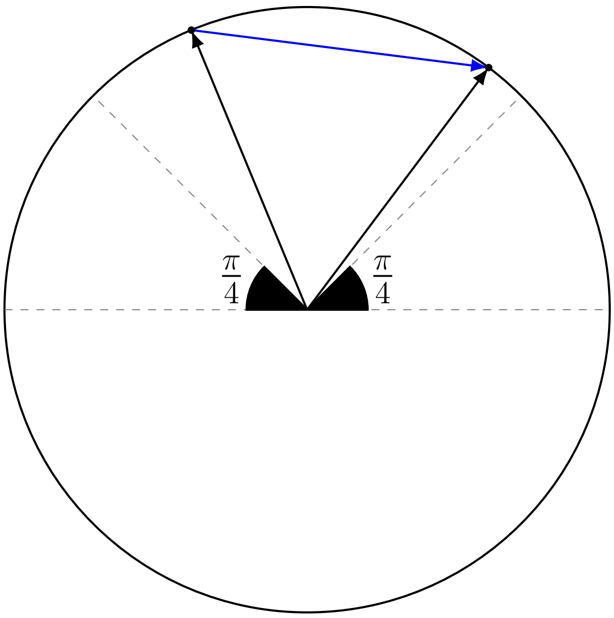


Figure 6.9: The difference vector between two underlying vectors.

Suppose we do have an non-empty intersection of $\mathrm{B}_n(w)_i$ and $\mathrm{B}_n(wt)_j$. Without loss of generality, we may assume $a \leq i \leq b$. We consider the three cases of $|i-j|=0$, $|i-j|=1$ and $|i-j| \geq 2$ separately.

If $|i-j|=0$, then $\mathrm{B}_n(wt)_i = \mathrm{B}_n(w)_i$ and hence they are non-equal, parallel edges and so do not intersect. If $i=a$ or $i=b$ then the vectors only intersect in their common vertices. All other vectors are equal and not in the critical region.

Next, we now show that if $|i-j|=1$ we still have no intersections in the critical region. Suppose, without loss of generality, that $\mathrm{B}_n(w)_i$ intersects $\mathrm{B}_n(wt)_{i+1}$ giving us the scenario described in Figure 6.10:

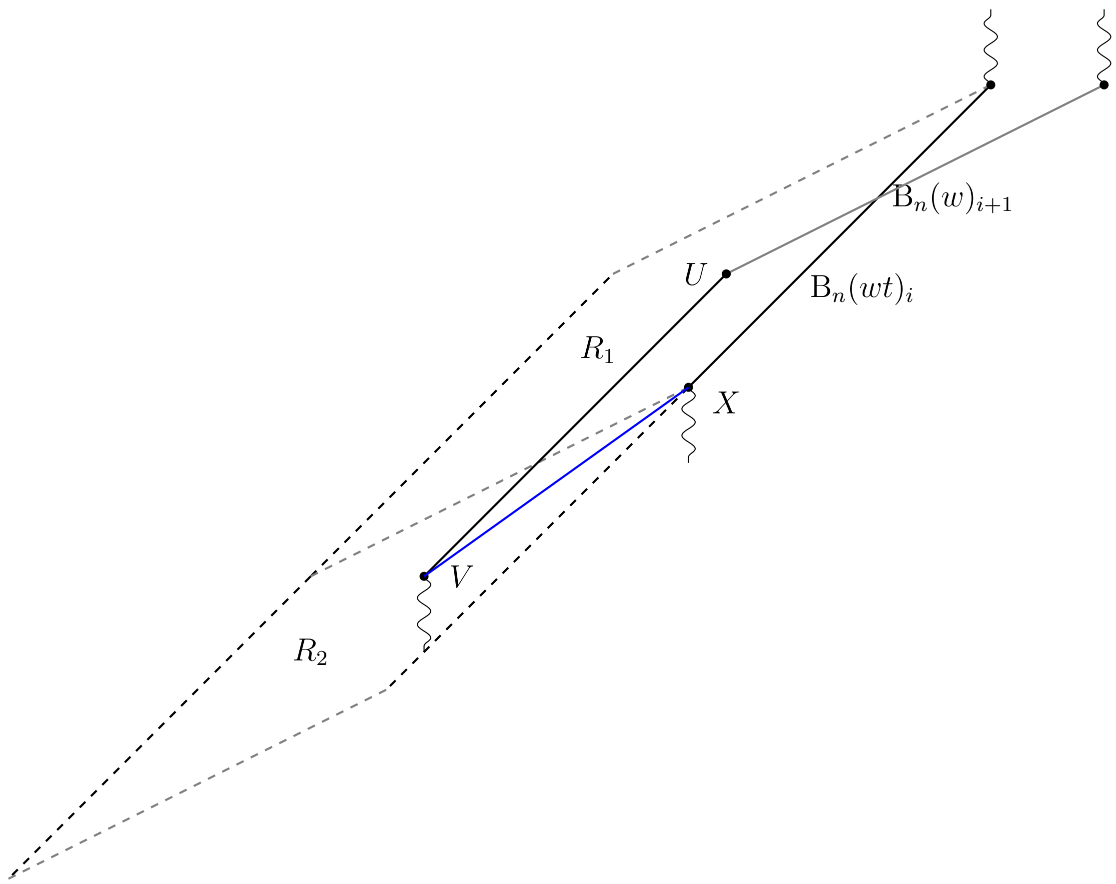


Figure 6.10: The rhombic region labelled R_2 for which the lower vertex of $\mathrm{B}_n(w)_i$ lies in if and only if $\mathrm{B}_n(w)_{i+1}$ and $\mathrm{B}_n(wt)_i$ intersect.

Let U and V be the upper and lower vertices of $\mathrm{B}_n(w)_{i+1}$ respectively and X be the lower vertex of $\mathrm{B}_n(wt)_i$. Note that U is in the rhombic region labelled R_1 if and only if $\mathrm{B}_n(w)_{i+1}$ and $\mathrm{B}_n(wt)_i$ intersect. Equivalently, this is true exactly when V is in the region labelled R_2. Observe that the line segment from V to X is equivalent to the difference vector of $\mathrm{B}_n(w)$ and $\mathrm{B}_n(wt)$. But then V is in R_2 if and only if the difference vector has angle from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ strictly between $\beta_n^{a^{-1}(w)}$ and $\beta_n^{b^{-1}(w)}$. But this is a contradiction as we saw the angles of $\beta_n^{a^{-1}(w)}$ and $\beta_n^{b^{-1}(w)}$ lie in $[\pi/4, 3\pi/4]$ whereas the angles of difference vectors lie in the disjoint, open interval $(-\pi/4, \pi/4)$.

For $|i-j| \geq 2$ we first consider $|i-j| = 2$ where the situation pictured in Figure 6.11 applies.

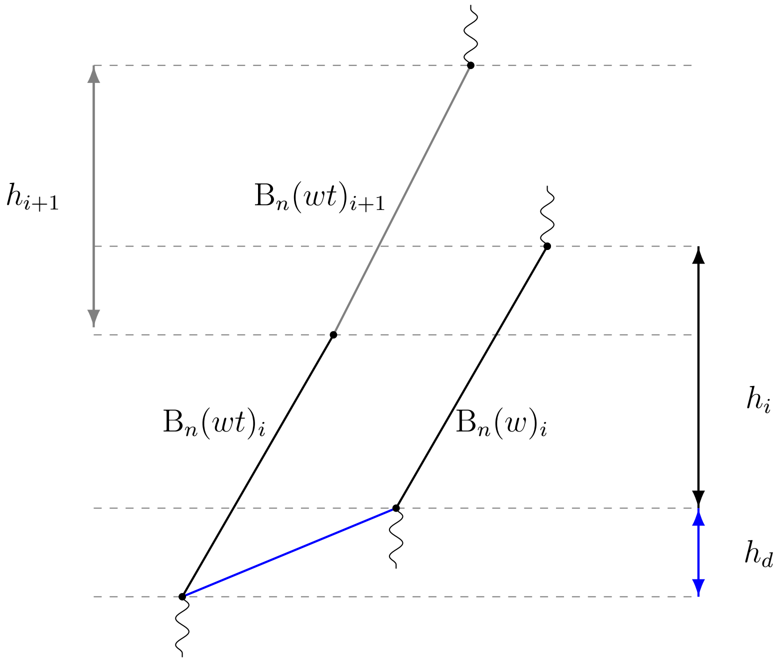


Figure 6.11: The heights concerning the $|i-j| \geq 2$ case.

Note that $\mathrm{B}_n(wt)_i$ is a translation of $\mathrm{B}_n(w)_i$ since i is in the critical strip. If the y-coordinate of $\mathrm{B}_n(w)_i$ is less than that of $\mathrm{B}_n(wt)_{i+2}$, then an intersection between $\mathrm{B}_n(wt)_{i+2}$ and $\mathrm{B}_n(w)_i$ is impossible. So we consider the inequality $h_d + h_i < h_i + h_{i+1}$, or equivalently, $h_d < h_{i+1}$. Note that h_d is bounded above by $1 - \frac{\sqrt{2}}{2}$ and h_{i+1} is bounded below by $\frac{\sqrt{2}}{2}$. But $1 - \frac{\sqrt{2}}{2} < \frac{\sqrt{2}}{2}$ and so $h_d \leq 1 - \frac{\sqrt{2}}{2} < \frac{\sqrt{2}}{2} \leq h_{i+1}$. Therefore, $\mathrm{B}_n(w)_i$ and $\mathrm{B}_n(wt)_{i+1}$ certainly do not intersect. But since the y-coordinate is strictly increasing in borders we know that no intersection can occur for all $|i-j| \geq 2$ also.

□

Taking $\alpha = \pi/4$ ensures that for the case $|i-j| = 1$, $\mathrm{B}_n(w)_i$ and $\mathrm{B}_n(wt)_j$ have an empty intersection. However if $\alpha < \pi/4$, a non-trivial intersection can occur for

some sufficiently large n. So $\alpha = \pi/4$ is sharp precisely in this sense. When $|i - j| \geq 2$, the least α needed to ensure that $\mathrm{B}_n(w)_i$ and $\mathrm{B}_n(wt)_j$ have an empty intersection in the critical region for all n is $\alpha = \pi/6$.
An important corollary evident from the proof of Theorem 6.0.1 is that $\mathrm{B}_n(w) \cap \mathrm{B}_n(wt)$ differ on exactly those points on edges in the so-called critical region. Furthermore, in that region, the points on $\mathrm{B}_n(wt)$ have an x-coordinate greater than their $\mathrm{B}_n(w)$ counterparts of the same y-coordinate. Putting this more precisely, we have Corollary 6.2.2.

Corollary 6.2.2. *Suppose $w, t \in \mathrm{Sym}(n)$ with $t \in T$ and $w <_B wt$. Write $t = (a, b)$ for some $1 \leq a < b \leq n$. Then $\mathrm{H}(\mathrm{B}_n(w), y) < \mathrm{H}(\mathrm{B}_n(wt), y)$ for all y satisfying*

$$\sum_{k=1}^{a-1} \sin\left(\frac{((k-1)w^{-1})(\pi - 2\alpha)}{n-1} + \alpha\right) < y < \sum_{k=1}^{b} \sin\left(\frac{((k-1)w^{-1})(\pi - 2\alpha)}{n-1} + \alpha\right),$$

(declaring $\sum_{k=1}^{a-1} \sin\left(\frac{((k-1)w^{-1})(\pi - 2\alpha)}{n-1} + \alpha\right) = 0$ for when $a = 1$), and $\mathrm{H}(\mathrm{B}_n(w), y) = \mathrm{H}(\mathrm{B}_n(wt), y)$ otherwise.

Chapter 7

E-embeddings and tiling bijections

In this chapter, we aim to use Theorem 6.2.1 as inspiration to create definitions that will be conducive to forming bijections between classes of reduced words and certain tilings of polygons for *all* finite irreducible Coxeter groups. Let (W, S) denote a finite irreducible Coxeter group. The main definition we examine is a map that sends comparable elements in the weak order of one Coxeter group to comparable maps in the Bruhat order of the symmetric group. This does not seem to appear in the literature.

Definition 7.0.1. *Suppose that $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ is an embedding. Then φ is an E-embedding for W if for all $w_1, w_2 \in W$*

$$w_1 <_R w_2 \text{ implies } \varphi(w_1) <_B \varphi(w_2).$$

We call these E-embeddings in homage of Elnitsky. There are two main theorems for this chapter. The first, Theorem 7.1.3, shows how an E-embedding associates reduced words to tilings of polygons. The second, Theorem 7.1.8, shows how the set of all such tilings associated to a given element is in bijection with the set of reduced words of said element up to a certain subset of braid relations.
This is a lightly edited version of more joint work with Professor Peter Rowley from [37]. We first state all the relevant definitions and theorems before proving them in the final section.

7.1 Definitions and results

We start developing various objects associated to an E-embedding.

Definition 7.1.1. *Let* $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ *be an E-embedding. Then we define* $\mathrm{B}^\alpha_\varphi(w)$ *to be* $\mathrm{B}^\alpha_n(\varphi(w))$ *and we set* $\mathrm{P}^\alpha_\varphi(u,v)$ *to be* $\mathrm{P}^\alpha(\varphi(u),\varphi(v))$ *for all* $u, v, w \in W$.

We define the *tiling* of a reduced expression.

Definition 7.1.2. *Let* $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ *be an E-embedding. Let* $s_{i_1}s_{i_2}\dots s_{i_k}$ *be a reduced expression of some* $w \in W$. *Then we define the tiling of* $s_{i_1}s_{i_2}\dots s_{i_k}$ *to be*

$$\mathrm{T}_\varphi(s_{i_1}s_{i_2}\dots s_{i_k}) = \bigcup_{j=0}^{k} \mathrm{B}^\alpha_\varphi(s_{i_1}\dots s_{i_j})$$

where it is agreed that when $j = 0$, $\mathrm{B}^\alpha_\varphi(s_{i_1}\dots s_{i_j}) = \mathrm{B}^\alpha_\varphi(id)$.

Our first main theorem is as follows.

Theorem 7.1.3. *Suppose* $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ *is an E-embedding and* $\alpha \in [\pi/4, \pi/2)$. *Then for all* $w \in W$ *and all reduced expressions* $w = s_{i_1}s_{i_2}\dots s_{i_k}$,

$$\mathrm{B}^\alpha_\varphi(id) \prec \mathrm{B}^\alpha_\varphi(s_{i_1}) \prec \mathrm{B}^\alpha_\varphi(s_{i_1}s_{i_2}) \prec \dots \prec \mathrm{B}^\alpha_\varphi(s_{i_1}\dots s_{i_k}).$$

The significance of this theorem is that what we call tilings in Definition 7.1.2 are indeed deserving of their name. Since $\mathrm{B}_\varphi(\mathrm{id}) \prec \mathrm{B}_\varphi(s_{i_1}) \prec \dots \prec \mathrm{B}_\varphi(s_{i_1}\dots s_{i_k})$ for each reduced word, each of $\mathrm{P}_\varphi(\mathrm{id}, s_{i_1}), \mathrm{P}_\varphi(s_{i_1}, s_{i_1}s_{i_2}), \dots, \mathrm{P}_\varphi(s_{i_1}\dots s_{i_{k-1}}, s_{i_1}\dots s_{i_k})$ forms a tile and set of the interiors of these tiles partition the interior of the polygon $\mathrm{P}_\varphi(s_{i_1}\dots s_{i_k})$.

Definition 7.1.4. *Let* $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ *be an E-embedding and* $w \in W$. *Let* $\mathcal{T}^\alpha_\varphi(w)$ *be the set consisting of all* $\mathrm{T}_\varphi(s_{i_1}s_{i_2}\dots s_{i_k})$ *for all reduced words* $s_{i_1}s_{i_2}\dots s_{i_k}$ *evaluating to* w.

Our next definition is an extension of relation set J from Chapter 5.

Definition 7.1.5. *Let* $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ *be an E-embedding. Let* $r, s \in S$, *then we define* J_φ *so that* $\{r, s\} \in J_\varphi$ *if and only if for all reduced words containing a consecutive subword of* $\underbrace{srs\dots}_{m_{s,r}}$, *we have*

$$\mathrm{T}_\varphi(s_{i_1}\dots s_{i_\ell}\underbrace{srs\dots}_{m_{s,r}}s_{i_r}\dots s_{i_j}) = \mathrm{T}_\varphi(s_{i_1}\dots s_{i_\ell}\underbrace{rsr\dots}_{m_{s,r}}s_{i_r}\dots s_{i_j}).$$

Surprisingly, J_φ admits an easily computed description. To describe this, we define the *support interval* of a permutation as follows.

Definition 7.1.6. *Let φ is an embedding of W in* $\mathrm{Sym}(n)$ *and $s \in S$. We define the* support interval *of s to be*

$$\mathrm{I}_\varphi(s) = \bigcup_{i=1}^{k} \{a_i, a_i+1, \ldots, b_i-1, b_i\}$$

where $\varphi(s) = \prod_{i=1}^{k}(a_i, b_i)$ with $a_i < b_i$.
Necessarily, there must exists a unique choice of subsets
$\{a'_1, \ldots, a'_{k'}\} \subseteq \{a_1, \ldots, a_k\}$, and $\{b'_1, \ldots, b'_{k'}\} \subseteq \{b_1, \ldots, b_k\}$ such that

$$\mathrm{I}_\varphi(s) = \bigcup_{m=1}^{k'} \{a'_m, a'_m+1, \ldots, b'_m\}$$

and that the intervals in this union are pairwise disjoint. We call this the disjoint form of $\mathrm{I}_\varphi(s)$ and a'_i and b'_i its i^{th} disjoint representatives.

Lemma 7.1.7. *Let $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ be a E-embedding. Then*

$$J_\varphi = \{\{r, s\} \subseteq S \mid \mathrm{I}_\varphi(s) \cap \mathrm{I}_\varphi(r) = \emptyset\}.$$

Recall that $\mathcal{R}_{J_\varphi}(w)$ is the set of all reduced words of w up to those braid relations for pairs in J_φ. Now we can state the final final theorem of this chapter.

Theorem 7.1.8. *Suppose that φ is a E-embedding of W. Then for all $w \in W$ there exists a bijection between $\mathcal{T}_\varphi(w)$ and $\mathcal{R}_{J_\varphi}(w)$.*

7.2 Proofs

We start with a proof of Theorem 7.1.3.

Proof. Suppose $w = s_{i_1} \ldots s_{i_k}$ is a reduced word for some element $w \in W$. Necessarily, we have

$$\mathrm{id} <_R s_{i_1} <_R \ldots <_R s_{i_1} \ldots s_{i_k}.$$

Since φ is an E-embedding, we know

$$\varphi(\mathrm{id}) <_B \varphi(s_{i_1}) <_B \ldots <_B \varphi(s_{i_1} \ldots s_{i_k})$$

and then Theorem 6.0.1 implies

$$\mathrm{B}_\varphi(\mathrm{id}) \prec \mathrm{B}_\varphi(s_{i_1}) \prec \ldots \prec \mathrm{B}_\varphi(s_{i_1} \ldots s_{i_k}).$$

□

We now prove an auxiliary lemma. This is an extension of Corollary 6.2.2.

Lemma 7.2.1. *Let* $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ *be an E-embedding. Let* $w \in W$ *and* $s \in S$ *be such that* $w <_R ws$. *Suppose* $\mathrm{I}_\varphi(s) = \bigcup_{m=1}^{k'} \{a'_m, a'_m + 1, \ldots, b'_m\}$ *in disjoint form. Then* $\mathrm{H}(\mathrm{B}_\varphi(w), y) < \mathrm{H}(\mathrm{B}_\varphi(ws), y)$ *if and only if for some* $m \in \{1, \ldots, k'\}$,

$$\sum_{i=1}^{a'_m - 1} \sin\left(\frac{((i-1)\varphi(w^{-1}))(\pi - 2\alpha)}{n-1} + \alpha\right) < y,$$
$$\sum_{i=1}^{b'_m} \sin\left(\frac{((i-1)\varphi(w^{-1}))(\pi - 2\alpha)}{n-1} + \alpha\right) > y,$$

and $\mathrm{H}(\mathrm{B}_\varphi(w), y) = \mathrm{H}(\mathrm{B}_\varphi(ws), y)$ *otherwise.*

Proof. Since $w <_R ws$ and φ is an E-embedding, $\varphi(w) <_B \varphi(ws)$. By Theorem 2.26 of [1], there exists a sequence of (not necessarily pairwise commuting) transpositions in $\mathrm{Sym}(n)$, $t_1, \ldots, t_\ell$, such that $\varphi(s) = t_1 \ldots t_\ell$ and

$$\varphi(w) <_B \varphi(w)t_1 <_B \ldots <_B \varphi(w)t_1 \ldots t_{\ell-1} <_B \varphi(w)t_1 \ldots t_\ell \quad (= \varphi(ws))$$

where $\ell(\varphi(w)t_1 \ldots t_j) = \ell(\varphi(w)) + j$.

Consider $\varphi(s)$ restricted to a given part of the discjoint form of $\mathrm{I}_\varphi(s)$, $\{a'_m, \ldots, b'_m\}$, and call this induced permutation $\varphi_{|m}(s)$. This is an element of the symmetric group $\mathrm{Sym}(\{a'_m, \ldots, b'_m\})$, itself a parabolic subgroup of $\mathrm{Sym}(n) = \mathrm{Sym}(\{1, \ldots, n\})$. Using Corollaries 1.4.4 and 1.4.8 of [1], we know that any reduced expression of $\varphi_{|m}(s)$ is the product of adjacent transpositions using only those in $\mathrm{Sym}(\{a'_m, \ldots, b'_m\})$. Furthermore, each t'_i is in $\mathrm{Sym}(\{a'_m, \ldots, b'_m\})$ also. Since $\{a'_m, \ldots, b'_m\}$ is an interval in the disjoint form of the interval support of s, for each $a'_m \leq c \leq b'_m$, there must exist some $t'_i = (a', b')$ with $a' \leq c \leq b'$.

Therefore, applying Corollary 6.2.2 to each t_i' for $i = 1, \ldots, \ell'$ tells us that

$$\sum_{i=1}^{a_m'-1} \sin\left(\frac{((i-1)\varphi(w^{-1}))(\pi - 2\alpha)}{n-1} + \alpha\right) < y \quad \text{and}$$
$$\sum_{i=1}^{b_m'} \sin\left(\frac{((i-1)\varphi(w^{-1}))(\pi - 2\alpha)}{n-1} + \alpha\right) > y.$$

Repeating this for all $m = 1, \ldots, k'$ gives the result.

□

We now prove Lemma 7.1.7.

Proof. Let $s, r \in S$ and take some reduced word containing $\underbrace{rsr\ldots}_{m_{s,r}}$ as a consecutive subword, $w = s_{i_1} \ldots s_{i_\ell} \underbrace{rsr\ldots}_{m_{s,r}} s_{i_r} \ldots s_{i_j}$. Since φ is an embedding, s and r are distinct if and only if $\varphi(s)$ and $\varphi(r)$ are distinct also. Moreover, we must have $\mathrm{B}_\varphi(us) \neq \mathrm{B}_\varphi(ur)$ for all $u \in W$.
Write $v = s_{i_1} \ldots s_{i_\ell}$. Lemma 7.2.1 shows that support intervals determine exactly the points at which $\mathrm{B}_\varphi(vs)$ and $\mathrm{B}_\varphi(vr)$ differ from $\mathrm{B}_\varphi(v)$ respectively. Let

$$\mathrm{I}_\varphi(s) = \bigcup_{m=1}^{k'} \{a_m', a_m' + 1, \ldots, b_m'\},$$
$$\mathrm{I}_\varphi(r) = \bigcup_{j=1}^{h'} \{c_j', c_j' + 1, \ldots, d_j'\}$$

be the disjoint forms of $\mathrm{I}_\varphi(s)$ and $\mathrm{I}_\varphi(r)$ respectively.
Suppose $\mathrm{I}_\varphi(s) \cap \mathrm{I}_\varphi(r) \neq \emptyset$. First we consider the case that $\{a_m', a_m' + 1, \ldots, b_m'\} = \{c_j', c_j' + 1, \ldots, d_j'\}$ for some m and j, and that the restricted permutations, $\varphi(s)_{|m}$ and $\varphi(r)_{|j}$, are equal. Then $\varphi(vrs)_{|m} = \varphi(v)_{|m}$. Now consider those edges of $\mathrm{B}_\varphi(v), \mathrm{B}_\varphi(vs)$ and $\mathrm{B}_\varphi(vsr)$ indexed by $\{a_m', a_m' + 1, \ldots, b_m'\}$: they are identical for $\mathrm{B}_\varphi(v)$ and $\mathrm{B}_\varphi(vsr)$. Hence it cannot be true that $\mathrm{B}_\varphi(v) \prec \mathrm{B}_\varphi(vs) \prec \mathrm{B}_\varphi(vsr)$, contradicting Theorem 7.1.3 since $v <_R vs <_R vsr$.
Now suppose there exists some m and j such that $\{a_m', a_m' + 1, \ldots, b_m'\} \cap \{c_j', c_j' + 1, \ldots, d_j'\} \neq \emptyset$ but that the restricted permutations, $\varphi(s)_{|m}$ and $\varphi(r)_{|j}$, are not equal (they may not even be restricted to the same set). Since they are not equal, we must have points at which they differ. More

specifically, Lemma 7.2.1 shows there must exist some y satisfying

$$\sum_{i=1}^{\min(a'_m, c'_j)-1} \sin\left(\frac{((i-1)\varphi(v))(\pi - 2\alpha)}{n-1} + \alpha\right) < y \quad \text{and}$$
$$\sum_{i=1}^{\min(b'_m, d'_j)} \sin\left(\frac{((i-1)\varphi(v))(\pi - 2\alpha)}{n-1} + \alpha\right) > y$$

where, without loss of generality, $\mathrm{H}(\mathrm{B}_\varphi(v), y) < \mathrm{H}(\mathrm{B}_\varphi(vs), y) < \mathrm{H}(\mathrm{B}_\varphi(vr), y)$. We know that $v <_R vs$ and $v <_R vr$.
Now consider the consequences if
$\mathrm{T}(s_{i_1} \dots s_{i_\ell} \underbrace{rsr\dots}_{m_{s,r}} s_{i_r} \dots s_{i_j}) = \mathrm{T}(s_{i_1} \dots s_{i_\ell} \underbrace{srs\dots}_{m_{s,r}} s_{i_r} \dots s_{i_j})$. There must be some prefix of the word $w = s_{i_1} \dots s_{i_\ell} \underbrace{rsr\dots}_{m_{s,r}} s_{i_r} \dots s_{i_j}$, say $w_\gamma := s_{i_1} \dots s_{i_\gamma}$, such that $\mathrm{H}(\mathrm{B}_\varphi(w_\gamma), y) = \mathrm{H}(\mathrm{B}_\varphi(vs), y)$. Necessarily, $\gamma \neq \ell, \ell+1$.
If $\gamma < \ell$, then this contradicts Theorem 7.1.3 as $w_\gamma <_R v$ but

$$\mathrm{H}(\mathrm{B}_\varphi(v), y) < \mathrm{H}(\mathrm{B}_\varphi(vs), y) \quad (= \mathrm{H}(\mathrm{B}_\varphi(w_\gamma), y)).$$

If $\gamma > \ell + 1$, then this contradicts Theorem 7.1.3 again as $vr <_R w_j$ but

$$(\mathrm{H}(\mathrm{B}_\varphi(w_\gamma), y) =) \quad \mathrm{H}(\mathrm{B}_\varphi(vs), y) < \mathrm{H}(\mathrm{B}_\varphi(vr), y).$$

So far, we have proved if $\{s, r\} \in \mathrm{J}_\varphi$, then $\mathrm{I}_\varphi(s) \cap \mathrm{I}_\varphi(r) \neq \emptyset$. To see that the converse is true, suppose $\mathrm{I}_\varphi(s) \cap \mathrm{I}_\varphi(r) = \emptyset$. Lemma 7.2.1 shows that for all y, only one of $\mathrm{H}(\mathrm{B}_\varphi(vs), y)$ and $\mathrm{H}(\mathrm{B}_\varphi(vr), y)$ can differ from $\mathrm{H}(\mathrm{B}_\varphi(v), y)$. Hence the order in which we place the tiles corresponding to $\mathrm{P}(v, vs)$ and $\mathrm{P}(v, vr)$ does matter and produces the same ultimate tiling of w. □

The proof of Lemma 7.1.7 shows that the $\{s, r\}$ braid relations of the Coxeter group either always preserve a tiling or always alter it, regardless of the choice of reduced words. It is now evident that the necessary bijection for Theorem 7.1.8 follows.

Chapter 8

Strategies for creating E-embeddings

This chapter consists of a first attempt at creating E-embeddings for all finite irreducible Coxeter groups. Unfortunately, we do not obtain a proof that what we create is indeed an E-embedding. However, we provide an algorithm for creating certain CPR graphs and show that they exhibit promising behaviour; they behave consistently with E-embeddings when examining a certain tiling.
The images in this chapter were produced using Wolfram Language in Mathematica ([49]). The colour scheme used in this chapter colours objects associated to different generators of a given Coxeter groups by different colours. In particular, Figure 8.1 serves as a guide.

Figure 8.1: The colours used to represent each generator in our Coxeter groups. The i^{th} colour (read from left to right) represents the generator s_i.

8.1 A Strategy for constructing new tilings

We start by trying to recognise some common patterns in the CPR graphs associated with Elnitsky's type A, B and D tilings. We do not prove these patterns do indeed hold true. Let (W, S) denote either a type A, B or D Coxeter group.

(i) There exists an embedding $\varphi : W \hookrightarrow \mathrm{Sym}(n)$ such that, if it were an E-embedding, then $\mathrm{B}_\varphi(ws_i)$ is obtained from $\mathrm{B}_\varphi(w)$ by appending the tile associated to s_i in Elnitsky's construction for each $s_i \in S$.

(*ii*) Viewing these embeddings as CPR graphs, we recognise that they are the action of the group permuting the cosets of some maximal proper parabolic subgroup of W. That is, there exists some $s \in S$ such that for $J = S \setminus \{s\}$, φ is the action of permuting the coset of W_J. This s is s_n, s_1 and s_1 for when W is type A, B or D respectively.

(*iii*) It is known that each coset of any parabolic subgroup (of any Coxeter group), W_I say, has a unique element of minimal length (Corollary 2.4.5 of [1]). These form a set of representatives of the cosets of W_I which we denote by W^I.

(*iv*) Now consider the nodes in the CPR graph of φ as the minimal representatives W^J.

(*v*) The labelling of the nodes in this CPR graph, is given by some function $\mathtt{L} : W^J \to [|W^J|]$. We strongly suspect (but do not prove) that labelling $\mathtt{L}$ associated to each of Elnitsky's tilings is such that $u <_B v$ implies $\mathtt{L}(u) < \mathtt{L}(v)$.

Our last observation imposes a total order on W^J by the function $\mathtt{L}$. Recall a total order is a reflexive, anti-symmetric, transitive, binary relation in which *all* pairs of elements are comparable. We say that this function is 'Bruhat preserving' exactly because $u <_B v$ implies $\mathtt{L}(u) < \mathtt{L}(v)$ for all $u, v \in W^J$.
Given these observations, we propose the folllowing strategy to create candidate E-embeddings:

(*i*) Create a total order on W, $\ll$, that also refines the Bruhat order.

(*ii*) Choose some proper $J \subset S$ and consider the CPR graph induced by W permuting the cosets of W_J by group multiplication.

(*iii*) Label the nodes by the relative position of their unique minimal length representatives in W^J with respect to $\ll$.

(*iv*) See if the induced embedding produces something resembling an appropriate tiling of a polygon when tested on some chosen reduced word.

Restricting ourselves to those CPR graphs that are the action of these cosets is confessedly an artificial choice.
Before moving on we should clarify and define some helpful notation.

Definition 8.1.1. *Let (W, S) be a finite irreducible Coxeter group.*
Let $<_S$ denote total order on S and let $\prec$ denote a total order on W. Implicitly, we will write $s_i <_S s_j$ if and only if $i < j$. We say $\prec$ is a refinement of the Bruhat order if $u <_B v$ implies $u \prec v$ for all $u, v \in W$.
Given some total order $\prec$ on W, let $\mathtt{L}_J : W^J \hookrightarrow \{1, \ldots, |W^J|\}$ be the bijection so that for all $w \in W^J$, $\mathtt{L}_J(w) = i$ if and only if w is the i^{th} least element with respect to $\prec$ when restricted to W^J. We call $\mathtt{L}_J$ the labelling *of W^J with respect to $\prec$.*
Finally, we define $\phi^J_\prec : W \hookrightarrow Sym(|W^J|)$ so that $(i)\phi^J_\prec(w) = j$ if and only if w sends that coset labelled i to that j by $\mathtt{L}_J$. We call $\phi^J_\prec$ the induced embedding of $\prec$ on J. If $J = \emptyset$ then we simply write $\phi^J_\prec$ as $\phi_\prec$.

We can now state a conjecture:

Conjecture 8.1.2. *$\phi_\prec : W \to \mathrm{Sym}(|W|)$ is an E-embedding if and only if $\prec$ is a refinement of the Bruhat order.*

After some computer experimentation, the above conjecture seems more sensible than it might at a first glance. It is our hope that if this is true, then a general characterisation of E-embeddings on CPR graphs will become apparent.

8.2 A labelling algorithm

We now proceed to describe an algorithm to produce a total order $\ll$. The reason for considering this total order is that it seems to refine the Bruhat order (though we do not prove that in this thesis), is somewhat natural, is consistent with the CPR graphs associated to Elnitsky's tilings, and is easily implemented by computer.
An overview of the algorithm is as follows:

(*i*) Let C denote the Cayley graph of (W, S) and $<_S$ be a total order on S.

(*ii*) We inductively define the total order $\ll$ on W by first defining *id* to be the least element.

(*iii*) Suppose we know the order of the k^{th} least elements in W with respect to $\ll$. To choose the $(k+1)^{th}$ least element, consider all those elements $w \in W$ such that $w = xs$ where $\mathtt{L}(x) \leq k < \mathtt{L}(w)$.

(*iv*) Amongst these candidates, consider those that $w = xs$ such that $s \in S$ such that s is maximal with respect to $<_S$.

(*v*) If more than one candidate satisfies this criterion, choose that $w = xs$ such that `L` is minimal.

We provide a more precise and computer-friendly description. In the algorithm we construct an ordered list of the elements W. From the list `L` we may induce the total order $\ll$ by saying $u \ll v$ if and only if the position (the integer i such that u appears as the i^{th} element in `L`) of u is less than that of v. We use `*` to denote the group binary operation.

Algorithm 8.2.1.

```
Inputs:

    (W,S): a finite irreducible Coxeter group generated by S

    <_S: a total order on S

Outputs:

    L: an ordered list of the elements of W

    T: a spanning tree of the Cayley graph of (W,S)

Algorithm:

    Step 0:

        Set L = [ id ], the ordered list of elements of W

        Set T = { }, the set of edges that will form our tree
```

```
Step i > 1:

    Set C = { [ g, s ] | g in L, s in S and g*s not in L }

Set r to be that maximum element in S with respect to
 <_S such that [ g, r ] in C for some g in L

Set h to be that minimal element of W with respect to
its position in L such that [ h, r ] in C

Append h*r to L

Add { h, h*r } to T

If i != |W| then

    Go to Step i+1

Else

    Stop
```

The algorithm is very elementary. We expect it must exist somewhere in the wider literature yet have not found it yet despite searching. It is only in our ignorance that we refer to it only as Algorithm 8.2.1. Note that, with some very minor alterations, it can be implemented on any finite graph with an edge-colouring.

We know that `T` is a tree notice the edges appended to `T` contain exactly one element in `L` . Implicitly, we produce a canonical word for each element of the group: since `T` is a tree there is a unique path from id to w for all $w \in W$. We denote that word as $\overline{\texttt{NF}}(w)$.

8.3 Examples of Algorithm 8.2.1

Example 8.3.1. *We give a detailed example for running Algorithm 8.2.1 on* Sym(3)*. For all* $\delta \in \mathrm{Sym}(n)$*, we write* δ *in one-line notation so that*

$$\delta = \delta(1)\ \delta(2)\ \dots\ \delta(n).$$

The Cayley graph of Sym(3) *is shown in Figure 8.2 where we represent each permutation in this form:*

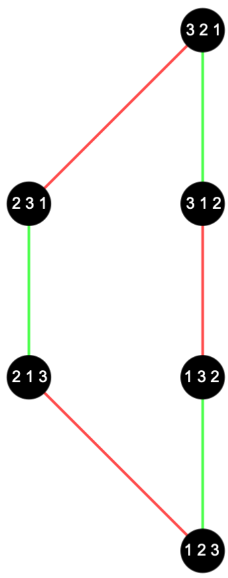


Figure 8.2: The Cayley graph of Sym(3) with the permutations viewed in one-line form.

We start with the step 0: Set $\mathtt{L} = [\,1\,2\,3\,]$ *and* $\mathtt{T} = \{\}$*. When an element is added to* L *we will label it on the Cayley graph by its position in* L*.*

For step 1: we start by setting $\mathtt{C} = \{[\,1\,2\,3\,, s_1], [\,1\,2\,3\,, s_2]\}$ *since the edges of the Cayley graph that contain exactly one labelled node are those of type* s_1 *and* s_2 *containing* 1 2 3 *. We have emboldened these edges in the left-most diagram of Figure 8.3.*

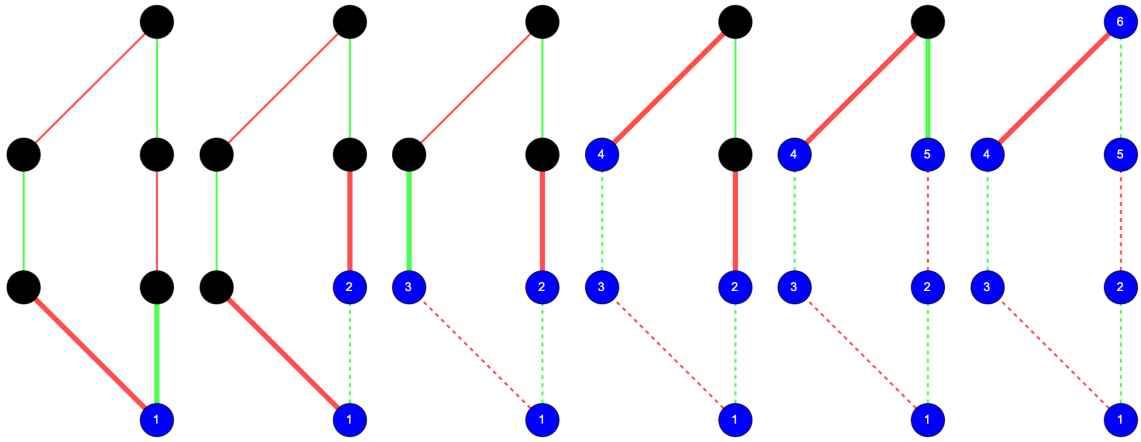


Figure 8.3: Applying Algorithm 8.2.1 to the Cayley graph of Sym(3), step-by-step.

Now $r = s_2$ *since* s_2 *is maximal (with respect to* $<_S$*) amongst those second entries in each element of* $\mathtt{C}$. *Since there is only one element of* $\mathtt{C}$ *whose second entry is* $r = s_2$, $[123, s_2]$, *we append* $123 * s_2 = 132$ *to* $\mathtt{L}$. *We also add the edge* $\{123, 132\}$ *to* $\mathtt{T}$ *and give* 132 *the label 2.*

For step 2, $\mathtt{C} = \{[123, s_1], [132, s_1]\}$. *Necessarily,* $r = s_1$ *since it is the only* $s \in S$ *that appears as the second entry of an element in* $\mathtt{C}$. *What is* h*? This time there are two elements of* $\mathtt{C}$ *whose second entry is* $r = s_1$. *So we choose that element of* $\mathtt{C}$ *whose first entry has lesser position in* $\mathtt{L}$ *(equivalently, lesser label on our Cayley graph). So* $h = 123$ *and we append* $h * r = 123 * s_1 = 213$ *to* $\mathtt{L}$, *label* 213 *with 3 on the Cayley graph and append* $\{123, 213\}$ *to* $\mathtt{T}$.

For steps 3, 4 *and* 5, *at the end of each step we have*

$$\mathtt{C} = [[132, s_1], [132, s_2]], \qquad \mathtt{L} = [123, 132, 213, 231]$$
$$\mathtt{C} = [[132, s_1], [231, s_1]], \qquad \mathtt{L} = [123, 132, 213, 231, 312]$$
$$\mathtt{C} = [[231, s_1], [312, s_2]], \qquad \mathtt{L} = [123, 132, 213, 231, 312, 321]$$

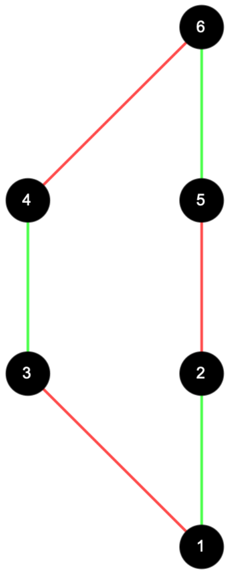
respectively. After the fifth and final step we have

$$\mathtt{T} = [\{123, 132\}, \{123, 213\}, \{213, 231\}, \{132, 312\}, \{312, 321\}].$$

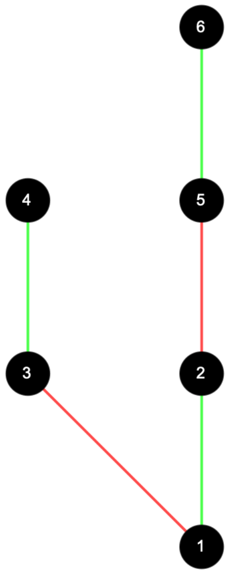
Running this algorithm has implicitly determined the $\ll$ *order, spanning tree and normal forms: the labels determine the order, the spanning tree consists of those edges selected at each step and the normal forms corresponds to paths from the identity to each element within this spanning tree. See Figure 8.3.1 below.*

$\mathtt{L}(w)$	w	$\overline{\mathtt{NF}}(w)$
1	1 2 3	
2	1 3 2	s_2
3	2 1 3	s_1
4	3 1 2	$s_1 s_2$
5	2 3 1	$s_2 s_1$
6	3 2 1	$s_2 s_1 s_2$

(a) The $\ll$-ordering on Sym(3). From left to right: the $\mathtt{L}$ value, the one-line form of the permutation and the $\ll$-normal form.



(b) The Cayley graph of Sym(3) labelled by $\mathtt{L}$.



(c) The $\ll$ spanning tree, $\mathtt{T}$, for Sym(3).

Our purpose for computing the order for $\ll$ is to try create $\phi^J_{\ll}$ for all $J \subseteq S$. In practice, taking some $J = S \setminus \{s\}$ for some $s \in S$ as this will produce permutation groups of a lesser degree. But thinking of our Cayley graph as the group action on $W_\emptyset$, we can construct $\phi_{\ll} : \mathrm{Sym}(3) \to \mathrm{Sym}(6)$ such that

$$s_1 \xrightarrow{\phi_{\ll}} (1,3)(2,5)(4,6) = a_2 a_1 a_4 a_3 a_2 a_5 a_4$$
$$s_2 \xrightarrow{\phi_{\ll}} (1,2)(3,4)(5,6) = a_1 a_3 a_5$$

where $a_i = (i, i+1)$ denotes a generator of the codomain.

We now repeat this process for some more selected Coxeter groups so that we can produce some suitable $\phi^J_{\ll}$.

For Sym(4) we have:

1	1 2 3 4		13	2 3 1 4	$s_2 s_1$
2	1 2 4 3	s_3	14	2 4 1 3	$s_2 s_1 s_3$
3	1 3 2 4	s_2	15	3 2 1 4	$s_2 s_1 s_2$
4	1 4 2 3	$s_2 s_3$	16	4 2 1 3	$s_2 s_1 s_2 s_3$
5	1 3 4 2	$s_3 s_2$	17	3 4 1 2	$s_2 s_1 s_3 s_2$
6	1 4 3 2	$s_3 s_2 s_3$	18	4 3 1 2	$s_2 s_1 s_3 s_2 s_3$
7	2 1 3 4	s_1	19	2 3 4 1	$s_3 s_2 s_1$
8	2 1 4 3	$s_1 s_3$	20	2 4 3 1	$s_3 s_2 s_1 s_3$
9	3 1 2 4	$s_1 s_2$	21	3 2 4 1	$s_3 s_2 s_1 s_2$
10	4 1 2 3	$s_1 s_2 s_3$	22	4 2 3 1	$s_3 s_2 s_1 s_2 s_3$
11	3 1 4 2	$s_1 s_3 s_2$	23	3 4 2 1	$s_3 s_2 s_1 s_3 s_2$
12	4 1 3 2	$s_1 s_3 s_2 s_3$	24	4 3 2 1	$s_3 s_2 s_1 s_3 s_2 s_3$

Table 8.1: The $\ll$-ordering on B_3. From left to right: the `L` value and the $\ll$-normal form.

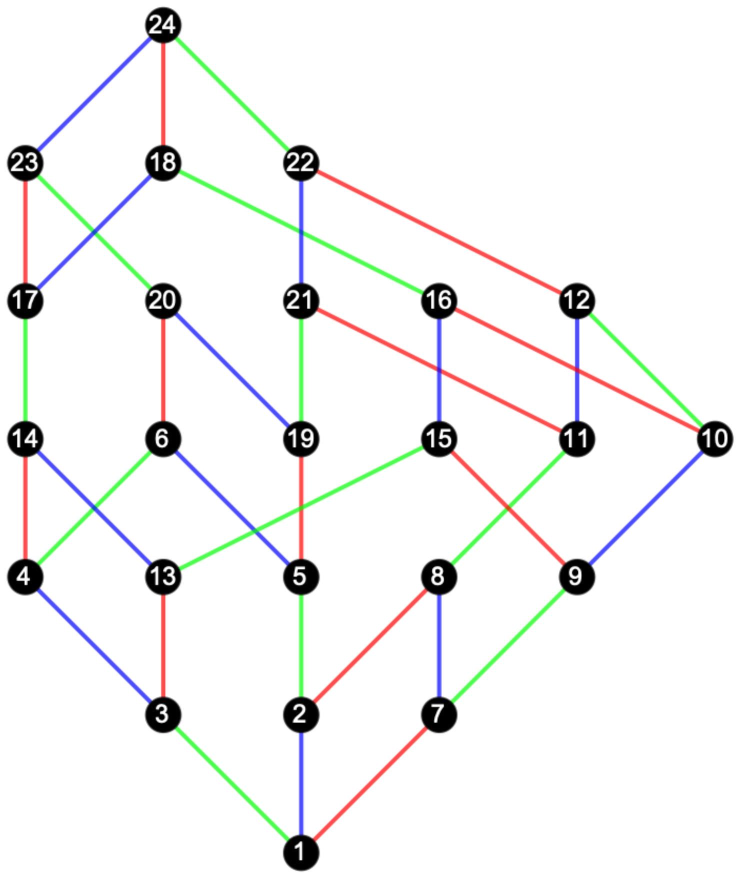


Figure 8.5: The Cayley graph of Sym(4) labelled by `L`.

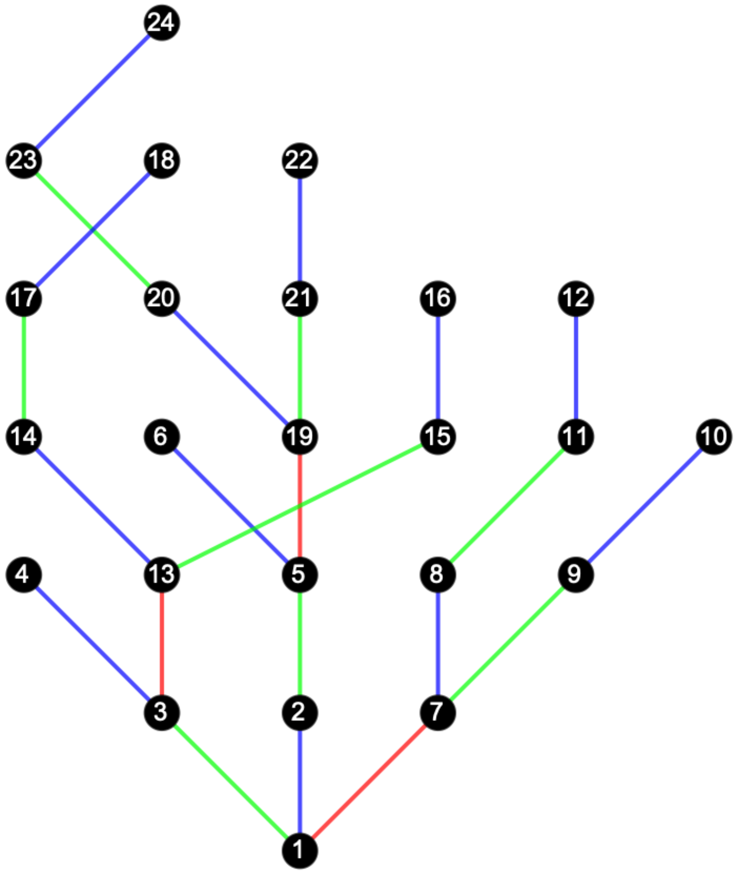


Figure 8.6: The ≪-spanning tree formed from applying Algorithm 8.2.1 to the Cayley graph of Sym(4).

1	1 2 3 4 5		11	1 4 2 5 3	$s_2 s_4 s_3$
2	1 2 3 5 4	s_4	12	1 5 2 4 3	$s_2 s_4 s_3 s_4$
3	1 2 4 3 5	s_3	13	1 3 4 2 5	$s_3 s_2$
4	1 2 5 3 4	$s_3 s_4$	14	1 3 5 2 4	$s_3 s_2 s_4$
5	1 2 4 5 3	$s_4 s_3$	15	1 4 3 2 5	$s_3 s_2 s_3$
6	1 2 5 4 3	$s_4 s_3 s_4$	16	1 5 3 2 4	$s_3 s_2 s_3 s_4$
7	1 3 2 4 5	s_2	17	1 4 5 2 3	$s_3 s_2 s_4 s_3$
8	1 3 2 5 4	$s_2 s_4$	18	1 5 4 2 3	$s_3 s_2 s_4 s_3 s_4$
9	1 4 2 3 5	$s_2 s_3$	19	1 3 4 5 2	$s_4 s_3 s_2$
10	1 5 2 3 4	$s_2 s_3 s_4$	20	1 3 5 4 2	$s_4 s_3 s_2 s_4$

21	1 4 3 5 2	$s_4s_3s_2s_3$	31	3 1 2 4 5	s_1s_2
22	1 5 3 4 2	$s_4s_3s_2s_3s_4$	32	3 1 2 5 4	$s_1s_2s_4$
23	1 4 5 3 2	$s_4s_3s_2s_4s_3$	33	4 1 2 3 5	$s_1s_2s_3$
24	1 5 4 3 2	$s_4s_3s_2s_4s_3s_4$	34	5 1 2 3 4	$s_1s_2s_3s_4$
25	2 1 3 4 5	s_1	35	4 1 2 5 3	$s_1s_2s_4s_3$
26	2 1 3 5 4	s_1s_4	36	5 1 2 4 3	$s_1s_2s_4s_3s_4$
27	2 1 4 3 5	s_1s_3	37	3 1 4 2 5	$s_1s_3s_2$
28	2 1 5 3 4	$s_1s_3s_4$	38	3 1 5 2 4	$s_1s_3s_2s_4$
29	2 1 4 5 3	$s_1s_4s_3$	39	4 1 3 2 5	$s_1s_3s_2s_3$
30	2 1 5 4 3	$s_1s_4s_3s_4$	40	5 1 3 2 4	$s_1s_3s_2s_3s_4$

41	4 1 5 2 3	$s_1s_3s_2s_4s_3$	51	2 4 1 3 5	$s_2s_1s_3$
42	5 1 4 2 3	$s_1s_3s_2s_4s_3s_4$	52	2 5 1 3 4	$s_2s_1s_3s_4$
43	3 1 4 5 2	$s_1s_4s_3s_2$	53	2 4 1 5 3	$s_2s_1s_4s_3$
44	3 1 5 4 2	$s_1s_4s_3s_2s_4$	54	2 5 1 4 3	$s_2s_1s_4s_3s_4$
45	4 1 3 5 2	$s_1s_4s_3s_2s_3$	55	3 2 1 4 5	$s_2s_1s_2$
46	5 1 3 4 2	$s_1s_4s_3s_2s_3s_4$	56	3 2 1 5 4	$s_2s_1s_2s_4$
47	4 1 5 3 2	$s_1s_4s_3s_2s_4s_3$	57	4 2 1 3 5	$s_2s_1s_2s_3$
48	5 1 4 3 2	$s_1s_4s_3s_2s_4s_3s_4$	58	5 2 1 3 4	$s_2s_1s_2s_3s_4$
49	2 3 1 4 5	s_2s_1	59	4 2 1 5 3	$s_2s_1s_2s_4s_3$
50	2 3 1 5 4	$s_2s_1s_4$	60	5 2 1 4 3	$s_2s_1s_2s_4s_3s_4$

61	3 4 1 2 5	$s_2s_1s_3s_2$	71	4 5 1 3 2	$s_2s_1s_4s_3s_2s_4s_3$
62	3 5 1 2 4	$s_2s_1s_3s_2s_4$	72	5 4 1 3 2	$s_2s_1s_4s_3s_2s_4s_3s_4$
63	4 3 1 2 5	$s_2s_1s_3s_2s_3$	73	2 3 4 1 5	$s_3s_2s_1$
64	5 3 1 2 4	$s_2s_1s_3s_2s_3s_4$	74	2 3 5 1 4	$s_3s_2s_1s_4$
65	4 5 1 2 3	$s_2s_1s_3s_2s_4s_3$	75	2 4 3 1 5	$s_3s_2s_1s_3$
66	5 4 1 2 3	$s_2s_1s_3s_2s_4s_3s_4$	76	2 5 3 1 4	$s_3s_2s_1s_3s_4$
67	3 4 1 5 2	$s_2s_1s_4s_3s_2$	77	2 4 5 1 3	$s_3s_2s_1s_4s_3$
68	3 5 1 4 2	$s_2s_1s_4s_3s_2s_4$	78	2 5 4 1 3	$s_3s_2s_1s_4s_3s_4$
69	4 3 1 5 2	$s_2s_1s_4s_3s_2s_3$	79	3 2 4 1 5	$s_3s_2s_1s_2$
70	5 3 1 4 2	$s_2s_1s_4s_3s_2s_3s_4$	80	3 2 5 1 4	$s_3s_2s_1s_2s_4$

81	4 2 3 1 5	$s_3s_2s_1s_2s_3$	91	3 4 5 1 2	$s_3s_2s_1s_4s_3s_2$
82	5 2 3 1 4	$s_3s_2s_1s_2s_3s_4$	92	3 5 4 1 2	$s_3s_2s_1s_4s_3s_2s_4$
83	4 2 5 1 3	$s_3s_2s_1s_2s_4s_3$	93	4 3 5 1 2	$s_3s_2s_1s_4s_3s_2s_3$
84	5 2 4 1 3	$s_3s_2s_1s_2s_4s_3s_4$	94	5 3 4 1 2	$s_3s_2s_1s_4s_3s_2s_3s_4$
85	3 4 2 1 5	$s_3s_2s_1s_3s_2$	95	4 5 3 1 2	$s_3s_2s_1s_4s_3s_2s_4s_3$
86	3 5 2 1 4	$s_3s_2s_1s_3s_2s_4$	96	5 4 3 1 2	$s_3s_2s_1s_4s_3s_2s_4s_3s_4$
87	4 3 2 1 5	$s_3s_2s_1s_3s_2s_3$	97	2 3 4 5 1	$s_4s_3s_2s_1$
88	5 3 2 1 4	$s_3s_2s_1s_3s_2s_3s_4$	98	2 3 5 4 1	$s_4s_3s_2s_1s_4$
89	4 5 2 1 3	$s_3s_2s_1s_3s_2s_4s_3$	99	2 4 3 5 1	$s_4s_3s_2s_1s_3$
90	5 4 2 1 3	$s_3s_2s_1s_3s_2s_4s_3s_4$	100	2 5 3 4 1	$s_4s_3s_2s_1s_3s_4$

101	2 4 5 3 1	$s_4s_3s_2s_1s_4s_3$	111	4 3 2 5 1	$s_4s_3s_2s_1s_3s_2s_3$
102	2 5 4 3 1	$s_4s_3s_2s_1s_4s_3s_4$	112	5 3 2 4 1	$s_4s_3s_2s_1s_3s_2s_3s_4$
103	3 2 4 5 1	$s_4s_3s_2s_1s_2$	113	4 5 2 3 1	$s_4s_3s_2s_1s_3s_2s_4s_3$
104	3 2 5 4 1	$s_4s_3s_2s_1s_2s_4$	114	5 4 2 3 1	$s_4s_3s_2s_1s_3s_2s_4s_3s_4$
105	4 2 3 5 1	$s_4s_3s_2s_1s_2s_3$	115	3 4 5 2 1	$s_4s_3s_2s_1s_4s_3s_2$
106	5 2 3 4 1	$s_4s_3s_2s_1s_2s_3s_4$	116	3 5 4 2 1	$s_4s_3s_2s_1s_4s_3s_2s_4$
107	4 2 5 3 1	$s_4s_3s_2s_1s_2s_4s_3$	117	4 3 5 2 1	$s_4s_3s_2s_1s_4s_3s_2s_3$
108	5 2 4 3 1	$s_4s_3s_2s_1s_2s_4s_3s_4$	118	5 3 4 2 1	$s_4s_3s_2s_1s_4s_3s_2s_3s_4$
109	3 4 2 5 1	$s_4s_3s_2s_1s_3s_2$	119	4 5 3 2 1	$s_4s_3s_2s_1s_4s_3s_2s_4s_3$
110	3 5 2 4 1	$s_4s_3s_2s_1s_3s_2s_4$	120	5 4 3 2 1	$s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4$

Table 8.2: The $\ll$-ordering on Sym(5). From left to right for each column: the `L` value, the one-line form of the permutation and the $\ll$-normal form.

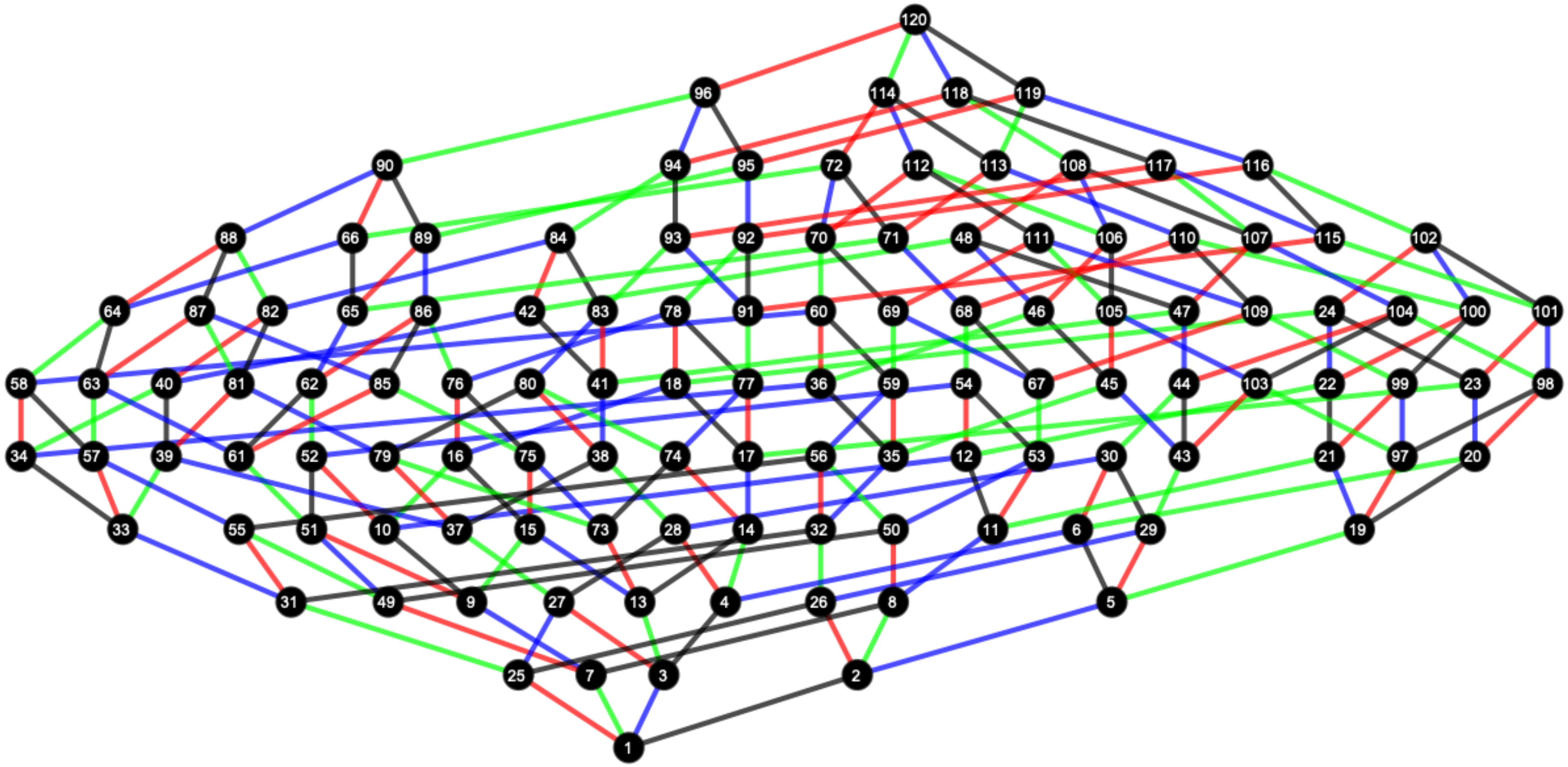


Figure 8.7: The Cayley graph of Sym(5) labelled by L.

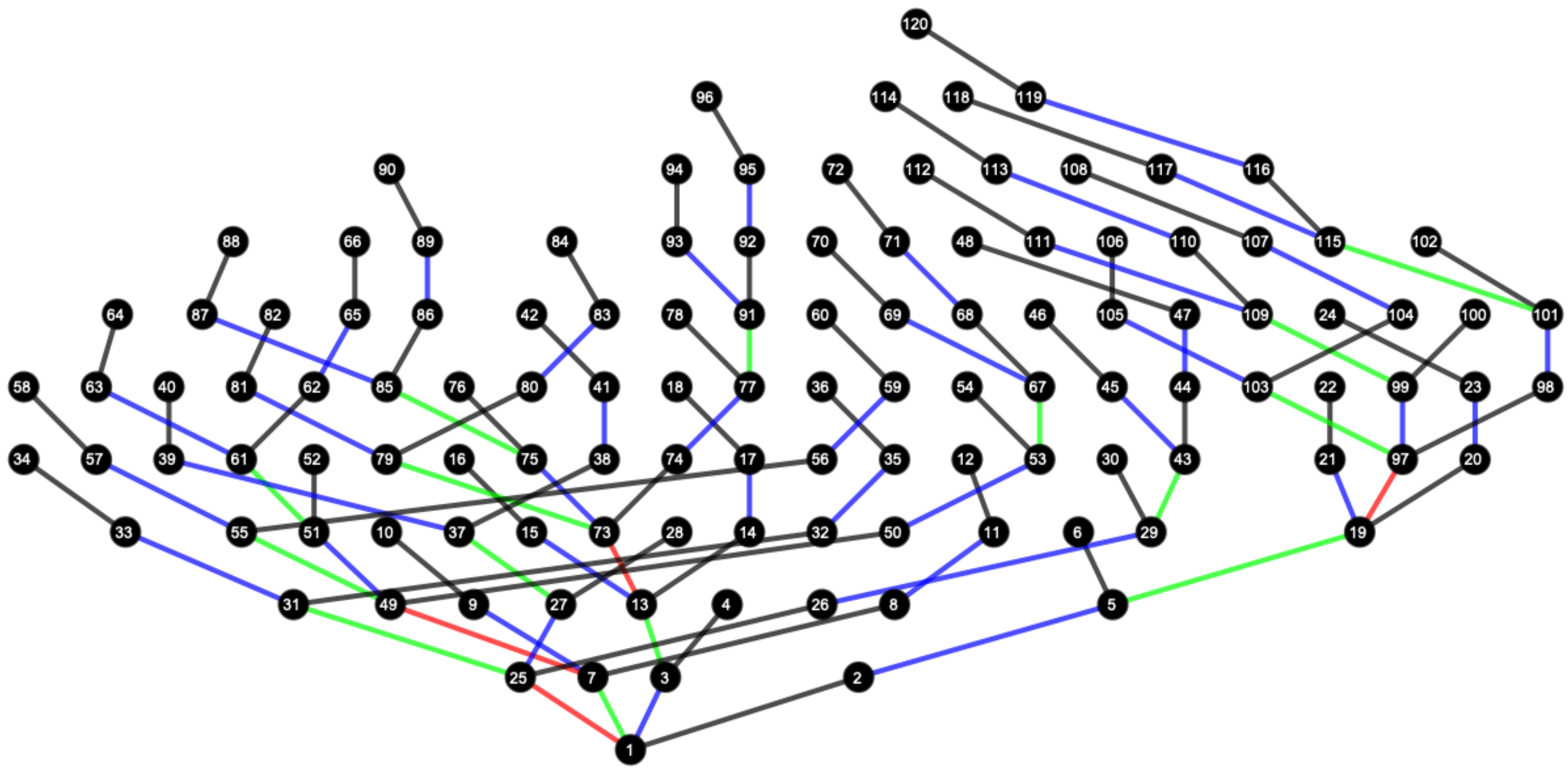


Figure 8.8: The $\ll$-spanning tree formed from applying Algorithm 8.2.1 applied to the Cayley graph of Sym(5).

1		25	$s_1 s_2 s_1$
2	s_3	26	$s_1 s_2 s_1 s_3$
3	s_2	27	$s_1 s_2 s_1 s_2$
4	$s_2 s_3$	28	$s_1 s_2 s_1 s_2 s_3$
5	$s_3 s_2$	29	$s_1 s_2 s_1 s_3 s_2$
6	$s_3 s_2 s_3$	30	$s_1 s_2 s_1 s_3 s_2 s_3$
7	s_1	31	$s_1 s_3 s_2 s_1$
8	$s_1 s_3$	32	$s_1 s_3 s_2 s_1 s_3$
9	$s_1 s_2$	33	$s_1 s_3 s_2 s_1 s_2$
10	$s_1 s_2 s_3$	34	$s_1 s_3 s_2 s_1 s_2 s_3$
11	$s_1 s_3 s_2$	35	$s_1 s_3 s_2 s_1 s_3 s_2$
12	$s_1 s_3 s_2 s_3$	36	$s_1 s_3 s_2 s_1 s_3 s_2 s_3$
13	$s_2 s_1$	37	$s_2 s_1 s_3 s_2 s_1$
14	$s_2 s_1 s_3$	38	$s_2 s_1 s_3 s_2 s_1 s_3$
15	$s_2 s_1 s_2$	39	$s_2 s_1 s_3 s_2 s_1 s_2$
16	$s_2 s_1 s_2 s_3$	40	$s_2 s_1 s_3 s_2 s_1 s_2 s_3$
17	$s_2 s_1 s_3 s_2$	41	$s_2 s_1 s_3 s_2 s_1 s_3 s_2$
18	$s_2 s_1 s_3 s_2 s_3$	42	$s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_3$
19	$s_3 s_2 s_1$	43	$s_1 s_2 s_1 s_3 s_2 s_1$
20	$s_3 s_2 s_1 s_3$	44	$s_1 s_2 s_1 s_3 s_2 s_1 s_3$
21	$s_3 s_2 s_1 s_2$	45	$s_1 s_2 s_1 s_3 s_2 s_1 s_2$
22	$s_3 s_2 s_1 s_2 s_3$	46	$s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3$
23	$s_3 s_2 s_1 s_3 s_2$	47	$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2$
24	$s_3 s_2 s_1 s_3 s_2 s_3$	48	$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_3$

Table 8.3: The $\ll$-ordering on B_3. From left to right: the `L` value, the one-line form of the permutation and their induced $\ll$-normal form.

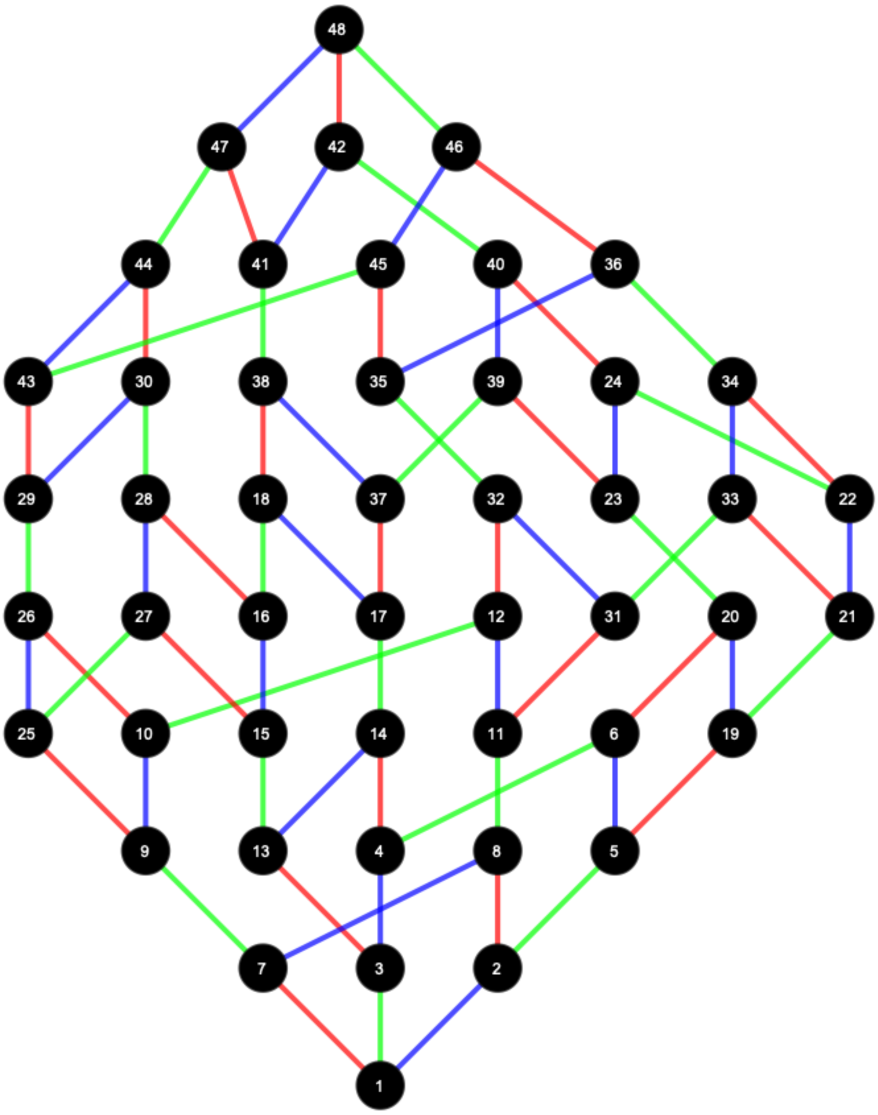


Figure 8.9: The Cayley graph of B_3 labelled by `L`.

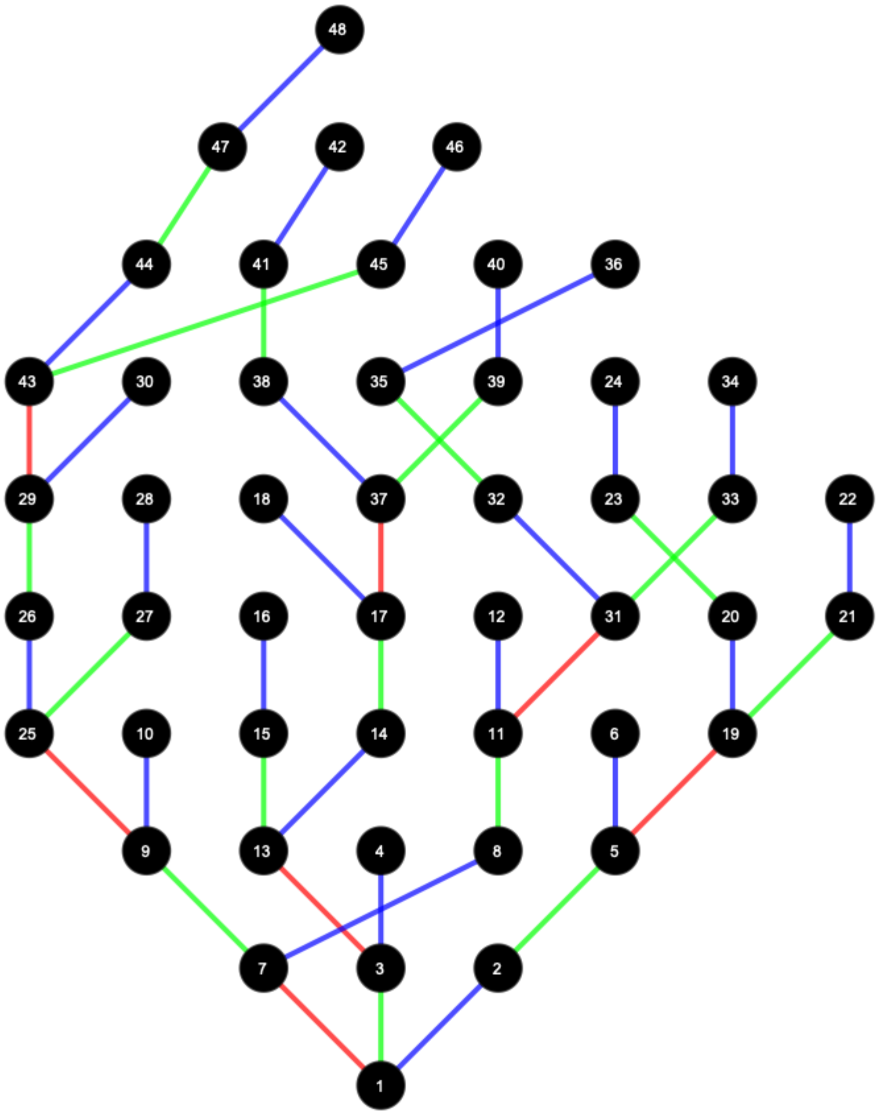


Figure 8.10: The ≪-spanning tree formed from applying Algorithm 8.2.1 applied to the Cayley graph of B_3.

1		36	$s_3s_2s_1s_3s_2s_3$
2	s_3	37	$s_3s_2s_1s_2s_3s_2$
3	s_2	38	$s_3s_2s_1s_2s_3s_2s_3$
4	s_2s_3	39	$s_3s_2s_1s_3s_2s_3s_2$
5	s_3s_2	40	$s_3s_2s_1s_3s_2s_3s_2s_3$
6	$s_3s_2s_3$	41	$s_2s_3s_2s_1$
7	$s_2s_3s_2$	42	$s_2s_3s_2s_1s_3$
8	$s_2s_3s_2s_3$	43	$s_2s_3s_2s_1s_2$
9	$s_3s_2s_3s_2$	44	$s_2s_3s_2s_1s_2s_3$
10	$s_3s_2s_3s_2s_3$	45	$s_2s_3s_2s_1s_3s_2$
11	s_1	46	$s_2s_3s_2s_1s_3s_2s_3$
12	s_1s_3	47	$s_2s_3s_2s_1s_2s_3s_2$
13	s_1s_2	48	$s_2s_3s_2s_1s_2s_3s_2s_3$
14	$s_1s_2s_3$	49	$s_2s_3s_2s_1s_3s_2s_3s_2$
15	$s_1s_3s_2$	50	$s_2s_3s_2s_1s_3s_2s_3s_2s_3$
16	$s_1s_3s_2s_3$	51	$s_3s_2s_3s_2s_1$
17	$s_1s_2s_3s_2$	52	$s_3s_2s_3s_2s_1s_3$
18	$s_1s_2s_3s_2s_3$	53	$s_3s_2s_3s_2s_1s_2$
19	$s_1s_3s_2s_3s_2$	54	$s_3s_2s_3s_2s_1s_2s_3$
20	$s_1s_3s_2s_3s_2s_3$	55	$s_3s_2s_3s_2s_1s_3s_2$
21	s_2s_1	56	$s_3s_2s_3s_2s_1s_3s_2s_3$
22	$s_2s_1s_3$	57	$s_3s_2s_3s_2s_1s_2s_3s_2$
23	$s_2s_1s_2$	58	$s_3s_2s_3s_2s_1s_2s_3s_2s_3$
24	$s_2s_1s_2s_3$	59	$s_3s_2s_3s_2s_1s_3s_2s_3s_2$
25	$s_2s_1s_3s_2$	60	$s_3s_2s_3s_2s_1s_3s_2s_3s_2s_3$
26	$s_2s_1s_3s_2s_3$	61	$s_1s_2s_3s_2s_1$
27	$s_2s_1s_2s_3s_2$	62	$s_1s_2s_3s_2s_1s_3$
28	$s_2s_1s_2s_3s_2s_3$	63	$s_1s_2s_3s_2s_1s_2$
29	$s_2s_1s_3s_2s_3s_2$	64	$s_1s_2s_3s_2s_1s_2s_3$
30	$s_2s_1s_3s_2s_3s_2s_3$	65	$s_1s_2s_3s_2s_1s_3s_2$
31	$s_3s_2s_1$	66	$s_1s_2s_3s_2s_1s_3s_2s_3$
32	$s_3s_2s_1s_3$	67	$s_1s_2s_3s_2s_1s_2s_3s_2$
33	$s_3s_2s_1s_2$	68	$s_1s_2s_3s_2s_1s_2s_3s_2s_3$
34	$s_3s_2s_1s_2s_3$	69	$s_1s_2s_3s_2s_1s_3s_2s_3s_2$
35	$s_3s_2s_1s_3s_2$	70	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_3$

71	$s_1s_3s_2s_3s_2s_1$	96	$s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3$
72	$s_1s_3s_2s_3s_2s_1s_3$	97	$s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2$
73	$s_1s_3s_2s_3s_2s_1s_2$	98	$s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2s_3$
74	$s_1s_3s_2s_3s_2s_1s_2s_3$	99	$s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2$
75	$s_1s_3s_2s_3s_2s_1s_3s_2$	100	$s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2s_3$
76	$s_1s_3s_2s_3s_2s_1s_3s_2s_3$	101	$s_2s_3s_2s_1s_3s_2s_3s_2s_1$
77	$s_1s_3s_2s_3s_2s_1s_2s_3s_2$	102	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3$
78	$s_1s_3s_2s_3s_2s_1s_2s_3s_2s_3$	103	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2$
79	$s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2$	104	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3$
80	$s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2s_3$	105	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2$
81	$s_2s_1s_3s_2s_3s_2s_1$	106	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3$
82	$s_2s_1s_3s_2s_3s_2s_1s_3$	107	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2$
83	$s_2s_1s_3s_2s_3s_2s_1s_2$	108	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2s_3$
84	$s_2s_1s_3s_2s_3s_2s_1s_2s_3$	109	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2$
85	$s_2s_1s_3s_2s_3s_2s_1s_3s_2$	110	$s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2s_3$
86	$s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3$	111	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1$
87	$s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2$	112	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3$
88	$s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2s_3$	113	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2$
89	$s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2$	114	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3$
90	$s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2s_3$	115	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2$
91	$s_3s_2s_1s_3s_2s_3s_2s_1$	116	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3$
92	$s_3s_2s_1s_3s_2s_3s_2s_1s_3$	117	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2$
93	$s_3s_2s_1s_3s_2s_3s_2s_1s_2$	118	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3s_2s_3$
94	$s_3s_2s_1s_3s_2s_3s_2s_1s_2s_3$	119	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2$
95	$s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2$	120	$s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_3s_2s_3$

Table 8.4: The $\ll$-order and $\ll$-normal form for elements of H_3.

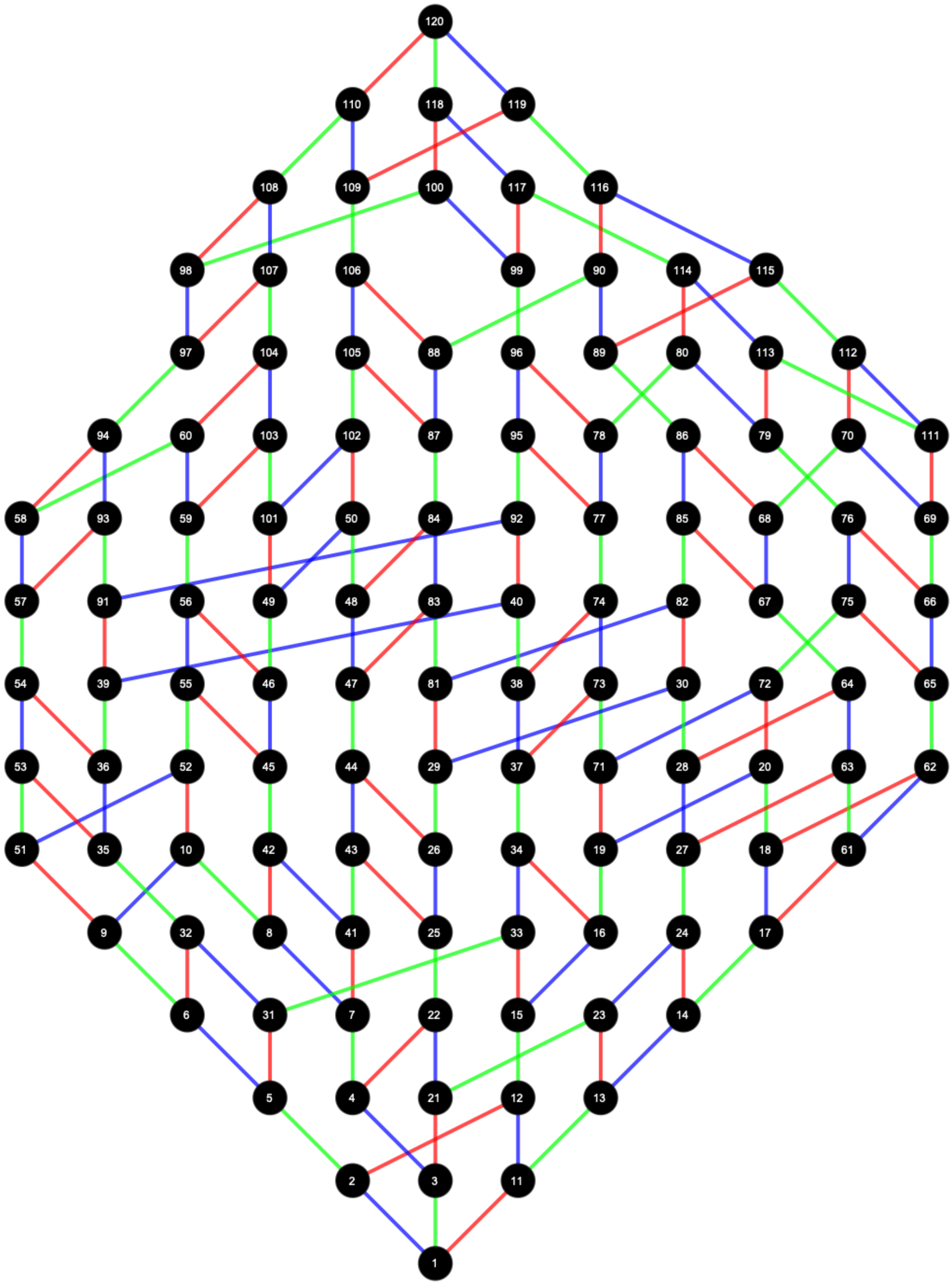


Figure 8.11: The Cayley graph of H_3 labelled by `L`.

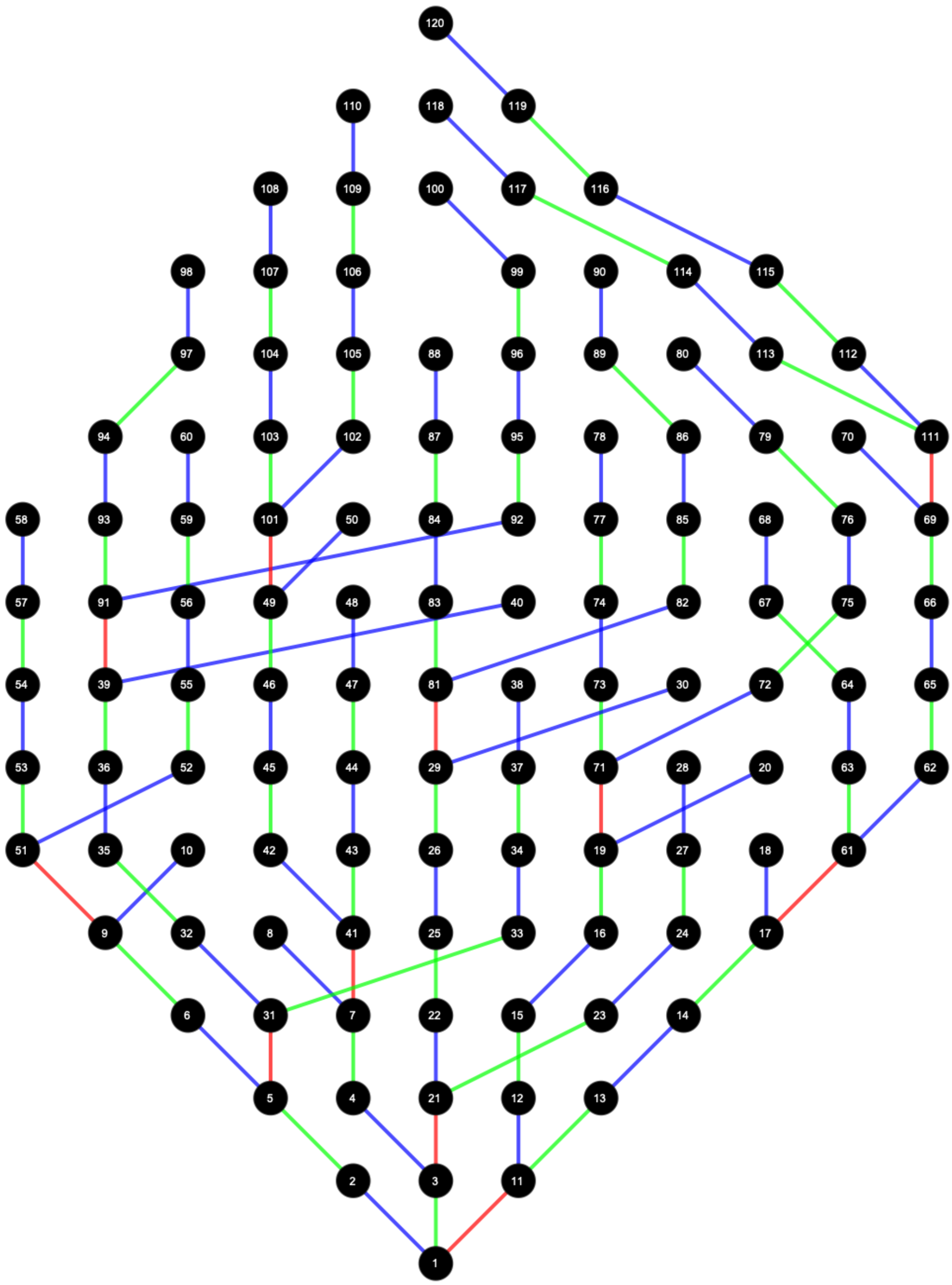


Figure 8.12: The $\ll$-spanning tree of H_3.

Some striking patterns emerge that we tentatively note down as speculation and conjecture. These are not statements we claim to have proven but are just those patterns which are apparent within these examples that we would aim to prove in further investigation:

(*i*) Repeating earlier claims: $\ll$ refines the Bruhat Order and $\phi^J_{\ll}$ is an E-embedding.

(*ii*) $\overline{\mathtt{NF}}(w)$ seems to be the lexicographic normal form for w^{-1} but read in reverse and with respect to the reverse order of $<_S$; equivalently, read from right-to-left, $\overline{\mathtt{NF}}(w)$ uses the $<_S$-maximal generator available in each position. Consequently, if this is genuinely true, then (see Section 3.4 of [1]) for all $w \in W$ there exists a unique choice of $x_i, \in W^{\{i+1,\ldots,n\}}_{\{i,\ldots,n\}}$ for $i = 1, \ldots, n$ such that $w = x_1 \ldots x_n$ and $\overline{\mathtt{NF}}(w) = \overline{\mathtt{NF}}(x_1) \ldots \overline{\mathtt{NF}}(x_n)$.

(*iii*) Let $\mathtt{L}_0 : W \to \mathbb{Z}$ be given by $\mathtt{L}_0(w) = \mathtt{L}(w) - 1$. That is, $\mathtt{L}_0(w) = i$ if and only if w is the $(i+1)^{th}$ least element in W with respect to $\ll$. Then

$$\mathtt{L}_0(w) = \mathtt{L}_0(x_1) + \mathtt{L}_0(x_2) + \ldots + \mathtt{L}_0(x_n)$$

surprisingly seems to holds for all $w \in W$ in our examples. No such analogous formula exists for the usual lexicographical order.

(*iv*) For all $w \in W$, $\mathtt{L}(w) + \mathtt{L}(\omega_0 w) = |W| + 1$. Equivalently, $\ll$ preserves the anti-automorphism of the Bruhat order of multiplication by ω_0.

(*v*) Algorithm 8.2.1 does not make use of the length function of the group.

(*vi*) If we extend Algorithm 8.2.1 to the Cayley graph of any Coxeter group then it will eventually label every node if and only if $W_{\{s_2,\ldots,s_n\}}$ (with respect to $<_S$) is finite. If this is indeed true then the algorithm terminates regardless of choice of $<_S$ if and only if every proper parabolic subgroup is finite; such groups are classified as being those finite, affine or compact hyperbolic Coxeter groups in [26].

8.4 Permutation representations derived from $\ll$

From $\ll$ we produce the permutation groups $\phi^J_{\ll}$ for the finite irreducible Coxeter groups. For each (W, S) we take $J = S \setminus \{s_n\}$. We will use these permutation groups to test if we produce anything resembling sensible tilings from them.

For Sym(5) we have

$$
\begin{aligned}
s_1 =&(4,5)\\
s_2 =&(3,4)\\
s_3 =&(2,3)\\
s_4 =&(1,2)
\end{aligned}
$$

Figure 8.13: The CPR graph of $\phi^J_{\ll}$ for Sym(5).

Note that this is the reverse order of the emebedding corresponding to Elnitsky's tiling.
For B_3 we have

$$
\begin{aligned}
s_1 =&(3,5)(4,6)\\
s_2 =&(2,3)(6,7)\\
s_3 =&(1,2)(3,4)(5,6)(7,8)
\end{aligned}
$$

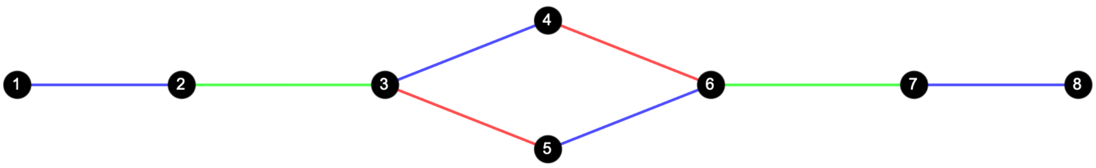


Figure 8.14: The CPR graph of $\phi^J_{\ll}$ for B_3.

For B_4 we have

$$
\begin{aligned}
s_1 =&(5,9)(6,10)(7,11)(8,12)\\
s_2 =&(3,5)(4,6)(11,13)(12,14)\\
s_3 =&(2,3)(6,7)(10,11)(14,15)\\
s_4 =&(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
\end{aligned}
$$

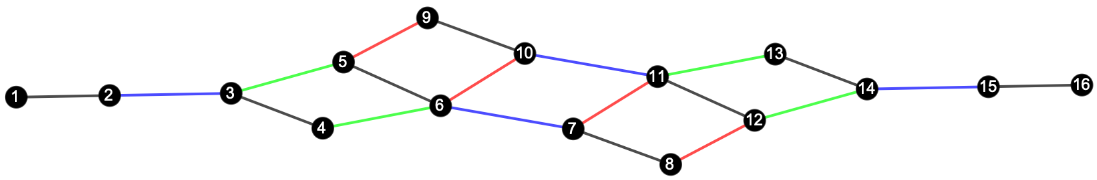


Figure 8.15: The CPR graph of $\phi^J_{\ll}$ for B_4.

For D_4 we have

$$s_1 = (3,5)(4,6)$$
$$s_2 = (2,3)(6,7)$$
$$s_3 = (3,4)(5,6)$$
$$s_4 = (1,2)(7,8)$$

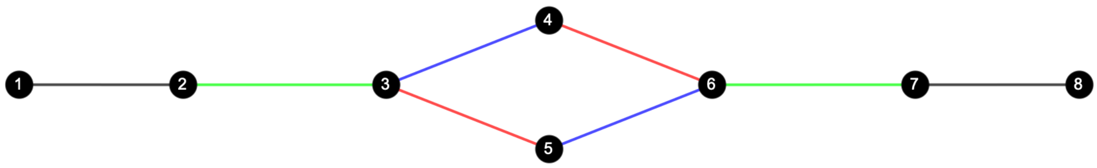


Figure 8.16: The CPR graph of $\phi^J_{\ll}$ for D_4.

For F_4 we have

$$s_1 = (4,7)(5,8)(6,9)(16,19)(17,20)(18,21)$$
$$s_2 = (3,4)(8,10)(9,11)(14,16)(15,17)(21,22)$$
$$s_3 = (2,3)(4,5)(7,8)(10,12)(11,14)(13,15)(17,18)(20,21)(22,23)$$
$$s_4 = (1,2)(5,6)(8,9)(10,11)(12,13)(14,15)(16,17)(19,20)(23,24)$$

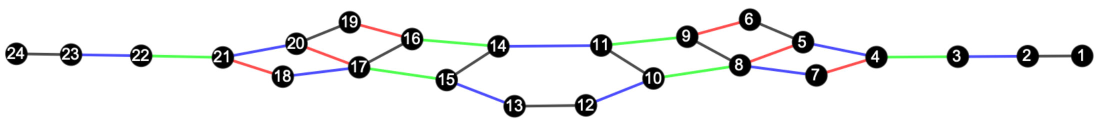


Figure 8.17: The CPR graph of $\phi^J_{\ll}$ for F_4.

For E_6 we have

$$\begin{aligned}
s_1 =&(5,11)(7,12)(8,13)(9,14)(10,15)(26,27)\\
s_2 =&(4,6)(5,7)(11,12)(19,22)(20,23)(21,24)\\
s_3 =&(4,5)(6,7)(13,16)(14,17)(15,18)(25,26)\\
s_4 =&(3,4)(7,8)(12,13)(17,19)(18,20)(24,25)\\
s_5 =&(2,3)(8,9)(13,14)(16,17)(20,21)(23,24)\\
s_6 =&(1,2)(9,10)(14,15)(17,18)(19,20)(22,23)
\end{aligned}$$

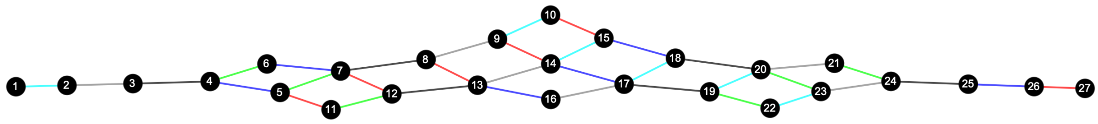


Figure 8.18: The CPR graph of $\phi^J_{\ll}$ for E_6.

For E_7 we have

$$\begin{aligned}
s_1 =&(6,13)(8,14)(9,15)(10,16)(11,17)(12,18)\\
&(39,45)(40,46)(41,47)(42,48)(43,49)(44,51)\\
s_2 =&(5,7)(6,8)(13,14)(23,29)(24,30)(25,31)\\
&(26,32)(27,33)(28,34)(43,44)(49,51)(50,52)\\
s_3 =&(5,6)(7,8)(15,19)(16,20)(17,21)(18,22)\\
&(35,39)(36,40)(37,41)(38,42)(49,50)(51,52)\\
s_4 =&(4,5)(8,9)(14,15)(20,23)(21,24)(22,25)\\
&(32,35)(33,36)(34,37)(42,43)(48,49)(52,53)\\
s_5 =&(3,4)(9,10)(15,16)(19,20)(24,26)(25,27)\\
&(30,32)(31,33)(37,38)(41,42)(47,48)(53,54)\\
s_6 =&(2,3)(10,11)(16,17)(20,21)(23,24)(27,28)\\
&(29,30)(33,34)(36,37)(40,41)(46,47)(54,55)\\
s_7 =&(1,2)(11,12)(17,18)(21,22)(24,25)(26,27)\\
&(30,31)(32,33)(35,36)(39,40)(45,46)(55,56)
\end{aligned}$$

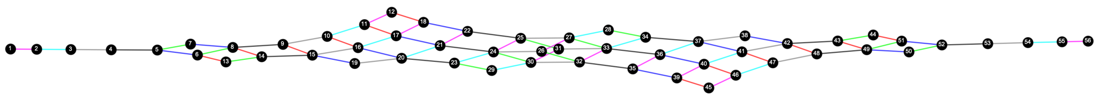


Figure 8.19: The CPR graph of $\phi^J_{\ll}$ for E_7.

For E_8 we have

$$s_1 = (7,15)(9,16)(10,17)(11,18)(12,19)(13,20)(14,21)(57,79)(58,80)(59,81)(60,82)(61,83)(62,84)(63,85)(64,86)(65,87)(66,88)(67,89)(68,90)(69,91)(70,92)(71,93)(72,100)(73,101)(74,102)(75,103)(76,104)(77,105)(78,163)(136,164)(137,165)(138,166)(139,167)(140,168)(141,169)(148,170)(149,171)(150,172)(151,173)(152,174)(153,175)(154,176)(155,177)(156,178)(157,179)(158,180)(159,181)(160,182)(161,183)(162,184)(220,227)(221,228)(222,229)(223,230)(224,231)(225,232)(226,234)$$

$$s_2 = (6,8)(7,9)(15,16)(27,37)(28,38)(29,39)(30,40)(31,41)(32,42)(33,43)(34,44)(35,45)(36,46)(67,72)(68,73)(69,74)(70,75)(71,76)(89,100)(90,101)(91,102)(92,103)(93,104)(94,106)(95,107)(96,108)(97,109)(98,110)(99,142)(131,143)(132,144)(133,145)(134,146)(135,147)(137,148)(138,149)(139,150)(140,151)(141,152)(165,170)(166,171)(167,172)(168,173)(169,174)(195,205)(196,206)(197,207)(198,208)(199,209)(200,210)(201,211)(202,212)(203,213)(204,214)(225,226)(232,234)(233,235)$$

$$s_3 = (6,7)(8,9)(17,22)(18,23)(19,24)(20,25)(21,26)(47,57)(48,58)(49,59)(50,60)(51,61)(52,62)(53,63)(54,64)(55,65)(56,66)(77,78)(89,94)(90,95)(91,96)(92,97)(93,98)(100,106)(101,107)(102,108)(103,109)(104,110)(105,136)(131,137)(132,138)(133,139)(134,140)(135,141)(143,148)(144,149)(145,150)(146,151)(147,152)(163,164)(175,185)(176,186)(177,187)(178,188)(179,189)(180,190)(181,191)(182,192)(183,193)(184,194)(215,220)(216,221)(217,222)(218,223)(219,224)(232,233)(234,235)$$

$$
\begin{aligned}
s_4 =&(5,6)(9,10)(16,17)(23,27)(24,28)(25,29)(26,30)(41,47)(42,48)(43,49)(44,50)\\
&(45,51)(46,52)(63,67)(64,68)(65,69)(66,70)(76,77)(85,89)(86,90)(87,91)\\
&(88,92)(98,99)(104,105)(106,111)(107,112)(108,113)(109,114)(110,131)\\
&(127,132)(128,133)(129,134)(130,135)(136,137)(142,143)(149,153)(150,154)\\
&(151,155)(152,156)(164,165)(171,175)(172,176)(173,177)(174,178)(189,195)\\
&(190,196)(191,197)(192,198)(193,199)(194,200)(211,215)(212,216)(213,217)\\
&(214,218)(224,225)(231,232)(235,236)
\end{aligned}
$$

$$
\begin{aligned}
s_5 =&(4,5)(10,11)(17,18)(22,23)(28,31)(29,32)(30,33)(38,41)(39,42)(40,43)(50,53)\\
&(51,54)(52,55)(60,63)(61,64)(62,65)(70,71)(75,76)(82,85)(83,86)(84,87)\\
&(92,93)(97,98)(103,104)(109,110)(111,115)(112,116)(113,117)(114,127)\\
&(124,128)(125,129)(126,130)(131,132)(137,138)(143,144)(148,149)(154,157)\\
&(155,158)(156,159)(165,166)(170,171)(176,179)(177,180)(178,181)(186,189)\\
&(187,190)(188,191)(198,201)(199,202)(200,203)(208,211)(209,212)(210,213)\\
&(218,219)(223,224)(230,231)(236,237)
\end{aligned}
$$

$$
\begin{aligned}
s_6 =&(3,4)(11,12)(18,19)(23,24)(27,28)(32,34)(33,35)(37,38)(42,44)(43,45)(48,50)\\
&(49,51)(55,56)(58,60)(59,61)(65,66)(69,70)(74,75)(80,82)(81,83)(87,88)\\
&(91,92)(96,97)(102,103)(108,109)(113,114)(115,118)(116,119)(117,124)\\
&(122,125)(123,126)(127,128)(132,133)(138,139)(144,145)(149,150)(153,154)\\
&(158,160)(159,161)(166,167)(171,172)(175,176)(180,182)(181,183)(185,186)\\
&(190,192)(191,193)(196,198)(197,199)(203,204)(206,208)(207,209)(213,214)\\
&(217,218)(222,223)(229,230)(237,238)
\end{aligned}
$$

$$
\begin{aligned}
s_7 = & (2,3)(12,13)(19,20)(24,25)(28,29)(31,32)(35,36)(38,39)(41,42)(45,46)(47,48) \\
& (51,52)(54,55)(57,58)(61,62)(64,65)(68,69)(73,74)(79,80)(83,84)(86,87) \\
& (90,91)(95,96)(101,102)(107,108)(112,113)(116,117)(118,120)(119,122) \\
& (121,123)(124,125)(128,129)(133,134)(139,140)(145,146)(150,151)(154,155) \\
& (157,158)(161,162)(167,168)(172,173)(176,177)(179,180)(183,184)(186,187) \\
& (189,190)(193,194)(195,196)(199,200)(202,203)(205,206)(209,210)(212,213) \\
& (216,217)(221,222)(228,229)(238,239)
\end{aligned}
$$

$$
\begin{aligned}
s_8 = & (1,2)(13,14)(20,21)(25,26)(29,30)(32,33)(34,35)(39,40)(42,43)(44,45)(48,49) \\
& (50,51)(53,54)(58,59)(60,61)(63,64)(67,68)(72,73)(80,81)(82,83)(85,86) \\
& (89,90)(94,95)(100,101)(106,107)(111,112)(115,116)(118,119)(120,121) \\
& (122,123)(125,126)(129,130)(134,135)(140,141)(146,147)(151,152)(155,156) \\
& (158,159)(160,161)(168,169)(173,174)(177,178)(180,181)(182,183)(187,188) \\
& (190,191)(192,193)(196,197)(198,199)(201,202)(206,207)(208,209)(211,212) \\
& (215,216)(220,221)(227,228)(239,240)
\end{aligned}
$$

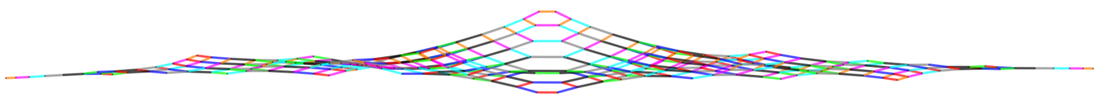


Figure 8.20: The CPR graph of $\phi^J_{\ll}$ for E_8 with the labelling of vertices omitted.

For H_3 we have

$$
\begin{aligned}
s_1 = & (3,4)(5,7)(6,8)(9,10) \\
s_2 = & (2,3)(4,5)(8,9)(10,11) \\
s_3 = & (1,2)(5,6)(7,8)(11,12)
\end{aligned}
$$

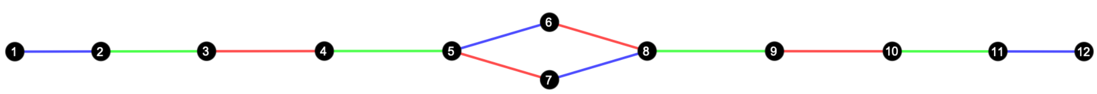


Figure 8.21: The CPR graph of $\phi^J_{\ll}$ for H_3.

For H_4 we have

$$s_1 =(4,5)(6,9)(7,10)(8,11)(12,15)(13,16)(14,17)(24,27)(25,28)(26,29)(33,39)$$
$$(34,40)(35,41)(36,42)(37,43)(38,44)(45,51)(46,52)(47,53)(48,55)(49,56)$$
$$(50,57)(54,67)(64,71)(65,72)(66,73)(68,74)(69,75)(70,76)(77,83)(78,84)$$
$$(79,85)(80,86)(81,87)(82,88)(92,95)(93,96)(94,97)(104,107)(105,108)$$
$$(106,109)(110,113)(111,114)(112,115)(116,117)$$

$$s_2 =(3,4)(5,6)(10,12)(11,13)(15,18)(16,19)(17,24)(22,25)(23,26)(27,30)(28,33)$$
$$(29,34)(31,35)(32,36)(41,45)(42,46)(43,48)(44,49)(47,50)(53,54)(55,58)$$
$$(56,59)(57,64)(62,65)(63,66)(67,68)(71,74)(72,77)(73,78)(75,79)(76,80)$$
$$(85,89)(86,90)(87,92)(88,93)(91,94)(95,98)(96,99)(97,104)(102,105)$$
$$(103,106)(108,110)(109,111)(115,116)(117,118)$$

$$s_3 =(2,3)(6,7)(9,10)(13,14)(16,17)(18,20)(19,22)(21,23)(24,25)(27,28)(30,31)$$
$$(33,35)(34,37)(36,38)(39,41)(40,43)(42,44)(46,47)(49,50)(52,53)(56,57)$$
$$(58,60)(59,62)(61,63)(64,65)(68,69)(71,72)(74,75)(77,79)(78,81)(80,82)$$
$$(83,85)(84,87)(86,88)(90,91)(93,94)(96,97)(98,100)(99,102)(101,103)$$
$$(104,105)(107,108)(111,112)(114,115)(118,119)$$

$$s_4 =(1,2)(7,8)(10,11)(12,13)(15,16)(18,19)(20,21)(22,23)(25,26)(28,29)(31,32)$$
$$(33,34)(35,36)(37,38)(39,40)(41,42)(43,44)(45,46)(48,49)(51,52)(55,56)$$
$$(58,59)(60,61)(62,63)(65,66)(69,70)(72,73)(75,76)(77,78)(79,80)(81,82)$$
$$(83,84)(85,86)(87,88)(89,90)(92,93)(95,96)(98,99)(100,101)(102,103)$$
$$(105,106)(108,109)(110,111)(113,114)(119,120)$$

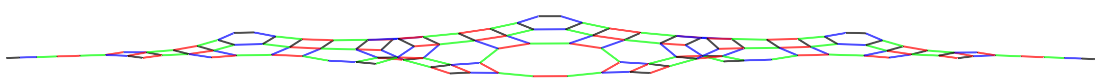


Figure 8.22: The CPR graph of $\phi^J_{\ll}$ for H_4.

8.5 Tilings produced

We mention here what conventions we will use for each such example; we will need a choice of embedding for each group and some reduced word for which we will display its associated tiling.

For the embedding: we use $\phi_J^{\leqslant}$ for each group from Section 8.4.

For the word: we take the bipartite alternating word of the longest element of the group.

Definition 8.5.1. *One can see directly from the classification that the Coxeter diagram for finite irreducible Coxeter group is bipartite. Partition the generators into these induces equivalence classes, $S = A \sqcup B$. Write $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$ and say that $s_1 \in A$. Then define $\alpha = a_1 \ldots a_p$ and $\beta = b_1, \ldots, b_q$ and fix some ordering for each to produce a reduced word for the elements. We say the* bipartite alternating word *for ω_0 is*

$$\omega_0 = \underbrace{\alpha\beta\alpha\ldots}_{h}$$

where h is the Coxeter number as defined in Section 3.16 of [26]. We know, by [42] for example, that this word is reduced for all finite irreducible Coxeter groups.

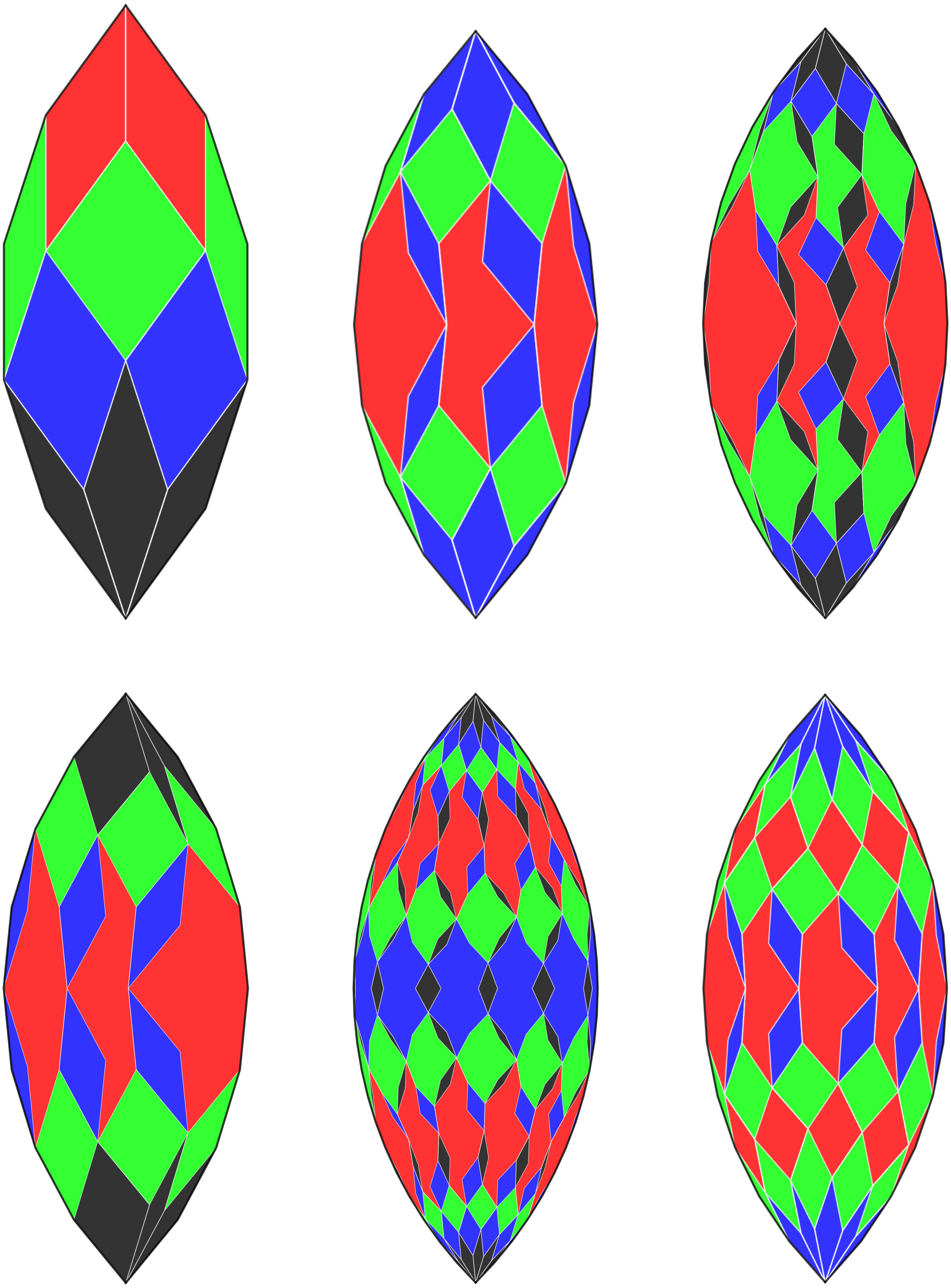


Figure 8.23: Tilings for Sym(5), B_3 and B_4 (above, left-to-right), and D_4, F_4 and, H_3 (below, left-to-right) with $\alpha = \pi/4$.

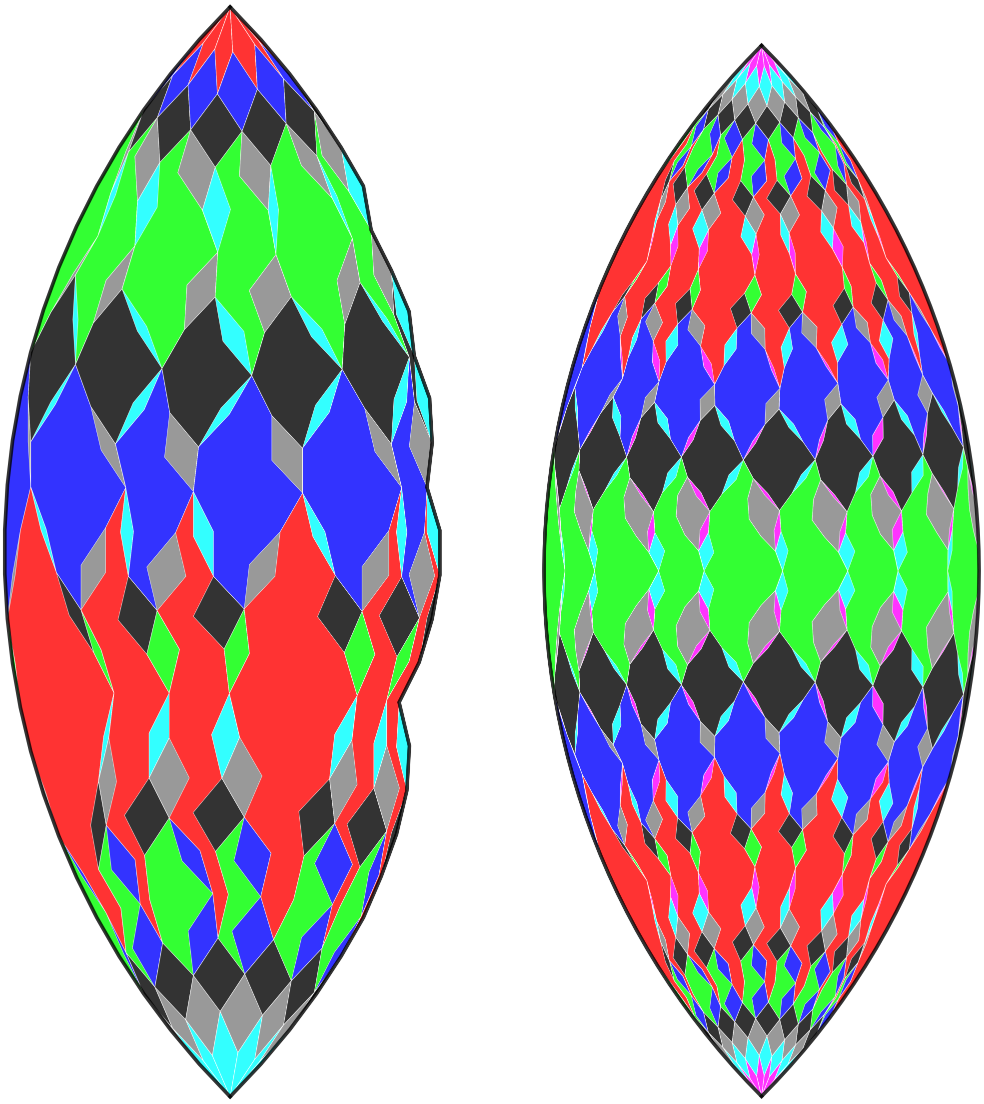


Figure 8.24: Tilings for (left) E_6 and (right) E_7 with $\alpha = \pi/4$.

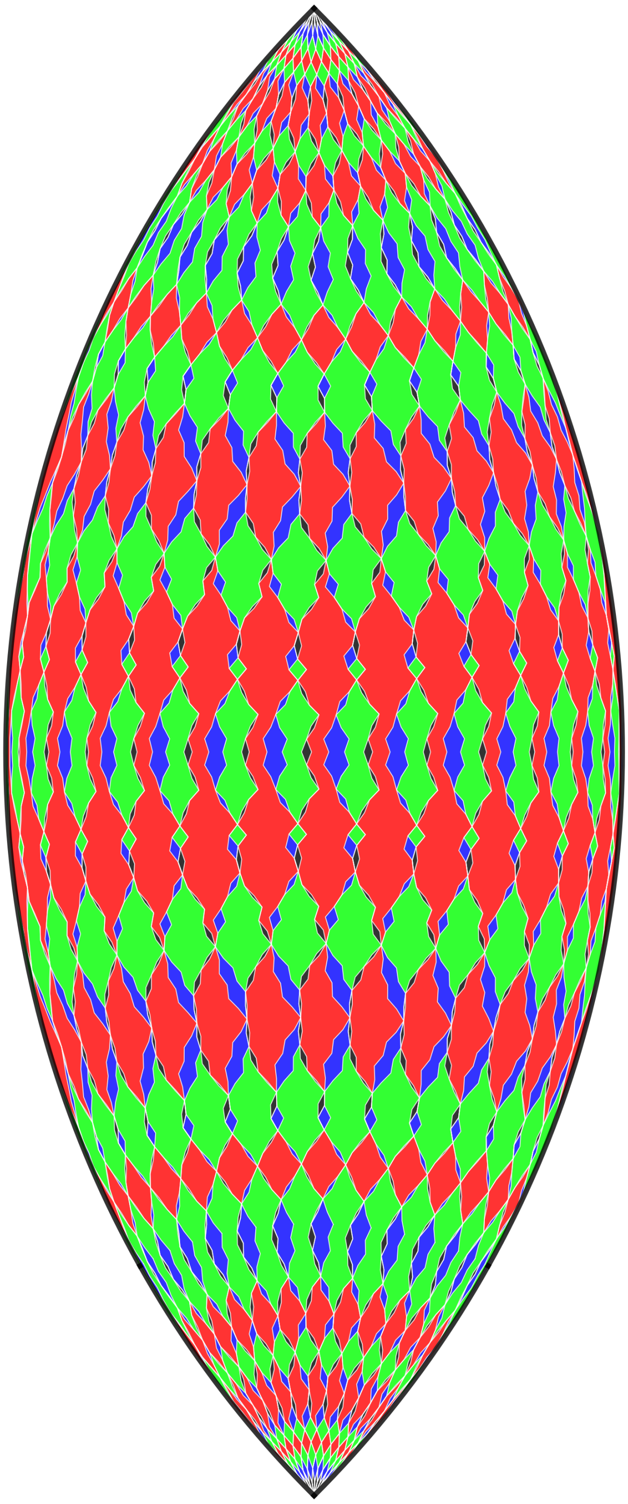


Figure 8.25: Tilings for H_4 with $\alpha = \pi/4$.

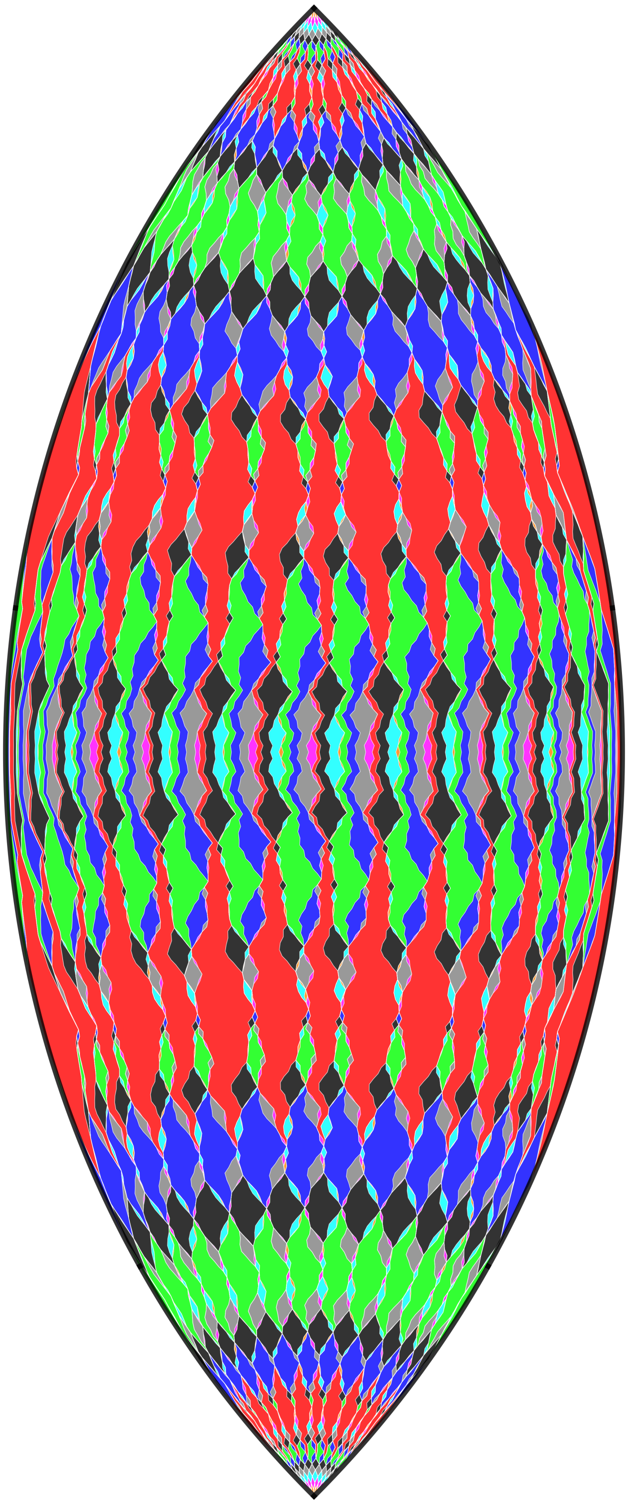


Figure 8.26: Tilings for E_8 with $\alpha = \pi/4$.

We also display the corresponding regular polygon constructions ($\alpha = \pi/n$).

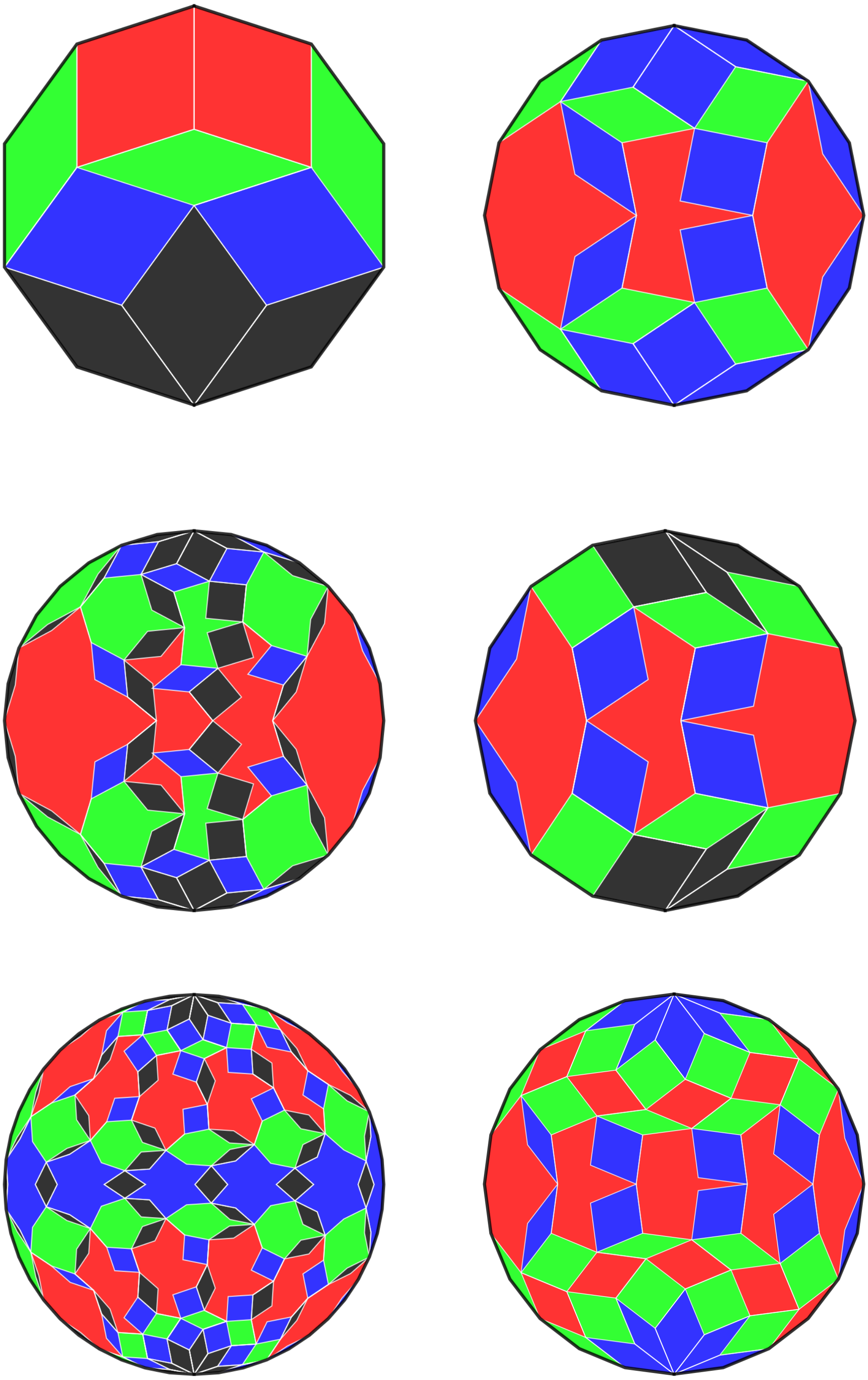


Figure 8.27: Tilings for Sym(5), B_3, B_4, D_4, F_4 and, H_3 (read left-to-right then top-to-bottom) with $\alpha = \pi/n$.

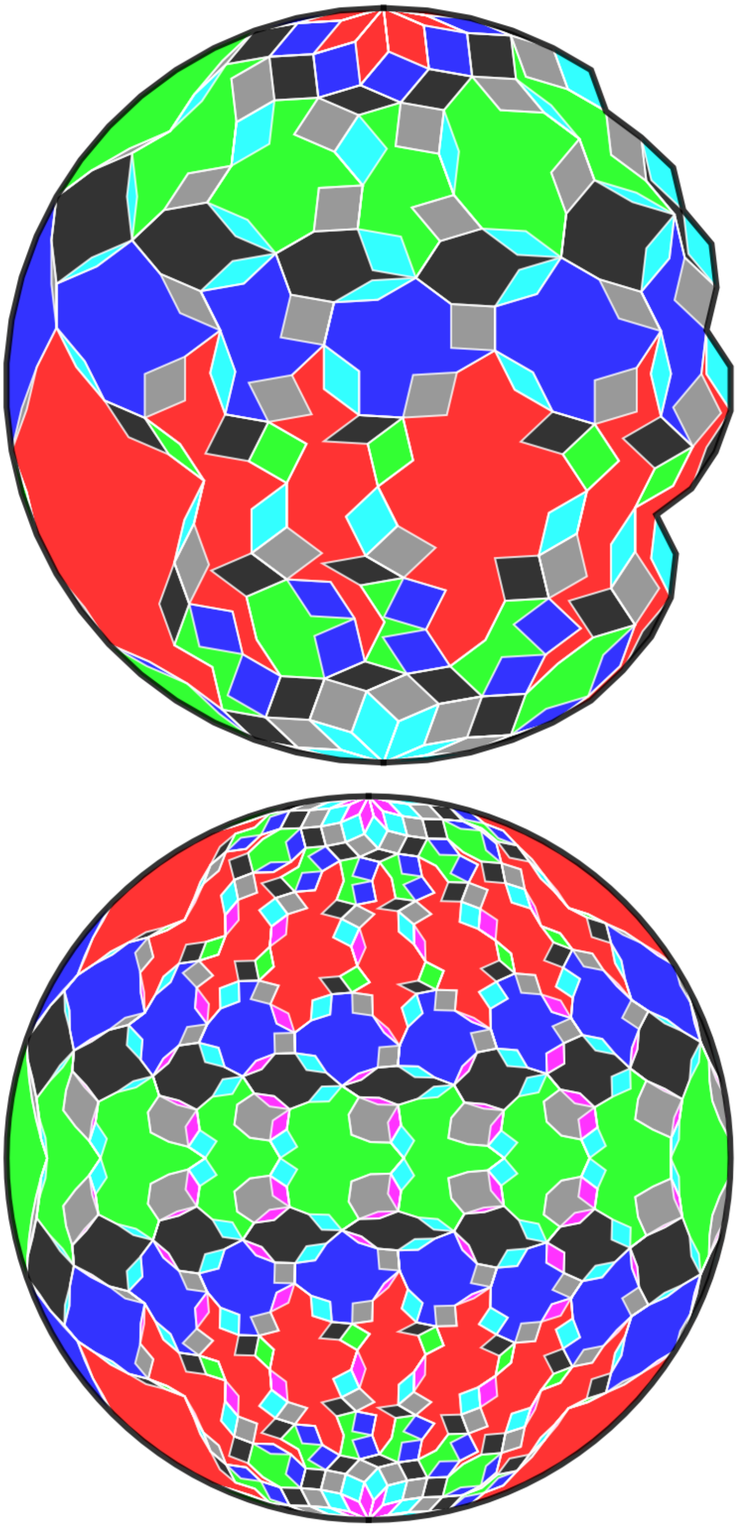


Figure 8.28: Tilings for (above) E_6 and (below) E_7 with $\alpha = \pi/n$.

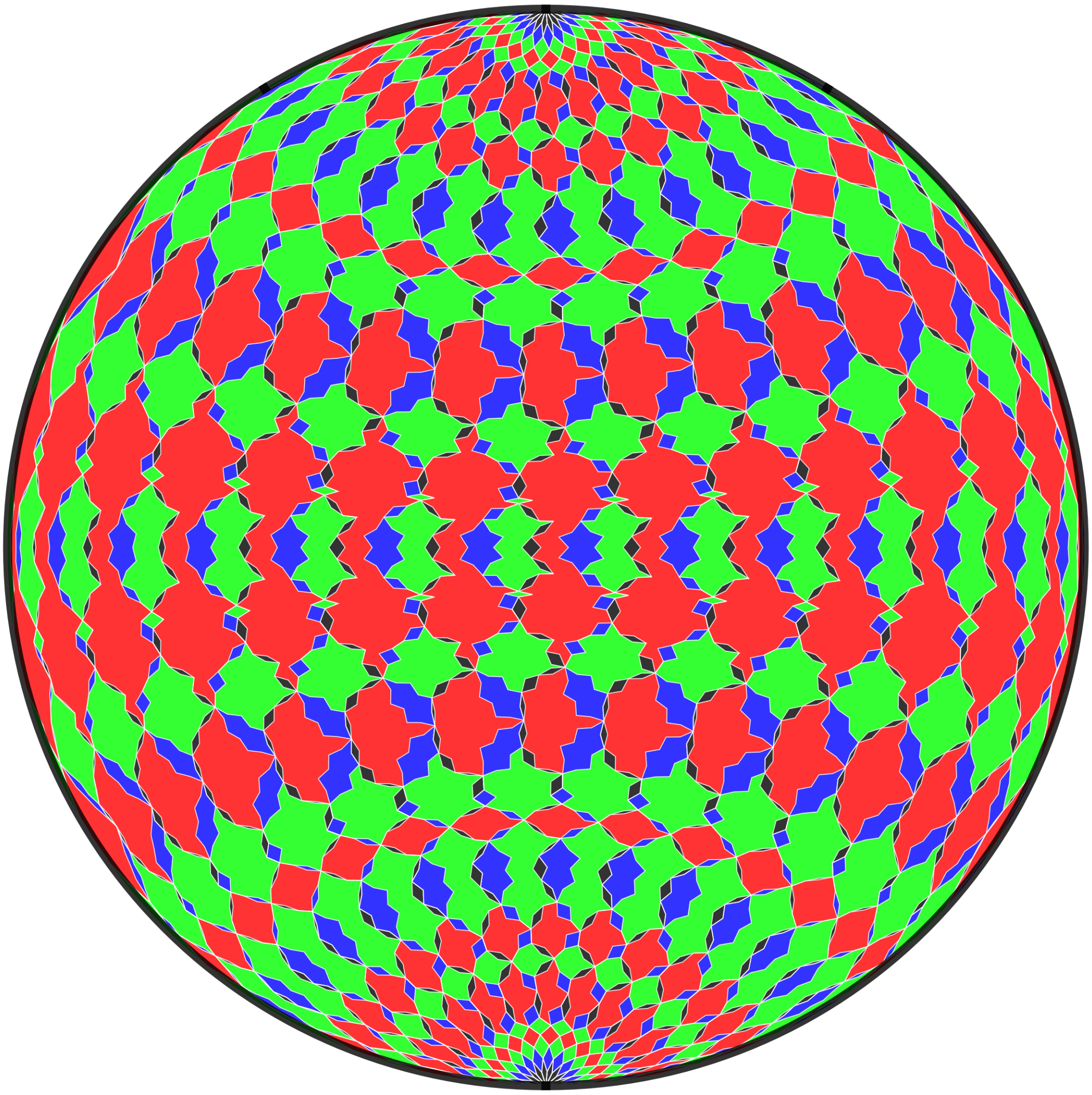


Figure 8.29: Tiling of H_4 with $\alpha = \pi/n$.

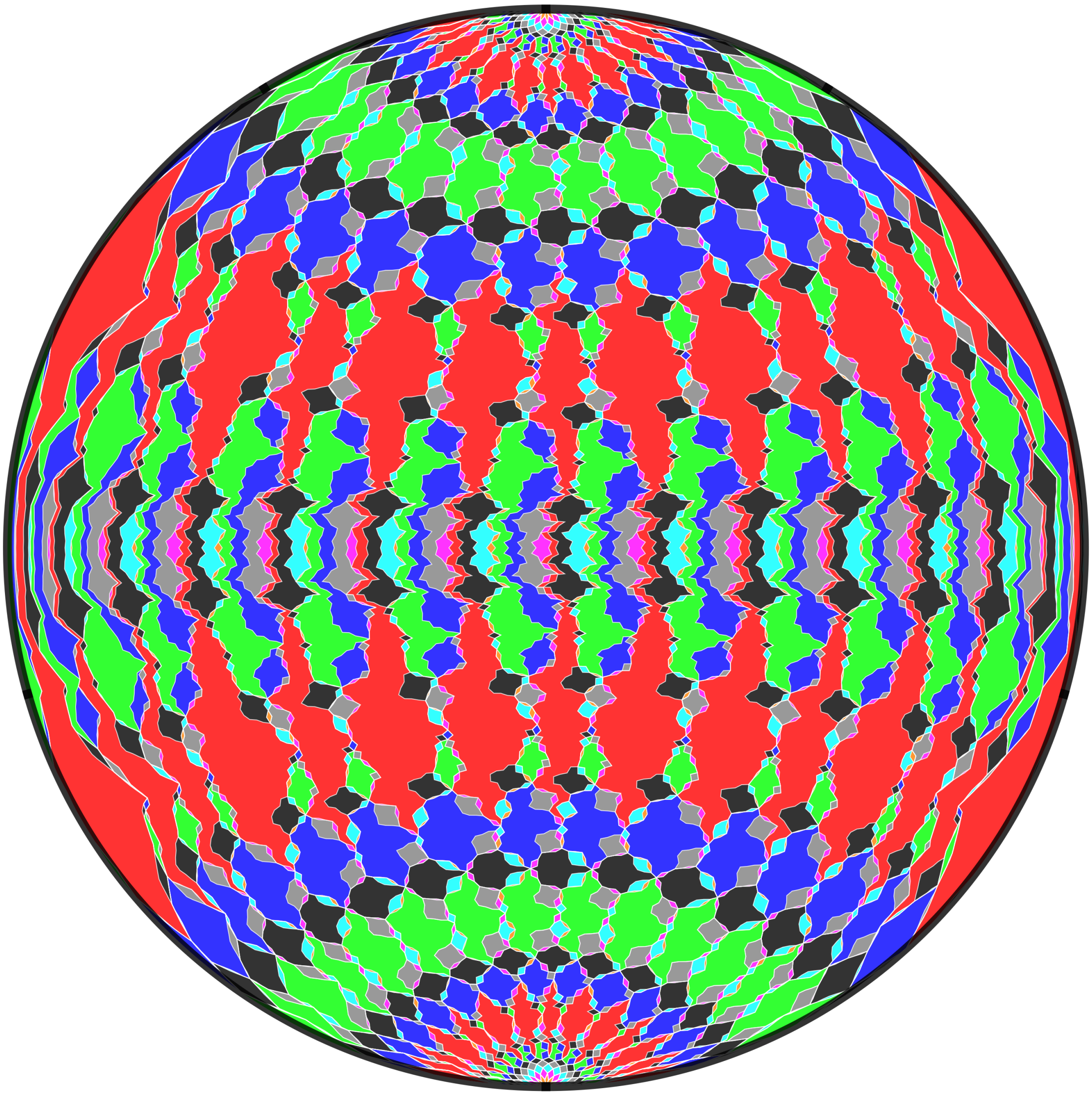


Figure 8.30: Tiling of E_8 with $\alpha = \pi/n$.

8.5.1 An Arbitrary Attempt

We note that these do visually look like tilings in the sense that we do not obviously see self-intersections. It is for this reason that we consider $\phi^J_{\ll}$ a sensible candidate to be an E-embedding. To appreciate what would happen if we arbitrarily label the nodes for the same CPR graph for F_4, we present example 8.5.2.

Example 8.5.2. *Consider the embedding, ψ, of F_4 conjugate to $\phi^J_{\ll}$ with the following CPR of F_4 graph:*

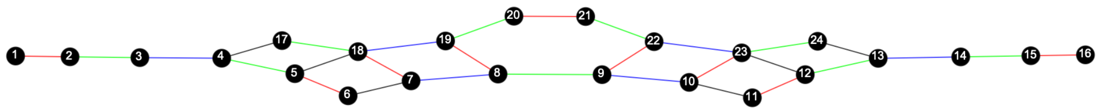


Figure 8.31: A somewhat arbitrarily chosen permutation representation for F_4, ψ, acting on W_J.

Write $\omega_0 = s_{i_1} \dots s_{i_{24}}$ *to be the bipartite alternating word of the longest element* $\omega_0 \in F_4$*. Displaying the sequence of tilings for the partial words* $T_\psi(s_{i_1}), T_\psi(s_{i_1}s_{i_2}), \dots T_\psi(s_{i_1} \dots s_{i_{24}})$ *(with the false assumption that* ψ *is an E-embedding) we produce the following:*

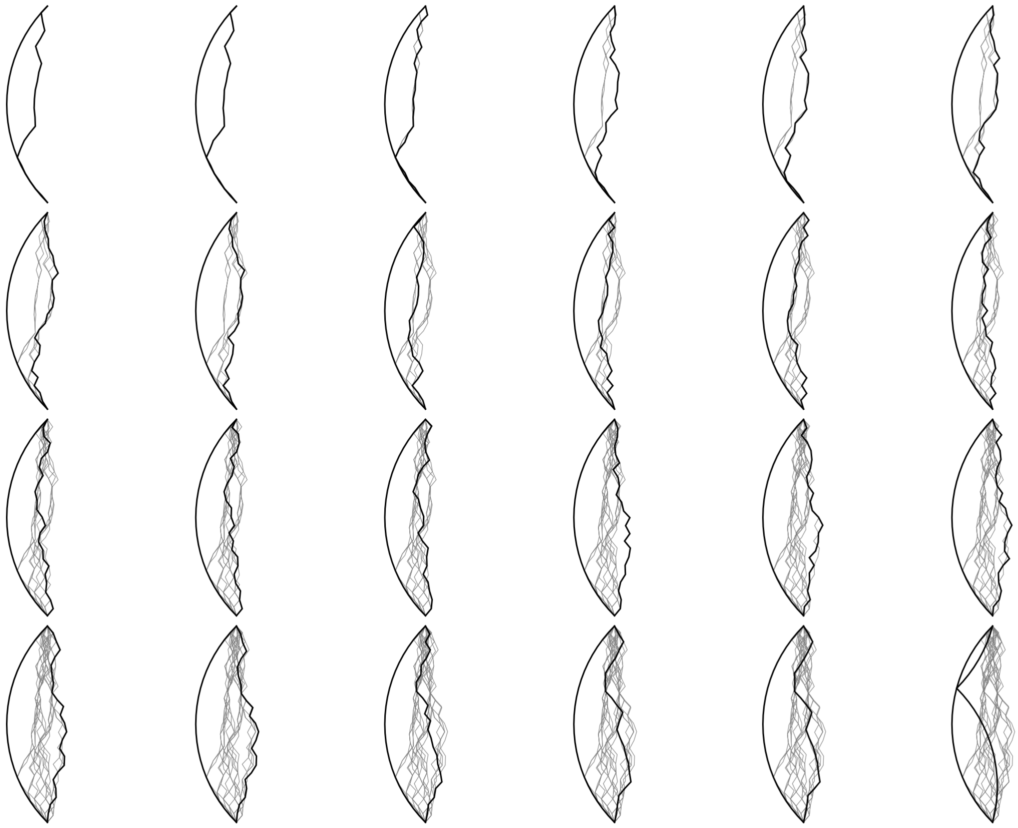


Figure 8.32: The 'tilings' produced for the bipartite alternating word by placing tile after tile (read from left-to-right then top-to-bottom) for the permutation representation of Figure 8.31.

Chapter 9

Subtilings and admissible partitions

Mühlherr's admissible partitions of Definition 1.2.7 and Table 1.1 necessarily play a crucial role. Again, this chapter consists of results from joint work with Professor Peter Rowley in [38]. It has been edited for this thesis to make use of the definition of an E-embedding, which does not appear in [38]. We will reuse the notation used in Definition 1.2.7.

Our main theorem for this section is Theorem 9.0.1. It shows how one can form a 'subtiling' of an E-embedding when we have an admissible partition.

If $\phi : W \hookrightarrow \mathrm{Sym}(n)$ is an E-embedding and Σ an admissible partition of W, then denote the embedding of $W_\Sigma \lessdot_\Sigma W$ by the map $\psi_\Sigma : W_\Sigma \to W$. Given the part $\Sigma_i \in \Sigma$, partition this again (with indexing set J, say) into subsets of S that are connected in the primitive Coxeter diagram of W, $\Sigma_i = \{\Omega_{i,j} \mid j \in J\}$. Let $\omega_{i,j}$ denote the longest element in $W_{\Omega_{i,j}}$. Then

$$\phi(\psi_\Sigma(s_{\Sigma_i})) = \prod_{j \in J} \phi(\omega_{i,j}).$$

Hence, if $(\phi \circ \psi_\Sigma)$ is an E-embedding then, the tiles associated to each s_{Σ_i} are formed by taking union the tiles associated each $\omega_{i,j}$. In the case that each $\Omega_{i,j}$ contains a single generator (equivalently, the generators in Σ_i mutually commute), then $\phi(\psi_\Sigma(s_{\Sigma_i}))$ is simply the union of the tiles associated to each $r \in s_{\Sigma_i}$. So the support interval of each s_{Σ_i} is given by the union of the support intervals of each

$r \in \Sigma_i$. That is,

$$\mathrm{I}_{(\phi\circ\psi_\Sigma)}(s_{\Sigma_i}) = \bigcup_{r\in\Sigma_i} \mathrm{I}_\phi(r).$$

Theorem 9.0.1. *Suppose that (W, S) has an admissible partition Σ, so that $W_\Sigma \lessdot_\Sigma W$. If $\phi : W \hookrightarrow \mathrm{Sym}(n)$ is an E-embedding then $(\phi \circ \psi_\Sigma)$ is an E-embedding also.*
Moreover, the relation set $J_{(\phi\circ\psi_\Sigma)}$ is given by

$$J_{(\phi\circ\psi_\Sigma)} = \{\{s_{\Sigma_i}, s_{\Sigma_j}\} \subseteq S_\Sigma \mid \textit{ for all } s \in \Sigma_i, r \in \Sigma_j,\ \{s, r\} \in J_\phi\}.$$

Proof. To prove that $(\phi \circ \psi_\Sigma)$ is an E-embedding, we need to show that for all $u, v \in W_\Sigma$, $u <_R v$ implies $(\phi \circ \psi_\Sigma)(u) <_B (\phi \circ \psi_\Sigma)(v)$. Suppose that $u <_R v$. Since Σ is an admissible partition, ψ_Σ preserves the Bruhat order ($w_1 <_B w_2$ implies $\psi_\Sigma(w_1) <_B \psi_\Sigma(w_2)$ for all $w_1, w_2 \in W_\Sigma$) and therefore the weak order too ($w_1 <_R w_2$ implies $\psi_\Sigma(w_1) <_R \psi_\Sigma(w_2)$ for all $w_1, w_2 \in W_\Sigma$). Hence $\psi_\Sigma(u) <_R \psi_\Sigma(v)$. Now, since ϕ is an E-embedding, $\phi(\psi_\Sigma(u)) <_B \phi(\psi_\Sigma(v))$, giving the desired result.
Now we address $J_{(\phi\circ\psi_\Sigma)}$. We know, by Lemma 7.1.7, the pair $\{s_{\Sigma_i}, s_{\Sigma_j}\}$ is in $J_{(\phi\circ\psi_\Sigma)}$ if and only if $\mathrm{I}_{(\phi\circ\psi_\Sigma)}(s_{\Sigma_i}) \cap \mathrm{I}_{(\phi\circ\psi_\Sigma)}(s_{\Sigma_j}) = \emptyset$. But

$$\begin{aligned}\mathrm{I}_{(\phi\circ\psi_\Sigma)}(s_{\Sigma_i}) \cap \mathrm{I}_{(\phi\circ\psi_\Sigma)}(s_{\Sigma_j}) &= \left(\bigcup_{s\in\Sigma_i} \mathrm{I}_\phi(s)\right) \cap \left(\bigcup_{r\in\Sigma_j} \mathrm{I}_\phi(r)\right) \\ &= \bigcup_{s\in\Sigma_i, r\in\Sigma_j} \mathrm{I}_\phi(s) \cap \mathrm{I}_\phi(r)\end{aligned}$$

whence the result. □

We note that the same result can be obtained without mention of E-embeddings if one wanted to just consider subtilings of Elnitsky's three tilings.
This gives us an easy way to make new tilings from old. For example, Theorem 9.0.1 tells us that every tiling for E_8 induces a subtiling for H_4, or that every tiling for a type A group induces a tiling for a type B group.
We focus only on those subtilings for Elnitsky's original tilings for the remainder of this chapter. We use x_0 to denote the longest element of W_Σ and $K = J_{(\phi\circ\psi_\Sigma)}$ for what follows. When displaying the subtilings in this chapter we will write the corresponding words so that $s_{i_1} \ldots s_{i_k}$ is expressed as $[\, i_1\ i_i\ \ldots i_k\,]$.

9.1 W_Σ of type B_n, W of type A_{2n-1}

First we reevaluate Elnitsky's tiling of type B and notice that it is formed from the following admissible partition:

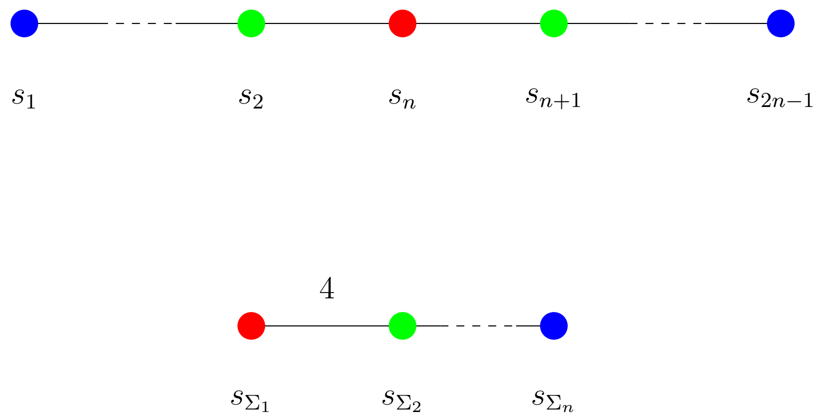


Figure 9.1: The admissible partition of B_n into A_{2n-1}.

So for our subtiling we are implicitly using the embedding, $s_{\Sigma_1} = s_n = (n, n+1)$ while $s_{\Sigma_{i+1}} = s_{n-i}s_{n+i} = (n-i, n-i+1)(n+i, n+i+1)$ for $i \in \{1, \ldots, n-1\}$. In this case, the relation set K is $\{\{s_{\Sigma_i}, s_{\Sigma_j}\} \mid |i-j| \geq 2\}$. This gives us the tilings of type A that are symmetric about the horizontal line equidistant from the upper and lowermost vertices of the polygon. These are exactly Elnitsky's tilings of type B in Section 6 of [11].

We demonstrate some of the examples for $\mathcal{T}(x_0)$ when $n = 3$.

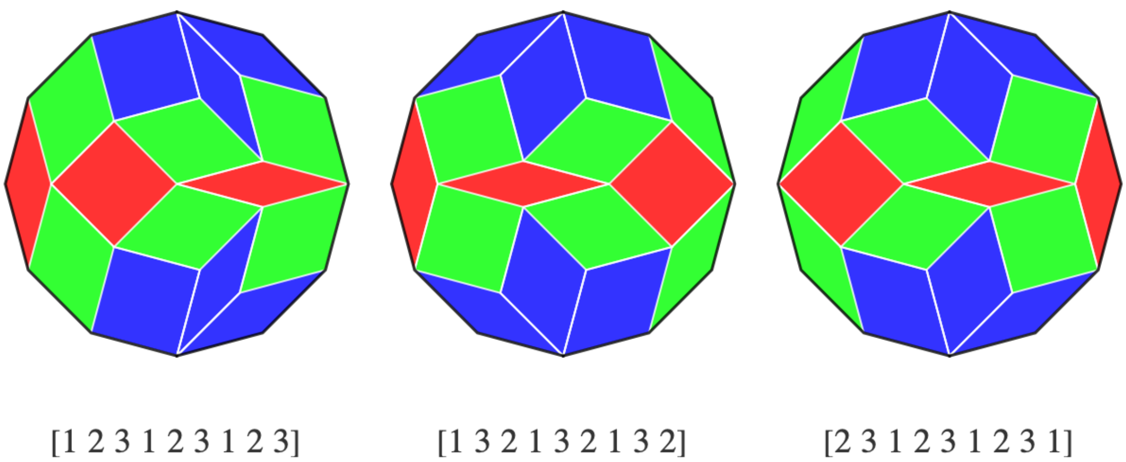


Figure 9.2: A tiling for B_3 viewed as a subtiling of A_5.

We present two more tilings for the type B groups. Neither of these seem to have appeared in the literature. In particular, these can be seen to be subtilings of Elnitsky's original tilings and could have, in hindsight, been discovered in a similar manor.

9.2 W_Σ of type B_n, W of type A_{2n}

The other admissible partition of type A groups that induce a Coxeter group of type B is the following. Here s_{Σ_1} is sent to the longest element in the parabolic subgroup of $\{s_n, s_{n+1}\}$, which is $s_{\Sigma_1} = s_n s_{n+1} s_n = s_{n+1} s_n s_{n+1} = (n, n+2)$, while the others are $s_{\Sigma_{i+1}} = s_{n-i} s_{n+1+i} = (n-i, n-i+1)(n+1+i, n+2+i)$ for $i = 1, \ldots, n-1$. See Figure 9.3 below.

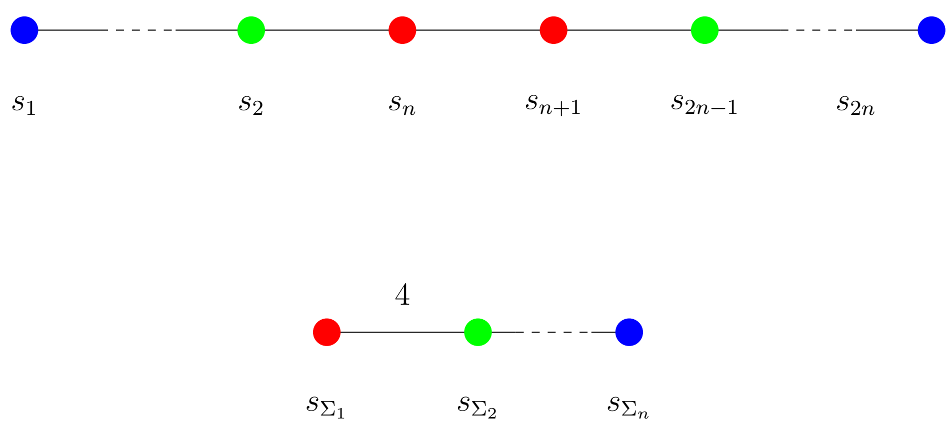


Figure 9.3: The admissible partition of B_n into A_{2n-1}.

Since s_{Σ_1} is not the product of disjoint transpositions, the corresponding tile is necessarily formed by placing the sequence of tiles corresponding to either $s_1 s_2 s_1$or $s_2 s_1 s_2$ in Elnitstky's type A tiling. We identifying the placement of these equivalent sequences as one so-called megatile which itself is a hexagon. Again, we observe that $K = \{\{s_{\Sigma_i}, s_{\Sigma_j}\} \mid |i-j| \geq 2\}$ and we again consider what $\mathcal{T}(x_0)$ looks like for the case $n = 3$ – see Figure 9.4.

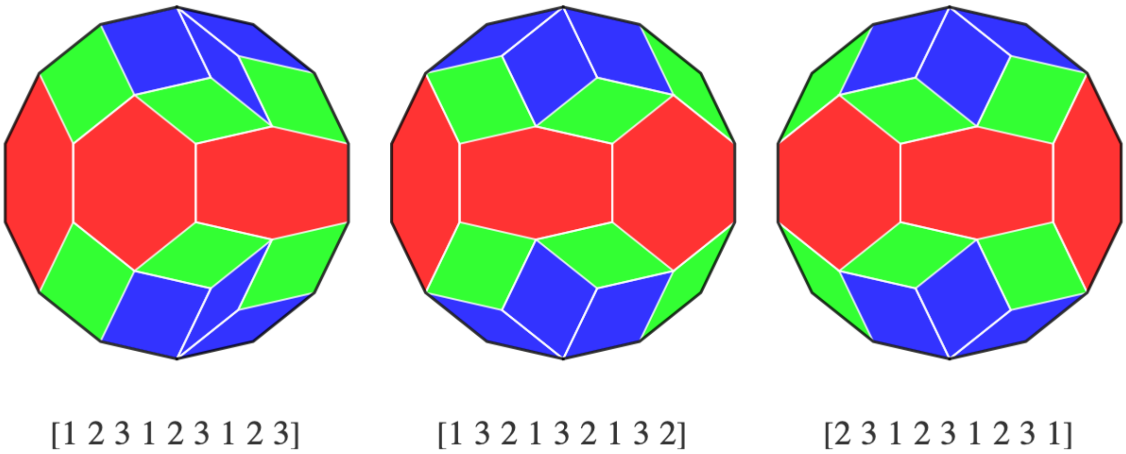


Figure 9.4: A tiling for B_3 viewed as a subtiling of A_6.

We observe that the existence of this particular tiling given that of Elnitsky's type B tiling is, in hindsight, very intuitive; as it has horizontal symmetry, if we were to insert the constant vertical edge in place of the middle vertex we will preserve the tiling and words. Similar observations can be found in [11] and [19], when studying strips.

9.3 W_Σ of type B_{n-1}, W of type D_n

We consider the final admissible partition that induces B_{n-1}. This time it is a partition of D_n as seen in Figure 9.5.

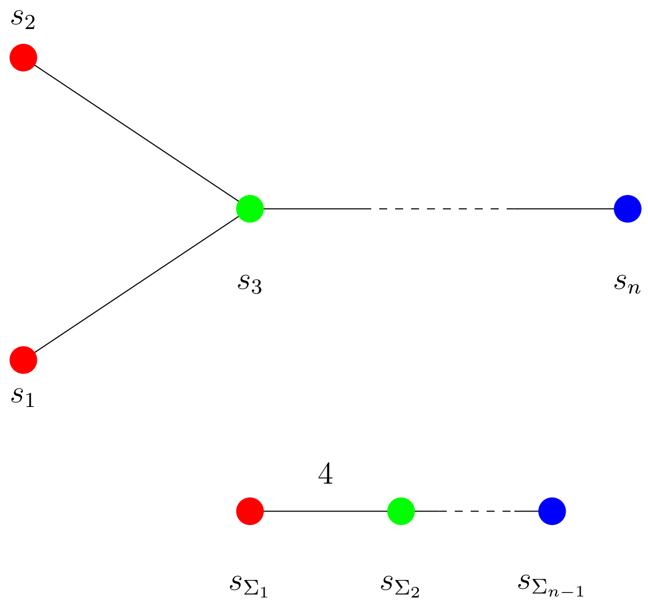


Figure 9.5: The admissible partition of B_{n-1} into D_n.

In this case we have $s_{\Sigma_1} = s_1 s_2 = (-1, 1)(-2, 2)$ along with $s_{\Sigma_{i+1}} = s_{i+2} = (i+1, i+2)(-i-1, -i-2)$ for $1 \leq i \leq n-2$. Yet again, we look at the reduced words of longest element of B_3. Note that the relation set K is $\{\{s_{\Sigma_i}, s_{\Sigma_j}\} \mid |i-j| \geq 2\}$.

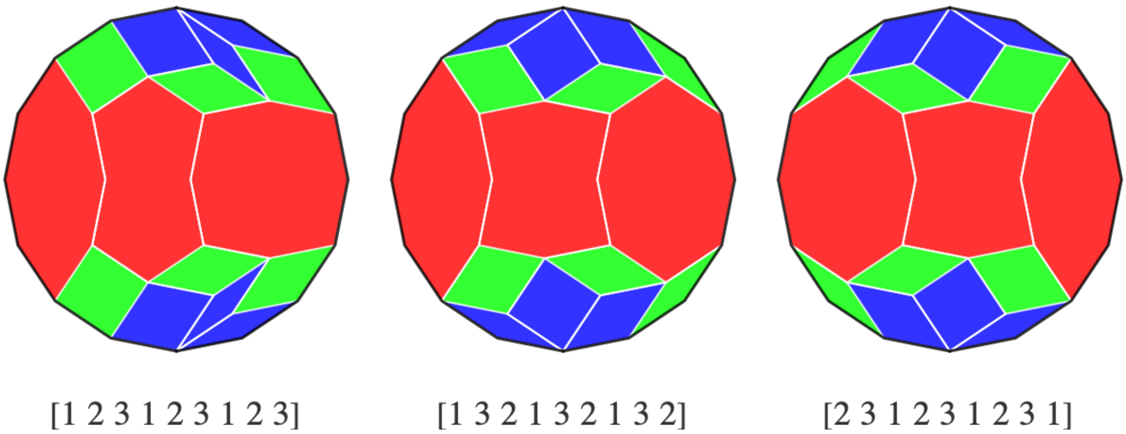


Figure 9.6: A tiling for B_3 viewed as a subtiling of the type D_4 tiling.

We note that if one were to transpose the labelling of s_1 and s_2 in the Coxeter diagram of D_n we would get an alternative tiling for this admissible partition. Interestingly, this would also change the relation set.

9.4 W_Σ of type H_3, W of type D_6

Finally, we consider the tiling for H_3 as a subtiling for D_6 induced from the following admissible partition.

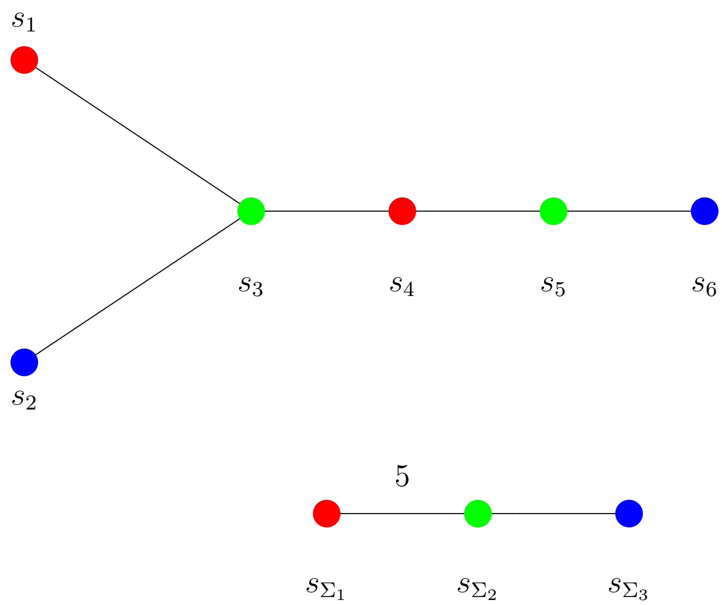


Figure 9.7: The admissible partition of D_6 into H_3.

For this tiling we have an empty relation set, which, as luck would have it, gives us genuine bijections between reduced words w of H_3 and subtilings in $\mathcal{T}(w)$. That is, $K = \emptyset$. There are 286 reduced words for the longest element of H_3, we highlight a selected sample of six corresponding tilings in Figure 9.8.

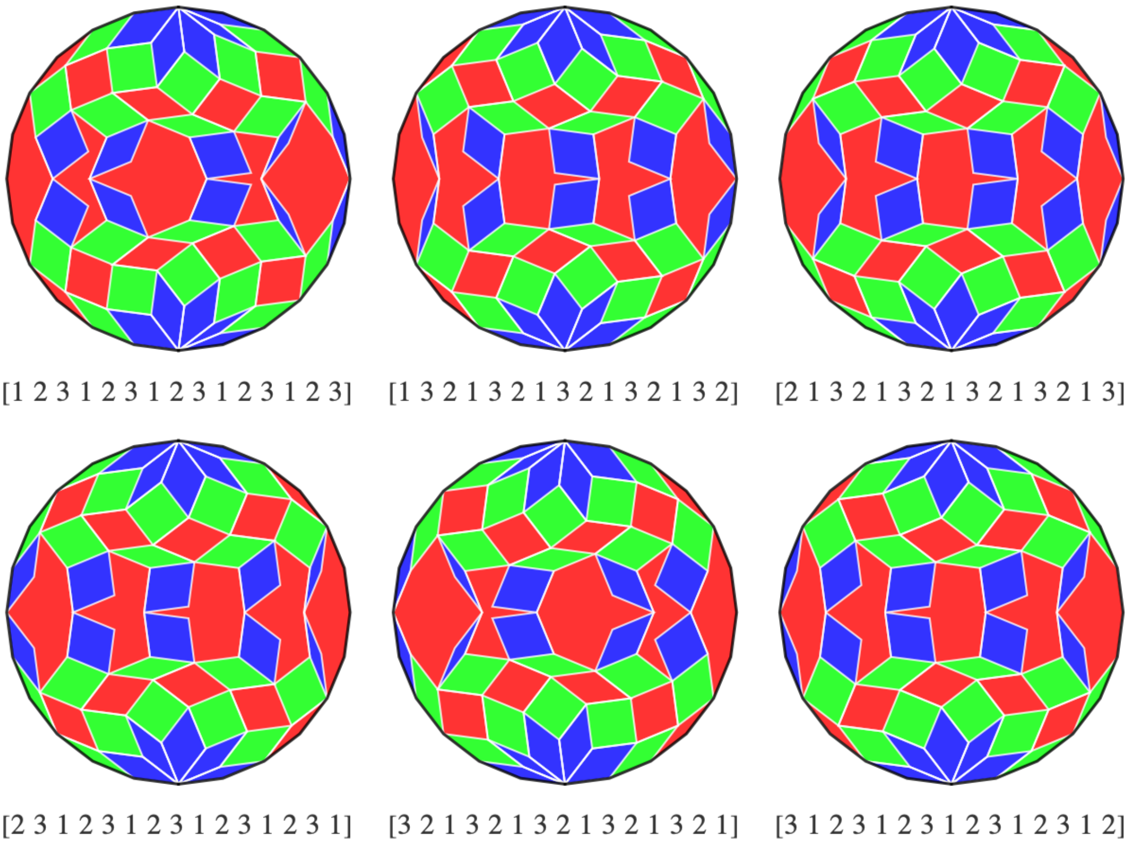


Figure 9.8: A tiling for H_3 viewed as a subtiling of the type D_6 tiling.

Chapter 10

Suggestions for further research

We conclude the thesis by indulging in some ideas for avenues of further work. As in Chapter 8, we do not aim to formally prove anything here. We first sketch a strategy for an alternative construction for Elnitsky's type A tilings in terms of abstract regular polytopes. This strategy seems ripe for generalisation so we want to include it and give an example of trying to use these ideas to define an affine analogue of the work of Chapter 7.

10.1 A sketch of an alternative construction for Elnitsky's tilings

For an alternative construction of Elnitsky's tilings we start with the following well-known observation: regular n-Simplex, as an abstract regular polytope has $\mathrm{Sym}(n)$ as it is automorphism group. Furthermore, its Hasse diagram is isomorphic to a (directed) n-hypercubic graph and its automorphism group is $\mathrm{Sym}(n)$. We saw an instance of this in Section 1.3.
Suppose the associated Hasse graph is embedded into $\mathbb{R}^n$ with the coordinates of its 2^n vertices having entries of the form $[0, 0 \ldots, 0], [1, 0 \ldots, 0], \ldots, [1, 1, \ldots, 1]$. Then Theorem 1.5.1 tells us then that $\mathrm{Sym}(n)$ acts regularly on the set of paths of from $[0, 0 \ldots, 0]$ to $[1, 1, \ldots, 1]$ where two paths are adjacent if and only if they differ only in one vertex. Hence adjacent flags in the $\mathcal{P}(\mathrm{Sym}(n))$ are in bijection with the 2D-faces of the n-hypercubic graph. This, in turn, gives a bijection between (commutation classes of) reduced words in $\mathrm{Sym}(n)$ and connected sets of faces of the n-cube. Then if one can find a suitable faithful projection on to some $\mathbb{R}^k$ for $1 \leq k \leq n$ then we obtain a bijection between these projections and our

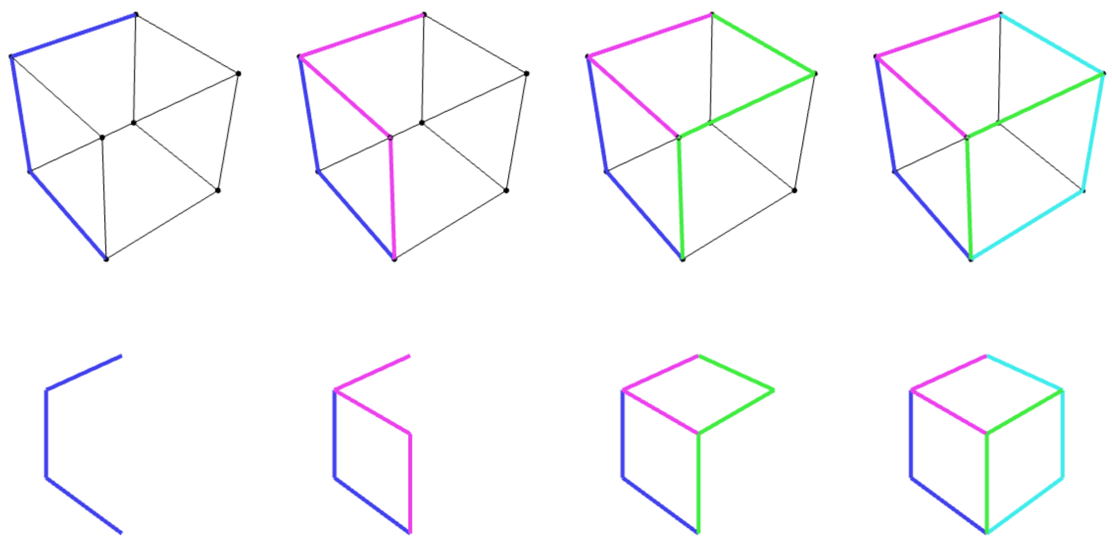


Figure 10.1: Projecting paths on the cube on to its Petrie polygon to form rhombic tilings.

words. These projections are where the difficulty is hidden. A Petrie polygon of a polytope (introduced properly in [9]) is some (possibly skew) polygon for which every consecutive $(n-1)$ edges lies on some rank $n-1$ face but no consecutive n edges do. Projections on to a carefully chose Petrie polygon seems to do just the trick needed though this is speculation only and where the complexity is hidden. We summarise this in Figure10.1.
This suggests two directions for generalisation:

1. Find those abstract regular polytopes whose Hasse graph is also the 1-skeleton of some real polytope equipped with some 'nice' projection.

2. Find those groups that have a regular group action on the Petrie polygon of some convex regular polytope.

A specific instance of (ii) that actually seems promising is to try define a natural affine generalisation. Let $\overline{A}_n$ denote the Coxeter group whose Coxeter diagram is the unlabelled circuit graph. The other affine Coxeter groups are classified in [26]. Then a natural definition to consider is:

Definition 10.1.1. *Suppose for irreducible, affine, W that $\phi : W \hookrightarrow \overline{A}_n$ is an embedding. Then ϕ is an* Affine E-Embedding *if, for all $u, v \in W$,*

$$u <_R v \quad \textit{implies} \quad \phi(u) <_B \phi(v).$$

In order to capture the geometric side of the definition one might suggest examining Petrie polygons on cubic lattices in place of paths on the cube. Then one might attempt to find to find 'nice' faithful projections in order to create similarly 'nice' tilings.
One might want bijections between (some classes of) reduced in $\overline{A}_n$ and some set of geometrical object. We naively suggest one such candidate by examining the Petrie polygon of an n-hypercubic Lattice and attempting to derive a regular group action on it. Then the examination of faithful projections is what is required to find meaningful bijections of such objects. Would we be able to have bijections between zonotopal surfaces in $\mathcal{R}^3$ and reduced words of affine groups? I expect this is too good to be true but something similar might work.
And one might want to examine extensions to C-groups in general: a first candidate might be to study those C-groups with a linear CPR graph. An interpretation of E-embeddings on to CPR graphs of Coxeter groups does certainly seem achievable. This might naturally extend to all CPR graphs as a natural extension.

10.2 R-Polynomials

One last question that is forthcoming and irresistible is this: can one interpret the R-polynomials on the type A tilings? The R-Polynomial of a Coxeter group is determined by the following initial conditions and recursions (Theorem 5.1.1. of [1]) and is of central importance in the combinatorics of Coxeter groups and an entry-point into studying the celebrated Kazhdan–Lusztig polynomials:

Theorem 10.2.1 (Theorem 5.1.1 in [1]). *There is a unique family of polynomials* $\{R_{u,v}(q)\}_{u,v} \in W \subseteq \mathbb{Z}[q]$ *satisfying:*

(i) $R_{u,v}(q) = 0$ *if* u *and* v *are not comparable in* $<_B$*,*

(ii) $R_{u,v}(q) = 1$ *if* $u = v$*,*

(iii) if $s \in I^-(v) \cap I^-(u)$ *then* $R_{u,v}(q) = R_{us,vs}(q)$*,*

(iv) if $s \in I^-(v) \setminus I^-(v)$ *then* $R_{u,v}(q) = qR_{us,vs}(q) + (q-1)R_{u,vs}(q)$*.*

Does there exist a direct characterisation of these polynomials in terms of the tilings?

Appendix A

Magma code

Here we attach some of the relevant code used in this thesis. In particular, we have included the algorithms implemented to find all the C-strings associated to a given group as well as checking that they are unravelled. The main algorithm we use to find all abstract regular polytopes is an implementation of that described in [25].

```
// Implementation of the depth-first algorithm as described in
   Hartley and Hulpkes POLYTOPES DERIVED FROM SPORADIC SIMPLE
   GROUPS paper (see references).
// Also including further code on testing unravelledness.
makeInvolutionsPermutations := function(G)
    // Function for extracting the action of a group G on its
   involutions by conjugation.
    // {@ @} denotes an ordered set in Magma.
    involutionsOfG := {@ g : g in G | Order(g) eq 2 @};
    // We index the involutions so that we act on these indices.
    involutionsOfGIndices :={1..#involutionsOfG};
    involutionsOfGIndicesxG := CartesianProduct(
   involutionsOfGIndices,G);
    // This creates the actual map.
    indexPermutationOfInvolutionsMap := map<involutionsOfGIndicesxG
     -> involutionsOfGIndices | x :-> Position(involutionsOfG,
   involutionsOfG[x[1]]^x[2])>;
    return  GSet(G, involutionsOfGIndices,
   indexPermutationOfInvolutionsMap);
end function;

makeInvolutionsPermutationsInvolutionsProvided := function(G,
   involutionsOfG)
```

```
    // The same function as before, except we may take the involutions as an input as we might have a more efficient way of computing them.
    involutionsOfG := {@g : g in involutionsOfG @};
    involutionsOfGIndices := {1..#involutionsOfG};
    involutionsOfGIndicesxG:= CartesianProduct(involutionsOfGIndices,G);
    indexPermutationOfInvolutionsMap:=map<involutionsOfGIndicesxG -> involutionsOfGIndices|x :-> Position(involutionsOfG, involutionsOfG[x[1]]^x[2])>;
    return GSet(G,involutionsOfGIndices, indexPermutationOfInvolutionsMap);
end function;

checkStringConditionInduction := function( ancestoryString, InvolutionsOfG )
    // An auxiliary function that will check the string condition for an (ordered) set of involutions.
    // The ancestoryString is an indexing set determining a set of involutions.
    // offspring is the last generator we have added to this list of involutions. By induction, we only need to check the string condition holds correctly for calculations involving the offspring.
    offspring := InvolutionsOfG[ancestoryString[#ancestoryString]];
    for i in [1..(#ancestoryString-1)] do
            // ord checks if our new generator commutes with the others by computing the order of their product.
            // Note that offspring^2 is just the identity. Its easy to access this way.
            ord := (InvolutionsOfG[ancestoryString[i]]*offspring)^2 ne offspring^2;
            // This checks the commutation requirements.
            if i le #ancestoryString-2 and ord then
                return false;
            end if;
    end for;
    return true;
end function;

checkStringConditionInductionIrreducible := function( ancestoryString, InvolutionsOfG )
```

```
    // This is essentially the same function as the previous one
   but also rules out adjacent involutions commuting for efficiency
   .
    offspring := InvolutionsOfG[ancestoryString[#ancestoryString]];
    for i in [1..(#ancestoryString-1)] do
            ord := (InvolutionsOfG[ancestoryString[i]]*offspring)^2
     ne offspring^2;
            if i le #ancestoryString-2 and ord then
                return false;
            end if;
            // Here is the additional condition we add.
                    if i eq #ancestoryString-1 ne ord then
                              return false;
                    end if;
    end for;
    return true;
end function;

checkIntersectionConditionInstance := function( I, J,
   ancestorString, setOfAllInvolutionsOfG, G )
        // This is an auxiliary function for testing the
   intersection condition on a specific instance of distinguished
   groups.
        // meet is an intersection function in Magma.
        if sub< G | [setOfAllInvolutionsOfG[ancestorString[i]] : i
   in I] > meet  sub< G | [setOfAllInvolutionsOfG[ancestorString[j
   ]] : j in J] > eq sub< G | [setOfAllInvolutionsOfG[
   ancestorString[ij]] : ij in I meet J]> then
                return true;
        end if;
        return false;
end function;

checkIntersectionConditionInduction := function(ancestorString,
   setOfAllInvolutions, G)
        // Using the previous function, we can produce an inductive
    test for the intersection condition based on the number of
   generators.
        // We only need to check the intersection condition with
   those sets containing the offspring.
        // In hindsight this could be made more efficient by
   excluding certain trivial cases.
```

```
        // This checks Intersection condition with just one appearance of the offspring.
        for I in Subsets({1..#ancestorString-1}) do
                for J in Subsets({1..#ancestorString-1}) do
                        if checkIntersectionConditionInstance(I, Include(J,#ancestorString), ancestorString,  setOfAllInvolutions, G) eq false then
                                return false;
                        end if;
                end for;
        end for;

        // Now we check when we have two appearances of the offspring.
        // SetToIndexedSet is an inbuilt function to extract the indexing set.
        for IJ in [SetToIndexedSet(IJ): IJ in Subsets(Subsets({1..#ancestorString-1}),2)] do
                if checkIntersectionConditionInstance(Include(IJ[1],#ancestorString), Include(IJ[2],#ancestorString), ancestorString, setOfAllInvolutions, G) eq false then
                        return false;
                end if;
        end for;
        return true;
end function;

cStringProcedure := procedure( ancestorString, ancestorStabiliser, G, involutionsOfG, ~setOfPolytopesFound)
        // A depth-first algorithm for finding all ordered subsets of involutions that are C-strings up to automorphism.
        // We will now assume that ancestorStabaliser is a subgroup of Aut(G) under a homo of looking at how Aut(G) permutes the indices of the involutions.
        // We usually use a faster (almost identical) algorithm searching just for those irreducible polytopes given below.
        // A procedure doesnt return a value but this is potentially altering the value of setOfPolytopesFound recursively.
        youngestInvolution := ancestorString[ #ancestorString ];
        newAncestorStabaliser := Stabiliser( ancestorStabiliser, youngestInvolution );
```

```
        // We make the new potential generators to add, assuring they are not redundant by choosing them up to automorphism.
        offsprings := {@ {@involution: involution in orbit | involution notin ancestorString @}[1] : orbit in Orbits( newAncestorStabaliser ) | orbit notsubset ancestorString @};
        // Now we see if adding a new potential generator satisfies the string and intersection conditions.
        // If it does, we recurse by adding yet more generators immediately.
        for newOffspring  in offsprings do
                newAncestorString := Include( ancestorString, newOffspring);
                newAncestorInvolutions := {@ involutionsOfG[k] : k in newAncestorString @};
                newAncestorStringDistinguishedGroup := sub< G | newAncestorInvolutions >;
                // Now we check the conditions.
                if checkStringConditionInduction( newAncestorString , involutionsOfG ) and checkIntersectionConditionInduction( newAncestorString,involutionsOfG, G) then
                        // Now we check if weve generated the whole group. If yes, then we keep this data. Else we recurse forward, increasing the depth of the search.
                        if #newAncestorStringDistinguishedGroup eq # G then
                                // Here is where we list the actual polytopes found. We print to see progress.
                                Append( ~setOfPolytopesFound, newAncestorString );
                                        print newAncestorString;
                        else
                                // Double Dollars refers to calling the same function recursively in Magma.
                                // The ~ symbol ensures that setOfPolytopesFound remembers the new polytopes found independent of each instance of calling itself.
                                $$( newAncestorString, newAncestorStabaliser, G, involutionsOfG, ~setOfPolytopesFound );
                        end if;
                end if;
        end for;
end procedure;
```

```
cStringProcedureIrreducible := procedure( ancestorString, ancestorStabiliser, G, involutionsOfG,~setOfPolytopesFound)
        // A slightly faster algorithm compared to the one above that finds only those irreducible polytopes. Its almost identical to the above procedure.
        youngestInvolution := ancestorString[ #ancestorString ];
        newAncestorStabaliser := Stabiliser( ancestorStabiliser, youngestInvolution );
        offsprings := {@ {@involution: involution in orbit | involution notin ancestorString @}[1] : orbit in Orbits( newAncestorStabaliser ) | orbit notsubset ancestorString @};
        for newOffspring  in offsprings do
                newAncestorString := Include( ancestorString, newOffspring);
                newAncestorInvolutions := {@ involutionsOfG[k] : k in newAncestorString @};
                newAncestorStringDistinguishedGroup := sub< G | newAncestorInvolutions >;
                if checkStringConditionInductionIrreducible( newAncestorString, involutionsOfG ) and checkIntersectionConditionInduction( newAncestorString, involutionsOfG, G) then
                        if #newAncestorStringDistinguishedGroup eq # G then
                                Append( ~setOfPolytopesFound, newAncestorString );
                                        print newAncestorString;
                        else
                                $$( newAncestorString, newAncestorStabaliser, G, involutionsOfG,~setOfPolytopesFound );
                        end if;
                 end if;
        end for;
end procedure;


findAllARPsOfGroupWithAutGActionImageProvided := function(G, AutGActionImage, involutionsOfG)
 // Here is one example of how we can use the algorithm to find all polytopes of a given automorphism group.
 //This is the most general (and slowest!) version taking in an input of a group, its automorphism group and involutions.
```

```
    // We assume that AutGActionImage is the permutation group defined by the homo of how autG permutes the indices of InvolutionsOfG.
    ordG := #G;
    setOfPolytopesFound := [];
    // Take representatives of involutions up to automorphism.
    involutionRepresentatives := {@ orbit[1] : orbit in Orbits(AutGActionImage) @};
    for involutionIndex in involutionRepresentatives do
        cStringProcedure( {@ involutionIndex @}, AutGActionImage, G, involutionsOfG, ~setOfPolytopesFound);
    end for;
    return setOfPolytopesFound;
end function;

findAllARPsOfGroupGivenAutGIsoToGIrreducible := function(G)
    // Another version of the above function that uses specialisations to be faster.
    // Uses nice facts about the Automorphism group only having inner automorphisms and allowing ourselves to only consider irreducible polytopes.
    ordG := #G;
    autG := G;
    setOfPolytopesFound := [];
    involutionsOfG := {@ g : g in G | Order(g) eq 2 @};
    involutionsGset := makeInvolutionsPermutations(G);
    AutGActionImage := ActionImage(G,involutionsGset);
    involutionRepresentatives := {@ orbit[1] : orbit in Orbits(AutGActionImage) @};
    for involutionIndex in involutionRepresentatives do
        cStringProcedureIrreducible( {@ involutionIndex @}, AutGActionImage, G, involutionsOfG, ~setOfPolytopesFound);
    end for;
    return setOfPolytopesFound;
end function;

checkIntersectionConditionQuotient:=function(G,N)
  // An auxiliary function for checking the intersection condition when checking if a group is unravelled.
  // Indexing here is only to be stylistically consistent with literature. This is essentially just taking the power sets of sets of generators.
```

```
        leftSubsets := [set : set in Subsets({0..Ngens(G)-1})| #set gt 0 ];
        rightSubsets := [set : set in Subsets({0..Ngens(G)-1})| #set gt 0 ];
        count := 2;
        for I in leftSubsets do
                for J in rightSubsets[count..#leftSubsets] do
    // Checks the intersection condition for all pairs without repetitions. This could be optimised better.
                        if sub<G|N,{G.(i+1) : i in I}> meet sub<G|N,{G.(i+1) : i in J}> ne sub<G|N,{G.(i+1) : i in (I meet J)}> then
                                return false;
                        end if;
                end for;
                count := count +1;
        end for;
  // Only returns true if it never returned false when checking each instance.
        return true;
end function;

quotientElementOrder := function(g, N)
  // An auxilliary finction for checking the order of elements in the quotient group.
  // This is used in the string condition test.
        power := 1;
        currentg := g;
        while power lt Order(g)+1 do
                if currentg in N then
                        return power;
                else
                currentg:= currentg*g;
                end if;
                power:=power+1;
        end while;
end function;

checkStringConditionQuotient := function(G,N)
        // Checking the string condition in the quotient.
  // Works similar to the original function but now makes use of the quotient order function.
  // First we check if the rank decreases in the quotient.
```

```
          for i in [1..(Ngens(G))] do
                    if G.i in N then
                              return false;
                    end if;
  // Now we just check the string condition.
            for j in [i..(Ngens(G))] do
                ord := quotientElementOrder(G.i*G.j,N);
                if Abs(i - j) ge 2 and not ord eq 2 then
                     return false;
                end if;
            end for;
        end for;
    return true;
end function;

checkCStringGroupQuotientGroup := function( polytopeGroup, N)
  // This amalgamates the previous functions to check if the quotient is a C group.
        if checkStringConditionQuotient(polytopeGroup, N) then
                if checkIntersectionConditionQuotient(polytopeGroup, N) then
                              return true;
                    end if;
        end if;
        return false;
end function;

checkUnravelledGroup := function(polytopeGroup)
  // Checks if a group is unravelled by cycling over the normal subgroups and checking if rhey are c strings.
          normalSubgroups := [N`subgroup : N in NormalSubgroups(polytopeGroup)];
  // If only two normal subgroups exist then its simple so we can ignore it.
          if #normalSubgroups eq 2 then
                    return true;
          else
    // Otherwise we need to check if we have a quoteint C string.
                    for i in [2..#normalSubgroups-1] do
                              N:= normalSubgroups[i];
                              if checkCStringGroupQuotientGroup(polytopeGroup,N) eq true then
                                        return false;
```

```
                                    end if;
                        end for;
            end if;
            return true;
end function;
```

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