Delocalisation of Laplacian Eigenfunctions on Large Genus Random Surfaces

A thesis submitted to the University of Manchester for the degree of Doctor of Philosophy in the Faculty of Science and Engineering

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Contents

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List of Figures

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Abstract

In this thesis, we seek to understand delocalisation properties exhibited by Laplacian eigenfunctions on closed hyperbolic surfaces of large genus. In particular, we study the shape and spread of the eigenfunctions over the surface. We then exhibit stronger results that hold with high probability for surfaces in the Weil-Petersson random surface model.

The first contribution presented here is a study of the L^p norms of the eigenfunctions. Understanding the magnitude of these norms offers insight into the shape of the eigenfunctions; for example, how large they can be at any point. Our results show that these norms decay with respect to a parameter involving geodesic loops on the surface. We then study this parameter probabilistically, leading to decay rates on the L^p norms logarithmic in the surface genus.

Next, we study the geometry of the surfaces themselves more precisely by introducing the tangle-free parameter of a surface. This looks at what types of subsurfaces can be embedded inside a surface. We demonstrate that knowledge of the size of this parameter translates to information on the structure of geodesics in the surface whose lengths are of a similar size. We then study the parameter probabilistically, showing that the local geometry of these surfaces is similar to that of regular graphs.

Using this tangle-free framework, we then study the extent to which eigenfunctions can concentrate on subsets of the surface. In particular, we show near full concentration can only happen on subsets of size at least exponential in the tangle-free parameter, or probabilistically, at least the genus to some power.

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Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Outline

This thesis is written in accordance with the university journal format style and this section will outline both the structure of the remainder of the thesis, as well as modifications to the articles included for consideration for the awarding of the doctoral degree.

The first chapter of the thesis serves as an introduction to the content of the work undertaken throughout the doctoral programme, and as such, reviews a wide range of relevant literature associated with the research aims of the doctoral project. In addition to this, some time is spent discussing the original results of this thesis, the methodology utilised to obtain these results, as well as how the results fit within the wider literature. The introductory chapter is concluded with an outlook on perspectives for the thesis and further research aims that relate to the current work conducted. As the style of this thesis is that of a journal format, the introductory chapter draws upon the introductions of the articles presented in the later chapters, thus resulting in some overlap of the ideas discussed. The purpose of the introductory chapter is to elaborate more on these details in a less technical manner as a means to unify the body of work, and more specific introductions are reserved for the articles themselves in their respective chapters.

Within the second chapter, greater detail on the methodologies used in the original contributions are included. This chapter can be seen as an overview of the fundamentals required for understanding more precisely the research conducted. Nothing presented in this chapter is therefore necessarily new in terms of content, and should therefore serve primarily as an aid to the reader as a source of references and common results that would (and indeed, have not) been included in a journal style article.

The final three chapters form the original contributions of the thesis. These chapters each contain an individual, stand-alone research article that either has been accepted for publication or is in the submission process at the time of writing this document. In order, these are

- Chapter 3: *Short Geodesic Loops and L^p Norms of Eigenfunctions on Large Genus Random Surfaces*, joint with Clifford Gilmore, Etienne Le Masson and Tuomas Sahlsten. Published in *Geometric and Functional Analysis*, Volume 31 (2021) 62–110, https://doi.org/10.1007/s00039-021-00556-6.

- Chapter 4: *The Tangle-free Hypothesis on Random Hyperbolic Surfaces*, joint with Laura Monk. Accepted for publication in *International Mathematical Research Notices (IMRN).*

- Chapter 5: *Delocalisation of Eigenfunctions on Large Genus Random Surfaces.* Accepted for publication in *Israel Journal of Mathematics.*

Each of these pieces of work have significant contributions from myself, with the latter being solo-authored. These articles are left in a largely unaltered state to their submitted or accepted version. There are some modifications to notation to make the presentation uniform throughout, as well as some minor corrections and changes to wording for greater clarity for the reader.

Acknowledgements

I would like to begin by thanking my supervisors Tuomas Sahlsten and Etienne Le Masson for their support and constant encouragement throughout the course of my PhD studies. I have thoroughly enjoyed the conversations we have had, both work and non-work related, over these past years, and I hope that they will continue into the future. In particular, I am especially thankful for their support with applications for post-PhD progression. Without a doubt, their guidance has benefited me immensely, and will aid me in my future endeavours.

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1 Introduction

This chapter serves as a general introduction to the thesis and the research articles that proceed in Chapters 3, 4 and 5. As such, we include a broad overview of the literature, and of the results, whilst seeking to emphasise the role that the original contributions in this thesis have in the wider area. More technical (and indeed, tailored) introductions to the research articles themselves are left in their unaltered states at the beginning of each of their respective chapters. Owed to this, there will be some crossover between what is written in this chapter, and what is written later on. Throughout, we will use the notations $A \lesssim B$ and $A = O(B)$ both to mean that there is a constant $C > 0$ independent of any parameters such that $A \leq CB$.

1.1 Motivations

To begin discussing the research undertaken in this thesis, it will help for us to first explore some motivation behind the themes of interest. If the title of the thesis is not to be deceitful, then one should expect that we will be exploring the spectral theoretic properties of the Laplacian operator, and their relationship with the geometry of the ambient space that we are considering. The spectral theoretic property that we will consider is predominantly the behaviour and shape of the eigenfunctions. It should be emphasised that the spectrum of the Laplacian and distribution of the eigenvalues themselves are also particularly interesting topics, and results on properties of the eigenfunctions (for example the sup-norms) have implications on the spectrum itself.

Let us now break down the word delocalisation in the context of this thesis with regards to the study of eigenfunctions. By definition, the word delocalisation is the act of spreading out, or dispersing over, the space that one considers. In a spectral theoretic sense, we can thus refer to eigenfunctions as delocalising, or not concentrating, if they 'spread' out over their domain. Thus, this eludes to studying the shape and magnitude of the eigenfunctions. Moreover, one could also ask whether the eigenfunctions do not deviate so much throughout their domain and in a sense 'look' rather uniform. If the geometry of the domain plays an important role in how eigenfunctions look like, then possessing a 'uniform' or 'homogeneous' geometry would lead one to conjecture something along these lines. In fact, it is precisely this thought process that serves as motivation for some of the results here. In the mathematical physics literature, it is believed that the Laplacian eigenfunctions should exhibit behaviours that depend solely upon geometric features of the ambient space and we will discuss some of these ideas now.

Various notions of eigenfunction delocalisation are expected to occur when one considers certain limiting aspects. An intriguing regime where this is typically considered is in the eigenvalue aspect, due to its relationship to quantum mechanics. Consider the setting of a compact Riemannian manifold X so that the Hilbert space $L^2(X)$ can be diagonalised by Laplacian eigenfunctions. That is, there exists an orthonormal basis of $L^2(X)$ consisting of eigenfunctions of the Laplacian $\{\psi_j\}_{j\geq 0}$ with

$$\Delta\psi_j = \lambda_j\psi_j,$$

and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. The quantum behaviour of a particle is then modelled by a function ψ on $X \times \mathbb{R}$ such that $\psi(\cdot, t) \in L^2(X)$, and $\|\psi(\cdot, t)\|_2 = 1$ for each $t \in \mathbb{R}$. In this sense, $|\psi(x,t)|^2 \mathrm{d}x$ can be seen as a probability model for the position of the particle at time t.

Time evolution of such a particle is then given by the Schrödinger operator

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2}\Delta\psi,$$

subject to an initial condition $\psi(x,0) = \psi(x)$. Here $\hbar$ is a small semiclassical parameter (in physics it is the reduced Planck's constant). Supposing that $\psi(x) \in L^2(X)$, one can use the decomposition of the L^2 space by Laplacian eigenfunctions to expand the initial condition in the form

$$\psi(x) = \sum_{j=0}^{\infty} a_j \psi_j(x).$$

By solving Schrödinger's equation, $\psi(x,t)$ is given by

$$\psi(x,t) = \sum_{j=0}^{\infty} a_j e^{it\lambda_j\hbar}\psi_j(x).$$

From quantisation of the geodesic flow, the energy states of the quantum system are given by eigenvalues of $\hbar^2\Delta$. If E_j is such an eigenvalue, then $\lambda_j = \hbar^{-2}E_j$ is an eigenvalue of the Laplacian and vice-versa. Thus, we can write the eigenvalues of the Laplacian in the form $\hbar^{-2}E_j$. Notice by the decomposition that if one takes the initial condition $\psi(x) = \psi_{\hbar^{-2}E_j}(x)$, then the time evolved solution becomes $\psi(x,t) = e^{\frac{itE_j}{\hbar}}\psi_{\hbar^{-2}E_j}(x)$. In this case, the density of the probability measure associated with $\psi(x,t)$ is then constant in time as the exponential term has unit norm. For this reason, the Laplacian eigenfunctions play a special role, and are called *stationary states* of the system.

Bohr's correspondence principle then states that the quantum behaviour should converge to the classical behaviour of a particle when the semiclassical parameter $\hbar$ tends to zero. Equivalently, this is when the eigenvalue of the Laplacian tends to ∞, hence giving rise to the eigenvalue aspect that we eluded to previously. The classic mechanical behaviour is governed by the dynamics of the geodesic flow of the manifold. Supposing that this flow is ergodic, like in the case of hyperbolic surfaces, one would expect that in the semiclassical limit ($\hbar \to 0$), the probability models for the particles converge to uniform probabilities. This means that the densities of the probability measures converge to a constant function on the space due to the equidistribution of classical orbits. In this sense, the eigenfunctions must become *delocalised*. This line of study is related to the so-called quantum unique ergodicity conjecture of Rudnick and Sarnak [96] which states that the full sequence of eigenfunction measures should converge to the Liouville measure (volume measure) in the large eigenvalue limit for surfaces of negative sectional curvature. This question has seen a plethora of activity in recent years with some very significant advancements (see for example [111, 115, 34, 53]) for a variety of different domains. In particular, we highlight the work of Lindenstrauss [70] in the setting of arithmetically defined hyperbolic surfaces, who showed that quantum unique ergodicity holds when one considers an orthonormal basis of joint eigenfunctions for both the Laplacian and all Hecke operators on the surface (these are certain operators acting on the L^2 space that are defined in terms of the arithmetic structure). The way this is proven is by studying the entropy of the microlocal lifts of the weak-* limit points of subsequences of the eigenfunction measures. These microlocal lifts are constructed by considering a quantisation, Op, of compactly supported smooth functions on the unit tangent bundle T^1X to obtain operators on $L^2(X)$. Then one can consider

$$a \mapsto \langle \mathrm{Op}(a)\psi_j, \psi_j \rangle,$$

with $a \in C_c^\infty(T^1X)$. A microlocal lift or quantum limit (sometimes also a semiclassical defect measure) is then a probability measure μ on T^1X for which there is a subsequence $\{\psi_{j_i}\}_i$ of the eigenfunctions so that

$$\langle \mathrm{Op}(a)\psi_{j_i}, \psi_{j_i}\rangle \to \int_{T^1X} a\mathrm{d}\mu,$$

as $i \to \infty$ for all $a \in C_c^\infty(T^1X)$. When a is a function of X only, the quantisation $\mathrm{Op}(a)$ reduces to a multiplication operator, and so we obtain a weak-* limit of the eigenfunction measures $|\psi_{j_i}|^2\mathrm{dVol}_X$ when projecting μ to a measure on X. The benefit of the microlocal lifts is that the weak-* limits arising in this way are invariant under the geodesic flow on T^1X. The entropy of these measures tells us about their complexity. If, as conjectured in the quantum unique ergodicity conjecture, the only microlocal lift is the Liouville measure, then the entropy of the limit measure should be maximal [68]. Lindenstrauss [70] and Lindenstrauss and Bourgain [23] show that each ergodic component of the quantum limits in the arithmetic setting have positive entropy and then ideas from earlier results on measure rigidity are used to demonstrate arithmetic quantum unique ergodicity (see [102] for a more thorough overview).

Studying the entropy of the quantum limits has also been done by Anantharaman [4], Anantharaman-Nonnenmacher [8], and Rivière [95] in the more general setting of Anosov systems. Their results demonstrate that the entropy of a quantum limit is strictly positive, and thus a quantum limit cannot be supported only on a closed geodesic as such a measure would have zero entropy. In fact, it is shown that the entropy is at least half of the maximal entropy, and this is sharp as demonstrated in certain quantisations of cat map systems [45, 44, 65]. To improve this in more specialised cases, such as those of hyperbolic surfaces, the authors conjecture that one must have further understanding about the system, such as information on the degeneracies in the spectrum or lengths of closed geodesics. In fact, in the counterexample of cat maps where the entropy bound is sharp, high degeneracies in the spectrum are observed [45] which is thought to be atypical in some sense.

More recently, Dyatlov and Jin [39] have studied the support of the quantum limits directly for compact hyperbolic surfaces. Their results can be seen as complementary to those on the entropy. Indeed, they show that any quantum limit has full support on T^1X, that is, any quantum limit assigns a positive measure to any open subset of T^1X. This means quantum limits supported on a subset of dimension slightly under 3 (the dimension of T^1X) are ruled out, despite such a limit measure potentially having entropy close to the maximal. On the contrary, it does not rule out the quantum limit having a very small Liouville measure component such as $\alpha\mu_{\text{Liouville}} + (1-\alpha)\delta_\gamma$ for some small $\alpha > 0$, where $\mu_{\text{Liouville}}$ is the Liouville measure and δ_γ is a delta mass on a closed geodesic γ. The entropy bounds of Anantharaman and Nonnenmacher would however force $\alpha \geq \frac{1}{2}$ since the delta mass component has zero entropy.

Some other interesting conjectures related to the Laplacian and the geometry (although not necessarily delocalisation properties of the eigenfunctions) are the Berry-Tabor [18] and Bohigas-Giannoni-Schmit [21] conjectures. These are conjectures about the universality of the quantum systems in the large eigenvalue aspect. Berry and Tabor consider systems that have integrable dynamics governing the classical system and conjecture that in the semiclassical limit, the spectral statistics of the associated quantum system should typically look Poissonian. Thus, regardless of the starting system we should see universal features in the limit solely due to the underlying classical dynamics. Similarly, Bohigas, Giannoni and Schmit consider classical systems with chaotic dynamics, and conjecture that typically in the semiclassical limit on the corresponding quantum system, the spectral statistics should resemble those of certain Hermitian Gaussian random matrix ensembles, dependent only upon features such as time-reversibility of the original system. Reasonable questions to ask regarding these are what is meant by a typical system, and what sort of spectral statistics should we expect to be universal? Let us begin with the second of these. A common statistic to study is the level spacing, which is a measure of the spacing between consecutive eigenvalues. The Weyl law for a compact hyperbolic surface X states that we have the following asymptotic for the number of eigenvalues less than some λ,

$$|\{\text{eigenvalues } \lambda_j \text{ of } X \text{ with } \lambda_j \leq \lambda\}| \sim \frac{\mathrm{Vol}(X)}{4\pi}\lambda,$$

as $\lambda \to \infty$. The mean spacing between the eigenvalues is thus $m = \frac{4\pi}{\mathrm{Vol}(X)}$. The consecutive level spacing distribution is then given by

$$P(s,N) = \frac{1}{N}\sum_{j=1}^{N} \delta(s - (\lambda_{j+1} - \lambda_j)),$$

for $\delta(x)$ the Dirac mass at x on the real line. In the case of integrable classical dynamics, Berry and Tabor then conjecture that there exists a weak-* limit distribution $P(s)$ as $N \to \infty$ (i.e when we look at the asymptotics in the eigenvalue aspect). Moreover, they conjecture that $P(s)$ should be of the form $P(s) = me^{-ms}$, which is the distribution for a sequence of uncorrelated levels with mean spacing m (i.e. Poissonian with mean m). Similarly, in the case where the classical dynamics are hyperbolic (for instance Anosov), Bohigas, Giannoni and Schmit conjecture that such a $P(s)$ should also exist and be given by the distribution of a suitable Gaussian matrix ensemble (typically the Gaussian Orthogonal Ensemble or Gaussian Unitary Ensemble). In both the integrable and chaotic cases, counterexamples are known to these conjectures (for example, certain billiards in the integrable case [18] and certain arithmetic surfaces in the chaotic case [20]). This is why it is expected that one should restrict to 'typical' systems in some sense. An approach of averaging over systems for this purpose was introduced by Zirnbauer [116] for pair correlations (this is where one considers spacing between all eigenvalue pairs rather than just consecutive ones). One could also consider a probability distribution on a collection of similar systems and understand what happens with high probability in these systems. For example, the Weil-Petersson probability measure on the moduli space of hyperbolic surfaces that we consider in this thesis may provide a means to do this. In the case of random graphs, the eigenvalue statistics have been extensively studied and notions of universality have been established. We discuss aspects of these later in this chapter.

In general complete answers to questions related to the eigenvalue aspect are often rather elusive. However, we notice that in the eigenvalue aspect at least, what the expected behaviour of the eigenfunctions should be - they should delocalise on the manifold in the way that we described previously. This really is a crucial observation since it informs the type of results that we should aim to look at. Due to the difficulty of looking directly at the eigenvalue aspect, it is often instructive to look and see what is possible in a somewhat simpler but related setting, namely that of finite graphs. For the setting of hyperbolic surfaces that we consider here, it is especially beneficial when we look at regular graphs, as they share many local geometric features with hyperbolic surfaces; this is explored more explicitly in Chapter 4. A significant difference between the combinatorial Laplacian on regular graphs compared with the compact Riemannian manifold case, is that the spectrum is finite and bounded in terms of the graph degree. As a result, one can not study delocalisation in the eigenvalue aspect directly. Instead, the most natural setting to look at is the geometric, or large vertex, regime. This aspect is not only interesting as a means to understand what sort of results may be possible for manifolds, but also in its own right due to its relation to notions of large networks in computer science, and aspects of universality eluded to above. Another interesting setting that is not explored here but may offer another route to understanding delocalisation on surfaces, is the setting of quantum graphs. By this we mean a one-dimensional CW-complex with each edge assigned a length ℓ, and then studying the Laplacian on each edge considered as the interval $[0, \ell]$ with suitable boundary conditions at the vertices. There are many results pertaining to quantum ergodicity, and eigenfunction and eigenvalue statistics in this setting – see [16, 6, 49] for a selection.

Back in the world of hyperbolic surfaces, the large vertex regime has a natural translation to the spatial, or volume aspect. That is, looking at surfaces of large volume and whether eigenfunctions delocalise in this regime. The hope then is that results in this geometric limit serve as a stepping stone in the direction of the eigenvalue aspect. For example, if one can establish strong delocalisation results for large volume surfaces, can one then pass to hybrid bounds that hold for large eigenvalues increasing with the volume? This is an interesting question, and we shall return to it towards the end of this chapter. Results of this form have recently been shown for certain types of hyperbolic surfaces with an arithmetic flavour. Such surfaces originate from principal congruence subgroups of some level, and this level is related to the volume of the surfaces. How eigenfunctions behave in this level aspect is of great interest in number theory, and there are many hybrid results that seek to understand the joint dependence of Laplacian eigenfunctions simultaneously in the level and eigenvalue regimes (see for example [108] and the references therein).

Aside from this, the large volume aspect is a fruitful one for understanding one way in which the geometry of a domain may influence the shape of eigenfunctions. Recent work of Abert, Bergeron and Le Masson [2], have used this viewpoint in a formulation of the random wave conjecture of Berry [17]. Roughly, this conjecture suggests that Laplacian eigenfunctions with large eigenvalues should behave like random combinations of plane waves. For certain models of this phenomena, the law of iterated logarithms along with bounds on sup norms of Fourier series means the Gaussian behaviour predicts that typically eigenfunctions with a large eigenvalue λ should be approximately of the order $\sqrt{\log \lambda}$ (see [55, Section 6], [100, Chapter 6], [101]). By rescaling the spatial domain, Abert, Bergeron and Le Masson present a conjecture where one can instead consider eigenfunctions on sequences of manifolds that Benjamini-Schramm converge (see Chapters 4 and 5 for details of this convergence). A key example of this is a tower of coverings of compact hyperbolic surfaces $\Gamma_n \backslash \mathbb{H}$ with $\Gamma_{n+1} \subseteq \Gamma_n$, so that the surfaces are increasing in volume, and we recover the aspect that we consider here.

Another, more speculative, motivation of this regime is its potential relation to the thermodynamic limit in statistical mechanics. Here, one considers N particles in a system of volume V and considers the simultaneous limit of $N, V \to \infty$ while keeping the density N/V constant. In our setting, this would resemble considering a volume amount of wave functions that model quantum particles on surfaces as the volume tends to infinity. In the work presented here, we only consider the case of a single particle as we elect to study only a single eigenfunction at a time.

In summary, in this thesis, we shall concern ourselves with understanding the relationship between features of delocalisation for Laplacian eigenfunctions and the volume of their domain in the setting of compact hyperbolic surfaces in the volume aspect.

1.2 Spatial Delocalisation on Graphs

Due to the relation between the geometries of surfaces and graphs, it is instructive for us to discuss some of the known results and literature in this setting to inform our perspectives for the spectral geometry of surfaces. Let us begin by precisely formulating the delocalisation properties that are of interest.

Suppose that G_N is a $(d+1)$-regular graph with N vertices (we use $d+1$ for ease of presentation of the results), with associated adjacency matrix A_N.

Spectrally, the adjacency matrix and the Laplacian, Δ_N, on such graphs are essentially the same since they are related via the equation

$$A_N = \Delta_N + (d+1)I_N.$$

In particular, they have the same eigenfunctions, and thus we can consider the adjacency matrix instead of the Laplacian. Notice that A_N is symmetric, with spectrum contained in $[-(d+1), d+1]$[1]; it is thus diagonalisable with an orthonormal basis $\{\psi_j\}_{j=1}^N$. Using this diagonalisation, we can form probability measures on the graph by weighting Dirac masses:

$$\sum_{v=1}^{N} |\psi_j(v)|^2 \delta_v,$$

where we have enumerated the vertices of the graph by $v \in \{1, \ldots, N\}$. These probability measures are analogous to those formed on the manifold previously. Spatial delocalisation then looks at comparing these probability measures to the uniform measure as N gets large. Recall that the uniform probability measure is just the sum of Dirac masses at each vertex weighted by $\frac{1}{N}$. Thus, to study the extent of delocalisation of the ψ_j on a large vertex graph, some points of interest will be the following.

1. L^∞ and L^p norms of eigenfunctions: By comparing the two probability measures, an instructive step forward would be to compare $|\psi_j(v)|$ to $\frac{1}{\sqrt{N}}$. This can be done by looking at $\|\psi_j\|_\infty$ and comparing it's order to $\frac{1}{\sqrt{N}}$ or more generally, looking at the L^p norm and comparing it's order to $N^{\frac{1}{p}-\frac{1}{2}}$. Such bounds would offer strong supporting evidence to the non-localisation of the eigenfunctions.

2. Scarring phenomena: Another direction that one could attack this prob-

[1]The constant vector is always an eigenfunction with eigenvalue $d+1$.

lem from is understanding to what extent eigenfunctions can concentrate on certain subsets of the graph. The term scarring is used in analogy to the phenomena of eigenfunction concentration (in particular, near periodic geodesic orbits) in the manifold case. If the probability measures obtained by the eigenfunctions are to look uniform, then the eigenfunctions cannot localise on strict subsets of the graph. Thus, one can look at the L^2 norms of the eigenfunctions restricted to subsets of the graph in relation to the size of such a subset.

Each of these problems have seen significant progress in recent years for regular (and indeed, non-regular) graphs. Let us outline some of the known results with regards to this now. Firstly for L^p norms, the best known results thus far for deterministic regular graphs are given by Brooks and Le Masson [27]. Their work establishes a relationship between the eigenfunctions and an auxiliary parameter stemming from the geometry of the graph. This parameter is a bound on the L^∞ norm of a certain operator that acts on the universal cover of the graph (namely the infinite $(d+1)$-regular tree). More precisely, consider the operators $\tilde{S}_m$ for $m \in \mathbb{N}$ acting on functions $f : T_{d+1} \to \mathbb{C}$, where T_{d+1} is the infinite $(d+1)$-regular tree, by

$$\tilde{S}_m f(v) = \frac{1}{d^{\frac{m}{2}}} \sum_{w \in T_{d+1} : d(v,w) = m} f(w).$$

Hence, $\tilde{S}_m$ is the average on the boundary of the m-neighbourhood of a vertex normalised by roughly the square root of the size of this boundary. In particular, we can construct a copy of the universal cover T_{d+1} from any fixed vertex v in G_N. The vertices of the cover will be the non-backtracking walks ω in G_N originating at v of any length. Here, a non-backtracking walk of length m is a sequence of $m+1$ vertices $v_0 v_1 \ldots v_m$, such that $v_i v_{i+1}$ is an edge in the graph, and $v_{i-1} \neq v_{i+1}$ for each i. Two such vertices ω_1 and ω_2 are then joined by an edge if ω_2 is an extension of the walk ω_1 by walking exactly one extra edge in G_N. The projection of a vertex in T_{d+1} to the graph G_N through the universal covering projection is then just the endpoint of the corresponding walk.

With this point of view, we can interpret $\tilde{S}_m f(v)$ for lifted functions from G_N in the following way. Let $v_0 \in G_N$ be a projection of v from the cover. Up to isomorphism of the cover, we can thus consider $v \in T_{d+1}$ as the vertex corresponding to the walk of length zero starting and ending at v_0. The vertices of distance m from v are thus distinct non-backtracking walks of length m from v_0. Hence, if we consider f to be a function on T_{d+1} lifted from G_N, the function $\tilde{S}_m f$ will be the sum of the values of f evaluated at all points in G_N for which there exists a length m non-backtracking walk ending at that vertex and originating at v_0, weighted by the number of distinct such walks. In fact, the operator may be projected to functions $f : G_N \to \mathbb{C}$ on G_N in this way: lift the function to the universal cover, apply $\tilde{S}_m$ and then project the function back down. This projected operator is denoted by S_m.

To obtain estimates on the L^p norms of eigenfunctions, Brooks and Le Masson assume an upper bound estimate on the projected operator S_m of the following form: there exists an $M > 0$ such that for all $\delta > 0$, there exists a constant $C(\delta) > 0$ (independent of M) so that for each $m \leq M$,

$$\|S_m\|_{L^1 \to L^\infty} \leq C(\delta) d^{-m\left(\frac{1}{2}-\delta\right)}. \tag{1.1}$$

Notice that to obtain such a bound, it is sufficient to instead obtain a bound on the number of non-backtracking walks between vertices of different lengths. Indeed, instead one can assume for the graph G_N that there exists an $M > 0$ such that for all $\delta > 0$, there exists a constant $C(\delta) > 0$ (independent of M)

such that for all $m \leq M$ and all pairs of vertices v and w in G_N,

$$\left| \begin{Bmatrix} \text{non-backtracking walks of length } m \text{ in } G_N \\ \text{from } v \text{ to } w. \end{Bmatrix} \right| \leq C(\delta) d^{m\left(\frac{1}{2}-\delta\right)}.$$

This will imply the relevant operator norm bounds for S_m. Indeed, suppose that $f : G_N \to \mathbb{C}$ has normalised L^1 norm. Then, we can uniquely lift it to a function $\tilde{f}$ on T_{d+1} such that $\tilde{f}(\tilde{v}) = v$, for $\tilde{v}$ a lift of some vertex $v \in G_N$ as in the construction of T_{d+1} above. We then see that,

$$\begin{aligned}
|S_m f(v)| &\leq \frac{1}{d^{\frac{m}{2}}} \sum_{\tilde{w} \in T_d : d(\tilde{v}, \tilde{w}) = m} |\tilde{f}(\tilde{w})| \\
&= \frac{1}{d^{\frac{m}{2}}} \sum_{\pi(\tilde{w}) : d(\tilde{v}, \tilde{w}) = m} \left| \begin{Bmatrix} \text{non-backtracking walks of length} \\ m \text{ in } G_N \text{ from } v \text{ to } \pi(\tilde{w}) \end{Bmatrix} \right| |f(\pi(\tilde{w}))| \\
&\leq \frac{1}{d^{\frac{m}{2}}} C(\delta) d^{m\left(\frac{1}{2}-\delta\right)} \|f\|_1 \\
&= C(\delta) d^{-m\delta} \|f\|_1,
\end{aligned}$$

which upon reparametrising gives the desired norm.

It turns out that this bound on the number of non-backtracking walks is typical for random regular graphs. More precisely, if one fixes a degree, $d+1$, and number of vertices, N, then one can place the uniform probability measure on the collection of all $(d+1)$-regular graphs with N vertices. Then, with probability tending to 1 as N tends to infinity, one can take $M = c \log(N)$ for some constant $c > 0$ (see for example [76]). In a deterministic setting, one can always take M to be slightly smaller than $\mathrm{InjRad}(G_N)$, where $\mathrm{InjRad}(G_N)$ is half the length of the shortest cycle in G_N. This is because there can only be at most one non-backtracking walk of length at most M, otherwise one could follow one of the walks, and then return along the other to obtain a cycle shorter than twice the injectivity radius.

Under this assumption, Brooks and Le Masson [27] obtain L^p norms on

L^2-normalised eigenfuntions ψ for the Laplacian for any $2 < p \leq \infty$ of the form

$$\|\psi\|_p \leq \frac{C(p)}{\sqrt{M}}.$$

For deterministic graphs, this bound is the strongest currently known (of interest may be similar results proven for weighted and irregular graphs [66]). It is proven by testing eigenfunctions against a carefully constructed operator, and then using Fourier analysis on the universal cover. This lets one directly relate estimates of the norms of the operator to the distribution of cycles in the graph, which in turn can be controlled by the norms of the S_m.

With regards to the scarring phenomena, Brooks and Lindenstrauss [29] obtained a strong result under a similar assumption to that used for the L^p norms (in fact, this work predates that of the L^p norms). Indeed, suppose that $M > 0$ is a constant for the graph for which the required bounds on the operator norm of S_m from equation (1.1) hold for at least one $0 < \delta < \frac{1}{2}$. Similar to before, it suffices to assume a bound on the number of non-backtracking walks between vertices for this to hold (with again only requiring the existence of a single $0 < \delta < \frac{1}{2}$). The result then considers the relation between subsets of the vertices of the graph, and the L^2 norm of an eigenfunction restricted to such a subset.

To be more specific, suppose that ψ is an L^2 normalised eigenfunction of the Laplacian on the $(d+1)$-regular graph G_N. Moreover, assume that for some $\varepsilon > 0$ and some subset of vertices E of G_N, one has

$$\|\psi \mathbf{1}_E\|_2^2 = \varepsilon.$$

In other words, assume that

$$\sum_{v\in E} |\psi(v)|^2 = \varepsilon.$$

Obviously, due to the L^2-normalisation of the eigenfunction, $\varepsilon \leq 1$. If ε were close to or equal to, one, then this would mean that the eigenfunction is strongly concentrated on this subset in the sense that outside of that set of vertices, the supremum of the eigenfunction is small. If we expect that the eigenfunctions become delocalised on the graph, then in the case of ε close to 1, we should expect that the size of E is close to N. In fact, Brooks and Lindenstrauss show that when the parameter M is sufficiently large, one has

$$|E| \geq C\varepsilon^2 d^{\alpha\varepsilon^2 M},$$

where C and α are constants depending upon δ, and the degree, $d+1$. Recently, Ganguly and Srivastava [47] improved this result to

$$|E| \geq C\varepsilon d^{\alpha\varepsilon M}.$$

Recall that with probability tending to one as $N \to \infty$, one can take $M = c\log(N)$ for some constant $c > 0$. This means that with high probability,

$$|E| \geq C\varepsilon N^{\varepsilon\alpha'}.$$

Such a result implies that the eigenfunctions cannot (with high probability) have near full concentration on sets whose number of vertices are smaller than the order $N^{\alpha'}$. Indeed, this would correspond to $\varepsilon = 1 - \delta$ for some $\delta > 0$ small.

Ganguly and Srivastava also offer another insight into what this result implies about the geometry of the graph. If there is an eigenfunction ψ that has $\varepsilon > 0$ of the square of it's L^2 mass on a set E, then the graph must contain a cycle of length

$$O\left(\frac{\log_d\left(\frac{|E|}{\varepsilon}\right)}{\varepsilon} + \frac{1}{\varepsilon}\right).$$

Note that the initial result of Brooks and Lindenstrauss provides the same result with ε^2 rather than ε. This result follows from the fact that one can always take $2M$ to be at least the length of the shortest cycle in the graph. In fact using this contrapositive statement, Ganguly and Srivastava show that their delocalisation result is sharp, up to an additive constant of $\log_d \frac{1}{\varepsilon}$. Indeed, they demonstrate the existence of a graph which has an eigenfunction providing a squared L^2 mass of ε to a subset E, but with a cycle of length at least $\log_d(|E|)/\varepsilon$.

Let us now shift our focus to what is known about random regular graphs with regards to the delocalisation properties. We mentioned how some of the previous deterministic results can be transformed into probabilistic results by considering high probability values of the parameter M. In fact, one can actually achieve much stronger results than this by using probabilistic methods in an essential way, rather than examining geometric events. We will now tour through some of the recent works in this regard to highlight some of these results.

First, one should note that there are several models of random graphs that one can consider. Each of these models have different benefits, such as allowing one to consider slightly different classes of random regular graphs (e.g. including multi-edges between vertices, including self-loops etc.). Many of the results that are known for the delocalisation of Laplacian eigenfunctions on these sorts of graphs have been informed by what is known for random

Wigner matrices. These are random $N \times N$ Hermitian matrices whose entries are independent and identically distributed random variables. This is perhaps not so surprising since the adjacency and Laplacian matrices for the graphs in a random graph model will be random symmetric matrices. The graph degree does however put dependence between the entries.

For Wigner matrices with some mild conditions on the distributions of their entries, Erdős, Schlein and Yau [42, Theorem 5.1] prove that *most* eigenvectors, whose eigenvalues are sufficiently far from the edges of the spectrum, are completely delocalised (sup norms of order $N^{-\frac{1}{2}}$) up to a logarithmic correction, with probability tending to one as $N \to \infty$. This was improved to *all* of the eigenvectors of these random matrices (with probability tending to one as $N \to \infty$) in [41, Theorem 1.2], but again only for the eigenvectors with eigenvalues sufficiently far from edges of the spectrum. For more general Wigner matrices, namely those whose upper triangular entries have mean zero and variance one, and are bounded almost surely by some $K \leq N^{\frac{1}{2}-\varepsilon}$ (recall also that these matrices are symmetric), Tao and Vu also proved some delocalisation properties of the sup-norms. Indeed, they show that the eigenvectors have sup norms of the order

$$O\left(\frac{K \log(N)^{\alpha}}{\sqrt{N}}\right),$$

for some $\alpha > 0$, both when the eigenvalues are away from the spectral edges [106, Theorem 62], and arbitrarily close to the spectral edges [107, Proposition 1.10]. The method of proof for these types of results was revolutionised by Erdős, Schlein and Yau in the aforementioned paper. It consists of comparing the Stieltjes transform of the empirical spectral distribution of the matrices to the Stieltjes transform of the semicircle law on certain scales (what is called a local semicircle law). Recall, the empirical spectral distribution of an $N \times N$

matrix is the measure obtained from the eigenvalues of the matrix: the measure of a subset of $\mathbb{R}$ is $\frac{1}{N}$ multiplied by the number of eigenvalues in that set. For a random matrix, the empirical spectral distribution is a random variable whose image is a probability measure on $\mathbb{R}$, such that each realisation is the empirical spectral distribution of the corresponding matrix realisation. The semicircle law is the limiting distribution for many types of random matrices, it is the probability measure on $\mathbb{R}$ with density given by

$$\frac{1}{2\pi}\sqrt{\max\{4-|x|^2,0\}}.$$

The Stieltjes transform is then a certain transform of measures that has the following crucial property: if the Stieltjes transform of a sequence of measures converges to the Stieltjes transform of another measure, then one can infer convergence results about the measures themselves.

Erdős, Knowles, Yau and Yin [40] then pioneered analogues of this method for analysing the adjacency matrix of random graphs. This first approach was not for the regular graphs that are of interest to us here, rather for Erdős-Rényi graphs. These graphs are constructed by designating a probability p, perhaps dependent upon the number of vertices N, and then constructing a graph by placing an edge between any two vertices with probability p. When $pN \gg \log(N)$, such graphs are almost surely connected as $N \to \infty$. We mention the results for these briefly since they serve as direct inspiration for the results that have arrived thereafter for regular graphs. Firstly, Erdős, Knowles, Yau and Yin [40, Theorem 2.16] demonstrate under the condition that $pN \gg \log(N)^C$ for some constant $C > 0$, eigenfunctions undergo complete delocalisation. More precisely, with probability tending to one as $N \to \infty$, their sup-norms are bounded by

$$O\left(\frac{(\log(N))^{\alpha}}{\sqrt{N}}\right),$$

for some $\alpha > 0$ that controls the rate of the probability. Such work improved some results appearing around the same time for these types of graphs, namely those of Tran, Vu and Wang [110, Theorem 1.17], that proved that under the condition $pN \gg \log(N)$, the sup-norms of the eigenfunctions are bounded by $(pN)^{-\frac{1}{2}}$. Note these are only of a comparable order to that of Erdős, Knowles, Yau and Yin when $p \gg \log(N)^{-\alpha}$. Moreover, these bounds of Tran, Vu and Wang only hold for eigenfunctions whose eigenvalues are away from the edges of the spectrum, where as those of Erdős, Knowles, Yau and Yin hold all the way up to the edges.

The results for random *regular* graphs have appeared much more recently. Let us begin with those of Bauerschmidt, Knowles and Yau [12]. Again, the method of proof relies on establishing a local semicircle type of law for the random matrices that one considers. The regularity of the graphs obviously imposes restrictions on the sum of the row entries in the adjacency matrices, they must sum to the degree. Thus, there arises some dependence between the entries that was absent in the aforementioned works, adding some extra difficulty. Nonetheless, the authors establish complete delocalisation for all eigenfunctions in three distinct models of the regular graphs, namely the uniform model, permutation model and the matching model. In each model, they require control on the degree of the graph, namely lower bounds of the form $d \gg (\log(N))^4$, and upper bounds of the form $N^{\frac{2}{3}}(\log(N))^{-\frac{4}{3}}$, in the uniform model, and $N^2(\log(N))^{-4}$ for the other models. More precisely, they show [12, Corollary 1.2] that for a parameter ξ, such that $\xi\log(\xi) \gg (\log(N))^2$, one has with probability at least $1-\exp(-\xi\log(\xi))$, that the eigenfunctions have their sup norms of order

$$O\left(\frac{\xi}{\sqrt{N}}\right).$$

Here we should take ξ as small as possible, and so roughly of the order $(\log(N))^2$.

Because this is a delocalisation result on the optimal scale, we can also infer optimal scale results about non-concentration on small sets in a similar vein to the aforementioned results of Brooks and Lindenstrauss. Indeed, if we are considering an eigenfunction ψ that is L^2 normalised, and assigns an L^2 squared mass of size $\varepsilon > 0$ to a subset $E \subseteq G_N$, then

$$\varepsilon = \sum_{v \in E} |\psi(v)|^2 \leq |E| \|\psi\|_\infty^2.$$

Using the high probability upper bound on the sup norms then provides

$$|E| \geq C\varepsilon \frac{N}{\log(N)^4},$$

which is the optimal lower bound up to the logarithmic correction. In particular, this shows that eigenfunctions cannot concentrate most of their mass on a subset whose size is of order less than N with a logarithmic correction (take $\varepsilon = 1-\delta$ for some small $\delta > 0$). Moreover, they cannot have partial localisation on a small subset (take ε small but independent of N). Note that one could use the above inequality with the sup-norm estimates of Brooks and Le Masson mentioned previously. This would provide both deterministic and probabilistic non-concentration results in terms of the parameter M (see above) and hence on the scale $\log(N)$. Both of these fall short of the results already obtained by Brooks and Lindenstrauss. One should note that Dumitriu and Pal [38] also obtained delocalisation results for random regular graphs in the Brooks and

Lindenstrauss flavour, by requiring that the degree of the graphs to tend to infinity at a logarithmic rate. Their results however are not as strong as those of Bauerschmidt, Knowles and Yau.

Let us end this section with a discussion of some results in the fixed degree case. The first of these is by Bauerschmidt, Huang and Yau [11] where they investigate these delocalisation properties for random regular graphs on the optimal scale, but for large fixed degree (that is, independent of N). Again, they obtain results for the three aforementioned random models for regular graphs, but this time the eigenfunctions that one considers must have eigenvalues away from the spectral edges (roughly a distance of the order $(\log(N))^{-1}$ from the edges). Rather than deducing this from a local semicircle law, they first establish a local Kesten-McKay law. Recall that the Kesten-McKay law is the limiting distribution of the empirical spectral distributions of the graphs when keeping the degree fixed, and taking the number of vertices [75] to infinity. This allows them to deduce (see [11, Theorem 1.2]) optimal scale sup-norms, with bounds of the form

$$\frac{(\log(N))^{\alpha}}{\sqrt{N}},$$

with probability tending to one as the number of vertices tends to infinity. Again, as above, for these eigenfunctions one can determine optimal scale non-concentration estimates in the vein of Brooks and Lindenstrauss up to a logarithmic correction for random regular graphs. More recently, Huang and Yau [57] have obtained improvements of this result with similar methods, by removing the condition on the degree, so for any $d \geq 3$.

We close by noting that actually the focus of many of these aforementioned papers is to establish results on the local spectral statistics of the matrices or graphs in question, and thus it is just a byproduct of these stronger results that they manage to obtain these very strong eigenfunction delocalisation properties. Additionally, it should be noted that these papers also obtain quantum unique ergodicity results in the large vertex regime for these graphs, a natural analogue of studying the eigenfunction measures eluded to in the previous section for regular graphs.

1.3 Spatial Delocalisation on Surfaces: Contributions of the Thesis

We end this chapter by discussing the questions that this thesis seeks to address, the methods with which we tackle them, and the achieved results. Delocalisation in the spatial aspect takes the form of comparing the eigenfunctions to the volume of the space. This is the direct analogue of the spatial aspect that is considered for graphs and as such, many of the questions about spatial delocalisation that were asked in that setting, also make sense here when one replaces the graph theoretic concepts with those appropriate for manifolds, in particular for the setting of compact hyperbolic surfaces. Recall also that due to the constant curvature of hyperbolic surfaces, in the presence of no boundary, the Gauss-Bonnet theorem indicates that the volume and genus are equivalent parameters, related by the formula

$$\mathrm{Vol}(X) = 4\pi(g-1).$$

Thus, the large volume aspect is equivalent to the large genus aspect, which turns out to be more natural for us here.

With this in mind, we will consider the following questions

1. L^∞ and L^p norms of eigenfunctions: How do the sup-norms of the eigenfunctions compare to the optimal delocalisation order $g^{-\frac{1}{2}}$? Similarly, one can also study the L^p norms, and compare them to the order $g^{\frac{1}{p}-\frac{1}{2}}$.

2. Scarring phenomena: Can the eigenfunctions have almost full concentration on subsets of small volume in the manifold, or partial concentration on subsets of large volume (recall by ellipticity of the Laplacian, the zeroes of an eigenfunction have measure zero in this setting)?

Let us begin with the first of these questions. One can, like with the graph case above, consider both deterministic surfaces and randomised surfaces, and we will discuss what is meant by the latter of these shortly. A key contribution of this thesis is that we provide explicit geometric bounds for the eigenfunction norms (albeit by losing a good dependence on the eigenvalue), inspired by the line of approach used by Brooks and Le Masson for graphs. Indeed, we first impose a similar geometric assumption on the geodesic loops present in the surface. Such an assumption gives rise to a certain parameter that one can compare the norms with, and then one can argue probabilistically to show that this parameter is of a certain order for typical large genus surfaces. To make things precise, suppose that $X = \Gamma \setminus \mathbb{H}$ is a realisation of a hyperbolic surface as a quotient of the universal cover $\mathbb{H}$ by a subgroup of isometries Γ. Given $z, w \in \mathbb{H}$, we will be interested in the cardinalities of the sets

$$|\{\gamma \in \Gamma : d(z, \gamma w) \leq r\}|,$$

for different values of $r > 0$.

Geometrically, this set counts the number of images of the point w under isometries in Γ that are at most a geodesic distance of r from z. By projecting geodesic segments between z and γw for γ in this set to the surface, one obtains geodesic arcs between the projections of z and w of length at most r. In particular, each γ gives rise to homotopically distinct (with fixed endpoints) geodesic arcs between the projections of z and w on the surface. They are homotopically distinct since the lift of any homotopy between them to the universal cover results in a continuous map $H : I \times I \to \mathbb{H}$, such that $H(t,1) = \gamma_t w$ for some $\gamma_t \in \Gamma$ for each $t \in [0,1]$ where γ_0 and γ_1 are distinct elements of the above set. Since Γ acts freely and properly discontinuously (and hence has discrete topology), the γ_t are independent of t,and so $\gamma_0 = \gamma_1$, providing a contradiction. The geometric property that we will thus be interested in, will be a bound on the cardinalities of these sets for certain lengths r. Indeed, for $R, C > 0$ we will say that the surface X is (R, C)-*admissible* if for any $\delta > 0$, there exists a constant $C_0(\delta) > 0$ (dependent only upon δ), such that for any $z, w \in \mathbb{H}$ one has

$$|\{\gamma \in \Gamma : d(z, \gamma w) \leq r\}| \leq CC_0(\delta)e^{\delta r},$$

whenever $r \leq R$. The values of R and C may depend on some geometric feature of the surface X itself such as its genus or injectivity radius and for this reason we may also write $R(X)$ and $C(X)$ respectively for R and C when we wish to emphasise this. Our results will rely on X being (R, C)-admissible for R large (in fact, growing with the genus), but with C small in relation to R.

In words, the above property asks that there is a sub-exponential growth in the number of homotopically distinct geodesic arcs between points on the surface up to a certain length. The idea is that the parameter R plays the role of the parameter M in the analogous property for graphs explained previously. Indeed, one can view the number of non-backtracking walks between two points as homotopically distinct geodesic arcs, since one can view backtracking sections of a path as being null-homotopic sub-loops in a geodesic (this can be made more precise if one thinks of simplicial homotopy, see for example [52]). One glaring difference between the two concepts is the existence of the constant C in the surface case. To ensure that a surface is (R, C)-admissible for

R appropriately large, we will need to take C dependent upon the geometry of the surface which causes some slight technicalities in comparison to the results for graphs. For the results that we prove here to be meaningful, they will need to be applied to (R, C)-admissible surfaces with C small in comparison to R. This is exhibited to be typical for surfaces of large genus later on.

Similar to the graph case, there is always a pair (R, C) for which a surface X is (R, C)-admissible. Indeed, every surface will be $(c\mathrm{InjRad}(X), 1)$-admissible for any $0 < c < 1$ since in this case there are at most two group elements γ for which $d(z, \gamma w) \leq r$ for any $r \leq c\mathrm{InjRad}(X)$. Indeed, if γ_1 and γ_2 both satisfied $d(z, \gamma_i w) \leq r$ for $r \leq c\mathrm{InjRad}(X)$, for any $0 < c < 1$, then

$$d(z, \gamma_1\gamma_2^{-1}z) \leq d(z, \gamma_1 w) + d(\gamma_1 w, \gamma_1\gamma_2^{-1}z) = d(z, \gamma_1 w) + d(z, \gamma_2 w) \leq 2r.$$

Unless, $\gamma_1\gamma_2^{-1} = \mathrm{id}$, the geodesic segment between z and $\gamma_1\gamma_2^{-1}z$ projects to a non-trivial closed geodesic loop on the surface with length $d(z, \gamma_1\gamma_2^{-1}z)$. Since any such loop must be of length at least $2\mathrm{InjRad}(X) > 2r$, we must have $\gamma_1 = \gamma_2$.

One of our main results will show (see Theorem 3.1.3) that for an (R, C)-admissible surface, the Laplacian eigenfunctions have sup-norms roughly bounded above by

$$\sqrt{\frac{C}{R}},$$

up to some multiplicative constant dependent upon the eigenvalue of the eigenfunction. Similarly, one can obtain L^p norm estimates of a similar order. This bound highlights the reason why we wish for an admissibility pair with C small compared to R. Such a result is proven using the Selberg theory for hyperbolic surfaces. This allows one to construct an operator on the surface whose eigenfunctions are those of the Laplacian, and whose eigenvalues are given by a function of the L^2 spectrum of the Laplacian. The idea then is to tailor this operator to a specific eigenfunction in such a way that one obtains a certain decay on the operator norm (and hence on the eigenfunction norm). The order of decay in the sup norms that one obtains via this method is directly related to understanding the number of distinct geodesic arcs of different lengths between points on the surface (further details can be found in Chapter 3).

As is the case with graphs, to make this result stronger (and be a result for surfaces of large volume), one should look to finding parameters $R = R(X)$ and $C = C(X)$ such that $R(X)$ grows with respect to the genus whilst $C(X)$ has little growth with respect to the genus and such that there is a collection of surfaces that are $(R(X), C(X))$-admissible that is probabilistically large (probability tending to one as $g \to \infty$). There are several ways to do this, by choosing different models of random surfaces. The focus of this thesis is on the Weil-Petersson random model on the moduli space of surfaces of a fixed genus. We elaborate in more detail the precise definitions of this in Chapter 2, but in summary, the moduli space of surfaces of a fixed genus g is the collection of closed hyperbolic surfaces of genus g considered up to isometry. This space can also be regarded as the quotient of the Teichmüller space of genus g by the mapping class group. The Teichmüller space carries a natural symplectic form that passes through the quotient to the moduli space, called the Weil-Petersson symplectic form. By a standard procedure, this gives rise to a volume form on the space with respect to which the moduli space has finite volume. One can then normalise the volume measure to obtain the Weil-Petersson probability measure of genus g. The Weil-Petersson random surface model thus consists of sampling genus g surfaces with respect to this measure. This emphasises the fact that the genus aspect is more natural for us to look at in this context than the volume.

With respect to this random model, we will show that there is a collection of surfaces that are $(c\log(g), \min\{1, \mathrm{InjRad}(X)\}^{-1})$-admissible for some $0 < c < 1$ that has probability tending to one as $g \to \infty$. The proof of this occupies a large portion of Chapter 3, and utilises the integral formula of Mirzakhani [78], and some new moduli space volume estimates (see Lemma 3.6.5) to compute upper bounds on the expected number of primitive geodesic loops that are based at a given point on the surface (the collection of surfaces that we consider is one with certain bounds on these geodesic loops). It turns out that this is sufficient to give bounds on the expected number of homotopically distinct geodesic arcs between points on the surface. This is analogous to looking at bounds on cycles in graphs to prove results on the number of non-backtracking walks. Recall also, that for our result to show decay in the norms of the eigenfunctions with respect to the genus, we must have that $\min\{1, \mathrm{InjRad}(X)\}^{-1}$ is small compared to $c\log(g)$. In the Weil-Petersson model, one must be careful since the injectivity radius can be arbitrarily small for surfaces even for a non-zero measure proportion of surfaces in the large genus limit. However, Mirzakhani [80] proved that one can have control over how small the injectivity radius can be in terms of the genus for typical surfaces. Using those results, we can actually obtain that with probability tending to one as $g \to \infty$, we can bound $\min\{1, \mathrm{InjRad}(X)\}^{-1}$ from above by $\log(g)^{\varepsilon}$ for any $\varepsilon > 0$. Combining these observations, we thus obtain L^p norm estimates (see Theorem 3.1.1) holding for surfaces with Weil-Petersson probability tending to 1 as the surface genus g tends to infinity, of the form

$$\|\psi_\lambda\|_p \leq \frac{C(\lambda, p)}{\log(g)^{\frac{1}{2}-\alpha}},$$

for any $\alpha > 0$ and for some constant $C(\lambda, p) > 0$ dependent only upon the eigenvalue λ and p.

In joint work with Laura Monk [84], we further explore the geometric hypotheses utilised to obtain the previously stated eigenfunction norms. In doing so, we obtain an alternative characterisation, which leads to a simplification of the computation of suitable admissibility pairs (R, C) that hold with high probability. For this, we introduce the concept of *tangle-free surfaces.* Precisely, given a parameter $L > 0$, a hyperbolic surface X is called L-tangle-free if every embedded pair of pants and one-holed torus in X has their geodesic boundary length at least $2L$. Recall that a pair of pants is a sphere with three simple boundary curves and the one-holed torus is a genus one surface with one simple boundary curve. In both cases, we consider embedded surfaces with geodesic boundaries, and by boundary length, we mean the sum of the boundaries on that given embedded surface. If X is not L-tangle-free, then we call it L-tangled. L-tangled surfaces contain a non-simple ('tangled') closed geodesic with length at most $2L$. It turns out that if a surface is L-tangle-free, then it is $(\frac{L}{4}, \min\{1, \mathrm{InjRad}(X)\}^{-1})$-admissible. This follows from a series of results regarding the geodesics on surfaces that are L-tangle-free, whose lengths are in $[0, L]$. In fact, we demonstrate that such geodesics do not have much flexibility in their geometry.

By definition, it is easy to see that a surface is always $\mathrm{InjRad}(X)$-tangle-free. We also show that any surface is $(4\log(g) + O(1))$-tangled, and so one cannot hope to improve the order of R greater than $\log(g)$ using the tangle-free hypothesis, even for random surfaces. Let us now make a brief interlude to highlight some consequences on the geometry of tangle-free surfaces that can be found in Chapter 4.

Suppose that a surface X is L-tangle-free. We prove that all closed geodesics of length less than L are simple (Corollary 4.4.5). This is rather crucial, since in relation to admissibility pairs (R, C), it greatly simplifies the structure of curves that one needs to consider when considering curves of length

$R \leq L$. In addition to this, we demonstrate that all closed geodesics of length less than $\frac{L}{2}$ are pairwise disjoint (see Corollary 4.4.1) and, if they have length less than $\frac{L}{4}$, then they are embedded in pairwise disjoint cylinders of width at least $\frac{L}{4}$ (see Theorem 4.4.1). So not only does the tangle-free assumption tell us about the structure of the closed geodesics, it also simplfies the topology of the neighbourhoods of such geodesics. In fact when put together, these three implications provide an improvement to the classical collar theorem in hyperbolic geometry which provides the existence of disjoint isometric hyperbolic cylinders surrounding simple closed geodesic in the surface. In the classical theorem, the width of these cylinders are large when the geodesics are short, but rapidly collapses in the case of long geodesics. For L-tangle-free surfaces, this width is improved for longer geodesics when the parameter L is sufficiently large. In fact, using the integral formula of Mirzakhani, we show that with probability tending to one as the genus tends to infinity, surfaces are $a\log(g)$-tangle-free for any $0 < a < 1$, in the Weil-Petersson random model (Theorem 4.3.2). Thus, this extension to the collar theorem is rather significant as the logarithmic scale is large for hyperbolic surfaces (the diameter of such a surface is with high probability bounded by $40\log(g)$ [80]). Moreover, this probabilistic result offers another route to the logarithmic decay in the genus for the L^p norms of the eigenfunctions.

Each of the results on the geometry of the geodesics on the tangle-free parameter scale, stem from a theorem about the way loops based at a certain point are related to the shortest geodesic loop based at that point. This is the content of Theorem 4.4.2. In summary, it states that there is a unique geodesic loop of shortest length based at any point on the surface if the injectivity radius of the surface at that point is shorter than the tangle-free parameter L. Moreover, it shows that any other geodesic loop based at that point whose length is at most $\frac{L}{2}$, is homotopic to a power of the shortest loop at that point.

This is a rather powerful tool as it highlights the local structure of geodesic loops in the surface around each point, not just the closed geodesics themselves, which is a crucial aspect of (R, C)-admissibility for a surface. Further details of this are explicated in Chapter 4.

Let us now turn to the results in the final chapter of this thesis. These concern the second delocalisation question that we raised: the inability for eigenfunctions to concentrate large amounts of mass on small volume subsets of the surface. First note that, as with the case of graphs, one can infer some results in this regard from upper bounds on the sup-norms of the eigenfunctions. Indeed, suppose that $E \subseteq X$ is such that $\|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon$, for some $\varepsilon > 0$, and some L^2-normalised Laplacian eigenfunction ψ_λ. Then,

$$\varepsilon = \int_E |\psi_\lambda(x)|^2 \mathrm{d}\,\mathrm{Vol}(x) \leq \mathrm{Vol}(E)\|\psi_\lambda\|_\infty^2.$$

Hence, one has both deterministic and probabilistic lower bounds on the volume of E which are at best logarithmic in the genus. We can however do better than this, and improve it to a power of the genus. Indeed, using a similar approach as used for the L^p norm results, one can construct suitable test operators for the eigenfunctions that demonstrate delocalisation properties on scales of order e^R, for an (R, C)-admissible surface X. Indeed, if E and ψ_λ are as above, then we show in Theorem 5.1.3 that there exists some constant $A > 0$ that is independent of the surface and all other parameters such that

$$\mathrm{Vol}(E) \geq \frac{A\varepsilon}{C} e^{d(\lambda)\varepsilon R},$$

where $d(\lambda) > 0$ is some constant dependent upon the eigenvalue. As mentioned earlier, if X is L-tangle-free then it will be $(\frac{L}{4}, \min\{1, \mathrm{InjRad}(X)\}^{-1})$-admissible. Using the fact that surfaces are $(c\log(g))$-tangle-free for any $0 < c < 1$ with probability tending to one as $g \to \infty$, as well as probability estimates on the injectivity radius, the above deterministic bound on (R, C)-admissible surfaces translates to

$$\mathrm{Vol}(E) \geq A\varepsilon g^{\varepsilon\alpha(\lambda)},$$

for typical random surfaces. Once again, $A > 0$ here is a constant independent of the surface and all other parameters, and $\alpha(\lambda) > 0$ is a constant dependent only upon the eigenvalue. As with the work of Brooks and Lindenstrauss in the graph case, we observe a drop in the power of the genus here on the lower bound of the volume of E. In other words, we can only rule out that eigenfunctions cannot concentrate large amounts of mass on subsets of the surface with size of order $O(g^{\alpha(\lambda)})$. Similarly, using the Ganguly and Srivastava observation for graphs, we can say that, if there is an eigenfunction that concentrates an ε amount of L^2 squared mass on a subset E in the surface, then there exists a pair of pants or one holed torus embedded in the surface whose total boundary length is bounded above up to a multiplicative constant dependent upon the eigenvalue by

$$\frac{1}{\varepsilon}\log\left(\frac{\mathrm{Vol}(E)}{\varepsilon\,\mathrm{InjRad}(X)}\right).$$

As with the random graph case, if optimal delocalisation of sup-norms of eigenfunctions could be shown, then one would immediately obtain optimal estimates on the eigenfunction concentration using the relation between sup-norms and the volume of E presented above. It thus would be instructive to first attempt to find analogous results to the stronger random graph results that have been achieved already.

1.3.1 Perspectives

Let us conclude by discussing various avenues for improvement in the results that are presented here. Clearly, in the case of random regular graphs, the significantly stronger (indeed optimal) delocalisation results are encouraging to suggest that such results are indeed possible for random surfaces of large genus. Furthermore, the fact that we have already managed to transfer the results of Brooks and Le Masson and Brooks and Lindenstrauss to the surface setting via proving similar typical geometric properties is also very instructive. There are however significant differences between the methods used in Brooks and Le Masson and Brooks and Lindenstrauss compared to those used to obtain the optimal delocalisation results for random regular graphs.

First, there is a difference in the mentality of the approach. In the former articles, the idea is to isolate some property regarding the geometry of the graphs, connect this to the eigenfunctions, and then show that the property is typical. In the latter, from the outset, the results are proved probabilistically and there is no one specific geometric property that is isolated for the graphs. Instead, work is done directly on the probability space itself and understanding key transformations on the probability space (switchings of the graphs) that preserve the probability measure. This provides an action for which to study eigenfunction properties under whilst simultaneously ensuring they hold with high probability. Second, the methods used to control properties of the eigenfunctions are rather different. The former constructs operators to test against the eigenfunctions using fundamental properties of the Fourier analysis of the universal cover, whereas the latter uses probabilistic techniques such as concentration of measure bounds to understand the Green's function, which connects directly to the eigenfunctions.

For the case of surfaces, the methods of the former results have more direct and clear analogues. Indeed, identical tools such as the Fourier analysis on the universal cover are available in this setting, which thus makes obvious the type of geometric assumptions that one can impose to obtain the results. The difficulty then lies in determining a suitable test operator, and proving that the geometric parameter is large for typical surfaces. In the case of the purely probabilistic methods utilising the Green's function, many more difficulties arise. For example, the probability model that we use appears to be far less accessible than those used for graphs. Indeed, the discrete nature of the random graph model, and its immediate relation to random matrices offers a plethora of tools at ones disposal. The fact also that one may easily compare the adjacency/Laplacian matrices of different graphs to one another is a great advantage for the graphs. On the contrary with the Weil-Petersson model, transformations in the spaces are more complicated, and tools for working with random variables thus far seem to be quite restrictive. For example, the Mirzakhani integral formula requires one to study random variables defined in a specific way in terms of simple closed geodesics. In addition to this, working with the Green's function directly for surfaces is rather more difficult due to it being singular on the diagonal. In fact, its spectral action is not even compatible with the usual Selberg theory that is the typical tool used to connect the Laplacian spectrum to the length spectrum in this setting.

With this in mind, there are a few avenues that one may take instead. One could attempt to develop probabilistic tools in this setting in a similar vein to the random graph results. This could perhaps be achieved better in a different random model such as the Brooks-Makover model, where the relation to random graph models is more clear. In fact in this model, concentration of measure results have already been utilised to, for example, compute the probabilities of such surfaces having genus close to the expected genus in the construction. We will briefly discuss other random models such as this in Chapter 2.

Another aspect that may be of use would be to try and understand the analogue of the graph degree on surfaces. The reason for this is that the development of the results for random graphs first followed for large growing degree. This growing degree offers greater flexibility in working with the probabilistic model. It would be interesting if a reasonable analogue of the degree for surfaces would begin to make clear what would be needed to work with the probability space for the surfaces. It would also be interesting to understand what such an analogue of degree means for surfaces when it is taken large for example, what sort of geometric or spectral features would it control?

Let us mention one final avenue that may be useful to consider as an extension. It would be instructive to try to relate results in the spatial aspect back to results in the eigenvalue aspect considered from quantum mechanics previously. This could for example be done by considering hybrid bounds in both eigenvalue and genus. By considering a growing eigenvalue in terms of the genus (for example in a regime where λ grows at some rate in the genus), one can hope to obtain some forms of delocalisation that work in this weaker eigenvalue aspect. The benefit of doing this is that one is able to input a feature of randomness to the eigenvalue limit which may allow for stronger eigenvalue norm bounds.

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2 Preliminary Notions

In this chapter, we will gather many of the preliminary notions from hyperbolic geometry, spectral theory of hyperbolic surfaces and random surface theory that are used in the proceeding chapters. A reduced form of this background can also be found in those chapters where the articles are left in a mostly unaltered state. For this reason, one may wish to skip either the preliminary section here and just make use of the appropriate sections in the later chapters. The extra details are included here for the sake of completeness and exposition. Note however, proofs are largely omitted with some details given in only a few cases as most results can be found in standard references (which are provided appropriately).

2.1 Hyperbolic Geometry

We start with recalling some notions from hyperbolic geometry and explain how surfaces arise as quotients of a hyperbolic model by a group of isometries. This will be important for us as it will form the basic setting that we shall work with for the rest of the thesis.

2.1.1 Hyperbolic Plane

Hyperbolic geometry can be described through several equivalent models. For our purposes, the hyperbolic plane model provides a suitable setting to visualise much of the geometry that we will consider.

Definition 2.1.1. The *hyperbolic plane* is the upper half plane

$$\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y \in (0, \infty)\},$$

equipped with the Riemannian metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

This metric gives rise to a constant curvature equal to -1. Moreover, there is a natural volume form that arises from the metric which in coordinates can be written as

$$\mathrm{dVol} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}.$$

Also as is standard with a Riemannian metric, but we emphasise for clarity, the topology induced by the hyperbolic distance function $d(\cdot, \cdot)$ on the plane coincides with the subspace topology on $\mathbb{H}$ as a subset of $\mathbb{C}$. The boundary of the plane will play an important role when we consider geodesics and so we make a note of it.

Definition 2.1.2. The *boundary* of the hyperbolic plane is given by

$$\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}.$$

Let us now discuss the isometries of the plane as these are crucial to the construction of hyperbolic surfaces. Let $\mathrm{SL}(2, \mathbb{R})$ denote the special linear group of 2×2 matrices with real entries and determinant one. This group has a natural action on the plane via Möbius transformations:

$$\mathrm{SL}(2, \mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

This action is indeed well-defined since

$$\mathrm{Im}(g(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2} > 0.$$

If we let I denote the identity matrix, then clearly both I and $-I$ act trivially upon $\mathbb{H}$, and so we may in fact consider the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ where

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}.$$

It turns out that $\mathrm{PSL}(2,\mathbb{R})$ acts transitively upon $\mathbb{H}$ and is the full set of orientation-preserving isometries of $\mathbb{H}$. The stabiliser of any point in the plane is then just the subgroup of matrices that rotates points centred at that point. The action can also be naturally extended to $\partial\mathbb{H}$.

One can classify these matrices by the number of points that they fix in $\mathbb{H} \cup \partial\mathbb{H}$ or equivalently, by their trace.

Definition 2.1.3. Suppose that $g \in \mathrm{PSL}(2,\mathbb{R})$ is non-identity, then

1. g is called *parabolic* if $|\operatorname{Tr}(g)| = 2$ or equivalently, g has precisely one fixed point on $\partial\mathbb{H}$ and none in $\mathbb{H}$.

2. g is called *hyperbolic* if $|\operatorname{Tr}(g)| > 2$ or equivalently, g has precisely two fixed points on $\partial\mathbb{H}$ and none in $\mathbb{H}$.

3. g is called *elliptic* if $|\operatorname{Tr}(g)| < 2$ or equivalently, g has precisely one fixed point in $\mathbb{H}$ and none on the boundary $\partial\mathbb{H}$.

By conjugating with elements in $\mathrm{PSL}(2,\mathbb{R})$ each type of action has a standard form:

1. A parabolic element can be conjugated to a translation of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

for some $b > 0$. The fixed point is ∞.

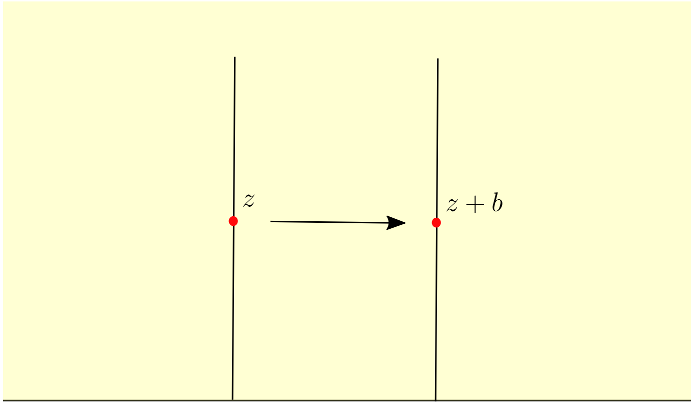


Figure 2.1: Parabolic translation.

2. A hyperbolic element can be conjugated to a dilation of the form

$$\begin{pmatrix} a^{\frac{1}{2}} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix},$$

for some $a > 1$. The fixed points are 0 and ∞.

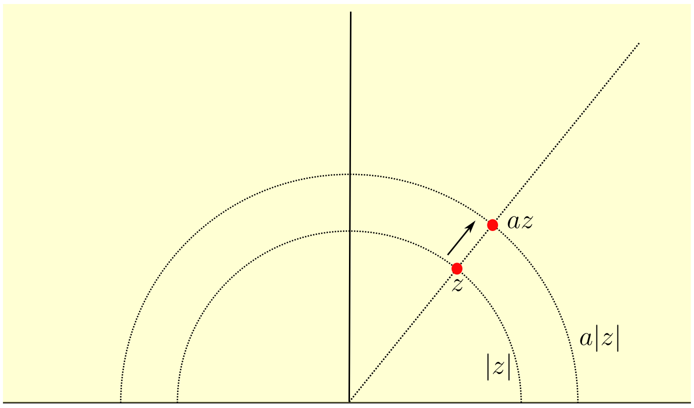


Figure 2.2: Hyperbolic translation.

3. An elliptic element can be conjugated to a rotation about i through an angle $\theta \in [0, 2\pi)$ of the form

$$\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

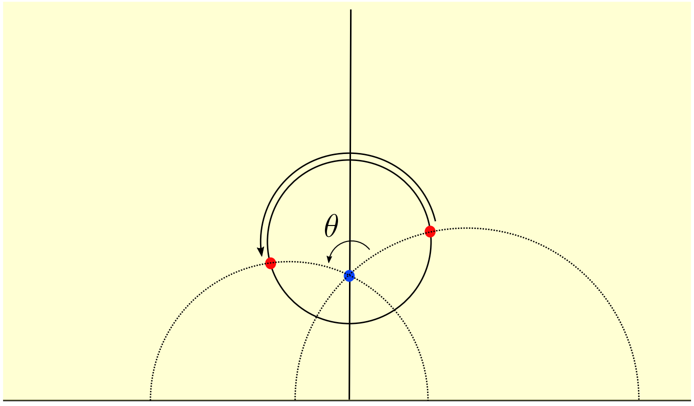


Figure 2.3: Elliptic translation.

Geodesics in the plane arising from the Riemannian metric take a particularly simple form.

Proposition 2.1.4 ([64, Theorem 1.2.1, Corollary 1.2.2])**.** *Between any two points in the plane there is a unique geodesic that joins them. In the case where the points have the same imaginary part, this geodesic is a segment of the straight line orthogonal to the real axis that passes through them. In the case where the points have different imaginary part, the geodesic joining them is a segment of the unique semicircle that passes through them and meets the real axis orthogonally.*

The distance between any two points in the plane is then just the length of the geodesic that joins them. We will make use of the following formula for this distance in the following chapters.

Proposition 2.1.5 ([64, Theorem 1.2.6])**.** *Given $z, w \in \mathbb{H}$,*

$$\cosh(d(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}.$$

2.1.2 Hyperbolic Surfaces

Recall that a hyperbolic surface (without boundary) is a two dimensional smooth manifold with charts whose domain are open subsets of $\mathbb{H}$, and whose transition maps between charts are isometries of the plane. A more useful characterisation of hyperbolic surfaces will be through quotients of the hyperbolic plane by subgroups of $\mathrm{PSL}(2, \mathbb{R})$.

Endowing $\mathrm{PSL}(2, \mathbb{R})$ with the natural topology it inherits as a quotient in $\mathbb{R}^4$, we can consider subgroups $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ that are discrete in this topology. Such subgroups are often called *Fuchsian groups*. If Γ contains no elliptic elements, then Γ acts freely on $\mathbb{H}$. In fact, when Γ is discrete it also satisfies a further condition.

Proposition 2.1.6 ([64, Theorem 2.21]). *Suppose that $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ is discrete. For all $z \in \mathbb{H}$, there exists a neighbourhood $z \in U \subseteq \mathbb{H}$ such that $\gamma U \cap U \neq \emptyset$ for only finitely many $\gamma \in \Gamma$.*

Thus, using the Quotient Manifold Theorem (see [69, Theorem 21.10]), the quotient $\Gamma \backslash \mathbb{H}$ is a smooth manifold when Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ that acts freely on $\mathbb{H}$ (contains no elliptic elements). The important case for us to consider is when such a surface is compact. A necessary condition for this to be the case is that Γ contains only hyperbolic elements and the identity. On the other hand, this condition is not sufficient since for example if one considers a single hyperbolic element such as the dilation $\gamma : z \mapsto az$, then $\Gamma = \langle \gamma \rangle$ is discrete and $\Gamma \setminus \mathbb{H}$ is not compact. Indeed, in this case, the surface is a hyperbolic cylinder with funnels for ends.

It is convenient when given a hyperbolic surface of the form $X = \Gamma \setminus \mathbb{H}$ to identify it with a subset of $\mathbb{H}$ called a *fundamental domain*. Recall that by definition of the quotient, X is just the collection of equivalence classes of the form $\Gamma z = \{\gamma z : \gamma \in \Gamma\}$ for each $z \in \mathbb{H}$ also known as the *orbits* of Γ.

Definition 2.1.7. Suppose that Γ is a Fuchsian group. A connected subset $D \subseteq \mathbb{H}$ is a *fundamental domain* of Γ if it contains precisely one element from each orbit of Γ.

Such fundamental domains readily exist for any Fuchsian group and are non-unique. One such example is the Dirichlet fundamental domain. To construct this, take any element $z \in \mathbb{H}$ that is not fixed by any non-trivial element of the group Γ (one always exists by discreteness). Then, consider for each $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ the half-planes

$$H_z(\gamma) = \{w \in \mathbb{H} : d(z, w) < d(\gamma z, w)\}.$$

A Dirichlet fundamental domain is then given by

$$D_\Gamma(z) = \bigcap_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} H_z(\gamma).$$

In the case where Γ gives rise to a compact hyperbolic surface, a fundamental domain as described above can be taken to be a compact and convex polygon with $4g$ geodesic sides contained in $\mathbb{H}$, where g is the genus of the resulting surface (details of this construction can be found in [52] for example).

2.2 Spectral Theory of Compact Hyperbolic Surfaces

We now briefly recall elements of spectral theory on compact hyperbolic surfaces. In particular, we will outline the Selberg transform and how it can be used to construct operators for analysing eigenfunctions of the Laplacian.

2.2.1 The Laplacian Operator

In coordinates, the Laplacian on the hyperbolic plane is given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The Laplacian is an important geometric operator since it commutes with isometries of the plane. That is, if $g \in \mathrm{PSL}(2, \mathbb{R})$ and T_g denotes the operator on functions of the hyperbolic plane given by $(T_g f)(z) = f(g(z))$, then

$$\Delta \circ T_g = T_g \circ \Delta,$$

for appropriately defined spaces of functions.

The Laplacian operator passes naturally to the surface via the quotient (or equivalently by looking in each chart). Functions on the surface can be identified with functions on a fundamental domain which we will fix and denote as D for the remainder of the thesis. Alternatively, they can be thought of as functions on $\mathbb{H}$ that are invariant under the action of Γ.

One can define the Laplacian operator more generally for any Riemannian manifold. In the case where the manifold is compact, such as with the compact hyperbolic surfaces that we consider here, the spectrum of the Laplacian is discrete, consists only of eigenvalues, and (with the sign chosen appropriately as above) is contained in $[0, \infty)$. It turns out that $\lambda_0 = 0$ is always an eigenvalue for the Laplacian on a compact Riemannian manifold and moreover it is simple - the only eigenfunctions with eigenvalue 0 are the constant functions.

Even more useful for us is the diagonalisation of the L^2 space of functions for a compact Riemannian manifold.

Proposition 2.2.1 ([63, Theorem 3.2.1])**.** *Given a compact Riemannian manifold M, there exists an orthonormal basis $\{\psi_{\lambda_i}\}_{i \geq 0}$ of $L^2(M)$ consisting of eigenfunctions of the Laplacian such that*

$$\Delta \psi_{\lambda_i} = \lambda_i \psi_{\lambda_i},$$

for each $i \geq 0$ and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$.

This result in particular highlights the importance of understanding the eigenfunctions of the Laplacian - any L^2 function on a compact hyperbolic surface can be decomposed as a linear combination of Laplacian eigenfunctions.

2.2.2 Invariant Integral Operators and the Selberg Transform

Next, we will define an important class of operators that, when constructed appropriately, can be used to isolate specific spectral properties of the eigenfunctions. In fact, we will make extensive use of this construction in the later chapters. The starting point for this is the notion of point-pair invariants on $\mathbb{H}$.

Definition 2.2.2. A bounded and measurable function $K : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is called a *point-pair invariant* if

1. For all $z, w \in \mathbb{H}$ and $g \in \mathrm{PSL}(2, \mathbb{R})$

$$K(g(z), g(w)) = K(z, w).$$

2. For all $z, w \in \mathbb{H}$, $K(z, w) = K(w, z)$.

These point-pair invariants are in bijection with *radial kernels*. These are bounded and measurable functions $k : [0, \infty) \to \mathbb{C}$. Indeed, given a radial kernel k, one may define a point-pair invariant via

$$K(z, w) = k(d(z, w)).$$

Conversely, given a point-pair invariant K, for any $\rho \in [0, \infty)$ there exists some $(z, w) \in \mathbb{H} \times \mathbb{H}$ with $d(z, w) = \rho$ (fix any z and travel a distance of r along any geodesic from it). Define the radial kernel by

$$k(\rho) = K(z, w).$$

This is well defined. Indeed, suppose that $(z', w') \in \mathbb{H} \times \mathbb{H}$ is also such that $d(z', w') = \rho$. Let $g \in \mathrm{PSL}(2, \mathbb{R})$ be such that $g(z) = z'$, and let $g' \in \mathrm{PSL}(2, \mathbb{R})$

be a rotation around z' that sends $g(w)$ to w' (this is possible as g is an isometry so both $g(w)$ and w' lie a distance ρ from z'). Then,

$$K(z', w') = K(g'g(z), g'g(w)) = K(z, w),$$

as required.

Now, when $K \in L^2(\mathbb{H} \times \mathbb{H})$ is a point-pair invariant, the integral operator $T_K : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ given by

$$(T_K f)(z) = \int_{\mathbb{H}} K(z, w) f(w) \mathrm{d}\mu(w),$$

is a Hilbert-Schmidt operator. Here, μ is the standard volume measure on $\mathbb{H}$ induced by the Riemannian metric. We shall actually impose a slightly stronger condition on our point-pair invariants namely, we shall assume that

$$|k(\rho)| = O\left(e^{-\rho(1+\delta)}\right), \tag{2.1}$$

for some $\delta > 0$, where k is the corresponding radial kernel. The reason we impose this extra decay condition is so that the corresponding operator that we can define on hyperbolic surfaces will converge in a sufficiently nice way for it to have many useful properties. Spectrally, these operators are closely related to the Laplacian. For example, eigenfunctions of the Laplacian are in fact eigenfunctions of T_K.

Theorem 2.2.3 ([15, Sections 3.3, 3.4] or [60, Theorem 1.14])**.** *Suppose that $k : [0, \infty) \to \mathbb{C}$ is a radial kernel satisfying the decay condition* (2.1)*. Suppose that ψ_λ is a C^∞ eigenfunction of the Laplacian on $\mathbb{H}$ with eigenvalue λ. Let s_λ be such that $s_\lambda^2 + \frac{1}{4} = \lambda$. Then, there exists a function $h : \mathbb{C} \to \mathbb{C}$ such that*

$$\int_{\mathbb{H}} k(z, w) f(w) \mathrm{d}\mu(w) = h(s_\lambda) f(w).$$

Note that the decay condition on k is required here, and that the eigenfunctions are just required to be C^∞ not the usual L^2. This result is proven by radialising the eigenfunction through rotational elliptic isometries. The remarkable fact is that the function h has a closed form and is called the Selberg transform of the radial kernel k. This is the reason that we introduced the spectral parameter s_λ in the previous theorem as it allows for a cleaner presentation of this closed form. One can follow a variety of sources such as [32, Section 9.3], [60, Chapter 1] or [15, Section 3.3] to obtain this closed form (or alternatively the original source [104]).

Theorem 2.2.4 ([104, 60, 15, 32]). *Suppose that $k : [0,\infty) \to \mathbb{C}$ is a radial kernel satisfying the decay condition* (2.1). *The function h associated to k as in Theorem 2.2.3 is given by the Selberg transform of k which is the Fourier transform*

$$h(r) := \mathcal{S}(k)(r) = \int_{-\infty}^{\infty} e^{iru} g(u) \mathrm{d}r,$$

of the function

$$g(u) = \sqrt{2} \int_{|u|}^{\infty} \frac{k(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} \mathrm{d}\rho.$$

In fact, one can also work the other way, and start with an appropriately chosen function h and recover a radial kernel. This is rather significant since for surfaces this will allow us to construct operators with a specifically chosen spectrum by choosing h particularly well.

Theorem 2.2.5 ([74]). *Suppose that $h : \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \frac{1}{2} + \varepsilon\} \to \mathbb{C}$ for some $\varepsilon > 0$ satisfies*

1. *h is analytic,*

2. *h is even,*

3. h satisfies the decay condition

$$|h(z)| = O((1+|z|^2)^{-1-\varepsilon}).$$

Then, the inverse Selberg transform of h is given

$$k(\rho) := \mathcal{S}^{-1}(\rho) = -\frac{1}{\sqrt{2}\pi}\int_\rho^\infty \frac{g'(u)}{\sqrt{\cosh(u)-\cosh(\rho)}}\mathrm{d}u,$$

where g is the inverse Fourier transform of h

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-isu}h(s)\mathrm{d}s.$$

Moreover, the function k is a radial kernel that satisfies the decay condition (2.1).

Let us now pass the theory to a compact hyperbolic surface $X = \Gamma\backslash\mathbb{H}$. To obtain a well-defined point-pair invariant that acts as the kernel of the integral operator, we can sum over the generating group Γ. To this end, given a point-pair invariant K satisfying the decay condition (2.1), define the *automorphic kernel*

$$K_\Gamma(z,w) = \sum_{\gamma\in\Gamma} K(z,\gamma w).$$

Here we are implicitly identifying the surface X with a compact fundamental domain $D \subseteq \mathbb{H}$ so that $K_\Gamma : D\times D \to \mathbb{C}$. The summation converges due to the decay condition (2.1) and the following growth estimate on the number of elements that are of a certain distance from one another.

Proposition 2.2.6 ([15, Lemma 3.17])**.** *Fix $z \in D$, then for each $m \in \mathbb{N}\cup\{0\}$*

the set

$$\Gamma_m = \{\gamma \in \Gamma : m \leq d(z, \gamma z) \leq m+1\},$$

satisfies $|\Gamma_m| = O(e^m)$ *as* $m \to \infty$.

We can then define an integral operator on the surface associated to a point-pair invariant similar to before. Indeed, let K be a point-pair invariant satisfying (2.1) and define an operator $T_K : L^2(X) \to L^2(X)$ by

$$(T_K f)(z) = \int_D K_\Gamma(z, w) f(w) \mathrm{d}\mu(w).$$

Consider now an L^2 eigenfunction of the Laplacian ψ_λ on X with eigenvalue λ. One can lift this to a C^∞ eigenfunction of the Laplacian on $\mathbb{H}$ via the covering map and thus by Theorem 2.2.3 we obtain the following result.

Theorem 2.2.7. *Let* $X = \Gamma \backslash \mathbb{H}$ *be a hyperbolic surface and* $k\colon [0, \infty] \to \mathbb{C}$ *a radial kernel. Suppose that* ψ_λ *is an eigenfunction of the Laplacian with eigenvalue* $\lambda = s_\lambda^2 + \frac{1}{4}$ *for* $s_\lambda \in \mathbb{C}$. *Then* ψ_λ *is an eigenfunction of the convolution operator* T_K *with point-pair invariant* K *and*

$$(T_K \psi_\lambda)(z) = \int_{\mathbb{H}} K(d(z, w)) \psi_\lambda(w) \, \mathrm{d}\mu(w) = h(s_\lambda) \psi_\lambda(z),$$

where $h(s_\lambda) = \mathcal{S}(k)(s_\lambda)$ *as given in Theorem 2.2.4.*

In particular, this result means that the L^2-spectrum of T_K on X is precisely $h\left(\mathrm{spec}\left(\sqrt{\Delta - \frac{1}{4}}\right)\right)$ due to the diagonalisation of $L^2(X)$ by the Laplacian. Thus, K_Γ is the integral kernel of the operator $h\left(\sqrt{\Delta - \frac{1}{4}}\right)$ defined via the functional calculus. This viewpoint is useful since one can once again use Theorem 2.2.5 to construct a test operator on the surface with specific spectral properties. This is made extensive use of to obtain the spectral results later in Chapters 3 and 5.

2.3 Geometry of Compact Hyperbolic Surfaces

We have now outlined the main spectral theoretic notions that pertain to compact hyperbolic surfaces that will be of use to us here. The next important step is to describe the main geometric features of the surfaces and in particular how this can be connected to the spectral theory. We will include a mention of the Selberg (pre-)trace formula to emphasise this connection.

2.3.1 Some Basic Geometric Concepts

Let us first describe properties of the geodesics on compact hyperbolic surfaces. Simply put, the geodesics on the surface are just the projections of geodesics in the plane to the surface, via the covering map. There are however many beautiful properties of the geometry of these geodesics, many of which are described in [32]. We reproduce some of them here as they will be used, often without direct mention, in the later chapters. Note that we will be using the concept of a hyperbolic surface with geodesic boundary which do not appear as quotients of the hyperbolic plane by a Fuchsian group as previously mentioned. Instead, their universal cover is a convex polygon in the plane whose boundaries are complete geodesics. Such a surface then arises as a quotient of this polygon by a certain Fuchsian group. The distinction won't be too important for us here, and it will be best to picture the results on the surfaces themselves rather than the universal cover.

Theorem 2.3.1. *Let X be a hyperbolic surface and let $A, B \subseteq X$ be disjoint closed boundary geodesics of X. Let $c : [a, b] \to X$ be a curve with $c(a) \in A$ and $c(b) \in B$. Then, there exists a unique geodesic γ in the homotopy class of c, where the homotopy map is allowed to have its endpoints glide along A and B respectively. The curve γ meets the boundary geodesics perpendicularly and all points other than its endpoints lie in the interior of X.*

Recall that a loop is a curve $c : [a, b] \to X$ such that $c(a) = c(b)$ i.e. the curve is closed. Such a loop is simple if there are no self-intersections of the curve other than the endpoint, or equivalently if $c|_{[a,b)}$ is injective.

Definition 2.3.2. Suppose that $c_1, c_2 : [a,b] \to X$ are closed loops. They are called *freely homotopic* if there exists a continuous map $H : [0,1] \times [a,b] \to X$ such that

$$\begin{aligned} H(0,t) &= c_1(t), \\ H(1,t) &= c_2(t), \\ H(s,a) &= H(s,b), \end{aligned}$$

for all $t \in [a,b]$ and $s \in [0,1]$.

So free homotopy is just a generalisation of homotopy for curves with designated endpoints, the latter condition ensures that each curve through the homotopy is also closed. Note that in general, when we parametrise a closed curve, we get an equivalent parametrisation if we simply start the curve from a different point. For this reason, we identify closed parametrised curves up to a reparametrisation of this form. A closed geodesic is thus an equivalence class of closed parametrised geodesics. We will also save the usage of the term closed geodesic for those that have a continuous derivative. When this is not the case at a single point, we refer to the curve as a (closed) geodesic loop with base point given by this point where it fails.

These arise in two natural ways on the surface. Recall when we work with a compact hyperbolic surface that the generating group Γ consists only of hyperbolic elements (and the identity). Each hyperbolic element $\gamma \in \Gamma$ has precisely two distinct fixed points in $\mathbb{R} \cup \{\infty\}$. The unique geodesic in $\mathbb{H}$ that joins these two endpoints is called the axis A_γ of the group element and it is precisely the set of points in $\mathbb{H}$ that realise the translation length of γ. This translation length is defined as

$$\ell(\gamma) := \inf_{z \in \mathbb{H}} d(z, \gamma z).$$

Consider now any $z \in A_\gamma$. The geodesic segment between z and γz lies entirely in A_γ, and it projects to a closed (not necessarily simple) geodesic on the surface of length $\ell(\gamma)$ through the covering map. The second kind of loop arises if one considers any point $z \notin A_\gamma$, the geodesic segment between z and γz also projects to a geodesic on the surface and this geodesic is closed but it does not necessarily close smoothly at z. Instead, we obtain a geodesic loop in this case. Such a loop will be freely homotopic to the closed geodesic originating from the axis of γ.

To see this, consider the unique shortest closed geodesic from z to the axis A_γ that meets it orthogonally. Let p be the meeting point on A_γ. The image of the geodesic under γ provides a geodesic segment that joins γz to γp, which also lies on A_γ. Parameterising the geodesic segment that joins z to p. For each $t \in [0,1]$, the corresponding point on the geodesic has its image under γ lying on the geodesic segment that joins γz and γp. Suppose for each such t we parametrise the geodesic joining these two points by $s \in [0,1]$. In doing so, one defines a map $H : [0,1] \times [0,1] \to \mathbb{H}$ where for fixed $t, s \in [0,1]$, $H(t,s)$ corresponds to the previously described parametrisation. By definition, $H(0,s)$ traverses the geodesic segment between z and γz and $H(1,s)$ traverses the portion of the axis A_γ between the points that are joined to the segment between z and γz. Thus, H is a homotopy between the geodesic segment joining z and γz and a portion of the axis A_γ. By construction this homotopy also satisfies $\gamma(H(t,0)) = H(t,1)$ for each $t \in [0,1]$, and thus it lifts to a homotopy of loops on the surface between the geodesic loop obtained as a projection of the geodesic segment between z and γz, to the closed geodesic corresponding to the projection of the axis A_γ.

Since any geodesic on the surface arises as the projection of geodesics in $\mathbb{H}$, closed geodesics and closed geodesic loops all will arise from identifying two points that are in the same Γ orbit in this way. Let us make the following definition that will actually allow us to make a stronger statement.

Definition 2.3.3. Suppose that X is a hyperbolic surface and γ is a closed geodesic on X. We call γ *primitive* if it is not the m-fold iterate of another closed geodesic on X for some $m \geq 2$. More generally, we call a geodesic loop *primitive* if it is the projection of a geodesic segment in the hyperbolic plane with endpoints differing by the action of a primitive group element.

Theorem 2.3.4 ([32, Theorem 1.5.3, 1.6.6])**.** *Suppose that X is a bordered compact hyperbolic surface whose boundary consists of primitive closed geodesics. Suppose that c is a closed homotopically non-trivial curve on X. Then,*

1. *c is freely homotopic to a unique closed geodesic γ.*
2. *γ is either contained in ∂X or is completely disjoint from the boundary.*
3. *If c is simple, then γ is simple (meaning, it contains no self-intersections).*
4. *γ is of minimal length amongst all curves in the free homotopy class.*

This means that closed geodesics are in bijection with free homotopy classes of closed curves on the surface.

Another important geometrical property of closed geodesics that we shall frequently employ is that they are minimally intersecting. Indeed, suppose that c_1 and c_2 are closed curves on the surface. Let $|c_1 \cap c_2|$ denote the minimal number of intersection points between $\tilde{c}_1$ and $\tilde{c}_2$, where $\tilde{c}_i$ is any curve freely homotopic to c_i. These intersections are counted with multiplicity. By this, we mean if $\tilde{c}_1$ and $\tilde{c}_2$ intersect at a point p and $\tilde{c}_i$ passes through p a total of m_i times, then p is counted as an intersection point $m_1 m_2$ times. We then obtain the following.

Theorem 2.3.5 ([32, Theorem 1.6.7])**.** *Suppose that X is a compact hyperbolic surface whose boundary consists of primitive closed geodesics. Let c_1 and c_2 be two closed curves on X. Then, the unique closed geodesics in the free homotopy classes of c_1 and c_2 realise the minimal number of intersections $|c_1 \cap c_2|$.*

In other words, closed geodesics are minimally intersecting with one another.

The lengths of these loops and geodesics are of paramount importance when trying to understand the spectral theory of the Laplacian, and so we make some final definitions in the subsection that allow us to talk about them.

Definition 2.3.6. Suppose that $X = \Gamma \setminus \mathbb{H}$ is a compact hyperbolic surface. For $z \in X$, and let $\tilde{z} \in \mathbb{H}$ be a point projecting to z. Define the *injectivity radius* of the surface at z by

$$\mathrm{InjRad}_X(z) = \frac{1}{2} \inf\{d(\tilde{z}, \gamma\tilde{z}) : \gamma \in \Gamma \setminus \{\mathrm{id}\}\}.$$

Moreover, define the *injectivity radius of the surface by*

$$\mathrm{InjRad}(X) = \inf_{z \in X} \mathrm{InjRad}_X(z).$$

The fact that the injectivity radius at a point is well-defined follows immediately from the fact that Γ acts isometrically on $\mathbb{H}$. In words, the injectivity radius at a point is half the length of the shortest geodesic loop that is based at that point on the surface. Equivalently, it is the radius of the largest ball one can fit around the point in the surface that is isometric to a ball of the same radius in the hyperbolic plane. The injectivity radius of the surface is thus half the length of the shortest geodesic loop on the surface or the largest radius of a ball one can fit around *every* point in the surface that is isometric to a ball of the same radius in the hyperbolic plane. For a compact hyperbolic surface, this quantity is always strictly positive.

Knowledge of the size of the injectivity radius of the surface implies some understanding of how large the surface must be. Indeed, if the surface has an embedded ball of radius $\mathrm{InjRad}(X)$, then it's volume must be at least the size of this ball which is of the order $\exp(\mathrm{InjRad}(X))$. In fact, such volume constraints also puts upper bounds on how large the injectivity radius can be. Recall that the volume of a genus g compact hyperbolic surface without boundary is given in terms of the Euler characteristic through the Gauss-Bonnet Theorem for constant curvature surfaces as

$$\mathrm{Vol}(X) = 4\pi(g-1).$$

Thus, the injectivity can be roughly only at most of the order $\log(g)$. We will see later that actually the injectivity radius of a non-trivial proportion of surfaces can be rather small.

Another important concept is the systole of a surface.

Definition 2.3.7. Suppose that X is a compact hyperbolic surface. The *systole* of the surface is the length of a shortest closed geodesic on the surface.

It turns out that the closed geodesics whose lengths realise the systole are always simple. Since every closed loop is freely homotopic to a length minimising closed geodesic on the surface that minimise self-intersections, we have that the systole length is twice the injectivity radius of the surface. We can also consider two types of systoles - the separating and non-separating variants. As their name suggests, the separating systole considers just the closed geodesics such that if one cuts along them, the surface becomes disconnected. The non-separating systole then considers the closed geodesics that when cut along do not disconnect the surface. It turns out, as we shall mention in the next section, that the separating systole is typically very large (in fact of order $\log(g)$).

One final concept we make note of is the *length spectrum* of the surface. This is the collection of lengths of the closed geodesics on the surface counted with multiplicities. One can also consider the primitive length spectrum which considers just the lengths of the primitive closed geodesics. An important bound to be aware of is one for the number of closed geodesics up to a certain length that exist on a surface. The fact that the number of such geodesics is finite is itself a key characteristic of the negative curvature. A rough upper bound on the number is given as follows.

Theorem 2.3.8 ([32, Lemmas 6.6.4, 9.2.7]). *Suppose that X is a compact hyperbolic surface of genus g and let $L > 0$. There exists a constant $C > 0$ independent of L and g such that there are at most $(g-1)\exp(L+6)+C(3g-3)$ closed geodesics of length at most L.*

In terms of the asymptotics of L this means that the number of closed geodesics of length at most L is $O(e^L)$ as $L \to \infty$. In fact, the prime number theorem for compact hyperbolic surfaces refines this to $O(L^{-1}e^L)$ as $L \to \infty$. Of course, the important asymptotics for us are contained in estimates for $g \to \infty$ potentially with L dependent upon g.

2.3.2 Selberg's (Pre-)Trace Formula

The reason why we emphasise the properties of the geodesics on these surfaces so greatly is because they are intimately linked to the Laplacian spectrum and offer the connection of the Laplacian eigenfunctions to the manifold geometry that we desire. We demonstrate a realisation of this connection here. Indirectly, this highlights some of the ideas behind how the L^p norms and non-concentration estimates are obtained in the final chapters of this thesis.

Recall that given a suitable function h as defined in Theorem 2.2.5, one can construct an integral operator whose spectrum is given through the spectrum of the Laplacian and the function h. In fact, due to the decay condition (2.1) that we place on the point-pair invariants that we use, the automorphic kernels are in $L^2(D \times D)$ for D a fundamental domain of the surface $X = \Gamma\backslash\mathbb{H}$. By general results in functional analysis, this means that the corresponding operator to a point-pair invariant K is a Hilbert-Schmidt operator and as the surface is compact, it is a trace-class operator. This means that we can compute the trace of the operator T_K by calculating

$$\mathrm{tr}(T_K) = \int_D \sum_{\gamma \in \Gamma} K(z, \gamma z) \mathrm{d}\mu(z).$$

However, using Theorem 2.2.7 we know that the Laplacian eigenfunctions form a complete system of $L^2(X)$. With convergence in $L^2(D \times D)$, we obtain

$$K_\Gamma(z, w) = \sum_{\gamma \in \Gamma} K(z, \gamma w) = \sum_{j \geq 0} h(s_{\lambda_j}) \psi_j(z) \psi_j(w),$$

where $\{\psi_j\}_{j \geq 0}$ are Laplacian eigenfunctions that form an orthonormal basis of $L^2(X)$ (here by abuse of notation we are identifying X and the fundamental domain D). Considering the case where $z = w$ then gives the pre-trace formula (pre-trace since we need to integrate to get the trace). That is,

$$\sum_{\gamma \in \Gamma} K(z, \gamma z) = \sum_{j \geq 0} h(s_{\lambda_j}) |\psi_j(z)|^2.$$

Making suitable (yet long) computations, one can find a nice form for $K(z, z)$ in terms of the Selberg transform h, and obtain the following pre-trace formula.

Theorem 2.3.9 (Selberg Pre-trace Formula [74, Theorem 3]). *Suppose that X is a compact hyperbolic surface and let $\{\psi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(X)$ consisting of Laplacian eigenfunctions. Suppose that h is a function* *satisfying the conditions of Theorem 2.2.5 and K is the associated point-pair invariant. Then, the following formula converges absolutely and uniformly in $z \in \mathbb{H}$.*

$$\sum_{j\geq 0} h(s_{\lambda_j})|\psi_j(z)|^2 = \frac{1}{4\pi}\int_{\mathbb{R}} h(\rho)\tanh(\pi\rho)\rho \mathrm{d}\rho + \sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} K(z,\gamma z),$$

where s_{λ_j} is the spectral parameter of λ_j given by $s_{\lambda_j}^2 + \frac{1}{4} = \lambda_j$.

We can now integrate both sides over the fundamental domain to obtain the trace formula.

Theorem 2.3.10 (Selberg Trace Formula [74, Theorem 4]). *Suppose that X is a compact hyperbolic surface and let $\{\psi_j\}_{j\geq 0}$ be an orthonormal basis of $L^2(X)$ consisting of Laplacian eigenfunctions. Suppose that h is a function satisfying the conditions of Theorem 2.2.5, K is the associated point-pair invariant, and g the inverse Fourier transform of h. Then the following converges absolutely.*

$$\sum_{j\geq 0} h(s_{\lambda_j}) = \frac{\mathrm{Vol}(X)}{4\pi}\int_{\mathbb{R}} h(\rho)\tanh(\pi\rho)\rho \mathrm{d}\rho + \sum_{\gamma\in\mathcal{P}(X)}\sum_{n=1}^{\infty}\frac{\ell(\gamma)}{2\sinh(n\ell(\gamma)/2)} g(n\ell(\gamma)),$$

where s_{λ_j} is the spectral parameter of λ_j given by $s_{\lambda_j}^2 + \frac{1}{4} = \lambda_j$, $\mathcal{P}(X)$ is the set of primitive closed geodesics on X and $\ell(\gamma)$ is the length of γ.

These two formulae demonstrate the remarkable connection between the Laplacian and the closed geodesics on the surface. Indeed, in the pre-trace formula, there is a direct connection between the eigenfunctions and the lengths of geodesic loops. This is because the point pair invariant K depends only on the distance between the inputs z and γz, which is the length of the geodesic loop on the surface obtained from projecting the geodesic segment between the points to the surface. Similarly, the trace formula offers a connection between the lengths of the geodesics and the spectrum of the integral operator associated to h.

In both cases, if one chooses h to be some function that localises around a certain eigenvalue of the Laplacian, then one can connect information about a given eigenfunction or eigenvalue to certain lengths of closed geodesics. For example, if one chose h to be a smoothed indicator function on some interval, then the right hand side of the trace formula will approximately count how many eigenvalues there are in that interval. The rate at which the smoothening takes place corresponds to what sort of length geodesics one needs accurate information about. Indeed, the relation between h, g and even k is controlled by a Fourier transform (through the Selberg transform). From general theory of this transform, it is known that one cannot localise a function and its Fourier transform simultaneously - localising strongly on one side causes a greater spread on the other.

We conclude this section by briefly mentioning a result of Huber that utilises the trace formula to emphasise that the length spectrum and Laplacian spectrum completely determine one another.

Theorem 2.3.11 (Huber's Theorem (see [32, Theorem 9.2.9])). *Two compact hyperbolic surfaces have the same Laplacian spectrum if and only if they have the same length spectrum.*

The idea of the proof is to use the Selberg trace formula with the heat kernel on the surface.

2.4 Models of Random Surfaces

We now conclude this chapter by describing some of the necessary background required for one of the key advances in this thesis; namely the random surface models. The primary focus will be describing the Weil-Petersson model as this is the model that we focus on for the results presented here. We will however provide some details of some other models that are of recent interest, especially with regards to understanding spectral theoretic results.

2.4.1 Building Hyperbolic Surfaces

Let us first describe the construction of the basic building blocks of compact hyperbolic surfaces.

Proposition 2.4.1. *Given lengths $a, b, c > 0$, there exists a right-angled hexagon in the hyperbolic plane with geodesic sides whose lengths are (in clockwise order) $\frac{a}{2}$,ℓ_1,$\frac{b}{2}$,ℓ_2,$\frac{c}{2}$,ℓ_3. Such a triangle is unique up to isometry.*

From this, the idea then is to glue two identical hexagons along the sides corresponding to the lengths ℓ_1, ℓ_2, ℓ_3 to obtain a surface with three closed geodesic boundaries of lengths a, b and c. This gluing is done by identifying the corresponding sides via isometries. In doing so, this puts a hyperbolic metric onto the surface. Such a surface is called a *pair of pants* and, due to the uniqueness of the hexagon up to isometry, the pair of pants with closed geodesic boundaries of lengths a, b and c is unique up to isometry.

To build a surface, we will wish to glue multiple pairs of pants together along common closed geodesic boundaries. For this to be possible, we require that the boundaries we wish to glue along are of the same length. The gluing is then done by choosing an isometry that maps one boundary to the other. Of course, there are many choices for this isometry, in fact, they may be parametrised by the unit circle: for each number t in the circle, we map a point on one boundary to the corresponding point that is a distance of t multiplied by the length of the boundary from it on the other boundary.

To obtain a compact surface with no boundary, we can use a 3-regular graph as a skeleton to glue pants along. Indeed, consider a connected 3-regular graph that has $2g - 2$ vertices and hence $3g - 3$ edges. To each vertex of this graph one may attach a pair of pants (with currently unspecified boundary lengths) such that precisely one closed geodesic boundary component lies over each emanating edge from the vertex. In the case where the edge is a self-loop of a vertex, two different boundaries will lie along the same edge but along the two different ends. To construct a genus g surface, we can then glue the pairs of pants along boundaries that lie over the same edge. For this recall that we need to specify that the boundaries have the same length (a parameter in $\mathbb{R}_{>0}$) and we require to specify an isometry. For the isometry we shall in fact prefer to specify a parameter in $\mathbb{R}$ rather than the unit circle. This isometry parameter will instead refer to the geodesic distance that one traverses around the boundary rather than the proportion. Because of this, we may wrap around the boundary geodesic several times resulting in the same gluing, but this will be preferential. So after specifying coordinates in

$$(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3},$$

we can obtain a (non-unique) hyperbolic surface of genus g.

We can actually say slightly more than this. Given a single degree 3 regular graph with $2g-2$ vertices, by varying over the parameter space above, we can obtain all of the genus g compact hyperbolic surfaces without boundary.

Theorem 2.4.2 ([32, Theorem 3.6.4])**.** *Given a fixed* 3*-regular graph, one can obtain all genus g closed hyperbolic surfaces by varying over the parameter space.*

2.4.2 Teichmüller space, Moduli space and the Weil-Petersson Model

The coordinate system that we described for specifying the surfaces is called the *Fenchel-Nielsen* coordinate system, and the space of all these surfaces is called the Teichmüller space, denoted by $\mathcal{T}_g$. This space has a more analytic description that we will now give, as this will be important for defining the Weil-Petersson random model. In fact, we will also consider surfaces with boundary as these will be required later. Let $g, n \geq 0$ be integers and let $\Sigma_{g,n}$ be a surface of genus g with n boundary components. We assign a length vector

$L = (L_1, \dots, L_n) \in \mathbb{R}^n_{\geq 0}$ to the surface such that the i^{th} boundary component has length L_i. If $L_i = 0$, then the component is thought of as a cusp or marked point on the surface.

Definition 2.4.3. The *Teichmüller space* of signature (g, n) and length vector $L \in \mathbb{R}^n_{\geq 0}$ is defined to be the space

$$\mathcal{T}_{g,n}(L) = \left\{ (X, f) : \begin{array}{l} X \text{ is a complete hyperbolic surface of genus } g\text{, with} \\ n \text{ geodesic boundary components with lengths} \\ \text{corresponding to } L\text{, and } f : \Sigma_{g,n} \to X \text{ is a} \\ \text{diffeomorphism.} \end{array} \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation defined by $(X, f) \sim (Y, k)$ if and only if there exists an isometry $h : X \to Y$ for which

$$k^{-1} \circ h \circ f \colon \Sigma_{g,n} \to \Sigma_{g,n}$$

is isotopic to the identity or equivalently, if $k \circ f^{-1} : X \to Y$ is isotopic to an isometry. Recall that an isotopy is just a homotopy via homeomorphisms. In this setting, two maps are homotopic if and only if they are isotopic (see Theorem 1.5.5 of [52]). In an element $[X, f]$, the mapping f is called a *marking* on X; it can be thought of as a hyperbolic structure on the base surface $\Sigma_{g,n}$ when one pulls back the metric via f. For notation, when L is the zero vector we denote $\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \dots, 0)$ and when there are no boundary components we simply write $\mathcal{T}_g$ for $\mathcal{T}_{g,0}$.

The equivalence of these two formulations of the Teichmüller space for $n = 0$ are expressed in [32, Theorem 6.2.7]. One can use this to obtain a real-analytic structure on the Teichmüller space by obtaining charts in $(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}$ for different choices of 3 regular graphs with $2g - 2$ vertices. One can in general define such a structure more generally for $n > 0$.

We noted that many of the surfaces in the Teichmüller space are isometric to one another. For example, varying the twist factor by the full length of a closed geodesic will give an isometric surface. For this reason, we wish to quotient out by the isometries of the surfaces. This space is called the moduli space denoted by $\mathcal{M}_{g,n}$:

$$\mathcal{M}_{g,n}(L) = \left\{ \begin{matrix} \text{complete hyperbolic surfaces of genus } g \text{ with} \\ n \text{ boundaries with length vector } L \end{matrix} \right\} \Big/ \{\text{isometries}\},$$

where the equivalence relation is up to isometries setwise preserving each boundary component. We can describe this more clearly through considering the mapping class group action on $\mathcal{T}_{g,n}(L)$. Recall the fixed base surface of the Teichmüller space is denoted by $\Sigma_{g,n}$.

Definition 2.4.4. The *mapping class group*[1] of $\Sigma_{g,n}$ is defined by

$$\mathrm{MCG}(\Sigma_{g,n}) = \mathrm{Diff}^+(\Sigma_{g,n})/\mathrm{Diff}_0^+(\Sigma_{g,n})$$

where

$$\mathrm{Diff}^+(\Sigma_{g,n}) = \left\{ \varphi : \Sigma_{g,n} \to \Sigma_{g,n} : \begin{matrix} \varphi \text{ is an orientation preserving} \\ \text{diffeomorphism that preserves the} \\ \text{boundaries of } \Sigma_{g.n} \text{ setwise and the} \\ \text{marked points pointwise} \end{matrix} \right\},$$

and $\mathrm{Diff}_0^+(\Sigma_{g,n})$ is the collection of those maps in $\mathrm{Diff}^+(\Sigma_{g,n})$ that are isotopic to the identity mapping.

The classical Dehn-Lickorish theorem states that this mapping class group is in fact finitely-generated (see for example [43, Theorem 4.1]).

The natural action of the mapping class group on the Teichmüller space is

[1] Sometimes this group is also referred to as the pure mapping class group.

then given by

$$[\varphi] \cdot [X, f] = [X, f \circ \varphi^{-1}]$$

Definition 2.4.5. The *moduli space* of signature (g, n) is given by

$$\mathcal{M}_{g,n}(L) = \mathcal{T}_{g,n}(L)/\mathrm{MCG}(\Sigma_{g,n})$$

As with the Teichmüller space, we use the notation $\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0)$ and $\mathcal{M}_g = \mathcal{M}_{g,0}$.

In addition to the real-analytic structure mentioned above, the Teichmüller space also has a complex-analytic structure that gives rise to a symplectic form $\omega_{g,n,L}$. This was proven by Goldman in [50]. It turns out that this form is invariant under the action of the mapping class group and descends in a natural way to a form on the moduli space also. This form is called the *Weil-Petersson form*. It is a remarkable theorem of Wolpert [112] that the Fenchel-Nielsen coordinates form symplectic charts for the Teichmüller and moduli spaces. In a standard way, we can obtain a volume form on the moduli space by taking the $(3g - 3 + n)$-fold wedge product:

$$\mathrm{dVol}_{g,n,L} = \frac{\bigwedge^{3g-3+n} \omega_{g,n,L}}{(3g - 3 + n)!}.$$

This volume form is called the *Weil-Petersson volume form* and it turns out that the moduli space has finite volume with respect to it. For notation, we shall write

$$V_{g,n}(L) = \mathrm{Vol}(\mathcal{M}_{g,n}(L)),$$

$$V_{g,n} = \mathrm{Vol}(\mathcal{M}_{g,n}),$$

$$V_g = \mathrm{Vol}(\mathcal{M}_g).$$

Since these are finite, we can normalise the volume form measure by the respective volumes to obtain a probability measure called the *Weil-Petersson probability measure.* In this thesis, we will only be computing probabilities of surfaces without boundary and so we simply use the notation

$$\mathbb{P}_g^{\mathrm{WP}} = \frac{\mathrm{dVol}_g}{V_g},$$

for this probability measure. The importance of introducing the Teichmüller and moduli spaces for the other types of spaces is that computing integrals of functions with respect to this probability measure, requires us to also work in the moduli spaces of surfaces with boundaries. With these preliminaries, we are in a position to define the *Weil-Petersson random surface model* as the probability model of random surfaces that samples points in the moduli space with respect to the Weil-Petersson probability measure. In particular, we shall be interested in calculating events of the form $\mathbb{P}_g^{\mathrm{WP}}(A)$ for some measurable subset $A \subseteq \mathcal{M}_g$ and computing the expectation $\mathbb{E}_g(f)$ of random variables $f : \mathcal{M}_g \to \mathbb{R}$. Recall, the expected value is just the integral

$$\mathbb{E}_g(f) = \frac{1}{V_g} \int_{\mathcal{M}_g} f(X)\mathrm{d}X,$$

where we use $\mathrm{d}X$ to denote integration with respect to the Weil-Petersson measure.

2.4.3 Mirzakhani Integral Formula

An extremely useful result that we will use for calculating integrals of certain functions over the moduli space will be *Mirzakhani's integral formula.* Such functions are defined in terms of oriented simple closed geodesics on the base surface of the Teichmüller space and then averaged over the mapping class group to define them on the moduli space. For their integration, we will require to understand how the surface looks like when it is cut along by a collection of simple closed geodesics. Recall that in the free homotopy class of an oriented simple closed curve on a hyperbolic surface, there exists a unique oriented simple closed geodesic minimising length amongst all curves in the homotopy class by Theorem 2.3.4. Thus, when we consider a simple closed curve, we can unambiguously think about the free homotopy class and simple closed geodesic representative in this class instead. In addition, we will not just be thinking about a single closed curve but instead a collection of them - these are called multicurves.

Definition 2.4.6. If $\gamma_1, \ldots, \gamma_k$ are homotopically distinct and disjoint oriented simple closed curves, we define their *multicurve* as the formal sum $\gamma = \sum_{i=1}^{k} \gamma_i$ which gives a union of curves in Σ_g.[2]

Fix a multicurve $\gamma = \sum_{i=1}^{k} \gamma_i$ and denote by $\Sigma_g \setminus \gamma$ the possibly disconnected surface with $q \geq 1$ connected components and $2k$ boundary components formed by cutting Σ_g along each of the γ_i, each cut producing two boundaries. Fix an order $\Gamma = (\gamma_1, \ldots, \gamma_k)$ for the curves in γ and suppose that $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$ denotes a length vector for Γ. Then, denote by

$$\mathcal{M}(\Sigma_g \setminus \gamma, \ell(\Gamma) = \mathbf{x}),$$

the moduli space of hyperbolic surfaces homeomorphic to $\Sigma_g \setminus \gamma$ such that the length of γ_i satisfies $\ell(\gamma_i) = x_i$. Moreover, set

$$V_g(\Gamma, \mathbf{x}) = \mathrm{Vol}(\mathcal{M}(\Sigma_g \setminus \gamma, \ell(\Gamma) = \mathbf{x})),$$

to be the volume of this moduli space. We may write $\Sigma_g \setminus \gamma$ as the disjoint

[2]We could also use weightings in the sum here but this is not needed for our purposes.

union of its connected components so that

$$\Sigma_g \setminus \gamma = \bigsqcup_{i=1}^{q} \Sigma_{g_i,n_i},$$

with $\sum_{i=1}^{q} n_i = 2k$. For the moduli space above, we then have that

$$\mathcal{M}(\Sigma_g \setminus \gamma, \ell_\Gamma = \mathbf{x}) \cong \prod_{i=1}^{q} \mathcal{M}_{g_i,n_i}(x_{i_1}, \ldots, x_{i_{n_i}}),$$

where the $i_1, \ldots, i_{n_i}$ are the indices corresponding to the multicurve components $\gamma_{i_1}, \ldots, \gamma_{i_{n_i}}$ that form the boundary of the corresponding connected component. The volume is then just

$$V_g(\Gamma, \mathbf{x}) = \prod_{i=1}^{q} V_{g_i,n_i}(x_{i_1}, \ldots, x_{i_{n_i}}).$$

Now to state Mirzakhani's integral formula, we will define the admissible geometric functions on the moduli space for which the formula can deal with. These are defined from a multicurve such as γ above. Indeed, let $F\colon \mathbb{R}_+^k \to \mathbb{R}_+$ be a symmetric measurable function, and define $F_\gamma\colon \mathcal{M}_g \to \mathbb{R}_+$ by

$$F_\gamma(X) := \sum_{\sum_{i=1}^{k} \alpha_i \in \mathrm{MCG}_g(\Sigma_g)\cdot\gamma} F(\ell_X(\alpha_1), \ldots, \ell_X(\alpha_k)).$$

Here $\sum_{i=1}^{k} \alpha_i$ is a multicurve on Σ_g obtained from mapping the multicurve γ by an element of the mapping class group. So, each α_i is the free homotopy class of the simple oriented closed curve that is the image of γ_i under a mapping class group element. Then, $\ell_X(\alpha_i)$ is the length of the simple closed geodesic in the free homotopy class of the image of α_i under the marking map on X. That is, we map the geodesic α_i from Σ_g to X via the marking map f in $[X, f]$ to obtain a simple closed curve, and then freely homotope it to its unique closed geodesic representative and calculate its length. Moreover, following Wright

[113], define

$$M(\gamma) = \left| \left\{ i = 1, \ldots, k : \begin{array}{l} \gamma_i \text{ separates a one-holed torus that does not} \\ \text{contain any other } \gamma_j \text{ from the surface} \end{array} \right\} \right|,$$

and

$$\mathrm{Sym}(\gamma) \coloneqq \mathrm{Stab}(\gamma) / \left\langle S, \bigcap_{i=1}^{k} \mathrm{Stab}^{+}(\gamma_i) \right\rangle. \tag{2.2}$$

Here $\mathrm{Stab}(\gamma)$ is the stabiliser of the multicurve γ under the mapping class group orbit, and $\mathrm{Stab}^{+}(\gamma_i)$ is the stabiliser of the single curve γ_i under mapping class group elements that preserve its orientation for $i = 1, \ldots, k$. Moreover, S is the kernel of the mapping class group orbit on the Teichmüller space, and it is trivial in the case of no boundaries and $g \geq 3$. Mirzakhani's integral formula is then stated as follows.

Theorem 2.4.7 (Mirzakhani's Integral Formula [78, Theorem 7.1])**.** *For any multicurve $\gamma = \sum_{i=1}^{k} \gamma_i$ and a symmetric measurable function $F \colon \mathbb{R}_{+}^{k} \to \mathbb{R}_{+}$, one has*

$$\int_{\mathcal{M}_g} F_{\gamma}(X) \mathrm{d}X = \frac{1}{2^{M(\gamma)} |\mathrm{Sym}(\gamma)|} \int_{\mathbb{R}_{+}^{k}} F(\mathbf{x}) V_g(\Gamma, \mathbf{x}) \mathbf{x} \cdot \mathrm{d}\mathbf{x}$$

where $\mathbf{x} \cdot \mathrm{d}\mathbf{x} = x_1 \cdots x_k \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$ and $\Gamma = (\gamma_1, \ldots, \gamma_k)$.

Thus, upon understanding how to calculate the volumes $V_g(\Gamma, \mathbf{x})$, the integral formula makes light work of integrating these sorts of functions. A crucial aspect of trying to prove spectral theoretic results is then isolating a relevant geometric condition on surfaces that can be stated in terms of multicurves for which one can use the Mirzakhani integral formula to prove its genericity. The tangle-free hypothesis that we introduce in Chapter 4 is one such property. Another is described in the second half of the next chapter for proving L^p norms.

2.4.4 Moduli Space Volumes

As highlighted, the usability of the Mirzakhani integral formula relies on ones ability to calculate the corresponding moduli space volumes of cut surfaces. In fact, explicit calculation for these is not entirely necessary and instead understanding their order for large genus will be sufficient. Indeed, recall that we are interested in the geometry of large genus surfaces, and we are primarily using the Mirzakhani integral formula for computing the probability of surfaces having some certain geometric property. If for some certain property that we characterise in the geometric function form, we can show that the probability is of some order tending to 1 as the genus tends to infinity, then we will have shown that it is typical for large genus surfaces in this model. For this, we will often only require volume estimates and asymptotics rather than explicit calculations. It should be noted however that exact volume recursion formulae are available if necessary and we will begin with stating those before stating some estimates.

In Mirzakhani's seminal paper [78] where the integral formula with respect to the Weil-Petersson form was introduced, Mirzakhani also deduced the volume recursion estimates that allow one to write the volumes of moduli space of type (g,n) in terms of volumes of moduli spaces of a type (i,j) with $3i+j<3g+n$. The starting point of this is then the base case of volumes of the moduli spaces of a pair of pants and of a one-holed torus. The first of these is one since there is precisely one pair of pairs with ascribed boundary lengths up to isometry. The one-holed torus is slightly trickier to evaluate, and was first obtained by Nakanishi and Näätänen [86] by finding a specific fundamental domain for the space. Mirzakhani also shows a calculation for this volume without using a fundamental domain in [78] which is testimony to the power of her methods not needing fundamental domains to calculate the volumes. In summary, the base surfaces have moduli space volumes given by

$$\mathrm{Vol}(\mathcal{M}_{0,3}(L_1, L_2, L_3)) = 1,$$
$$\mathrm{Vol}(\mathcal{M}_{1,1}(L)) = \frac{\pi^2}{6} + \frac{L^2}{24}.$$

The recursion formula in terms of volumes of moduli spaces with types as described above are then obtained by considering the topology of different surfaces that one may result in when removing embedded pairs of pants from a surface. For brevity (and the fact that we never will apply these formulae in succeeding chapters) we briefly state the recursion formula as

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L_1, \ldots, L_n) = \hat{\mathcal{A}}^{\mathrm{con}}_{g,n}(L_1, \ldots, L_n) + \hat{\mathcal{A}}^{\mathrm{dcon}}_{g,n}(L_1, \ldots, L_n) + \mathcal{B}_{g,n}(L_1, \ldots, L_n).$$

The first term roughly corresponds to volumes of the resulting surfaces that one obtains when one removes a pair of pants that has one boundary on the boundary of the original surface, and two boundaries in its interior configured in a way such that when cut along, they separate only the pants and leave the remaining surface connected. The second term is similar but the pair of pants disconnects the surface. Finally, the third term corresponds to pants for which only one boundary is on the interior of the surface. In this case, it won't separate the rest of the surface, only the pants itself. In addition, Mirzakhani shows that each of these terms can be expressed as polynomials in the boundary lengths $L_1, \ldots, L_n$ and as result the moduli space volume can be also. For full details, see the original article [78].

Let us now describe some further results that allow one to estimate the volumes of moduli spaces. Many of these results are again due to Mirzakhani. Firstly, one extremely useful tool is to remove the dependence on the lengths of the boundaries of the surfaces.

Lemma 2.4.8 ([80, Equation 3.7]). *Given any $g, n \geq 0$ with $2g-2+n > 0$, and $L \in \mathbb{R}^n_{\geq 0}$,*

$$V_{g,n}(2L) \leq e^{|L|} V_{g,n},$$

where $|L| = L_1 + \cdots + L_n$.

Recall that $V_{g,n}$ is just used as notation for $V_{g,n}(0,\ldots,0)$ and the condition on g and n is simply to make the surface have a negative Euler characteristic (so in particular it has a hyperbolic structure). This result is particularly useful since it then allows one to remove the volume term from the integrand of Mirzakhani's integral formula. After having used this, one can then compare the volumes of these moduli spaces with ones of different topological type using a variety of useful inequalities. For example, we have the following inequalities that are asymptotically sharp for large genus.

Lemma 2.4.9 ([80, Equation 3.20]). *Given $g, n \geq 0$ with $2g-2+n > 0$ and $0 \leq i \leq n/2$,*

$$\begin{aligned} V_{g,n} &\leq C_1 V_{g+i,n-2i}, \\ V_{g,n} &\leq C_2 \frac{V_{g,n+1}}{2g-2+n}. \end{aligned}$$

where the implied constants are independent of g, n and i.

Since we are interested in the case of surfaces of large genus, the asymptotics of these volumes in the genus can be very useful to us. Indeed, we make use of the following asymptotic expansion of $V_{g,n}$ by Mirzakhani and Zograf [82].

Theorem 2.4.10 ([82, Theorem 1.2]). *There exists a universal constant $C \in$*

$(0,\infty)$ *such that for any given* $n \geq 0$*,*

$$V_{g,n} = \frac{C}{\sqrt{g}}(2g-3+n)!(4\pi^2)^{2g-3+n}\left(1+O\left(\frac{1}{g}\right)\right),$$

as $g \to \infty$*. In particular,*

$$V_g = \frac{C}{\sqrt{g}}(2g-3)!(4\pi^2)^{2g-3}\left(1+O\left(\frac{1}{g}\right)\right),$$

as $g \to \infty$*.*

One should note that actually Mirzakhani and Zograf provide an asymptotic expansion to any polynomial order in $\frac{1}{g}$, but the first order expansion is sufficient for us here. It is also important to note that the coefficients of the powers of $\frac{1}{g}$ are actually polynomials in n and so caution should be exercised when one may have the number of boundary components dependent in some way upon the genus. In fact in general, when the number of boundary components is dependent upon the genus, not so much is known about precise asymptotics. If one does wish to consider the case where $n(g)$ depends upon the genus (as is the case here), one can use Lemma 2.4.9 to move $n(g)$ into the genus. Indeed, we can note in particular that there will always exist a genus g', and a number of boundary components $0 \leq n' \leq 3$ such that $V_{g,n(g)} \leq V_{g',n'}$. We make precise use of this observation later in proving Lemma 3.6.5, where $n(g)$ is of the form $o(g^a)$ for some $0 < a < 1$.

If one wishes to work with non-zero boundary lengths then there aren't known asymptotics of this form. In fact, a slight refinement of Lemma 2.4.8 is the best that is currently known, and we will make use of it in later chapters.

Lemma 2.4.11 ([81, Proposition 3.1])**.** *Suppose that* $g, n \geq 0$ *with* $2g-2+n >$

0 *and let* $L = (L_1, \ldots, L_n) \in \mathbb{R}^n_{\geq 0}$*, then*

$$\frac{V_{g,n}(2L)}{V_{g,n}} = \prod_{i=1}^{n} \frac{\sinh(L_i)}{L_i} \left(1 + O\left(\frac{\prod_{i=1}^{n} L_i}{g}\right)\right).$$

Notice that in this case, the constant originating from the asymptotics has no hidden dependence on n or the boundary length. Another useful and commonly occuring situation that one often finds oneself in when using the integral formula of Mirzakhani, is when there is uncertainty on the topological decomposition of the cut surface obtained when cutting with the multicurve. For this reason, it is often necessary to bound the integral by the sum over all of the different topological decompositions that one can obtain. For this, we are faced with estimating sums of products of moduli space volumes. A first result of this kind was offered by Mirzakhani in [80] where the multicurve that is considered decomposes the surfaces into a surface of n_1 boundaries and another of n_2 boundaries, so that $n_1 + n_2 = n$. In this case, the sum over all genera combinations of products of volumes of the corresponding moduli spaces of the possible subsurfaces with n_1 and n_2 boundaries can be controlled using the volume $V_{g,n}$.

Lemma 2.4.12 ([80, Lemma 3.3])**.** *Suppose that* $g, n \geq 0$ *and* $n_1, n_2 > 0$ *are such that* $2g - 2 + n > 0$*. Then,*

$$\sum_{\{(g_1, g_2)\}} V_{g_1,n_1} V_{g_2,n_2} = O\left(\frac{V_{g,n}}{g}\right),$$

where the summand runs over all combinations of $g_1, g_2 \geq 0$ *such that* $2g_i + n_i - 2 > 0$ *for* $i = 1, 2$ *and* $2(g_1 + g_2) + (n_1 + n_2) - 4 = 2g + n - 2$*.*

Here the particular restrictions on the g, n, g_i and n_i that are present originate from the Euler characteristic of the surfaces considered needing to be negative so that they have a hyperbolic metric, and such that the two subsurfaces in the decomposition do indeed glue back to the original surface (their Euler characteristics must sum to that of the original surface). One should be careful to note here that the implied constant on the right hand side is dependent upon both n_1 and n_2. This result was extended to the more general case of when the multicurve separates the surface into some q components by Mirzakhani and Petri [81].

Lemma 2.4.13 ([81, Lemma 3.2])**.** *Suppose that $g, n, q \geq 0$ and $n_1, \ldots, n_q > 0$ are such that $2g - 2 + n > 0$. Then*

$$\sum_{\{(g_i)\}} \prod_{i=1}^{q} V_{g_i,n_i} = O\left(\frac{V_{g,n}}{g^{q-1}}\right),$$

where the summand runs over all combinations of genera $g_1, \ldots, g_q \geq 0$ such that $2g_i + n_i - 2 > 0$ for $i = 1, \ldots, q$ and

$$2\sum_{i=1}^{q} g_i + \sum_{i=1}^{q} n_i - 2q = 2g + n - 2.$$

Once again, the implied constant will depend upon the number of boundary components on each subsurface. A key result of this thesis is generalising this result further to the case where the n_i can have some dependence on the genus of the surface, see Lemma 3.6.5 later for further details regarding this. We conclude this section by noting that a further generalisation of this type of result that also allows for the number of boundary components to depend upon the genus has been developed by Nie, Wu and Xue in [87, Lemma 22] using similar ideas to Lemma 3.6.5.

2.4.5 Typical Geometry in the Weil-Petersson Model

With the volume asymptotics and bounds discussed in the previous subsection, the Mirzakhani integral formula becomes very useful in determining typical properties regarding geodesics on typical surfaces in the Weil-Petersson model.

Indeed, in the proceeding chapters, this will be emphasised more when we present some original results that connect the geometry of these geodesics to properties of the eigenfunctions.

Let us begin with describing the geometry of the systole of a typical surface. Recall that the systole is the length of a shortest closed geodesic on the surface, and we distinguish two types of systole: the separating and non-separating variants.

Let us briefly illustrate with a simple example how one can obtain information about the geometry of geodesics in general by considering the case of a single simple closed geodesic. Suppose that Σ_g is a base surface of genus g as introduced previously. Then, a separating simple closed geodesic on Σ_g will have a topological decomposition of $\Sigma_{g_1,1} \cup \Sigma_{g_2,1}$ where $g_1 + g_2 = g$. suppose that $F : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $F(x) = \mathbf{1}_{[0,L]}(x)$, and γ is a simple closed geodesic that separates Σ_g in this way. Then, F_γ is the function that assigns to a surface $X \in \mathcal{M}_g$ the number of simple closed geodesics on X with lengths in $[0, L]$, that separate it into two subsurfaces of genus g_1 and g_2 respectively. This is because F_γ is the sum of F evaluated at the length function of the image of γ and all curves in its mapping class group orbit on X, and by the following result, this orbit covers all of the simple separating closed geodesics on the surface that cut in the same way.

Lemma 2.4.14 ([43, Subsection 1.3.1])**.** *There exists an orientation preserving homeomorphism of Σ_g that takes one simple closed curve to another if and only if the corresponding cut surfaces are homeomorphic.*

Now, recall Markov's inequality for a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If F is a non-negative random variable on F then for any $a > 0$, we have

$$\mathbb{P}(F \geq a) \leq \frac{1}{a}\mathbb{E}(F).$$

Then, we can see that the probability that a surface has a simple geodesic that separates a surface into two subsurfaces of genera g_1 and g_2 respectively with length in $[0, L]$ is equal to $\mathbb{P}_g(F_\gamma(X) \geq 1)$. By Markov's inequality we may bound this by

$$\mathbb{E}_g(F(X)) = \frac{1}{V_g}\int_{\mathcal{M}_g} F_\gamma(X)\mathrm{d}X = \frac{1}{V_g}\int_{\mathbb{R}} \mathbf{1}_{[0,L]}(x)V_g(\Gamma, x, x)x\mathrm{d}x.$$

Here one may use $V_g(\Gamma, x, x) \leq e^x V_{g_1,1}V_{g_2,1}$ from Lemma 2.4.8. One may then use a combination of the results in Lemma 2.4.9 to compare these volumes with V_g to obtain some decay in the genus. For example, suppose that $g_1 = 1$ and $g_2 = g-1$. Then, $V_{g-1,1} \leq \frac{C}{2g-3}V_{g-1,2} \leq \frac{C}{2g-3}V_g$ with the constant C changing between inequalities. Moreover, $V_{1,1} = \frac{\pi^2}{6}$ by the base case of the recursive formulae. Hence, the integral is bounded up to a constant independent of genus by

$$\frac{1}{V_g}\int_0^L xe^x V_{g-1,1}\mathrm{d}x = O\left(\frac{Le^L}{g}\right).$$

Thus, choosing $L < a\log(g)$ for some $0 < a < 1$ results in the integral being of order

$$O\left(\frac{\log(g)}{g^{1-a}}\right),$$

and so the probability of having a geodesic separating the surface in this way tends to zero as the genus tends to infinity.

Now that the general regime of how one can approach these sorts of problems has been highlighted, let us state some results that follow in a similar way. Firstly, one can exclude 'short' simple closed separating geodesics that give an arbitrary topological decomposition into two pieces by using Lemma 2.4.12. To this end, let $\ell_{\mathrm{sys}}(X)$ and $\ell_{\mathrm{sys}}^{\mathrm{sep}}(X)$ denote the systole and separating systole lengths on X respectively so in particular, $\ell_{\text{sys}}(X) \leq \ell_{\text{sys}}^{\text{sep}}(X)$.

Theorem 2.4.15 ([80, Theorem 4.4])**.** *For any* $0 < a < 2$*,*

$$\mathbb{P}_g^{\text{WP}}(\ell_{\text{sys}}^{\text{sep}}(X) < a\log(g)) = O\left(\frac{\log(g)^3 g^{\frac{a}{2}}}{g}\right),$$

as $g \to \infty$*.*

In other words, the probability of a surface having a simple closed geodesic of length smaller than $a\log(g)$ that separates the surface tends to zero, as $g \to \infty$. As noted by Mirzakhani, a result of [97, Theorem 1.3] shows that $\ell_{\text{sys}}^{\text{sep}}(X) \leq C\log(g)$ for some constant $C > 0$ for all compact hyperbolic surfaces X, and hence coupling this with the above result shows that there also exists a constant $B > 0$ such that

$$B\log(g) \leq \mathbb{E}_g(\ell_{\text{sys}}^{\text{sep}}(X)) \leq C\log(g),$$

for sufficiently large genus. Note that the constant B can be taken as close to 2 as one would like by the previous theorem. For slightly stronger estimates in the large genus limit, there are recent results of [87], and subsequently [92], that improve the constant B to 2 and then show this is optimal.

Theorem 2.4.16 ([87, Theorem 1])**.** *For any* $\varepsilon > 0$*,*

$$\mathbb{P}_g^{\text{WP}}((2-\varepsilon)\log(g) < \ell_{\text{sys}}^{\text{sep}}(X) < 2\log(g)) \to 1,$$

as $g \to \infty$*.*

Theorem 2.4.17 ([92, Theorem 1])**.**

$$\lim_{g\to\infty} \frac{\mathbb{E}_g(\ell_{\text{sys}}^{\text{sep}}(X))}{2\log(g)} = 1.$$

The reason that this expectation result is not immediate from the previous probability bound is the fact that the random variable $\ell_{\mathrm{sys}}^{\mathrm{sep}}(X)$ is unbounded on $\mathcal{M}_g$.

Unlike the separating systole, the systole itself can be rather small. In fact, Mirzakhani proved that there is a positive probability that it can be arbitrarily small with the probability proportional to the square of any systole length bound.

Theorem 2.4.18 ([80, Theorem 4.2])**.** *There exist constants $C > 0$ and $\varepsilon_0 > 0$ independent of the genus g, such that for any $0 < \varepsilon < \varepsilon_0$ independent of the genus,*

$$\frac{1}{C}\varepsilon^2 \leq \mathbb{P}_g^{\mathrm{WP}}(\ell_{\mathrm{sys}}(X) \leq \varepsilon) \leq C\varepsilon^2,$$

as $g \to \infty$.

The constant $\varepsilon_0 > 0$ in this result is a length scale such that any two simple closed geodesics of length less than ε_0 on any hyperbolic surface can not meet. If desired, an explicit value could be obtained from the Collar Theorem. The upper bound of Theorem 2.4.18 remains true if one allows ε to have some dependence upon genus. Indeed, if one chooses $\varepsilon = \varepsilon(g) < \varepsilon_0$ then the upper bound will still hold with the constant C independent of the genus. This is important since in later chapters we will wish to take $\varepsilon(g) = \log(g)^{-a}$ or $\varepsilon(g) = g^{-a}$ for some $a > 0$. This will allow us to work with surfaces whose injectivity radius is controlled by a shrinking lower bound at a rate that can be controlled.

Deterministically, one can obtain an upper bound on the length of the systole. This follows by noting that the injectivity radius at any point must be bounded by $\log(4g - 2)$, otherwise there will be a contradiction with the volume of the surface compared to the volume of an embedded ball in the surface. This leads to an upper bound on the systole by $2\log(4g-2)$. For an upper bound on the systole length that is typical for surfaces, Mirzakhani and Petri have a result that shows the systole length in general is not larger than scales of order $\log(g)$.

Theorem 2.4.19 ([81, Theorem 2.8])**.** *There exists constants $c, d > 0$ such that for any sequence $\{c_g\}_{g\geq 0}$ satisfying $c_g < c\log(g)$,*

$$\mathbb{P}_g^{\mathrm{WP}}(\ell_{\mathrm{sys}}(X) > c_g) < de^{-c_g}.$$

Aside from these probabilities, we can actually say a remarkable amount about the expected size of the systole length function for large genus surfaces. In fact, Mirzakhani and Petri [81] consider random variables $N_{g,[a,b]}(X)$ that assign to an $X \in \mathcal{M}_g$ the number of primitive closed geodesics on X with length in $[a,b]$, where $[a,b]$ is a sub-interval of $[0,\infty)$ with endpoints independent of genus. They then show that the sequence of random variables $\{N_{g,[a,b]}\}_{g\geq 0}$ converges in distribution to a random variable that is Poisson distributed with a certain explicit mean. Let us briefly recall some of these terms just for the sake of self-containment.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X, X_1, \ldots, X_n : \Omega \to \mathbb{N}\cup\{0\}$ be random variables. We say that X is Poisson distributed with mean λ if for any $k \in \mathbb{N}\cup\{0\}$

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Moreover, we say that the sequence $(X_n)_{n\geq 0}$ converges in distribution to X if for all $x \in \mathbb{N}\cup\{0\}$

$$\mathbb{P}(X_n = x) \to \mathbb{P}(X = x),$$

as $n \to \infty$. This can be generalised to random vectors by considering the joint distribution function instead. Recall also that we consider a random vector $(X_1, \ldots, X_n)$ to be independent or mutually independent if for each $k \leq n$, and $x_1, \ldots, x_k \in \mathbb{N}$,

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_k \leq x_k) = \prod_{i=1}^{k} \mathbb{P}(X_i \leq x_i).$$

Mirzakhani and Petri prove a result on the joint convergence in distribution of the random variables defined above.

Theorem 2.4.20 ([81, Theorem 4.1])**.** *Suppose that* $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$ *are disjoint subintervals of* $[0, \infty)$*. Then, the random vectors*

$$\{(N_{g,[a_1,b_1]}, \ldots, N_{g,[a_n,b_n]})\}_{g \geq 0},$$

converge jointly in distribution to a vector of independent Poisson distributed random variables with means

$$\lambda_{[a_i,b_i]} = \int_{a_i}^{b_i} \frac{e^t + e^{-t} - 2}{2t} \mathrm{d}t.$$

This result is proven using the method of moments which provides sufficient conditions for the random vectors to converge to a Poisson distribution (see [81, Theorem 2.7] for more details). In later chapters, we will develop some more geometric information about geodesics in terms of the tangle-free parameter of the surfaces (see Definition 4.1.1) which in particular, offers knowledge of geodesics on large volume surfaces also.

We will conclude this subsection by noting a few more points of interest regarding the geometry and its connection to the spectral theory of the Laplacian. As has been noted in the introduction, there is great interest between the geometry and the spectrum of the Laplacian itself. Such results are not the main focus of the thesis, however we collate some of them here. This is because greater knowledge of the spectrum leads to stronger results on eigenfunctions. Indeed, see Chapter 3 where slightly stronger results can be obtained depending on the knowledge of how large the spectral gap is for the surface (that is, the size of the first non-zero eigenvalue). Conversely, knowledge of the eigenfunctions can also provide some information on the spectrum. For example, the multiplicity of an eigenvalue is related to the sup-norms of eigenfunctions via the following:

$$m(\lambda) \leq \mathrm{Vol}(X) \sup\{\|\psi\|_\infty^2 : \Delta\psi = \lambda\psi, \|\psi\|_2 = 1\},$$

where $m(\lambda)$ is the multiplicity of the λ-eigenspace. An easy way to see this relationship is as follows. If $\{\psi_1, \ldots, \psi_{m(\lambda)}\}$ is an orthonormal basis of the λ-eigenspace, then

$$m(\lambda) = \int_X \sum_{i=1}^{m(\lambda)} |\psi_i(x)|^2 \mathrm{d}\,\mathrm{Vol}(x).$$

Positivity of the integrand implies that there is some $x_0 \in X$ for which

$$\sum_{i=1}^{m(\lambda)} |\psi_i(x_0)|^2 \geq \frac{m(\lambda)}{\mathrm{Vol}(X)}.$$

If we define a new eigenfunction in the λ-eigenspace by

$$\psi(x) = \sum_{i=1}^{m(\lambda)} \overline{\psi_i(x_0)}\psi_i(x),$$

then it is clear from the fact that the ψ_i form an orthonormal basis, that $\|\psi\|_2 = \sqrt{\sum_{i=1}^{m(\lambda)} |\psi_i(x_0)|^2}$. Moreover,

$$\psi(x_0) = \sum_{i=1}^{m(\lambda)} |\psi_i(x_0)|^2 \geq \frac{m(\lambda)}{\mathrm{Vol}(X)}.$$

Hence, it follows immediately that

$$\frac{\|\psi\|_\infty}{\|\psi\|_2} \geq \sqrt{\sum_{i=1}^{m(\lambda)} |\psi_i(x_0)|^2} \geq \sqrt{\frac{m(\lambda)}{\mathrm{Vol}(X)}}.$$

Since $\psi/\|\psi\|_2$ has L^2-norm equal to one, the left hand side is bounded by the supremum of the sup-norms of all L^2-normalised λ-eigenfunctions and the result follows.

As mentioned, a particular interest is with how small non-zero eigenvalues can be. The first non-zero eigenvalue which we denote by $\lambda_1(X)$ can be directly related to the surface geometry through the Cheeger constant. This constant essentially states how difficult it is to separate a surface.

Definition 2.4.21. Suppose that X is a connected and compact hyperbolic surface without boundary. The *Cheeger isoperimetric constant* of X is the quantity

$$h(X) = \inf \left\{ \frac{\ell(\Gamma)}{\mathrm{Vol}(A)} : \begin{array}{l} \Gamma \text{ ranges over all collections of separating} \\ \text{piecewise smooth multicurves and } \mathrm{Vol}(A) \text{ is} \\ \text{the smallest volume amongst the connected} \\ \text{components of } X \setminus \Gamma. \end{array} \right\}$$

Cheeger's inequality [32, Theorem 8.3.3] then states that $\lambda_1(X) \geq \frac{1}{4}h(X)^2$. Mirzakhani proved that for typical large surfaces, the Cheeger constant is bounded uniformly below.

Theorem 2.4.22 ([80, Theorem 4.8])**.**

$$\mathbb{P}_g^{\mathrm{WP}} \left\{ h(X) \geq \tfrac{\ln(2)}{2\pi + \ln(2)} \right\} \to 1,$$

as $g \to \infty$*.*

This lower bound thus gives a uniform spectral gap for surfaces with high probability through Cheeger's inequality. Such a lower bound can oftentimes be useful albeit not close to what is conjectured to be optimal probabilsitically ($\lambda_1(X) \geq \frac{1}{4} - \varepsilon$). Another approach to understanding the bottom of the spectrum is by looking at bounds on the counting function for eigenvalues in an interval. In a deterministic direction there are some informative results. Otal and Rosas [89] prove that there are at most $2g-2$ eigenvalues in the interval $[0, \frac{1}{4}]$. This is sharp in the sense that Randoll [93] and Buser [31] demonstrate for each $\varepsilon > 0$, the existence of surfaces that have $2g-2$ eigenvalues in $[0, \varepsilon]$. Moreover, Buser [31] shows that for any ε, there is no bound in terms of a function of genus for the number of eigenvalues in $[0, \frac{1}{4} + \varepsilon]$ for an arbitrary compact hyperbolic surface. In the case of probabilistic estimates, one can do slightly better. Indeed, recently Monk [83] showed that for any $b \leq \frac{1}{4}$ with probability tending to one as $g \to \infty$, the number of eigenvalues in $[0, b]$ is of order $O\left(\frac{g^{1-2^{-15}(\frac{1}{4}-b)^2}}{\log(g)^{\frac{3}{2}}}\right)$. Another striking result has recently been announced by Wu and Xue [114], and independently by Lipnowski and Wright [71], which shows with probability tending to one as $g \to \infty$, that surfaces have their first non-zero eigenvalue being at least $\frac{3}{16} - \varepsilon$ for any $\varepsilon > 0$.

2.4.6 A Few Words on Other Random Models

Let us conclude this chapter with a few words about some other models of random surfaces and some interesting spectral results that have been shown to hold for them.

We begin with the Brooks-Makover model constructed in [26]. This model is constructed from the random configuration graph model for 3-regular graphs by using a configuration as a skeleton for gluing together $2N$ ideal triangles (these are geodesic triangles whose sides are infinite length cusps). Recall that the configuration model for 3-regular graphs on $2N$ vertices is obtained by considering the uniform probability measure on partitions of the set $\{1, \ldots, 6N\}$ into pairs. Given a partition P, consider $2N$ points (which will be the graph vertices) each with 3 emanating half-edges (so we have $2N$ tripods). Running through through the tripods, label the half-edges with the numbers $\{1, \ldots, 6N\}$ so that the first tripod has labels $1, 2, 3$, the second $4, 5, 6$ and so on until the last tripod has labels $6N-2, 6N-1, 6N$ each written in a clockwise manner. To obtain a graph, glue together the half-edges that are in the same pair in the partition P. For a surface, associate to each vertex a simplicial triangle so that each side covers a half-edge, and introduce an orientation on it via the cyclic ordering of the half-edges. A surface is obtained by gluing together the triangles along sides on common graph edges via orientation reversing simplicial maps. To get hyperbolic surfaces, one can instead use the ideal triangles mentioned above. When glued together, one obtains a cusped surface which can then be compactified in a conformal manner. The parameter N is related to the genus of the surface constructed and equals $\frac{N}{2}$ up to a logarithmic correction with probability tending to 1 as $N \to \infty$ (the probability here comes from the configuration model probability). Thus as $N \to \infty$, we also obtain a random surface model for surfaces of large genus. This model is distinct from the Weil-Petersson model that we constructed previously. In this model, we can however see some similar notions regarding their geometry. For example, it is shown in [26] that the following holds.

Theorem 2.4.23 ([26, Theorem 2.2])**.** *There exists constants $C_i > 0$ for $i = 1, \ldots, 4$ such that*

1. *The first eigenvalue of a Brooks-Makover surface is bounded below by C_1 with probability tending to 1 as $N \to \infty$.*
2. *The Cheeger constant of a Brooks-Makover surface is bounded below by C_2 with probability tending to 1 as $N \to \infty$.*
3. *The systole of a Brooks-Makover surface is bounded below by C_3 with probability tending to 1 as $N \to \infty$.*

4. *The diameter of a Brooks-Makover surface is bounded above by* $C_4 \log(N)$ *with probability tending to* 1 *as* $N \to \infty$.

Notice a significant difference for the systole here to the Weil-Petersson model is that it is uniformly bounded and cannot be arbitrarily small with high probability.

Another random model for compact hyperbolic surfaces is the random cover model. Once again, this model provides a way of accessing hyperbolic surfaces of large genus. The idea for this model is to fix a base hyperbolic surface and then for each n, look at degree n covers of that surface. Since there are only finitely many such covers, one can equip the collection of degree n covers with a uniform probability. Recall that a degree n cover of a surface is one such that the pre-image of each point on the surface under the covering map contains precisely n points. For this reason, the Euler characteristic of the cover is precisely n times that of the base surface. If the base surface has no boundary, the genus of the cover is $2 + 2n(g - 1)$ where g is the base surface genus, and hence it grows linearly in n (from a fixed base g). The idea then is to understand spectral and geometric properties about random covers as $n \to \infty$. Magee, Naud and Puder [73] have investigated these sorts of problems in relation to the bottom of the spectrum. They show that if one starts with a base surface X, then with probability tending to 1 as $n \to \infty$, an n-cover of X has no new eigenvalues smaller than $\frac{3}{16} - \varepsilon$. In particular, if one starts with a base surface for which the smallest non-zero eigenvalue is greater than $\frac{3}{16}$, then one can obtain sequences of surfaces with increasing genus that also have no eigenvalues smaller than $\frac{3}{16} - \varepsilon$, for any $\varepsilon > 0$. This result is similar to (and in fact inspires) the previous result mentioned by Wu and Xue [114].

In both of these models, there is not yet currently any work on properties of the eigenfunctions. Using the spectral theoretic techniques that are demonstrated in Chapters 3 and 5, it would suffice to show that a property such as the tangle-free hypothesis is typical for surfaces in these models (see Chapter 4 for the definition of this property) to obtain information about the L^p norms of the eigenfunctions. Thus, one could work entirely geometrically to obtain further spectral theoretic results.

3 Short Geodesic Loops and L^p Norms of Eigenfunctions on Large Genus Random Surfaces

Abstract

We give upper bounds for L^p norms of eigenfunctions of the Laplacian on compact hyperbolic surfaces in terms of a parameter depending on the growth rate of the number of short geodesic loops passing through a point. When the genus $g \to +\infty$, we show that random hyperbolic surfaces X with respect to the Weil-Petersson volume have with high probability at most one such loop of length less than $c \log g$ for small enough $c > 0$. This allows us to deduce that the L^p norms of L^2 normalised eigenfunctions on X are $O(1/\sqrt{\log g})$ with high probability in the large genus limit for any $p > 2+\varepsilon$ for $\varepsilon > 0$ depending on the spectral gap $\lambda_1(X)$ of X, with an implied constant depending on the eigenvalue and the injectivity radius.

3.1 Introduction

3.1.1 Background and main result

In the setting of a compact n-dimensional Riemannian manifold (M, g), a deep understanding of the shape and asymptotics of eigenfunctions of the

Laplacian is intimately linked to underlying geometric properties of the space itself. One means to realise this connection is through studying the L^p norms of the eigenfunctions themselves. Indeed, as an example, primitive estimates show that the multiplicity of the eigenvalues are influenced by the sup norms of the eigenfunctions as well as the volume of the space through

$$m(\lambda) \leq \mathrm{Vol}(M) \sup\{\|\psi\|_\infty^2 : \Delta\psi = \lambda\psi, \|\psi\|_2 = 1\},$$

where $m(\lambda)$ is the multiplicity of the eigenvalue λ (see for example the proof of Proposition 2.1 in [37]).

Eigenfunctions of the Laplacian feature prominently in quantum mechanics since they are precisely the states for which the probability measures $|\psi(x,t)|^2 \,\mathrm{d}\,\mathrm{Vol}_M(x)$ are constants, where $\psi(x,t)$ is the free quantum evolution of a wavefunction $\psi(x)$. In this setting, a widely studied problem is to understand the properties of the eigenfunctions in the high energy, or large eigenvalue, limit, aiming to recover some characteristics of the classical dynamics, for example in the study of Quantum (Unique) Ergodicity [111, 115, 34, 96, 70, 53].

In the large eigenvalue aspect, Sogge's [105] seminal work identified the link between the growth of L^p norms of eigenfunctions and their L^2 norms in terms of their eigenvalue. In particular, if $\Delta\psi = \lambda\psi$ then

$$\|\psi\|_p \lesssim_M \lambda^{\sigma(n,p)} \|\psi\|_2,$$

where

$$\sigma(n,p) = \begin{cases} \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}, & \text{if } \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{4}\left(\frac{1}{2} - \frac{1}{p}\right), & \text{if } 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}$$

Here we use the notation $A \lesssim B$ to denote that there is a universal constant C for which $A \leq CB$. If there are subscripts on the symbol $\lesssim$, then we mean that the constant C possibly depends upon the parameters in the subscript. These bounds are sharp on the sphere and attained for high p by zonal spherical harmonics (concentration of the mass around a point) and for low p by Gaussian beams (concentration along closed geodesics). However, in the case of manifolds of non-positive curvature (or without conjugate points), Bérard [25] had previously obtained a logarithmic improvement of the sup norm. This was more recently extended to values of $p > \frac{2(n+1)}{n-1}$ by Hassell and Tacy [54].

The implied constant in the Sogge bound was investigated by Donnelly [37] to reveal that the underlying geometry again plays an important role. More specifically, the constant depends upon bounds on the injectivity radius and the sectional curvature of the manifold.

In this paper we restrict our attention to hyperbolic surfaces and investigate the influence of the geometry on L^p norms. Rather than seeking bounds in terms of eigenvalues, we focus on their dependence on the growth rate of short geodesic loops (see (3.1) below). Our goal is to understand this geometric connection with random hyperbolic surfaces, using integration tools on the moduli space developed by Mirzakhani [78, 79, 80, 82]. In [80], Mirzakhani initiated a theory of large genus random surfaces with respect to the Weil-Petersson volume (see Section 3.5 for background). An important success of these methods was the proof by Mirzakhani and Petri [81] that the length of short geodesics on random surfaces follow a Poisson distribution in the large genus limit. From there, it is natural to try to connect the behaviour of closed geodesics to the spectrum of random surfaces via Selberg's theory (see for example [15] for background on Selberg's trace formula). We present in this paper one of the first attempts at such a connection between the geometry of random surfaces and eigenfunctions of the Laplacian.

A central motivation in our work is to understand the delocalisation properties of eigenfunctions on large volume manifolds. In a recent article [67], Le Masson and Sahlsten proposed a version of quantum ergodicity for hyperbolic surfaces of large genus.[1] The theorem is a delocalisation result analogous to the quantum ergodicity theorem of Šnirel'man [111], Zelditch [115] and Colin de Verdière [34], but valid in the large volume limit, and for eigenfunctions in a bounded spectral interval. This result was inspired by corresponding theorems on regular graphs [7, 28], viewed as discrete analogues of hyperbolic surfaces. We will follow a similar heuristic to push the graph methods of Brooks and Le Masson [27] to the continuous setting. This deterministic aspect will be combined with new estimates on short geodesics of random surfaces, to obtain bounds on L^p norms for random surfaces. Recently, there have been major breakthroughs in the study of eigenvectors on random regular graphs, from optimal sup-norm bounds [11] to the proof of their Gaussian behaviour [10]. Our hope is to have provided a stepping stone towards the adaptation of these more advanced results.

Before we state our main theorem, let us define the model of random surfaces we are considering. For any $g \geq 2$, we denote by $\mathcal{M}_g$ the moduli space of compact hyperbolic surfaces of genus g. It can be seen as a quotient $\mathcal{M}_g = \mathcal{T}_g/\mathrm{MCG}_g$ of the Teichmüller space $\mathcal{T}_g$ by the mapping class group MCG_g (see Section 3.5 for definitions). The Teichmüller space $\mathcal{T}_g$ is equipped with a symplectic form ω_g called the Weil-Petersson form that is invariant under the action of MCG_g. The associated volume form then descends to the quotient $\mathcal{M}_g$, which is of finite total volume. Denoting by $\mathrm{Vol}_{wp}(A)$ the Weil-Petersson volume of a measurable set $A \subset \mathcal{M}_g$, we obtain the probability

[1] Note that by the Gauss-Bonnet theorem the genus g and the volume of a compact hyperbolic surface $|X|$ are related by the formula $|X| = 2\pi(2g-2)$, and are therefore equivalent parameters in this context.

measure

$$\mathbb{P}_g^{\mathrm{WP}}(A) = \frac{\mathrm{Vol}_{wp}(A)}{\mathrm{Vol}_{wp}(\mathcal{M}_g)}.$$

One of the remarkable achievements of Mirzakhani was to compute the Weil-Petersson volume of $\mathcal{M}_g$, and more generally of the moduli spaces of surfaces with boundaries and punctures, making it possible to estimate such probabilities. Note that an alternative model of compact random surfaces has been developed by Brooks and Makover [26]. This model is not equivalent in general and we will only work here with the Weil-Petersson model.

For a compact hyperbolic surface X, we denote by $\mathrm{InjRad}(X)$ its injectivity radius, which is half the length of its shortest geodesic loop. The main theorem we prove is the following.

Theorem 3.1.1. *Let X be a random compact hyperbolic surface of genus g distributed according to $\mathbb{P}_g^{\mathrm{WP}}$. There exists a universal constant $\delta > 0$ such that for any $c > 0$ and $0 < b < \frac{1}{2}$, we have the following bounds with probability $1 - O\left(g^{-\frac{1}{2}+\delta(c+b)} + g^{-2b}\right)$. For an eigenfunction ψ_λ of the Laplacian with eigenvalue $\lambda \geq \frac{1}{4}$,*

$$\|\psi_\lambda\|_p \lesssim_{c,p,\lambda} \frac{1}{\sqrt{\min\{1, \mathrm{InjRad}(X)^3\} \log(g)}} \|\psi_\lambda\|_2,$$

for any $2 + 4\beta < p \leq \infty$, where $\beta \in [0, \frac{1}{2})$ is such that the smallest non-zero eigenvalue of the Laplacian on X is at least $\frac{1}{4} - \beta^2$. Moreover, if ψ_λ is an eigenfunction of the Laplacian with eigenvalue $\lambda \in [0, \frac{1}{4} - \varepsilon)$ for some $\varepsilon > 0$ then

$$\|\psi_\lambda\|_p \lesssim \frac{1}{\min\{1, \mathrm{InjRad}(X)\} \left(g^{c\sqrt{\varepsilon}} - 1\right)^{1-\frac{2}{p}}} \|\psi_\lambda\|_2,$$

for any $2 < p \leq \infty$.

Remark. (1) The parameters c and b in Theorem 3.1.1 can be chosen sufficiently small such that the probability tends to 1 as $g \to +\infty$. We interpret this result as saying that for a random surface of large fixed genus, the given bounds are true *with high probability.* The implied constant in the first inequality is continuous in the eigenvalue and independent of the genus. Therefore if we fix any bounded interval $I \subset (1/4, +\infty)$, we have a decay of $\|\psi_\lambda\|_p / \|\psi_\lambda\|_2$ for any eigenvalue $\lambda \in I$ when $g \to +\infty$. However we emphasise that this is not a result on sequences of independent random surfaces in the sense of the product probability $\prod_k \mathbb{P}_{g_k}^{\mathrm{WP}}$ with $(g_k)_{k \in \mathbb{N}}$ such that $g_k \to +\infty$ when $k \to +\infty$.

(2) The implied constants on the eigenfunctions norms are both independent of b. The parameter b arises from a geometric constraint on the collection of surfaces that we consider (see the comments below this remark and see the proof of Theorem 3.6.8) rather than the norms of their eigenfunctions. However, for the probability of surfaces to tend to 1 as the genus tends to infinity, it is clear from the exponent $-\frac{1}{2} + \delta(c + b)$ in the probability estimate that c and b can not be considered in isolation from one another.

(3) Note the different behaviour between the two parts of the spectrum: $[1/4, +\infty)$, to which we will often refer as the *tempered spectrum*, and $(0, 1/4)$ called the *untempered spectrum.*

(4) By Theorem 4.2 of [80], we have that for any $\alpha > 0$

$$\mathbb{P}_g(X : \mathrm{InjRad}(X) \geq \log(g)^{-\alpha}) \geq 1 - O(\log(g)^{-2\alpha}).$$

This means one can remove the injectivity radius constants in the above result and obtain in the tempered case,

$$\|\psi_\lambda\|_p \lesssim_{p,\lambda,c} \frac{1}{\log(g)^{\frac{1}{2}(1-\alpha)}} \|\psi_\lambda\|_2,$$

and in the untempered case,

$$\|\psi_\lambda\|_p \lesssim \frac{\log(g)^\alpha}{\left(g^{c\sqrt{\varepsilon}}-1\right)^{1-\frac{2}{p}}}\|\psi_\lambda\|_2,$$

for any $\alpha > 0$ and $c > 0$ as before both occurring with probability tending to 1 as $g \to +\infty$ by a union bound.

The probability bound in Theorem 3.1.1 is governed by the measure of a certain subset of $\mathcal{M}_g$, within which surfaces satisfy the norm inequalities of the theorem. This subset which we denote by $\mathcal{A}_g^{b,c}$ depends upon two parameters $b, c > 0$ chosen independently of the genus and the construction of this set is the subject of the next result we describe. On its own, this result is a statement about short geodesic loops on random surfaces existing in the g^{-b}-thick part of the moduli space for some $b > 0$ to be chosen (independently of g), and so we isolate it as it can in itself be of interest. We remark that by a geodesic loop we mean the projection of a geodesic segment on the hyperbolic plane whose endpoints become identified when projected to a closed curve on the surface. These curves will be geodesic, but may not necessarily be smooth at a single point (the gluing of the identified ends of the geodesic segment) known as the base of the geodesic loop. They naturally arise as the projection of the geodesic joining a point in the plane to its image under an isometry in the surface deck transformation group. A closed geodesic in the usual sense (smooth with no distinguished base point) arises when the point lies on the axis of the isometry.

By the a-thick part of the moduli space, we mean the collection of $X \in \mathcal{M}_g$ whose injectivity radius is at least a; this space is often denoted by $(\mathcal{M}_g)_{\geq a}$. Again by Theorem 4.2 of [80], we have that

$$\mathbb{P}_g((\mathcal{M}_g)_{\geq g^{-b}}) \geq 1 - O(g^{-2b}).$$

This accounts for the origin of the parameter b in the set $\mathcal{A}_g^{b,c}$. The parameter c originates from a condition on the number of primitive geodesic loops of length at most $c\log(g)$ (for suitably chosen $c > 0$ independent of the genus g) that can be based at any point on the surface. For this we introduce the following random variable. For any $X \in \mathcal{M}_g$, let us denote by $N_L(X, x)$ the number of primitive geodesic loops γ (not necessarily simple) of length $\ell_X(\gamma) \leq L$ based at a point $x \in X$, and set

$$N_L(X) = \sup_{x \in X} N_L(X, x).$$

The set $\mathcal{A}_g^{b,c}$ is then the collection of surfaces in the g^{-b}-thick part of the moduli space that also have $N_{c\log(g)}(X) \leq 1$ for appropriately chosen constants b and c that are implicit and described in more detail in Section 3.6. In other words,

$$\mathcal{A}_g^{b,c} = \left\{ X \in (\mathcal{M}_g)_{\geq g^{-b}} : N_{c\log(g)}(X) \leq 1 \right\}.$$

We have the following result about the probability for a random surface to be in this set.

Theorem 3.1.2. *There exists $\delta > 0$ such that for all $c > 0$ and $0 < b < \frac{1}{2}$,*

$$\mathbb{P}_g \left(X \in (\mathcal{M}_g)_{\geq g^{-b}} : N_{c\log g}(X) \leq 1 \right) \geq 1 - O\left(g^{-\frac{1}{2}+\delta(c+b)} + g^{-2b} \right) \quad \text{as } g \to \infty,$$

and therefore for b and c small enough, this probability tends to 1 when $g \to +\infty$.

Thus, the rate at which the probability holds in Theorem 3.1.1 is given by the rate in Theorem 3.1.2. The previous theorem says that, with high probability when $g \to +\infty$, at any point of a random surface in the g^{-b}-thick part of the moduli space there is no more than one primitive geodesic loop of length less than $c\log g$ based at this point. This implies in particular that if there is one, this loop is necessarily simple since the shortest geodesic loop based at a point is always simple. Related countings of the number of short geodesics of a given length on random surfaces are done in [81]. We use similar ideas to this work, but the dependence of the length of the loops we consider on the genus (as opposed to just being uniformly bounded) and the fact that we consider geodesic loops rather than just closed geodesics requires us to develop more delicate and quantitative tools that we detail in Section 3.6.

Theorem 3.1.1 relies on Theorem 3.1.2 together with a deterministic theorem about L^p norms. This deterministic result requires us to consider a condition on the surfaces and it is precisely Theorem 3.1.2 that allows us to dispense of this condition in favour of a result holding with high probability. Let $X = \Gamma\backslash\mathbb{H}$ be a genus g compact hyperbolic surface with fundamental domain D. Given $R, C > 0$ we will say that X is (R, C)-admissible if for any $\delta > 0$ there exists $C_0(\delta) > 0$ (independent of R, C and X), such that

$$\sup_{z,w\in D} |\{\gamma \in \Gamma \mid d(z, \gamma w) \le r\}| \le C C_0(\delta)\, e^{\delta r} \qquad \text{for any } r \le R. \tag{3.1}$$

Thus if X is (R, C)-admissible, it means we will have good control, up to the parameter C, on the number of geodesics between any two points on the surface with lengths at most R. Sometimes we will write $R(X)$ and $C(X)$ rather than R and C to make it clear that they are an admissible pair for the surface X. It is easy to see that every surface is $(\mathrm{InjRad}(X), 1)$-admissible[2]. The crucial point is that we want a surface to be (R, C)-admissible for R large and C small compared to R, as can be seen from the following deterministic result holding for any (R, C)-admissible surface.

Theorem 3.1.3. *Suppose that $X = \Gamma\backslash\mathbb{H}$ is an (R, C)-admissible compact hyperbolic surface whose smallest non-zero eigenvalue of the Laplacian is at*

[2]See the discussion around (5.1) in Chapter 5.

least $\frac{1}{4} - \beta^2$ for some $\beta \in [0, \frac{1}{2})$. For an eigenfunction ψ_λ of the Laplacian with eigenvalue $\lambda \geq \frac{1}{4}$, we have that

$$\|\psi_\lambda\|_p \lesssim_{p,\lambda} \frac{\sqrt{A(X)}}{\sqrt{R}} \|\psi_\lambda\|_2,$$

for any $2 + 4\beta < p \leq \infty$ where

$$A(X) = \frac{C}{\min\{1, \mathrm{InjRad}(X)^2\}}.$$

Moreover, if ψ_λ is an eigenfunction of the Laplacian with eigenvalue $\lambda = \frac{1}{4} - \varepsilon$ for some $0 < \varepsilon \leq \frac{1}{4}$ then for any $\delta > 0$,

$$\|\psi_\lambda\|_p \lesssim_\delta \frac{C}{\left(e^{(1-\delta)\sqrt{\varepsilon}R} - 1\right)^{1-\frac{2}{p}}} \|\psi_\lambda\|_2,$$

for $2 < p \leq \infty$.

In Section 3.6, we will show that if for some $L > 0$ we have $N_L(X) \leq 1$, and $X \in (\mathcal{M}_g)_{\geq g^{-b}}$ for some $0 < b < \frac{1}{2}$, then the surface X will be (R, C)-admissible for $R = L$ and $C = \frac{1}{\min\{1, \mathrm{InjRad}(X)\}}$. From Theorem 3.1.2 we thus see that there is a collection of surfaces with probability tending to 1 as $g \to \infty$ that are $(c \log(g), \min\{1, \mathrm{InjRad}(X)\}^{-1})$-admissible surfaces, and use this to obtain the eigenfunction norms bounds for random surfaces.

Optimal bounds

One can ask what is the best bound on L^p norms that can be obtained in the large genus limit. Clearly for any function $\psi \colon X \to \mathbb{R}$

$$\|\psi\|_\infty \geq \frac{\|\psi\|_2}{\sqrt{\mathrm{Vol}(X)}} \tag{3.2}$$

with equality if and only if ψ is constant almost everywhere. Eigenfunctions of non-zero eigenvalue are not constant and therefore some correction is required.

On large random regular graphs, the following was proved by Bauerschmidt, Huang and Yau [11, Theorem 1.2]. Let $\mathrm{G}_{N,d}$ be the set of random regular graphs of degree d on N vertices. We put the uniform probability measure on $\mathrm{G}_{N,d}$. There exists d_0 very large but fixed such that for $d \geq d_0$ and with probability tending to 1 when $N \to +\infty$, any eigenvector v with eigenvalue in the tempered spectrum satisfies

$$\|v\|_\infty \lesssim \frac{(\log N)^\alpha}{\sqrt{N}} \|v\|_2,$$

for some $\alpha > 0$ depending on the distance of the eigenvalue from the boundaries of the tempered spectrum.

Inspired by this graph result we can formulate the following conjecture.

Conjecture 3.1.4. *Let X be a compact hyperbolic surface of genus g chosen uniformly at random with respect to the Weil-Petersson volume. Then for any $\varepsilon > 0$ and any eigenfunction ψ_λ with eigenvalue $\lambda \in \left(\frac{1}{4} + \varepsilon, +\infty\right)$ we have*

$$\|\psi_\lambda\|_\infty \lesssim \frac{(\log g)^{\alpha(\varepsilon)}}{\sqrt{g}} \|\psi_\lambda\|_2$$

for some function $\alpha(\varepsilon) > 0$ of ε, with probability tending to 1 when $g \to +\infty$.

Such a result on the sup norm would give a strong form of delocalisation. In particular it would prevent concentration of eigenfunctions on sets of volume less than $g/\log(g)^{2\alpha}$.

Arithmetic surfaces

In the compact arithmetic setting, and for a Hecke eigenfunction ψ_λ, stronger bounds exist both in terms of the eigenvalue, due to Iwaniec and Sarnak [61], and in terms of the genus (or more precisely the congruence level), due to Saha and Hue-Saha [99, 56]. In the eigenvalue aspect, for a given compact arithmetic surface the following bound holds [61, Theorem 0.1]: for any eigenfunction ψ with $\Delta\psi = \lambda\psi$ and any $\varepsilon > 0$

$$\|\psi\|_\infty \lesssim_\varepsilon \lambda^{5/24+\varepsilon}\|\psi\|_2.$$

In the level aspect the bound is more complex and depends on the arithmetic properties of the level but it has a power decay in terms of the genus of the form

$$\|\psi\|_\infty \lesssim_\lambda g^{-\alpha}\|\psi\|_2$$

for some exponent $\alpha > 0$. Note that similar level aspect bounds have been obtained previously in the non-compact case of congruence covers of the modular surface [19, 51].

Hybrid bounds

The bounds we obtain depend implicitly on the eigenvalue. The dependence can be made explicit in our proof but is much worse than Sogge's bounds. It would be interesting to have better combined dependence both in terms of eigenvalue and genus. Such hybrid bounds were obtained in the arithmetic setting for Maass cusp forms by Templier and Saha [108, 98]. Developing such a theory on random surfaces could for example allow one to improve eigenvalue bounds for a positive measure set of surfaces. Alternatively, in a similar way as the work of Bauerschmidt, Huang and Yau [11] requires graphs of very large degree, we could expect that Conjecture 3.1.4 could be easier to approach if we assume the eigenvalue λ to be large.

Multiplicities

As we have observed at the beginning of the introduction, the sup norm of an L^2-normalised eigenfunction ψ_λ with eigenvalue λ can be linked to the multiplicity of λ by

$$\frac{m(\lambda)}{g} \lesssim \|\psi_\lambda\|_\infty^2. \tag{3.3}$$

Through this inequality our result is connected to the problem of limit multiplicities in representation theory initiated by DeGeorge and Wallach [35, 36]. Bounds for multiplicities in arithmetic settings have been studied by Sarnak and Xue [103]. Recently [1], it was proved that for a general Benjamini-Schramm converging sequence of compact hyperbolic surfaces (X_n) with associated genus $g_n \to +\infty$, for any $\lambda > 0$ the ratio $m(\lambda)/g \to 0$ when $g_n \to +\infty$. Note that a sequence of random compact hyperbolic surfaces of increasing genus converges in the sense of Benjamini-Schramm to the hyperbolic plane with high probability ([80, Section 4.4]). In this case our theorem provides a rate via (3.3) and Remark 3.1.1(3).

Corollary 3.1.5. *Let X be a random compact hyperbolic surface of genus g chosen according to the probability $\mathbb{P}_g$. Denote by $m(\lambda)$ the multiplicity of an eigenvalue $\lambda \in (0, +\infty)$. Then there exists a universal constant $d > 0$ such that for any $\alpha > 0$ the following bounds occur with probability $1 - O\left((\log g)^{-2\alpha}\right)$.*

$$\frac{m(\lambda)}{g} \lesssim_{d,\lambda} \frac{1}{(\log g)^{1-\alpha}},$$

for tempered eigenvalues $\lambda \in (\frac{1}{4}, +\infty)$ and,

$$\frac{m(\lambda)}{g} \lesssim_{d} \frac{(\log g)^{2\alpha}}{g^{2d\sqrt{\varepsilon}}},$$

for untempered eigenvalues $\lambda \in (0, \frac{1}{4} - \varepsilon)$.

It is also possible here to take $\alpha = 0$ if we multiply by a factor $\frac{1}{\min\{1, \mathrm{InjRad}(X)^3\}}$ in the tempered spectrum bound and $\frac{1}{\min\{1, \mathrm{InjRad}(X)^2\}}$ for the untempered spectrum by directly using Theorem 3.1.1. This would also modify the probability to that given in Theorem 3.1.1.

The constant $d > 0$ here originates from the constant c in the length $c\log(g)$ of closed geodesic loops that we can control (see Theorem 3.1.2). In our case c, and hence d, can be very small and is not explicit. To make it explicit and optimise it, we would need a more careful analysis of the product in Lemma 3.6.5, which in turns requires more precise estimates than the ones in [82].

Optimal spectral gap

In the case of untempered eigenvalues, we expect that for any $\varepsilon > 0$, and $\lambda \in (0, \frac{1}{4} - \varepsilon)$, the multiplicity of $m(\lambda)$ tends to 0 when $g \to +\infty$, implying that the spectral gap is close to being optimal with high probability in the large genus limit (see [113, Section 10.4]). This can be seen as a random surfaces analogue of Selberg's $\frac{1}{4}$ conjecture. It is likely that a more quantitative understanding of Theorem 3.1.2 — and therefore a more explicit constant c — is required to prove such a result on random surfaces (see for example how such properties on short loops are used to prove an analogous theorem on regular graphs [22]). However, improving the sup norm bound for untempered eigenfunctions can only give at best $m(\lambda) \leq 1$ by an inequality such as (3.3), due to the absolute lower bound on sup norms (3.2). On the other hand, an optimal spectral gap theorem for random surfaces would improve Theorem 3.1.1 by extending the validity of the bound down to $p > 2$. To our knowledge, the only known lower bound for the spectral gap currently in the Weil-Petersson model is due to Mirzakhani [80] who proved that the first non zero eigenvalue is greater than $(\frac{\ln 2}{\pi + \ln 2})^2/4$ with probability tending to 1 when $g \to +\infty$. In the Brooks-Makover model, a non-explicit uniform lower bound on the spectral gap has also been shown to exist with high probability [26]. More recently in a model based on random coverings, strong explicit spectral gaps have been obtained by Magee and Naud [72] for non-compact surfaces.

3.1.2 Outline of the chapter

Aside from the introduction, this chapter consists of five other sections organised as follows.

1. Section 3.2: An overview of the preliminaries of the harmonic analysis used in the proof of the deterministic results.

2. Section 3.3: The proof of Theorem 3.1.3 in the case of a hyperbolic surface with optimal spectral gap.

3. Section 3.4: The proof of Theorem 3.1.3 in the case of a hyperbolic surface with an arbitrary spectral gap.

4. Section 3.5: An overview of the preliminaries of the Teichmüller and random surface theory used in the proof of the probabilistic results.

5. Section 3.6: The proofs of Theorem 3.1.2 and Theorem 3.1.1.

3.2 Harmonic Analysis on Hyperbolic Surfaces

In this section, we introduce some background necessary for our investigation. Much of what is found here is standard and we refer to Katok [64] for the background on hyperbolic geometry and Bergeron [15] and Iwaniec [60] for the background on invariant integral operators and the Selberg transform.

We will work with the Poincaré upper half-plane as a model for the hyperbolic plane

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

which is equipped with the standard hyperbolic Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The distance between two points $z, z' \in \mathbb{H}$ with respect to the metric is denoted by $d(z, z')$ and the associated hyperbolic volume is given by

$$d\mu(z) = \frac{dx\,dy}{y^2}.$$

We identify the group of orientation-preserving isometries of $\mathbb{H}$ with the projective special linear group $\mathrm{PSL}(2, \mathbb{R})$, which contains the 2×2 matrices, with real entries, that have determinant 1 modulo $\pm I_2$, where I_2 the 2×2 identity matrix. The group acts transitively on points $z \in \mathbb{H}$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R})$.

A *hyperbolic surface* can be seen as a quotient $X = \Gamma \backslash \mathbb{H}$, where $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ is a fixed point free Fuchsian group. In other words, Γ is a fixed point free discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Denote by $D \subseteq \mathbb{H}$ a fundamental domain associated with Γ. The Riemannian metric on $\mathbb{H}$ is then naturally inherited by the quotient in the standard way as a Riemannian manifold quotient since the group acts isometrically.

The *injectivity radius* on the surface $X = \Gamma \backslash \mathbb{H}$ at a point z is defined as

$$\mathrm{InjRad}_X(z) = \frac{1}{2} \inf \{ d(z, \, \gamma z) : \gamma \in \Gamma \setminus \{\pm \mathrm{id}\} \}$$

and this gives the largest $R > 0$ such that the ball $B_X(z, R)$ is isometric to a ball of radius R in the hyperbolic plane. In the case when the surface X is compact, there exists a universal positive lower bound for the injectivity radius at each of the points. This allows for the injectivity radius of a compact surface X to be defined as

$$\mathrm{InjRad}(X) = \inf_{z \in X} \mathrm{InjRad}_X(z) > 0.$$

We say that a bounded measurable kernel $K\colon \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is invariant under the diagonal action of Γ if for any $\gamma \in \Gamma$ we have

$$K(\gamma \cdot z, \gamma \cdot w) = K(z, w), \quad (z, w) \in \mathbb{H} \times \mathbb{H}.$$

Such kernels are also referred to as *point-pair invariants*.

A *radial kernel* $k\colon [0, +\infty] \to \mathbb{C}$ is a bounded, measurable, function. Given such a kernel, the mapping $K\colon \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ given by

$$(z, w) \mapsto k(d(z, w))$$

is an invariant kernel for $(z, w) \in \mathbb{H} \times \mathbb{H}$. Conversely, an invariant kernel gives rise to a radial kernel in an obvious way and so the two can be identified.

To construct an invariant integral operator on the surface $\Gamma\backslash\mathbb{H}$, we firstly note that functions on X are naturally identified with Γ-periodic functions on a fundamental domain $D \subseteq \mathbb{H}$. Given an invariant kernel K, we then define an associated automorphic kernel on $D \times D$ by

$$K_\Gamma(z, w) = \sum_{\gamma \in \Gamma} K(z, \gamma w).$$

This summation converges if one imposes an appropriate decay condition on the kernel k, such as the existence of some $\delta > 0$ such that

$$|k(\rho)| = O\left(e^{-(1+\delta)\rho}\right).$$

With this, we may define an associated invariant integral operator A on the surface X by

$$Af(z) = \int_D \sum_{\gamma \in \Gamma} K(z, \gamma w) f(w) \,\mathrm{d}\mu(w)$$

for any Γ-invariant function f and $z \in D$.

The importance of the radial operators is derived from their connection to the Laplacian. The Laplacian Δ on $\mathbb{H}$ is given in coordinates $z = x + iy$ by the differential operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

and since the Laplacian commutes with isometries it can be considered as a differential operator on any hyperbolic surface $\Gamma \backslash \mathbb{H}$.

In the case that $X = \Gamma \backslash \mathbb{H}$ is a compact surface, the spectrum of the Laplacian on X denoted by $\sigma_X(\Delta)$ is discrete and contained in the interval $[0, \infty)$. Moreover, the eigenfunctions corresponding to the eigenvalue 0 are all constant functions and thus in particular, the corresponding eigenspace is one dimensional. From the general theory of the Laplacian on compact Riemannian manifolds, there exists a sequence $0 = \lambda_0 \leq \lambda_1 \leq \ldots \to \infty$ and an orthonormal basis $\{\psi_{\lambda_i}\}_{i \geq 0}$ of $L^2(\Gamma \backslash \mathbb{H}) \cong L^2(D)$ such that

$$\Delta \psi_{\lambda_i} = \lambda_i \psi_{\lambda_i},$$

that is, ψ_{λ_i} is an eigenfunction corresponding to the eigenvalue λ_i. In the case of a hyperbolic surface, it is instructive to partition the spectrum into two parts: the *tempered spectrum* which corresponds to the portion of the spectrum inside $[\frac{1}{4}, \infty)$ and the *untempered spectrum* corresponding to $[0, \frac{1}{4})$. When a surface has $\sigma_X(\Delta) \subseteq \{0\} \cup [\frac{1}{4}, \infty)$, we say that it has *optimal spectral gap*[3].

[3] The word optimal is used in reference to conjectures on the size of the spectral gap in the genus aspect. However, for a given surface the spectral gap could be much larger than $\frac{1}{4}$.

We recall that any eigenfunction of the Laplacian is also an eigenfunction of an invariant integral operator. The corresponding eigenvalue of the integral operator is determined by the *Selberg transform* $\mathcal{S}(k)$ of the radial kernel $k\colon [0,\infty] \to \mathbb{C}$, which is defined as the Fourier transform

$$\mathcal{S}(k)(r) = h(r) = \int_{-\infty}^{+\infty} e^{iru} g(u)\,\mathrm{d}u$$

of the function

$$g(u) = \sqrt{2}\int_{|u|}^{+\infty} \frac{k(\rho)\sinh\rho}{\sqrt{\cosh\rho - \cosh u}}\,\mathrm{d}\rho.$$

Conversely, given a suitable function h, one can construct a kernel k via taking an inverse Selberg transform of h such that the associated automorphic kernel K_Γ defined above converges. More specifically, if $h : \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \frac{1}{2} + \varepsilon\} \to \mathbb{C}$ for some $\varepsilon > 0$ satisfies

1. h is analytic,

2. h is even,

3. h satisfies a decay condition of the form

$$|h(z)| = O\left(1 + |z|^2\right)^{-1-\varepsilon}\right),$$

then the inverse Selberg transform of h is defined through

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-isu} h(s)\,\mathrm{d}s$$

and then

$$k(\rho) = -\frac{1}{\sqrt{2}\pi}\int_{\rho}^{+\infty} \frac{g'(u)}{\sqrt{\cosh u - \cosh\rho}}\,\mathrm{d}u.$$

We note that under these conditions (see for example [74]), k satisfies the decay condition mentioned previously with $\delta = \varepsilon$. We will utilise these decay conditions on such a function h within this paper. With h satisfying these conditions, we obtain the following.

Theorem 3.2.1 ([15, Sections 3.3, 3.4] or [60, Theorem 1.14]). *Let $X = \Gamma\backslash\mathbb{H}$ be a hyperbolic surface and $k\colon [0,\infty] \to \mathbb{C}$ a radial kernel. Suppose that ψ_λ is an eigenfunction of the Laplacian with eigenvalue $\lambda = s_\lambda^2 + \frac{1}{4}$ for $s_\lambda \in \mathbb{C}$. Then ψ_λ is an eigenfunction of the convolution operator A with invariant kernel k and*

$$(A\psi_\lambda)(z) = \int_{\mathbb{H}} k(d(z,w))\psi_\lambda(w)\,\mathrm{d}\mu(w) = h(s_\lambda)\psi_\lambda(z),$$

where $h(s_\lambda) = \mathcal{S}(k)(s_\lambda)$.

3.3 Deterministic Bounds for Surfaces with Optimal Spectral Gap

Consider a compact hyperbolic surface $X = \Gamma\backslash\mathbb{H}$ with $D \subseteq \mathbb{H}$ a fundamental domain of X and assume that X is $(R(X), C(X))$-admissible for some $R(X) \geq 2$ and $C(X) > 0$. In this section, we additionally assume that X has an optimal spectral gap so that $\sigma_X(\Delta) \subseteq \{0\} \cup [\frac{1}{4}, \infty)$. In this case, by letting $\lambda = s_\lambda^2 + \frac{1}{4}$ be the parametrisation of the eigenvalue λ of the Laplacian as described in Section 3.2, then s_λ is either in $[0,\infty)$ or is equal to $\frac{1}{2}i$, with the latter case occurring when $\lambda = 0$.

The extra assumption on the surfaces here provides for slightly stronger results as emphasised in Theorem 3.1.3. Moreover, the crux of the methodology that we use to prove the result can be demonstrated without the additional technicalities that are brought about by the small eigenvalues. In fact, for this reason we also defer the proof of the result for untempered eigenfunctions to Section 3.4 and focus solely on the tempered portion of the spectrum.

3.3.1 Outline of the proof

The L^p norm bounds for tempered eigenfunctions in Theorem 3.1.3 is proven through the following methodology. Denote $R = R(X)$. We will use Selberg's theory to build a convolution operator $W_{R,\lambda}$ satisfying on the spectral side

$$\|W_{R,\lambda}\psi_\lambda\|_p \gtrsim_\lambda R\|\psi_\lambda\|_p$$

for any eigenfunction ψ_λ of eigenvalue $\lambda \geq \frac{1}{4}$, and on the geometric side

$$\|W_{R,\lambda}\|_{L^2(X)\to L^p(X)} \lesssim_p \sqrt{R},$$

with R being given by (3.1). The latter inequality will be obtained via a TT^* argument:

$$\|W_{R,\lambda}\|^2_{L^2(X)\to L^p(X)} = \|W_{R,\lambda}W^*_{R,\lambda}\|_{L^q(X)\to L^p(X)}.$$

For this purpose:

1. We firstly define via the inverse Selberg transform a family of operators P_t that can be seen as a smoothened version of the wave cosine kernel $\cos(t\sqrt{\Delta})$ and which will be used as a building block for our operator $W_{R,\lambda}$.

2. Preparing for the TT^* argument, we next prove a linearisation formula of the type
$$P_tP_s^* = \frac{1}{2}\left(Q_{t+s} + Q_{|t-s|}\right),$$
where Q_t is a family of operators studied previously by Brooks and Lindenstrauss [30]. This is done looking at the spectral action of the operators via the Selberg transform. (Lemma 3.3.1)

3. We use relevant bounds obtained in [30] (reproduced in Lemma 3.3.2) to bound the operator norms of Q_t for $t \leq \frac{1}{4}(R-1)$. (Lemma 3.3.3)

4. The operator $W_{R,\lambda}$ is then defined roughly as

$$W_{R,\lambda} = \int_0^R \cos(s_\lambda t) P_t \, \mathrm{d}t.$$

5. We realise the TT^* argument to finally bound $\|W_{R,\lambda}\|_{L^2(X)\to L^p(X)}$, and combine this with a lower bound on the spectral action of $W_{R,\lambda}$ to obtain our deterministic result. (Lemma 3.3.4 and Theorem 3.3.5)

3.3.2 Proof of Theorem 3.1.3 for optimal spectral gap surfaces

We begin by constructing a family of integral operators to analyse the eigenfunctions of the Laplacian. To this end, we define for $t \geq 0$ and r in the same range as the $s_\lambda \in \mathbb{C}$, the functions j_t given by

$$j_t(r) = \frac{\cos(rt)}{\sqrt{\cosh\left(\frac{\pi r}{2}\right)}}.$$

Using the Selberg transform, one may associate to j_t a radial kernel $\ell_t(z,w) = \ell_t(d(z,w))$ for an integral operator P_t acting on functions of $\mathbb{H}$ given by

$$P_t f(z) = \int_{\mathbb{H}} \ell_t(z,w) f(w) \mathrm{d}\mu(w).$$

The kernel ℓ_t is in fact real valued, which can be seen by the fact that the Selberg transform of the complex conjugate of ℓ_t coincides with j_t, since j_t is real valued for the specified r. Indeed,

$$\overline{j_t(r)} = \sqrt{2} \int_{-\infty}^{\infty} e^{irv} \int_{|v|}^{\infty} \overline{\ell_t(\rho)} \frac{\sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(v)}} \mathrm{d}\rho \mathrm{d}v.$$

The formal adjoint of P_t then takes the form

$$P_t^* f(z) = \int_{\mathbb{H}} \ell_t(z,w) f(w) \mathrm{d}\mu(w),$$

since the kernel ℓ_t is real and symmetric in z and w. These operators may then be defined on the surface via the fundamental domain and using the automorphic kernel formed from the group Γ generating the surface X as described in the previous section to give

$$P_t f(z) = \int_D \sum_{\gamma \in \Gamma} \ell_t(z, \gamma w) f(w) \mathrm{d}\mu(w),$$

and, due to the previous discussion,

$$P_t^* f(z) = \int_D \sum_{\gamma \in \Gamma} \ell_t(z, \gamma w) f(w) \mathrm{d}\mu(w).$$

On the surface, these operators are then bounded as operators from $L^2(X)$ to $L^p(X)$ and from $L^q(X)$ to $L^2(X)$ respectively with $p \geq 2$ (in fact we can consider a wider range of p but this is not necessary here) and p and q conjugate indices, using the decay conditions on the j_t.

In Section 3.4 we will see that the desired result in fact holds trivially for the constant eigenfunctions, thus we will only use P_t to analyse the eigenfunctions corresponding to the eigenvalues away from zero. Thus, when testing our operator against an arbitrary function, we will remove the component of the function corresponding to the zero eigenspace. To this end, we then define the operator $\Pi \colon L^q(X) \to L^q(X)$ by

$$f \mapsto f - \fint_D f(z) \mathrm{d}\mu(z) =: f - \bar{f},$$

where $\fint$ denotes the average:

$$\fint_D f(z) \mathrm{d}\mu(z) = \frac{1}{\mathrm{Vol}(D)} \int_D f(x) \mathrm{d}\mu(z).$$

Next we begin to understand the pertinent properties of the operators P_t. One crucial property that they possess is a linearisation formula under composition with their adjoint.

Lemma 3.3.1. *The integral kernel of the composition operator $P_tP_s^*\colon L^q(X) \to L^p(X)$ for $t, s \geq 0$ and p and q conjugate indices is given by*

$$\frac{1}{2}\left(k_{t+s} + k_{|t-s|}\right),$$

where k_t is the associated radial kernel through the Selberg transform with the function

$$h_t(r) = \frac{\cos(rt)}{\cosh(\frac{\pi r}{2})}.$$

In particular, if $Q_t\colon L^q(X) \to L^p(X)$ is the associated integral operator for the kernel k_t, then

$$P_tP_s^*\Pi = \frac{1}{2}\left(Q_{t+s}\Pi + Q_{|t-s|}\Pi\right). \tag{3.4}$$

Proof. This is essentially a consequence of trigonometric relations of the cosine function. Notice firstly that the kernel of $P_tP_s^*$ on $\mathbb{H}$ is given by the convolution kernel

$$m_{t,s}(z, w) = \int_{\mathbb{H}} \ell_t(z, w')\ell_s(w, w')\mathrm{d}\mu(w'),$$

which is itself a radial kernel by invariance of the measure $\mathrm{d}\mu$ under isometries. Let $M_{t,s}(d(z, w)) = m_{t,s}(z, w)$ denote the associated function on $\mathbb{R}$ that generates $m_{t,s}$. By Theorem 2.2.7, for an eigenfunction ψ of the Laplacian on $\mathbb{H}$ with corresponding eigenvalue $\lambda = \frac{1}{4} + r^2$, for r the eigenvalue parameter from before, we obtain

$$P_tP_s^*\psi = \mathcal{S}(M_{t,s})(r)\psi,$$

where $\mathcal{S}(M_{t,s})$ denotes the Selberg transform of the function $M_{t,s}$. On the other hand, by applying each of the operators in turn,

$$P_t P_s^* \psi = j_t(r) j_s(r) \psi,$$

and hence

$$j_t(r) j_s(r) = \mathcal{S}(M_{t,s})(r).$$

Notice then for real r, that one has

$$\begin{aligned} j_t(r) j_s(r) &= \frac{\cos(r(t+s))}{2\cosh(\frac{\pi r}{2})} + \frac{\cos(r|t-s|)}{2\cosh(\frac{\pi r}{2})} \\ &= \frac{1}{2}(h_{t+s}(r) + h_{|t-s|}(r)). \end{aligned}$$

Similarly, when $r = bi$ for $b \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned} j_t(r) j_s(r) &= \frac{\cosh(b(t+s))}{2\cos\left(\frac{\pi b}{2}\right)} + \frac{\cosh(b(t-s))}{2\cos\left(\frac{\pi b}{2}\right)} \\ &= \frac{1}{2}(h_{t+s}(r) + h_{|t-s|}(r)). \end{aligned}$$

By applying the inverse Selberg transform, it follows that $m_{t,s} = \frac{1}{2}(k_{t+s} + k_{|t-s|})$, where k_t is as given in the statement of the lemma, and therefore

$$P_t P_s^* = \frac{1}{2}\left(Q_{t+s} + Q_{|t-s|}\right).$$

By composing with Π, we obtain (3.4). $\square$

We remark that the function $h_t(r)$ in the previous lemma is precisely the Selberg transform considered by Brooks and Lindenstrauss [30]. It was introduced previously in the article of Iwaniec and Sarnak [61], where its Fourier transform was used to define a kernel to obtain sup norm bounds of eigenfunctions of the Laplacian on arithmetic surfaces. Thus much is already known regarding estimates on the kernel induced by this function through the Selberg transform. Indeed, Brooks and Lindenstrauss [30] have obtained the following bounds that are crucial in our investigation.

Lemma 3.3.2 (Brooks and Lindenstrauss [30])**.** *With k_t as above denoting the kernel associated via the Selberg transform with the function h_t, we have the following estimates. A sup norm bound of*

$$\|k_t\|_\infty \lesssim e^{-t/2}, \tag{3.5}$$

holding for all $t \geq 0$, and a hybrid bound holding for all $\rho \geq 4t$ of the form

$$|k_t(\rho)| \lesssim e^{t-\frac{3}{2}\rho}. \tag{3.6}$$

Next we consider the operator Q_t as defined in Lemma 3.3.1. We combine the bounds of Lemma 3.3.2 with the condition (3.1) assumed of our surfaces to obtain suitable bounds on the operator norm of $Q_t\Pi$ in terms of the parameter t.

Lemma 3.3.3. *Suppose that Q_t and Π are defined as above. For $t \leq \frac{1}{4}(R(X)-1)$, one may bound the $L^q(X) \to L^p(X)$ operator norm by*

$$\|Q_t\Pi\|_{L^q(X)\to L^p(X)} \lesssim A(\delta, X)e^{-\alpha_p t},$$

where α_p can be chosen to equal $(\frac{1}{2}-\delta)(1-\frac{2}{p})$ for any $\delta > 0$, q and p are conjugate indices, and

$$A(\delta, X) = \frac{C_0(\frac{\delta}{4})C(X)}{\min\{1, \mathrm{InjRad}(X)^2\}}.$$

Proof. We will proceed by interpolation, first calculating the norm

$\|Q_t\Pi\|_{L^1(X)\to L^\infty(X)}$. We have by the definition of the automorphic kernel integral operator that

$$\|Q_t\|_{L^1(X)\to L^\infty(X)} \leq \sup_{z,w\in D}\sum_{\gamma\in\Gamma}|k_t(d(z,\gamma w))|.$$

This summation can then be split into two parts corresponding to propagation at times shorter and longer than $\lceil 4t\rceil \leq R(X)$. Indeed, for any $z,w\in D$,

$$\sum_{\gamma\in\Gamma}|k_t(d(z,\gamma w))| \leq \sum_{\substack{\gamma\in\Gamma\\ d(z,\gamma w)\leq\lceil 4t\rceil}}|k_t(d(z,\gamma w))| + \sum_{m=\lceil 4t\rceil}^{\infty}\sum_{\substack{\gamma\in\Gamma\\ m\leq d(z,\gamma w)\leq m+1}}|k_t(d(z,\gamma w))|.$$

The first summation is dealt with using $(R(X),C(X))$-admissibility of X and the sup-norm bound on k_t in the first part of Lemma 3.3.2,

$$\sum_{\substack{\gamma\in\Gamma\\ d(z,\gamma w)\leq\lceil 4t\rceil}}|k_t(d(z,\gamma w))| \leq C(X)C_0(\tfrac{\delta}{4})e^{-(\frac{1}{2}-\delta)t},$$

for any $\delta>0$. For the second summation, we control the kernel using the pointwise estimate of Lemma 3.3.2, combined with an estimate on the number of γ satisfying $m\leq d(z,\gamma w)\leq m+1$. For this, we note that the InjRad(X) neighbourhoods of each element in the orbit Γw are by definition disjoint. Moreover, if $d(z,\gamma w)\leq m+1$, then the $\mathrm{InjRad}(X)$ neighbourhood of γw lies in the $m+1+\mathrm{InjRad}(X)$ neighbourhood of z. Hence, we can bound

$$\begin{aligned}|\{\gamma\in\Gamma : m\leq d(z,\gamma w)\leq m+1\}| &\leq \frac{\mathrm{Vol}(\text{ball of radius } m+1+\mathrm{InjRad}(X))}{\mathrm{Vol}(\text{ball of radius } \mathrm{InjRad}(X))}\\ &\leq \frac{Ce^m}{\min\{1,\mathrm{InjRad}(X)^2\}},\end{aligned} \tag{3.7}$$

for some absolute constant $C>0$ independent of all other parameters and the surface X. The latter inequality can be seen as follows. If $\mathrm{InjRad}(X)\leq 1$, then Vol(ball of radius InjRad(X)) $\geq \frac{1}{2}$InjRad$(X)^2$, so that

$$\frac{\text{Vol(ball of radius } m+1+\text{InjRad}(X))}{\text{Vol(ball of radius InjRad}(X))} \leq \frac{2e^2 e^m}{\text{InjRad}(X)^2}.$$

On the other hand, if InjRad$(X) \geq 1$, then

$$\text{Vol(ball of radius InjRad}(X)) \geq \frac{1}{4}\cosh(\text{InjRad}(X)) \geq \frac{1}{8}e^{\text{InjRad}(X)},$$

so that

$$\frac{\text{Vol(ball of radius } m+1+\text{InjRad}(X))}{\text{Vol(ball of radius InjRad}(X))} \leq \frac{8e^{m+1+\text{InjRad}(X)}}{e^{\text{InjRad}(X)}} \leq 8ee^m.$$

The inequality in (3.7) follows from combining the two cases with $C = 8e^2$. Thus, using the second part of Lemma 3.3.2,

$$\begin{aligned}\sum_{m=\lceil 4t\rceil}^{\infty} \sum_{\substack{\gamma\in\Gamma \\ m\leq d(z,\gamma w)\leq m+1}} |k_t(d(z,\gamma w))| &\leq \sum_{m=\lceil 4t\rceil}^{\infty} \frac{Ce^{t-\frac{1}{2}m}}{\min\{1, \text{InjRad}(X)^2\}} \\ &\leq \frac{C'e^{-t}}{\min\{1, \text{InjRad}(X)^2\}}.\end{aligned}$$

Putting all of this together we obtain

$$\sum_{\gamma\in\Gamma} |k_t(d(z,\gamma w))| \lesssim A(\delta, X)e^{-t(\frac{1}{2}-\delta)},$$

for any $\delta > 0$, where

$$A(\delta, X) = \frac{C_0(\frac{\delta}{4})C(X)}{\min\{1, \text{InjRad}(X)^2\}}.$$

Hence,

$$\|Q_t\|_{L^1(X)\to L^\infty(X)} \lesssim A(\delta, X)e^{-t(\frac{1}{2}-\delta)}.$$

Incorporating the operator Π, we then obtain

$$\begin{aligned}
\|Q_t\Pi f\|_\infty &= \|Q_t(f-\bar{f})\|_\infty \\
&\leq \|Q_t\|_{L^1\to L^\infty}(\|f\|_1 + \|\bar{f}\|_1) \\
&\lesssim A(\delta, X)e^{-t(\frac{1}{2}-\delta)}\|f\|_1,
\end{aligned}$$

and thus we get the bound

$$\|Q_t\Pi\|_{L^1\to L^\infty} \lesssim A(\delta, X)e^{-t(\frac{1}{2}-\delta)}.$$

Next we calculate the $L^2(X) \to L^2(X)$ norm. We note that the operators Q_t and Π acting on $L^2(X)$ to $L^2(X)$ are both self-adjoint. Indeed, the former has a real and symmetric kernel and the latter is a projection. In addition, the operators Q_t and Π commute with each other since for any $f \in L^2(X)$,

$$\begin{aligned}
\Pi Q_t f(z) &= Q_t f(z) - \fint_D Q_t f(w)\,\mathrm{d}\mu(w) \\
&= Q_t f(z) - \frac{1}{\mathrm{Vol}(D)}\int_D f(w')\int_D \sum_{\gamma\in\Gamma} k_t(w, \gamma w')\,\mathrm{d}\mu(w)\mathrm{d}\mu(w') \\
&= Q_t f(z) - h_t\left(\frac{1}{2}i\right)\bar{f} \\
&= Q_t(f-\bar{f})(z) \\
&= Q_t\Pi f(z).
\end{aligned}$$

This means that $Q_t\Pi$ is a self-adjoint operator from $L^2(X)$ to $L^2(X)$ and its norm is equal to its spectral radius. It follows from the projection operator, Theorem 2.2.7 and the fact that X has optimal spectral gap, the norm is given by

$$\|Q_t\Pi\|_{L^2(X)\to L^2(X)} = \sup_{r\in[0,\infty)} |h_t(r)| \leq 1.$$

Finally, we apply the Riesz-Thorin interpolation theorem to get the desired bound. □

We now construct an operator specific to an eigenvalue $\lambda \geq \frac{1}{4}$ of the Laplacian of X. To do this, we wish to combine our propagators P_t along values of t for which the bounds obtained in Lemma 3.3.3 are valid. In doing so, we are able to exhibit the dependence upon the parameter R of the surface. To this end, fix $T \leq \frac{1}{8}(R(X)-1)$ and let $W_{T,\lambda}\colon L^2(X) \to L^p(X)$ to be the operator defined for any $p \geq 2$ by

$$W_{T,\lambda} f(z) = \int_0^T \cos(s_\lambda t) P_t \Pi f(z) \, \mathrm{d}t, \tag{3.8}$$

where s_λ is the spectral parameter in the parametrisation $\lambda = s_\lambda^2 + \frac{1}{4}$ of the eigenvalue.

To calculate the $L^2(X) \to L^p(X)$ operator norm we will employ a TT^* argument, that is we use the fact that

$$\|W_{T,\lambda}\|^2_{L^2(X)\to L^p(X)} = \|W_{T,\lambda} W^*_{T,\lambda}\|_{L^q(X)\to L^p(X)},$$

where q is the conjugate index of p.

Lemma 3.3.4. *Let $\lambda \geq \frac{1}{4}$ be an eigenvalue of Δ on X and $T \leq \frac{1}{8}(R(X)-1)$. If $W_{T,\lambda}$ is defined as in* (3.8), *then*

$$\|W_{T,\lambda}\|_{L^2(X)\to L^p(X)} \lesssim_p \sqrt{A(X)T},$$

where $A(X)$ is given by

$$A(X) = \frac{C(X)}{\min\{1, \mathrm{InjRad}(X)^2\}}.$$

Proof. We compute through an application of Minkowski's integral inequality that

$$\|W_{T,\lambda}W_{T,\lambda}^*\|_{L^q(X)\to L^p(X)} = \left\|\int_0^T\int_0^T \cos(s_\lambda t)\cos(s_\lambda s)P_t\Pi P_s^*\,\mathrm{d}s\mathrm{d}t\right\|_{L^q(X)\to L^p(X)}$$
$$\leq \int_0^T\int_0^T \|P_t\Pi P_s^*\|_{L^q(X)\to L^p(X)}\,\mathrm{d}s\mathrm{d}t.$$

It thus suffices to consider the norm $\|P_t\Pi P_s^*\|_{L^q(X)\to L^p(X)}$.

Notice that by a similar argument to that used in the proof of Lemma 3.3.3, we can see that Π commutes with the adjoint P_s^*. We then use Lemma 3.3.1 to deduce that

$$\|P_t\Pi P_s^*\|_{L^q(X)\to L^p(X)} \lesssim \|Q_{t+s}\Pi\|_{L^q(X)\to L^p(X)} + \|Q_{|t-s|}\Pi\|_{L^q(X)\to L^p(X)}.$$

Now since $t+s\leq 2T\leq \frac{1}{4}(R(X)-1)$, it follows from Lemma 3.3.3 that

$$\|P_t\Pi P_s^*\|_{L^q(X)\to L^p(X)} \lesssim A(\delta,X)(e^{-\alpha_p(t+s)}+e^{-\alpha_p|t-s|})$$
$$\lesssim A(\delta,X)e^{-\alpha_p|t-s|}.$$

Fix $\delta<\frac{1}{2}$ so that $\alpha_p>0$ one may substitute this bound back into the integral to obtain

$$\|W_{T,\lambda}W_{T,\lambda}^*\|_{L^q(X)\to L^p(X)} \lesssim A(X)\int_0^T\int_0^T e^{-\alpha_p|t-s|}\,\mathrm{d}s\mathrm{d}t$$
$$= A(X)\int_0^T\int_0^t e^{-\alpha_p(t-s)}\,\mathrm{d}s\mathrm{d}t + \int_0^T\int_t^T e^{-\alpha_p(s-t)}\,\mathrm{d}s\mathrm{d}t$$
$$\lesssim_p A(X)T,$$

where $A(X)$ is given by

$$A(X) = \frac{C(X)}{\min\{1,\mathrm{InjRad}(X)^2\}}.$$

Notice that the dependence of the implicit constant on δ is now removed due to the fixing of a given $\delta > 0$. The bound

$$\|W_{T,\lambda}\|_{L^2(X)\to L^p(X)} \lesssim_p \sqrt{A(X)T}$$

is then immediate. $\square$

With this upper bound, we turn to examining the spectral action of $W_{T,\lambda}$ on an eigenfunction with eigenvalue λ. For this, we use the explicit form of the Selberg transform to obtain our desired result.

Theorem 3.3.5. *Suppose that X is an $(R(X), C(X))$-admissible compact hyperbolic surface such that $\sigma_X(\Delta) \subseteq \{0\} \cup [\frac{1}{4}, \infty)$. If ψ_λ is an eigenfunction of Δ with eigenvalue $\lambda \geq \frac{1}{4}$, then*

$$\|\psi_\lambda\|_p \lesssim_{\lambda,p} \frac{\sqrt{A(X)}}{\sqrt{R(X)}}\|\psi_\lambda\|_2,$$

with $A(X)$ given by

$$A(X) = \frac{C(X)}{\min\{1, \mathrm{InjRad}(X)^2\}}.$$

The dependence of the upper bound on λ is uniform on compacta.

Proof. We consider the action of the test operator $W_{T,\lambda}$, given by (3.8), on ψ_λ, but with $T = \frac{1}{8}(R(X) - 1) \geq \frac{1}{16}R(X)$. By Lemma 3.3.4, one immediately obtains

$$\|W_{T,\lambda}\psi_\lambda\|_p \leq \|W_{T,\lambda}\|_{L^2(X)\to L^p(X)}\|\psi_\lambda\|_2 \lesssim_p \sqrt{A(X)T}\|\psi_\lambda\|_2.$$

On the other hand, applying Theorem 3.2.1 to the operator P_t provides that

$$\|W_{T,\lambda}\psi_\lambda\|_p = \frac{1}{\sqrt{\cosh\left(\frac{\pi s_\lambda}{2}\right)}} \int_0^T \cos^2(s_\lambda t)\,\mathrm{d}t \|\psi_\lambda\|_p \gtrsim_\lambda T\|\psi_\lambda\|_p.$$

Dividing through then gives

$$\|\psi_\lambda\|_p \lesssim_{\lambda,p} \frac{\sqrt{A(X)}}{\sqrt{R}}\|\psi_\lambda\|_2.$$

□

3.4 Deterministic Bounds for Surfaces with an Arbitrary Spectral Gap

We now consider the case where the spectrum of the Laplacian on the $(R(X), C(X))$-admissible compact hyperbolic surface X takes values in the full range $[0, \infty)$. As before, we assume that $R(X) \geq 2$. To deal with the larger spectrum, we utilise two separate methods for the eigenfunctions belonging to the different parts of the spectrum.

For the untempered spectrum, we demonstrate a far stronger bound on the norms of eigenfunctions than previously obtained in the optimal spectral gap case above. Indeed, we show that the norm has some exponential decay in the parameter $R(X)$. This is carried out via a rescaled ball averaging operator of functions on the surface, which was previously used by Le Masson and Sahlsten [67]. The pertinent information required here is the spectral action of this operator on eigenfunctions, which is given through the Selberg transform.

For the portion of the spectrum lying above $\frac{1}{4}$, we may use an identical technique to the optimal spectral gap case to obtain the relevant bounds. However, due to the introduction of eigenfunctions in the untempered spectrum the result is weakened slightly and is only valid for values of p bounded below by a function dependent on the spectral gap of the surface. We begin by providing an outline of the proof.

3.4.1 Outline of proof

The methodology for the proof is similar to that in the optimal spectral gap case, so we emphasise the main differences.

1. Firstly we show the stronger exponential decay result for the L^p norms of the untempered portion of the spectrum. This is done via a rescaled averaging operator over hyperbolic balls on the surface to obtain the L^∞ norm and then a simple interpolation of this with the trivial L^2 norm bound provides the result for general $p > 2$. (Theorem 3.4.1 and Corollary 3.4.2)

2. For the tempered portion of the spectrum, we utilise the same method as in Section 3.3. The main difference is that the existence of untempered eigenfunctions, other than constants, put restrictions upon the values of p for which the bounds are valid dependent upon the spectral gap. These come from a technicality in the computation of the $L^2 \to L^2$ operator norm of the propagation operator since the convolution operator eigenvalue for untempered eigenfunctions of the Laplacian exhibits exponential growth in the propagation parameter. (Theorem 3.4.3)

3.4.2 Untempered eigenfunctions deterministic bound proof

We start by defining the required ball averaging operator on the surface. Let $(B_t)_{t\geq 0}$ denote the family of operators

$$B_t f(z) = \frac{1}{\sqrt{\cosh(t)}} \int_{B(z,t)} f(w) \mathrm{d}\mu(w),$$

acting on appropriate functions of $\mathbb{H}$. We pass this to an operator on the surface $X = \Gamma\backslash\mathbb{H}$ by considering functions defined upon a fundamental domain D and using the automorphic kernel, so that

$$B_t f(z) = \frac{1}{\sqrt{\cosh(t)}} \int_D \sum_{\gamma \in \Gamma} \mathbf{1}_{\{d(z,\gamma w)<t\}} f(w) \mathrm{d}\mu(w). \tag{3.9}$$

It then follows immediately that the kernel of this operator is induced by the function

$$k_t(\rho) = \frac{\mathbf{1}_{\{\rho<t\}}}{\sqrt{\cosh(t)}},$$

whose Selberg transform is given by

$$\mathcal{S}(k_t)(r) = 4\sqrt{2} \int_0^t \cos(ru) \sqrt{1 - \frac{\cosh(u)}{\cosh(t)}}\, \mathrm{d}u.$$

We now prove the required bounds in order to deduce our desired result for eigenfunctions in the untempered spectrum. In doing so, we complete the result for the optimal spectral gap case in the previous section, since we then have the required bound for the constant eigenfunctions. We initially prove a slightly stronger result than required, namely that a real linear combination of real-valued Laplacian eigenfunctions corresponding to eigenvalues in the untempered spectrum have strong sup norm decay. The case of an arbitrary untempered eigenfunction follows immediately from a simplification of the proof of this result.

Theorem 3.4.1. *Let $\varepsilon > 0$, $0 < \delta < \sqrt{\varepsilon}$ and $n \in \mathbb{N}$. Suppose that*

$$f = \sum_{j=1}^{n} \alpha_j \psi_j$$

is a finite real linear combination of mutually orthogonal untempered real-valued eigenfunctions $\{\psi_j\}_{j=1}^n$ of the Laplacian with corresponding eigenvalues

$\{\lambda_j\}_{j=1}^n \subseteq [0, \frac{1}{4} - \varepsilon)$. Then,

$$\|f\|_\infty \lesssim \frac{C_0(\delta)C(X)}{e^{(\sqrt{\varepsilon}-\delta)R(X)} - 1}\|f\|_2.$$

Proof. The eigenfunctions are smooth so it follows that f is smooth. The compactness of the surface gives that there exists $x \in D$ such that $|f(x)| = \|f\|_\infty$. For each $j = 1, \ldots, n$ define

$$\beta_j = \begin{cases} \alpha_j & \text{if } f(x) \geq 0, \\ -\alpha_j & \text{if } f(x) < 0. \end{cases}$$

By construction, we then have

$$\sum_{j=1}^n \beta_j \psi_j(x) = |f(x)| = \|f\|_\infty.$$

Let $J \subseteq \{1, \ldots, n\}$ be the collection of indices for which $\beta_j \psi_j(x) > 0$. Define

$$\tilde{f} = \sum_{j \in J} \beta_j \psi_j.$$

Then,

$$\tilde{f}(x) = \sum_{j \in J} \beta_j \psi_j(x) \geq \sum_{j=1}^n \beta_j \psi_j(x) = \|f\|_\infty.$$

Moreover, using the orthogonality of the eigenfunctions, $\tilde{f}$ satisfies

$$\|\tilde{f}\|_2^2 = \sum_{j \in J} \|\beta_j \psi_j\|_2^2 \leq \sum_{j=1}^n \|\beta_j \psi_j\|_2^2 = \sum_{j=1}^n \|\alpha_j \psi_j\|_2^2 = \|f\|_2^2.$$

We will thus work with $\tilde{f}$. Consider the ball-averaging operators defined in (3.9) for radii $t \leq R(X)$, where $R(X)$ is such that X is $(R(X), C(X))$-admissible and $R(X) \geq 2$. The fact $t \leq R(X)$ means that by definition, the number of terms in the summation in the automorphic kernel of B_t is bounded by $e^{\gamma t}$ for any $\gamma > 0$.

We now use the Selberg transform of the associated kernel function of B_t to analyse the action of B_t on $\tilde{f}$ about the point x,

$$B_t\tilde{f}(x) = \sum_{j\in J} \mathcal{S}(k_t)(s_{\lambda_j} i)\beta_j\psi_j(x), \tag{3.10}$$

where $s_{\lambda_j} \in [\sqrt{\varepsilon}, \frac{1}{2}]$ is the eigenvalue parameter of λ_j, where $\lambda_j = \frac{1}{4} - s_{\lambda_j}^2$. We now demonstrate that the values of $\mathcal{S}(k_t)(s_{\lambda_j} i)$ are in fact non-negative and bounded below for when $t \geq 2$ by an exponentially growing term. Notice that

$$\begin{aligned}
\mathcal{S}(k_t)(s_{\lambda_j} i) &\gtrsim \int_0^t \cos(s_{\lambda_j} iu)\sqrt{1 - \frac{\cosh(u)}{\cosh(t)}}\,\mathrm{d}u \\
&= \int_0^t \cosh(s_{\lambda_j} u)\sqrt{1 - \frac{\cosh(u)}{\cosh(t)}}\,\mathrm{d}u,
\end{aligned}$$

and hence the values are non-negative. For $u \in [0, t]$ we have

$$\sqrt{1 - \frac{\cosh(u)}{\cosh(t)}} \geq 1 - \frac{\cosh(u)}{\cosh(t)},$$

so that

$$\begin{aligned}
\mathcal{S}(k_t)(s_{\lambda_j} i) &\gtrsim \int_0^t \cosh(s_{\lambda_j} u)\,\mathrm{d}u - \frac{1}{2}\int_0^t \frac{\cosh((s_{\lambda_j}+1)u)}{\cosh(t)} + \frac{\cosh((s_{\lambda_j}-1)u)}{\cosh(t)}\,\mathrm{d}u \\
&= \frac{\sinh(s_{\lambda_j} t)}{s_{\lambda_j}} - \frac{1}{2}\left(\frac{\sinh((s_{\lambda_j}+1)t)}{(s_{\lambda_j}+1)\cosh t} + \frac{\sinh((s_{\lambda_j}-1)t)}{(s_{\lambda_j}-1)\cosh t}\right).
\end{aligned}$$

This expression increases for all values of t in the parameter s_{λ_j} and hence we may bound this expression below with s_{λ_j} replaced by $\sqrt{\varepsilon}$. In addition, when $t \geq 2$ this expression is bounded below[4] by $\sinh(\sqrt{\varepsilon}t)$. Thus for $t \geq 2$ and each

[4]Full details of this can be found in Lemma 5.4.2.

$j \in J$,

$$\mathcal{S}(k_t)(s_{\lambda_j} i) \gtrsim \sinh(\sqrt{\varepsilon} t).$$

This lower bound on the spectral action thus provides

$$\begin{aligned} B_t \tilde{f}(x) = \sum_{j \in J} \beta_j \mathcal{S}(k_t)(s_{\lambda_j} i) \psi_j(x) &\gtrsim \sinh(\sqrt{\varepsilon} t) \sum_{j \in J} \beta_j \psi_j(x) \\ &= \sinh(\sqrt{\varepsilon} t) \tilde{f}(x) \\ &\geq \sinh(\sqrt{\varepsilon} t) \|f\|_\infty . \end{aligned}$$

Conversely, notice for $t \leq R(X)$ that there are at most $C(X) C_0(\delta) e^{\delta t}$ non-zero terms in the summation for the automorphic kernel of B_t for any $\delta > 0$, and hence we have

$$\begin{aligned} \left\| \sum_{\gamma \in \Gamma} \frac{\mathbf{1}_{\{d(x,\gamma\cdot) \leq t\}}}{\sqrt{\cosh(t)}} \right\|_2^2 &= \int_D \left| \sum_{\gamma \in \Gamma} \frac{\mathbf{1}_{\{d(x,\gamma w) \leq t\}}}{\sqrt{\cosh(t)}} \right|^2 \mathrm{d}\mu(w) \\ &\leq \frac{C_0(\delta)^2 C(X)^2 e^{2\delta t}}{\cosh(t)} \mathrm{Vol}(\text{Ball of radius } t) \\ &\leq C_0(\delta)^2 C(X)^2 e^{2\delta t}. \end{aligned}$$

By the Cauchy-Schwarz inequality, we obtain for all $\delta > 0$ that

$$\begin{aligned} |B_t \tilde{f}(x)| &= \left| \int_D \sum_{\gamma \in \Gamma} \frac{\mathbf{1}_{\{d(x,\gamma w) \leq t\}}}{\sqrt{\cosh(t)}} f(w) \, \mathrm{d}\mu(w) \right| \\ &\leq \left\| \sum_{\gamma \in \Gamma} \frac{\mathbf{1}_{\{d(x,\gamma\cdot) \leq t\}}}{\sqrt{\cosh(t)}} \right\|_2 \|\tilde{f}\|_2 \\ &\leq C_0(\delta) C(X) e^{\delta t} \|f\|_2 . \end{aligned}$$

We may then combine the upper and lower bounds on $B_t \tilde{f}(x)$ so that for any

$\delta > 0$ and $2 \le t \le R$,

$$\|f\|_\infty \lesssim \frac{C_0(\delta)C(X)e^{\delta t}}{\sinh(\sqrt{\varepsilon}t)}\|f\|_2.$$

It follows that

$$\|f\|_\infty \lesssim \frac{C_0(\delta)C(X)}{e^{(\sqrt{\varepsilon}-\delta)t}-1}\|f\|_2,$$

and taking $t = R$ then gives the result. $\square$

By using the same argument as in the above proof and applying an interpolation argument on the norms, we obtain the desired eigenfunction bound for any eigenfunction corresponding to an eigenvalue in the untempered spectrum.

Corollary 3.4.2. *Suppose that ψ_λ is an eigenfunction of the Laplacian for an $(R(X), C(X))$-admissible surface X with eigenvalue $\lambda \in [0, \frac{1}{4})$. Then for any $\varepsilon > 0$ for which $\lambda \in [0, \frac{1}{4} - \varepsilon)$ we have*

$$\|\psi_\lambda\|_p \le \frac{C_0(\delta)C(X)}{(e^{(\sqrt{\varepsilon}-\delta)R}-1)^{1-\frac{2}{p}}}\|\psi_\lambda\|_2,$$

for any $\delta > 0$.

Proof. Once again, by compactness of D there exists some $x \in D$ for which $|\psi_\lambda(x)| = \|\psi_\lambda\|_\infty$. Using the ball averaging operator then gives that

$$|B_t\psi_\lambda(x)| = |\mathcal{S}(k_t)(s_\lambda i)||\psi_\lambda(x)|.$$

For $t \ge 2$, we obtain as in Theorem 3.4.1 that

$$|B_t\psi_\lambda(x)| \ge \sinh(\sqrt{\varepsilon}t)\|\psi_\lambda\|_\infty.$$

Analysing the upper bound as before then results in

$$\|\psi_\lambda\|_\infty \leq \frac{C_0(\delta)C(X)}{e^{(\sqrt{\varepsilon}-\delta)R}-1}\|\psi_\lambda\|_2,$$

for any $\delta > 0$. We now use interpolation to see that

$$\begin{aligned}\|\psi_\lambda\|_p &\leq \|\psi_\lambda\|_2^{\frac{2}{p}}\|\psi_\lambda\|_\infty^{1-\frac{2}{p}} \\ &\leq \frac{C_0(\delta)C(X)}{(e^{(\sqrt{\varepsilon}-\delta)R(X)}-1)^{1-\frac{2}{p}}}\|\psi_\lambda\|_2.\end{aligned}$$

□

3.4.3 Proof of Theorem 3.1.3

For the tempered eigenfunctions, we can use the same method as in the optimal spectral gap case. The smaller spectral gap associated with the surface weakens the values of p for which the result holds, however at worst we obtain that the bounds are valid for at least $p > 4$.

Theorem 3.4.3. *Suppose that X is an $(R(X), C(X))$-admissible compact hyperbolic surface whose smallest non-zero eigenvalue of the Laplacian is at least $\frac{1}{4}-\beta^2$ for some $\beta \in [0, \frac{1}{2})$. For a tempered eigenfunction ψ_λ of the Laplacian with eigenvalue $\lambda \geq \frac{1}{4}$, we have the following bound*

$$\|\psi_\lambda\|_p \lesssim_{p,\lambda} \frac{\sqrt{A(X)}}{\sqrt{R(X)}}\|\psi_\lambda\|_2,$$

for any $2+4\beta < p \leq \infty$, where

$$A(X) = \frac{C(X)}{\min\{1, \mathrm{InjRad}(X)^2\}}.$$

Proof. We utilise the operator $W_{T,\lambda}$ as defined by (3.8). As in Lemma 3.3.4, the calculation of the $L^2(X) \to L^p(X)$ norm of $W_{T,\lambda}$ is reduced to computing the operator norms

$$\|Q_t\Pi\|_{L^1(X)\to L^\infty(X)} \quad \text{and} \quad \|Q_t\Pi\|_{L^2(X)\to L^2(X)},$$

where Q_t is the operator defined in Lemma 3.3.1. Using the same argument as in Lemma 3.3.3, with $t \le \frac{1}{4}(R(X)-1)$ we obtain that

$$\|Q_t\Pi\|_{L^1(X)\to L^\infty(X)} \lesssim_\delta A(\delta, X)e^{-t(\frac{1}{2}-\delta)},$$

for any $\delta > 0$. For the $L^2(X) \to L^2(X)$ norm, we notice that in this case there is an exponential growth in the spectral radius. Indeed, we now have

$$\|Q_t\Pi\|_{L^2(X)\to L^2(X)} = \sup_{\substack{r\in[0,\infty),\\ \text{or } r=ai,\ a\in[0,\beta]}} \left|\frac{\cos(rt)}{\cosh(\pi r/2)}\right| \le e^{\beta t}.$$

Applying the Riesz-Thorin Interpolation Theorem, we then obtain for the conjugate exponent q of p and any $\delta > 0$ that

$$\begin{aligned}\|Q_t\Pi\|_{L^q(X)\to L^p(X)} &\le \|Q_t\Pi\|_{L^1(X)\to L^\infty(X)}^{1-\frac{2}{p}} \|Q_t\Pi\|_{L^2(X)\to L^2(X)}^{\frac{2}{p}} \\ &\lesssim_\delta A(X)e^{-t(\frac{1}{2}-\delta-\frac{1}{p}+\frac{2}{p}\delta-\beta\frac{2}{p})}.\end{aligned}$$

When $p > 2 + \frac{4\beta}{1-\delta}$ (assuming $\delta < 1$), the norm exhibits exponential decay. Since this is true for all $0 < \delta < 1$, it follows that there is exponential decay whenever $p > 2 + 4\beta$ and in this case, we can show as in Lemma 3.3.4 that

$$\|W_{T,\lambda}\|_{L^2(X)\to L^p(X)} \lesssim_p \sqrt{A(X)T}.$$

Since the spectral action of $W_{T,\lambda}$ on ψ_λ is identical to that considered in Theorem 3.3.5, we also recover the lower bound

$$\|W_{T,\lambda}\psi_\lambda\|_p \gtrsim_\lambda T\|\psi_\lambda\|_p.$$

Combining these two estimates gives the desired result. $\square$

Theorem 3.1.3 is then obtained by combining Theorem 3.3.5, Corollary 3.4.2 and Theorem 3.4.3.

3.5 Teichmüller Theory and Random Surfaces

This section gathers much of the background required and notation utilised when formulating and working with probabilistic statements on surfaces in this paper. Further details on the foundational material on Teichmüller theory, geodesics and mapping class groups can be found in [59], [32] and [43].

Let $g, n \geq 0$ be integers. We will denote by $\Sigma_{g,n}$ a surface of genus g with n boundary components; if $n = 0$ this is simply written as Σ_g. Given the n boundary components, one can associate a length vector $L = (L_1, \ldots, L_n) \in \mathbb{R}^n_{\geq 0}$ to the surface such that the i^{th} boundary component has length L_i. If $L_i = 0$, then the component is thought of as a cusp or marked point on the surface.

The *Teichmüller space* of signature (g, n) and length vector $L \in \mathbb{R}^n_{\geq 0}$ is defined to be the space

$$\mathcal{T}_{g,n}(L) = \left\{ (X, f) : \begin{array}{l} X \text{ is a complete hyperbolic surface of genus } g \\ \text{and with } n \text{ geodesic boundary components with} \\ \text{lengths corresponding to } L \text{ and } f : \Sigma_{g,n} \to X \text{ is a} \\ \text{diffeomorphism.} \end{array} \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation defined by $(X, f) \sim (Y, g)$ if and only if there exists an isometry $h : X \to Y$ for which

$$g^{-1} \circ h \circ f \colon \Sigma_{g,n} \to \Sigma_{g,n}$$

is isotopic to the identity or equivalently, if $g \circ f^{-1} : X \to Y$ is isotopic to an isometry. In an element $[X, f]$, the mapping f is called a *marking* on X. For notation, when L is the zero vector we denote $\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \ldots, 0)$ and when there are no boundary components we simply write $\mathcal{T}_g$ for $\mathcal{T}_{g,0}$.

There exists a natural group action on the Teichmüller space that acts by changing the marking. The group is called the *mapping class group* $\mathrm{MCG}_{g,n}(\Sigma_{g,n})$ and is defined as the collection of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ that fix the boundary components setwise identified up to isotopy to the identity mapping. If $[\varphi] \in \mathrm{MCG}_{g,n}(\Sigma_{g,n})$ then the action on an element $[X, f] \in \mathcal{T}_{g,n}(L)$ is given by

$$[\varphi] \cdot [X, f] = [X, f \circ \varphi^{-1}].$$

Equivalently, $\mathrm{MCG}_{g,n}(\Sigma_{g,n})$ can be defined as the group of orientation preserving homeomorphisms fixing the boundary components, up to homotopy. This is due to the fact that on a compact surface, any homeomorphism is isotopic to a diffeomorphism, and two orientation preserving homeomorphisms are homotopic iff they are isotopic.

The *moduli space* $\mathcal{M}_{g,n}(L)$ is then the space obtained through identification of points in the Teichmüller space up to the mapping class group action. That is,

$$\mathcal{M}_{g,n}(L) = \mathcal{T}_{g,n}(L)/\mathrm{MCG}_{g,n}(\Sigma_{g,n}).$$

As with the Teichmüller space, we use the shorthand notation

$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0)$ and $\mathcal{M}_g = \mathcal{M}_{g,0}$.

As well as a group action, there is an associated symplectic form on $\mathcal{T}_{g,n}(L)$ called the *Weil-Petersson form* denoted by $\omega_{g,n}$ which is invariant under the action of the mapping class group (see Goldman [50]). Due to this invariance, the form passes also to the moduli space and hence provides a volume form on $\mathcal{M}_{g,n}(L)$ called the *Weil-Petersson volume*

$$\frac{\wedge^{3g+n-3}\omega_{g,n}}{(3g+n-3)!}.$$

In particular, we write

$$V_{g,n}(L) = \int_{\mathcal{M}_{g,n}(L)} \frac{\wedge^{3g+n-3}\omega_{g,n}}{(3g+n-3)!},$$

for the volume of $\mathcal{M}_{g,n}(L)$ and use the shorthand notation $V_{g,n} = V_{g,n}(0, \ldots, 0)$ and $V_g = V_{g,0}$.

Some particularly important results concerning volumes of moduli spaces that will be made use of here are from Mirzakhani [80] and Mirzakhani and Zograf [82] and we reproduce them for the convenience of the reader. The first allows one to relate the volumes $V_{g,n}(L)$ to $V_{g,n}$.

Lemma 3.5.1 (Mirzakhani [80, Equation 3.7])**.** *Given any $g, n \in \mathbb{N}$ and $L \in \mathbb{R}^n_{\geq 0}$,*

$$V_{g,n}(2L) \leq e^{|L|} V_{g,n},$$

where $|L| = L_1 + \cdots + L_n$.

The second result shows a relationship between volumes with different genus and boundary components. For $g \to \infty$, the relation is asymptotically sharp.

Lemma 3.5.2 (Mirzakhani [80, Equation 3.20])**.** *Given $g, n \in \mathbb{N} \cup \{0\}$ with $2g - 2 + n \geq 0$ and $0 \leq i \leq n/2$,*

$$V_{g,n} \lesssim V_{g+i,n-2i},$$

where the implied constant is independent of g, n and i.

The last volume estimate result we need provides growth estimates for moduli space volumes in the large genus limit.

Theorem 3.5.3 (Mirzakhani and Zograf [82, Theorem 1.2])**.** *There exists a universal constant $C \in (0, \infty)$ such that for any given $n \geq 0$,*

$$V_{g,n} = \frac{C}{\sqrt{g}}(2g - 3 + n)!(4\pi^2)^{2g-3+n}\left(1 + O\left(\frac{1}{g}\right)\right),$$

as $g \to \infty$. In particular,

$$V_g = \frac{C}{\sqrt{g}}(2g - 3)!(4\pi^2)^{2g-3}\left(1 + O\left(\frac{1}{g}\right)\right),$$

as $g \to \infty$.

Notice in particular that the volume of the moduli space is finite and hence there is a probability measure on the moduli space called the *Weil-Petersson probability measure*, $\mathbb{P}^{\mathrm{WP}}_{g,n}$. If $A \subseteq \mathcal{M}_{g,n}$ we will write

$$\mathbb{P}^{\mathrm{WP}}_{g,n}(A) = \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \mathbf{1}_A(X) \mathrm{d}X,$$

where we use $\mathrm{d}X$ as shorthand for the Weil-Petersson volume measure and X for an element of the moduli space. Moreover, one can determine the expectation of a measurable function $F\colon \mathcal{M}_{g,n} \to \mathbb{R}$ with respect to this measure in the usual way through the expression

$$\mathbb{E}_{g,n}(F) = \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} F(X)\mathrm{d}X.$$

An extremely useful result that we will use for calculating integrals of certain functions over moduli space will be *Mirzakhani's integral formula.* For this, we need to introduce the notion of cutting open a surface along a system of curves. To this end, recall that in the free homotopy class of a simple closed curve on a hyperbolic surface, there exists a unique simple closed geodesic minimising length amongst all curves in the homotopy class. When we consider a simple closed curve, we will always be considering the free homotopy class or simple closed geodesic representative in this class. In the following we will consider the notion of *multicurves.*

Definition 3.5.4. If $\gamma_1, \ldots, \gamma_k$ are homotopically distinct and simple closed curves, we define their *multicurve* as the formal sum $\gamma = \sum_{i=1}^{k} \gamma_i$ which gives a union of curves in Σ_g.

We now seek to understand how such a multicurve cuts the surface. Fix a multicurve $\gamma = \sum_{i=1}^{k} \gamma_i$ and denote by $\Sigma_g \setminus \gamma$ the possibly disconnected surface with $q \geq 1$ connected components and $2k$ boundary components formed by cutting Σ_g along the γ_i. Each such curve component γ_i in γ provides precisely two boundary components on $\Sigma_g \setminus \gamma$. Fix an order $\Gamma = (\gamma_1, \ldots, \gamma_k)$ for the curves in γ and suppose that $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$. Denote by

$$\mathcal{M}(\Sigma_g \setminus \gamma, \ell(\Gamma) = \mathbf{x}),$$

the moduli space of hyperbolic surfaces homeomorphic to $\Sigma_g \setminus \gamma$ such that the length of γ_i satisfies $\ell(\gamma_i) = x_i$. Moreover, set

$$V_g(\Gamma, \mathbf{x}) = \mathrm{Vol}(\mathcal{M}(\Sigma_g \setminus \gamma, \ell(\Gamma) = \mathbf{x})),$$

to be the volume of this moduli space. We may write $\Sigma_g \setminus \gamma$ as the disjoint union of its connected components so that

$$\Sigma_g \setminus \gamma = \bigsqcup_{i=1}^{q} \Sigma_{g_i,n_i},$$

with $\sum_{i=1}^{q} n_i = 2k$. For the moduli space above, we then have that

$$\mathcal{M}(\Sigma_g \setminus \gamma, \ell_\Gamma = \mathbf{x}) \cong \prod_{i=1}^{q} \mathcal{M}_{g_i,n_i}(x_{i_1}, \ldots, x_{i_{n_i}}),$$

where the $i_1, \ldots, i_{n_i}$ are the indices corresponding to the multicurve components $\gamma_{i_1}, \ldots, \gamma_{i_{n_i}}$ that form the boundary of the corresponding connected component. The volume is then just

$$V_g(\Gamma, \mathbf{x}) = \prod_{i=1}^{q} V_{g_i,n_i}(x_{i_1}, \ldots, x_{i_{n_i}}).$$

For Mirzakhani's integral formula, one considers the integral of so-called geometric functions on the moduli space. These are defined from a multicurve such as γ above. Indeed, let $F\colon \mathbb{R}_+^k \to \mathbb{R}_+$ be a symmetric measurable function, and define $F_\gamma\colon \mathcal{M}_g \to \mathbb{R}_+$ by

$$F_\gamma(X) \coloneqq \sum_{\sum_{i=1}^{k} \alpha_i \in \mathrm{MCG}_g(\Sigma_g)\cdot\gamma} F(\ell_X(\alpha_1), \ldots, \ell_X(\alpha_k))$$

where $\ell_X(\alpha_i)$ is the length of the simple closed geodesic in the free homotopy class of the image of α_i under the marking on X. Moreover, define

$$M(\gamma) = \left| \left\{ i = 1, \ldots, k : \begin{array}{l} \gamma_i \text{ separates a one-holed torus that does not} \\ \text{contain any other } \gamma_j \text{ from the surface} \end{array} \right\} \right|,$$

and

$$\mathrm{Sym}(\gamma) \coloneqq \mathrm{Stab}(\gamma) / \left\langle S, \bigcap_{i=1}^{k} \mathrm{Stab}^{+}(\gamma_i) \right\rangle . \tag{3.11}$$

Here $\mathrm{Stab}(\gamma)$ is the stabiliser of the multicurve γ, $\mathrm{Stab}^{+}(\gamma_i)$ is the stabilisier of the single curve γ_i that preserves its orientation for $i = 1, \ldots, k$ and S is the kernel of the action of the mapping class group on Teitchmüller space (it is trivial when $g \geq 3$). Mirzakhani's integral formula is then stated as follows.

Theorem 3.5.5 (Mirzakhani [78, Theorem 7.1])**.** *For any multicurve* $\gamma = \sum_{i=1}^{k} \gamma_i$ *and a symmetric measurable function* $F\colon \mathbb{R}_+^k \to \mathbb{R}_+$*, one has*

$$\int_{\mathcal{M}_g} F_\gamma(X) \mathrm{d}X = \frac{1}{2^{M(\gamma)} |\mathrm{Sym}(\gamma)|} \int_{\mathbb{R}_+^k} F(\mathbf{x}) V_g(\Gamma, \mathbf{x}) \mathbf{x} \cdot \mathrm{d}\mathbf{x}$$

where $\mathbf{x} \cdot \mathrm{d}\mathbf{x} = x_1 \cdots x_k \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$ *and* $\Gamma = (\gamma_1, \ldots, \gamma_k)$*.*

3.6 Short Geodesic Loops on Random Surfaces

Consider a compact hyperbolic surface $X = \Gamma \backslash \mathbb{H}$ with a fundamental domain D. Recall that the assumption on the surfaces we considered is that they are (R, C)-admissible for some parameter R and C. In this section, we will demonstrate that with probability tending to 1 as $g \to \infty$, a surface X chosen with respect to the Weil-Petersson probability measure will be (R, C)-admissible with $R = c \log(g)$ for sufficiently small c independent of the genus g if the surface X has injectivity radius bounded below by g^{-b} for some $b > 0$ to be determined (also independently of g), and $C = \min\{1, \mathrm{InjRad}(X)\}^{-1}$. We do this by exhibiting that a certain geometric event holds with probability tending to 1 as $g \to \infty$ and that surfaces satisfying this condition are (R, C)-admissible with the aforementioned R and C parameters. For this, we will require to show that about any point $z \in D$ there is at most one primitive geodesic loop on the surface based at z with length at most $c\log(g)$ with probability tending to one as $g \to \infty$; that is, we show Theorem 3.1.2. An outline of the proof of this result is given as follows.

3.6.1 Outline of the proof

Theorem 3.1.2 is proven using contradiction via the following methodology. The general idea is close to the methods used by Mirzakhani and Petri [81]: we want to deduce from the presence of two distinct geodesic loops at one point the existence of a separating multicurve, and show that the probability for such a multicurve to exist in the large genus limit tends to 0. An important difference with [81] is that we deal here with geodesic loops instead of closed geodesics, which most notably behave differently in terms of self-intersections (Lemma 3.6.2). Our main contribution is then a generalised volume product formula (Lemma 3.6.5) based on finer estimates of volumes of moduli spaces from Mirzakhani and Zograf [82]. The dependence of all the constants on the genus renders the analysis considerably more involved, and we can also highlight in particular that in the proof of Theorem 3.6.8 we need additional steps to reduce our analysis to a certain topological type of multicurves that we call *minimally separating*.

We now give a detailed outline.

1. Given two primitive geodesics loops of length at most $c\log(g)$ on a hyperbolic surface X of genus g passing through the same point, we determine an upper bound on the number of self-intersections that these two curves can have between one another and with themselves. (Lemma 3.6.2)

2. The bound determined then provides an upper bound on the number of components in a multicurve obtained by taking a regular neighbourhood of the original curves and we show that for large enough genus g, this multicurve is separating and has total length at most $4c\log(g)$. (Lemmas

3.6.3 and 3.6.4)

3. We next prove an estimate on the order of growth on the sum of the products of the volumes of moduli spaces obtained by cutting along a multicurve as above, over all possible configurations of subsurface genera that such a multicurve could cut into given the number of components in the curve. (Lemma 3.6.5)

4. Using this estimate, we show that asymptotically as $g \to \infty$ such a multicurve does not exist on the surface with probability tending to one as $g \to \infty$. This is done by computing an upper bound on the expected number of separating multicurves with a bounded number of simple closed geodesic constituents (computed using item 2.) with length at most $4c\log(g)$ that can exist on a surface and showing it asymptotically tends to zero as $g \to \infty$. (Theorem 3.6.8)

5. One can then conclude that with probability tending to 1 as $g \to +\infty$, given a surface of genus g there is at most one primitive geodesic loop of length at most $c\log(g)$.

3.6.2 Bounds on $N_{c\log(g)}(X)$ are sufficient for condition (3.1)

Before starting the proof of Theorem 3.1.2, we provide a simple argument to demonstrate that a surface for which $N_R(X) \leq n$ for some $n \in \mathbb{N}$ is $(R, C(X))$-admissible with the $C(X)$ dependent upon the injectivity radius of the surface and n. Recall that a geodesic loop based at a point is primitive if it is the projection of a geodesic segment with endpoints identified by a primitive element of the group Γ.

Lemma 3.6.1. *Suppose that $X = \Gamma/\mathbb{H}$ is a compact hyperbolic surface for which there exists an $R > 0$ such that $N_R(X) \leq n$ some $n \in \mathbb{N}$. Then for each*

$z, w \in D$,

$$\left|\left\{\gamma \in \Gamma : d(z, \gamma w) \leq \frac{r}{2}\right\}\right| \leq \frac{2nr}{\mathrm{InjRad}(X)} + 2, \text{ for all } r \leq R.$$

Proof. Suppose that $N_r(X) \leq n$; we will first count the number of non-identity $\gamma \in \Gamma$ for which $d(z, \gamma z) \leq r$ for a given $z \in \mathbb{H}$. By definition, each primitive element $\gamma \in \Gamma$ that satisfies $d(z, \gamma z) \leq r$ produces a primitive geodesic loop based at z on the surface of length at most r. From the assumption that $N_r(X) \leq n$, this means there can be at most n primitive elements that satisfy this distance bound.

Given such a primitive γ, the powers γ^i will also satisfy the distance bound $d(z, \gamma^i z) \leq r$ if they generate short enough geodesic loops. As z does not necessarily lie on the axis of γ, the geodesic loop arising from the projection of the geodesic between z and $\gamma^i z$ onto the surface may have length shorter than i times the distance $d(z, \gamma z)$. By definition however, it has length at least the injectivity radius of the surface and thus at most the powers

$$\gamma^{\pm i}, \quad \text{for } i = 1, \ldots, \left\lfloor \frac{r}{\mathrm{InjRad}(X)} \right\rfloor,$$

can also satisfy $d(z, \gamma^i z) \leq r$. Such powers will however account for all of the possible group elements with $d(z, \gamma z) \leq r$. Indeed, if γ' is an element of Γ with $d(z, \gamma' z) \leq r$ then it is either the identity or a power of a primitive element, say γ. This follows from the fact that the surface is compact, and so all elements are hyperbolic and powers of primitives [15, Lemma 5.4]. In this latter case, we necessarily have that

$$d(z, \gamma z) \leq d(z, \gamma' z).$$

To see this, first suppose that γ is a dilation of the form $\gamma \colon z \mapsto az$ for some

$a > 0, a \neq 1$. Then, $d(z, \gamma z) \leq d(z, \gamma^m z)$ for all $z \in \mathbb{H}$ and $m \in \mathbb{Z}$ by the monotone increasing property of the cosh function and the explicit formula for the hyperbolic distance between two points given by

$$\cosh(d(z,w)) = 1 + \frac{(\mathrm{Re}(z) - \mathrm{Re}(w))^2 + (\mathrm{Im}(z) - \mathrm{Im}(w))^2}{2\mathrm{Im}(z)\mathrm{Im}(w)},$$

to compare both sides of the inequality. For a general $\gamma \in \Gamma$, we can use the fact that since γ is hyperbolic, there is some $g \in \mathrm{PSL}(2, \mathbb{R})$ for which $\gamma = g\delta g^{-1}$ where δ is a dilation as above. Then,

$$d(z, \gamma z) = d(g^{-1}z, \delta g^{-1} z) \leq d(g^{-1}z, \delta^m g^{-1} z) = d(z, \gamma^m z),$$

for any $m \in \mathbb{Z}$, proving the desired inequality. Thus, this means that if $d(z, \gamma' z) \leq r$, then we also have $d(z, \gamma z) \leq r$ for γ the primitive of γ', and hence γ' is accounted for as a power of one of the primitive elements that satisfy the distance inequality. With this in mind, we obtain the bound

$$|\{\gamma \in \Gamma \setminus \{\mathrm{id}\} : d(z, \gamma z) \leq r\}| \leq 2n \left\lfloor \frac{r}{\mathrm{InjRad}(X)} \right\rfloor.$$

For fixed $z, w \in \mathbb{H}$ we now count the number of $\gamma \in \Gamma$ with $d(z, \gamma w) \leq \frac{r}{2}$. Suppose there were at least $m = 2n\lfloor \frac{r}{\mathrm{InjRad}(X)} \rfloor + 2$ distinct non-identity elements with this property labelled $\gamma_1, \ldots, \gamma_m$. The elements $\gamma_j \gamma_1^{-1}$ are then distinct non-identity elements in Γ for $j = 2, \ldots, m$. Moreover, for each j we have

$$d(\gamma_1 w, (\gamma_j \gamma_1^{-1})(\gamma_1 w)) \leq d(\gamma_1 w, z) + d(\gamma_j w, z) < r.$$

This means that we have found $m - 1$ distinct, non-identity elements in Γ for which $d(\gamma_1 w, \gamma(\gamma_1 w)) \leq r$ which is a contradiction to the above counting argument. This means that there can be at most $m - 1$ such elements, and so including the identity we obtain

$$\left|\left\{\gamma \in \Gamma : d(z, \gamma w) \leq \frac{r}{2}\right\}\right| \leq \frac{2nr}{\mathrm{InjRad}(X)} + 2.$$

□

Note that the bound is in the form of $(R, C(X))$-admissibility when one observes that

$$\frac{2nr}{\mathrm{InjRad}(X)} + 2 \leq \frac{12ne^{\frac{1}{\delta}}e^{\delta r}}{\min\{1, \mathrm{InjRad}(X)\}},$$

for any $\delta > 0$. We will next work to show that $N_{c\log(g)}(X) \geq 1$ with high probability so that X is $(R(X), C(X))$-admissible with $R(X) = c\log(g)$ and $C(X)$ equal to

$$C(X) = \frac{1}{\min\{1, \mathrm{InjRad}(X)\}}.$$

3.6.3 Geometry of loops and extracting a separating multicurve

Suppose now that $N_r(X) > 1$. Then, there exists some $z \in X$ that has at least two primitive geodesic loops passing through it of length at most r. With $r \leq c\log(g)$ for some $c > 0$ to be determined, we next demonstrate that two such loops give rise to a certain separating multicurve on the surface X for large enough genus g. This result requires an improvement on the technique of Mirzakhani and Petri [81, Proposition 4.5] to allow for the curve lengths to have some dependence on the genus g and for the curves themselves to be geodesic loops rather than closed geodesics. We will first require the following lemma to determine the number of intersections between two such loops that have a finite number of intersections. Note the case where the two have an infinite number of intersections happens only when one loop is a subloop of the other and we will use primitivity of the loops to deal with this later in Lemma 3.6.4.

Lemma 3.6.2. *Suppose that α and β are geodesic loops of lengths $\ell(\alpha)$ and $\ell(\beta)$ respectively which have a finite number of intersections between them. Then,*

$$i(\alpha,\beta) \leq \left\lceil \frac{2\ell(\alpha)}{\mathrm{InjRad}(X)} \right\rceil \left\lceil \frac{2\ell(\beta)}{\mathrm{InjRad}(X)} \right\rceil,$$

where $i(\alpha,\beta) = \#(\alpha \cap \beta)$ denotes the number of intersections between the two curves.

Proof. Consider a geodesic segment $\bar{\alpha}$ of α of length $\rho = \frac{1}{2}\mathrm{InjRad}(X)$ which has the maximal number of intersections with β amongst all geodesic segments of α of length ρ. In this way, one obtains the upper bound

$$i(\alpha,\beta) \leq \left\lceil \frac{\ell(\alpha)}{\rho} \right\rceil i(\bar{\alpha},\beta).$$

Similarly, dividing β into geodesic segments of length ρ, we can bound this latter intersection by the number of such segments multiplied by the intersection number between $\bar{\alpha}$ and the segment of β with the most intersections with $\bar{\alpha}$, say $\bar{\beta}$ so that

$$i(\alpha,\beta) \leq \left\lceil \frac{\ell(\alpha)}{\rho} \right\rceil \left\lceil \frac{\ell(\beta)}{\rho} \right\rceil i(\bar{\alpha},\bar{\beta}).$$

Suppose that $\bar{\alpha}$ and $\bar{\beta}$ intersect at some point p. Then, by construction, both of these geodesic segments lie in $B_\rho(p)$. This ball however is an embedded ball in the surface by definition of ρ, and so they cannot intersect at another point in the ball (otherwise we would have distinct geodesics in the plane intersecting in more than one place). This gives $i(\bar{\alpha},\bar{\beta}) \leq 1$ and the result follows. □

Notice that the previous result can be easily modified to show that the same bounds hold on the number of self-intersections of a single loop with multiplicity. By multiplicity, we mean that if the loop intersects itself in the same point multiple times then we count each of these occurrences individually. For example, if at a self-intersection point there are 6 emanating curve segments then it will mean the curve has crossed through that point three times and hence intersected itself twice so this will be counted as two intersections. In summary this means that the total number of intersections between two curves of length at most $c\log(g)$ for some $c > 0$ and themselves is

$$O((c\log(g))^2 \mathrm{InjRad}(X)^{-2}).$$

Recall that we assumed that $\mathrm{InjRad}(X) \geq g^{-b}$ for some $b > 0$ to be chosen later (independently of g). With this condition, the number of intersections will be $O(c^2 g^{2b}(\log(g))^2)$.

In constructing our desired multicurve from the geodesic loops, we will be taking a regular neighbourhood and thus need to be able to deduce properties about the resulting subsurface that is bounded by the components of the neighbourhood. For this, we will need the following result, that is an adaptation to surfaces with boundaries and non-simple curves of [9, Lemma 2.1]. We will say that two curves α, β on a surface $\Sigma_{g,n}$ of genus g with n boundaries are *filling* if $\Sigma_{g,n} \setminus (\alpha \cup \beta)$ is a disjoint union of topological disks and annuli, such that each annulus is homotopic to a boundary component of $\Sigma_{g,n}$.

Lemma 3.6.3. *Suppose that α and β are two curves that fill $\Sigma_{g,n}$ whose intersection with one another and themselves (if there are any) are transversal, then*

$$i(\alpha, \beta) + i(\alpha, \alpha) + i(\beta, \beta) \geq 2g + n - 2,$$

where $i(\alpha_1, \alpha_2)$ is the number of intersections of the curves α_1 and α_2; recall as above that when $\alpha_1 = \alpha_2$ then this is counted with multiplicity.

Proof. Because α and β are filling, $\Sigma_{g,n}\backslash(\alpha\cup\beta)$ is a disjoint union of topological disks and annuli. We can form the 1-skeleton of a cellular decomposition of $\Sigma_{g,n}$ by considering the graph $\mathcal{G}(\alpha,\beta)$ whose vertex set is the set of intersection points both between α and β, and amongst themselves, and then adjoin to $\mathcal{G}(\alpha,\beta)$ one additional vertex and two additional edges per annuli. Let $i_0(\alpha,\beta)$ denote the number of intersection points between α and β and the curves themselves counted without multiplicity, or in other words, the number of vertices in $\mathcal{G}(\alpha,\beta)$.

In the graph of $\mathcal{G}(\alpha,\beta)$ adjoined with the extra components, there will be $i_0(\alpha,\beta)+n$ vertices and

$$2n+\frac{1}{2}\sum_{v\in\mathcal{G}(\alpha,\beta)}\deg_{\mathcal{G}(\alpha,\beta)}(v)$$

edges where $\deg_{\mathcal{G}(\alpha,\beta)}(v)$ denotes the degree of the vertex v in the graph $\mathcal{G}(\alpha,\beta)$. Now, the sum of these degrees in $\mathcal{G}(\alpha,\beta)$ will be

$$\begin{aligned}4i_0(\alpha,\beta)+2(i(\alpha,\beta)+i(\alpha,\alpha)+i(\beta,\beta)-i_0(\alpha,\beta))&\\ =2(i(\alpha,\beta)+i(\alpha,\alpha)+i(\beta,\beta)+i_0(\alpha,\beta)).&\end{aligned}$$

To see this, notice that each vertex appearing in the graph $\mathcal{G}(\alpha,\beta)$ will have degree 4 plus an extra 2 edges will emanate from a vertex for every additional crossing of one of the curves at that point. Hence the first term on the left hand side accounts for the base degree of 4 at each vertex and the second term accounts for the total number of additional crossings at all vertices in $\mathcal{G}(\alpha,\beta)$. This total number of additional crossings will be precisely the total number of crossings which is

$$i(\alpha,\beta)+i(\alpha,\alpha)+i(\beta,\beta),$$

minus the number of crossings that are the initial crossings of the curves (with each other or themselves) which is $i_0(\alpha, \beta)$. This means, that the total number of edges in this adjoined graph will be

$$2n + i(\alpha, \beta) + i(\alpha, \alpha) + i(\beta, \beta) + i_0(\alpha, \beta).$$

Let D be the number of 2-cells in this cellular decomposition of the surface so that $D \geq n$.

Then the Euler characteristic of $\Sigma_{g,n}$ is

$$\begin{aligned}\chi(\Sigma_{g,n}) = 2 - 2g - n = i_0(\alpha, \beta) + n - (2n + i(\alpha, \beta) + i(\alpha, \alpha) + i(\beta, \beta)\\ + i_0(\alpha, \beta)) + D\\ \geq -(i(\alpha, \beta) + i(\alpha, \alpha) + i(\beta, \beta))\end{aligned}$$

By rearranging, we obtain

$$i(\alpha, \beta) + i(\alpha, \alpha) + i(\beta, \beta) \geq 2g - 2 + n.$$

$\square$

We can now show that two geodesic loops with length at most $c\log(g)$ for some $c > 0$ based at the same point imply the existence of a separating multicurve for large enough g.

Lemma 3.6.4. *Suppose that α and β are primitive geodesic loops in the surface $X = \Sigma_g$ based at the same point with lengths bounded by $c\log(g)$ for some constant $c > 0$. Moreover, assume that for some $0 < b < \frac{1}{2}$, $\mathrm{InjRad}(X) > g^{-b}$. Then, there exists a separating multicurve γ on Σ_g consisting of $O(c^2 g^{2b}(\log(g))^2)$ simple closed geodesics whose total length is bounded by $4c\log(g)$ for g sufficiently large.*

Proof. Given the two curves α and β, there are two possibilities. Firstly, the two loops will have a finite number of intersections between them (at least one since they intersect at the base point of the loop). In this case we have a bound on the total number of intersections both between the two curves and their self-intersections by Lemma 3.6.2 of order $O(c^2 g^{2b}(\log(g))^2)$ using the condition on the injectivity radius. The second possibility is that one of the loops is a subloop of the other. In this case, we consider just the longer of the two curves. Again this curve has at least one self-intersection since for it to be distinct from the other curve it must contain more than one subloop. The total number of self-intersections of this curve is also again of order $O(c^2 g^{2b}(\log(g))^2)$ using Lemma 3.6.2.

Consider a regular neighbourhood of the curves in either possibility described above in Σ_g. The boundary of this neighbourhood will be a collection of disjoint simple closed curves and we consider the multicurve γ consisting of the simple closed geodesics that are freely homotopic to the boundary curves (discarding any such repeated curves). By construction when taking the neighbourhood of the set, each boundary component will be homotopic to simple closed segments of $\alpha \cup \beta$ (or just one of the curves in the second case) with each such segment appearing exactly twice (the portion of the neighbourhood either side of the union of the curves). Since the geodesics in the free homotopy classes are length minimising, their total sum must then be at most twice the total sum of the curves α and β from this double counting and so the total length of the multicurve constructed is bounded by $4c\log(g)$.

If one considers the graph whose vertices are the points of intersection of the curve(s) and edges being the geodesic segments between the curves, then one may homotope this graph to a wedge of circles. In each possibility, the primitivity of the curves ensures that we have at least two distinct circles in this wedge and so the regular neighbourhood bounds a non-trivial hyperbolic

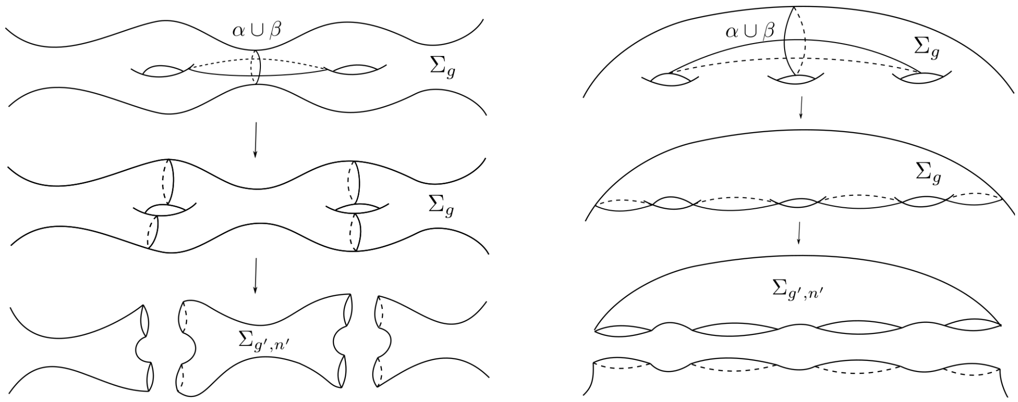


Figure 3.1: Some possibilities of the formation of the subsurface $\Sigma_{g',n'}$ from the regular neighbourhood of $\alpha \cup \beta$ in Σ_g. One begins by taking the regular neighbourhood of the union $\alpha \cup \beta$ and homotoping the boundary components to their geodesic representations. Then one cuts along these geodesics to obtain the subsurface $\Sigma_{g',n'}$.

surface. It is clear by construction that α and β are together filling curves for the subsurface constructed by this regular neighbourhood as it is non-trivial. Thus, if (g', n') is the signature of this subsurface $\Sigma_{g',n'}$ we have by Lemma 3.6.3 that

$$2g' + n' - 2 \leq I,$$

where $I = i(\alpha, \beta) + i(\alpha, \alpha) + i(\beta, \beta)$. By Lemma 3.6.2 applied to each of the intersections, we obtain that $I = O(c^2 g^{2b} (\log(g))^2)$ in either case by hypothesis on the surface.

If also α and β filled the surface Σ_g then by the same argument one would have that

$$2g - 2 \leq I,$$

which for g sufficiently large is not possible since $I = o(g)$ as $b < \frac{1}{2}$, and so γ must be separating when g is large enough. Two possibilities of how this multicurve could separate the surface are given in Figure 3.1. The number of components in γ is given by n' which from the inequality $n' \leq 2g' + n' \leq I + 2$ is seen to be of order $O(c^2 g^{2b}(\log(g))^2)$ as required. $\square$

So with this result, from two primitive geodesic loops based at a point of length at most $c\log(g)$, we obtain for large enough genus a separating multicurve with total length at most $4c\log(g)$ consisting of disjoint simple closed geodesics. We will next investigate how such a multicurve can be realised on a surface and show that with probability tending to one as $g \to \infty$, such a multicurve can not exist on a random surface X with injectivity radius bound given previously. This will mean that $N_r(X) \leq 1$ for all $r \leq c\log(g)$ for some $c > 0$ and that we have a suitable bound on the number of group elements desired, both with high probability.

3.6.4 Proving Theorems 3.1.1 and 3.1.2

We require an estimate on the product of volumes of moduli spaces of the subsurfaces obtained from cutting along the multicurve when the lengths of the curves can depend on the genus. In particular, we wish to see how the sum of such products can grow over all possible genera configurations on the subsurfaces with a given number of boundary components on each subsurface. We will only require a special case of this result for when the number of subsurfaces is two since we will later refine the multicurve to one that cuts the surface in a very specific way, but we include the more general result here as it is of interest in its own right. The starting point for this is the relation between different volumes given in Mirzakhani [80, Lemma 3.2] which has been reproduced here in Lemma 3.5.2 and the growth estimate on volumes of moduli spaces from Mirzakhani and Zograf [82] stated in Theorem 3.5.3.

Lemma 3.6.5. *Suppose that $q, k(g), n_1(g), \ldots, n_q(g) \in \mathbb{N}$ with $2 \leq q \leq k(g) +$*

1, $\sum_{i=1}^{q} n_i(g) = 2k(g)$ and $k(g) = O(g^d)$ for some $0 < d < 1$, then

$$\sum_{\{g_i\}} \prod_{i=1}^{q} V_{g_i,n_i(g)} = O\left(\frac{V_g D^{k(g)}\sqrt{k(g)}}{g^{\frac{1}{2}(q-1)}}\right),$$

as $g \to \infty$ where the sum is over all ordered sets of $\{g_i\}_{i=1}^{q} \subseteq \mathbb{Z}_{\geq 0}$ satisfying $\sum_{i=1}^{q} g_i = g + q - k(g) - 1$ and $2g_i - 3 + n_i \geq 0$ for all $i = 1, \ldots, q$ and D is some universal constant independent of all the parameters.

Proof. By Lemma 3.5.2, one has

$$V_{g_i,n_i(g)} \lesssim \begin{cases} V_{g_i+n_i(g)/2,0} & \text{for } n_i(g) \text{ even,} \\ V_{g_i+n_i(g)/2-1/2,1} & \text{for } n_i(g) \text{ odd.} \end{cases}$$

In either case, by Theorem 3.5.3

$$V_{g_i,n_i(g)} \lesssim \frac{C(2g_i + n_i(g) - 3)!(4\pi^2)^{2g_i+n_i(g)-3}}{\max\{1, \sqrt{g_i + n_i(g)/2 - 1}\}}\left(1 + O\left(\frac{1}{g_i + n_i(g)/2}\right)\right).$$

This latter remainder term can be bounded by some C' independent of g_i and $n_i(g)$ and so we have

$$\frac{1}{V_g}\sum_{\{g_i\}} \prod_{i=1}^{q} V_{g_i,n_i(g)} \lesssim D^{k(g)} \sum_{\{g_i\}} \frac{\prod_{i=1}^{q} \frac{1}{\max\{1,\sqrt{g_i}\}}(2g_i - 3 + n_i(g))!(4\pi^2)^{2g_i-3+n_i(g)}}{\frac{1}{\sqrt{g}}(2g-3)!(4\pi^2)^{2g-3}},$$

for some constant D independent of the n_i, g, k and q. To tackle the factorial terms, we use Stirling's approximation to infer that $n! \asymp \sqrt{n}\left(\frac{n}{e}\right)^n$ so that the summand is bounded up to a constant uniform in g, q, the $n_i(g)$ and $k(g)$ by

$$\frac{\sqrt{g}\prod_{i=1}^{q}(2g_i - 3 + n_i(g))^{2g_i - \frac{5}{2} + n_i(g)}\left(\frac{4\pi^2}{e}\right)^{2g_i-3+n_i(g)}}{\left(\frac{4\pi^2}{e}\right)^{2g-3}(2g-3)^{2g-\frac{5}{2}}\prod_{i=1}^{q}\max\{1, \sqrt{g_i}\}}.$$

Notice that

$$\left(\frac{4\pi^2}{e}\right)^{\sum_{i=1}^{q}(2g_i-3+n_i(g))-2g+3} = \left(\frac{4\pi^2}{e}\right)^{1-q} \leq 1,$$

since $q \geq 2$. Next,

$$\frac{\sqrt{g}}{\prod_{i=1}^{q}\max\{1,\sqrt{g_i}\}} = \frac{\sqrt{k(g)+1-q+\sum_{i=1}^{q} g_i}}{\prod_{i=1}^{q}\max\{1,\sqrt{g_i}\}} = O(\sqrt{k(g)}).$$

Lastly, one can observe that

$$\prod_{i=1}^{q}(2g_i-3+n_i(g))^{2g_i-\frac{5}{2}+n_i(g)} \leq \prod_{i=1}^{q}\left(2g_i-\frac{5}{2}+n_i(g)\right)^{2g_i-\frac{5}{2}+n_i(g)}.$$

Thus up to a constant independent of $k(g)$, g and q the sum of the products is bounded by

$$\sqrt{k(g)}\sum_{\{g_i\}}\frac{\prod_{i=1}^{q}\left(2g_i-\frac{5}{2}+n_i(g)\right)^{2g_i-\frac{5}{2}+n_i(g)}}{(2g-3)^{2g-\frac{5}{2}}}.$$

We now bound this summation by the number of possible ordered sets $\{g_i\}$ subject to the given Euler characteristic constraints multiplied by an upper bound on the summand itself. The former is clearly bounded above by the number of possible tuples $(g_1,\ldots,g_q) \in \mathbb{Z}_{\geq 0}^q$ that are solutions to

$$g_1+\ldots+g_q = g+q-k(g)-1,$$

using the first Euler characteristic constraint. However, the number of solutions to this is equal to

$$\binom{g+2(q-1)-k(g)}{q-1} \leq (g+2(q-1)-k(g))^{q-1} \leq (2g-3)^{q-1},$$

for g sufficiently large, with the latter inequality coming from the fact that

$q \leq k(g)+1$ and $k(g) \leq g-3$ when g is sufficiently large. For the maximum of the summand, we require an upper bound on the product term in the summand. Notice that by the Euler characteristic constraints, we are seeking the maximum of

$$\prod_{i=1}^{q} \left(2g_i - \frac{5}{2} + n_i(g)\right)^{2g_i - \frac{5}{2} + n_i(g)},$$

subject to

$$\sum_{i=1}^{q} 2g_i - \frac{5}{2} + n_i(g) = 2g - \frac{q}{2} - 2.$$

A product of this form attains the maximum value when all but one of the terms in the product are equal to 1 and the last component is determined by the summation condition. Hence, it is bounded above by

$$\left(2g - 2 - \frac{q}{2} - (q-1)\right)^{2g-2-\frac{q}{2}-(q-1)} \leq (2g-3)^{2g-\frac{3}{2}q-1},$$

where the latter inequality comes from the fact that $q \geq 2$. Combining these, we obtain that

$$\begin{aligned}
\frac{1}{V_g} \sum_{\{g_i\}} \prod_{i=1}^{q} V_{g_i, n_i(g)} &= O\left(\frac{D^{k(g)}\sqrt{k(g)}(2g-3)^{2g-\frac{3}{2}q-1+(q-1)}}{(2g-3)^{2g-\frac{5}{2}}}\right) \\
&= O\left(\frac{D^{k(g)}\sqrt{k(g)}}{g^{\frac{1}{2}(q-1)}}\right).
\end{aligned}$$

□

We now show that a separating multicurve as in Lemma 3.6.4 existing on a surface Σ_g tends to zero in the Weil-Petersson probability asymptotically as $g \to \infty$. To this end, let $K(g)$ denote the maximal number of components in the separating multicurve, so that by Lemma 3.6.4, we shall look at $K(g)$ of the form $K(g) = O(c^2 g^{2b}(\log(g))^2)$ for some $0 < b < \frac{1}{2}$ and $c > 0$ to be chosen. In fact, for the sake of simplifying the exposition of the proof we will consider $K(g) = O(g^d)$ with d sufficiently small. In our case, we can take $d = 2b + \varepsilon$ for any $\varepsilon > 0$ if one considers g large enough.

Suppose that we have such a multicurve γ then either γ is *minimally separating* in the sense that any sub-multicurve does not separate the surface or, we can find a sub-multicurve[5] that separates the surface and trivially satisfies the same conditions on the length and number of curve components as γ. By recursively extracting sub-multicurves in this manner we will arrive at one that is minimally separating due to the fact that the number of simple closed geodesics in γ is finite, and at least one simple closed geodesic is required to separate the surface. Thus we can show that a separating multicurve of the form we describe does not exist with probability tending to one as $g \to \infty$ by showing that a minimally separating multicurve with the same length and curve component restrictions occur with probability tending to zero as $g \to \infty$.

The reason that we reduce to these minimally separating multicurves is because their geometry is particularly accessible to us. Indeed, we can understand exactly how they cut a surface with the following result.

Lemma 3.6.6. *A minimally separating multicurve γ with k components separates the surface into exactly two connected subsurfaces with k boundary components each.*

Proof. Consider the dual graph to the multicurve γ. This is the graph whose vertex set is the connected components of the cut surface weighted with the genus and number of boundary components of that surface, and the edge set consists of an edge between two vertices for each component of the multicurve that creates a common boundary between the two surface components

[5] By sub-multicurve of γ we mean a multicurve whose simple geodesics are all present in γ.

represented by these vertices when one cuts the surface with that multicurve component. If this graph had more than two vertices, then one could fix any two vertices that are connected by at least one edge and consider the multicurve associated to all edges of the graph aside from the edges joining these two vertices. This multicurve by construction is a sub-multicurve of γ and separates the surface, since it would disconnect the two fixed vertices (and hence the corresponding connected components) from the other vertices in the graph which contradicts the fact that the multicurve is minimally separating.

There are k boundary components on each of the subsurfaces. Indeed, any further boundary components on one of them (and hence less on the other since there are $2k$ boundaries in total) would originate from some of the curves in the multicurve cutting open holes on this connected component, and so such curves could be removed from the multicurve producing a smaller multicurve that still disconnects the surface – a contradiction to the minimal separating property. In terms of the dual graph to the multicurve described above, this is equivalent to there being no self-loops at either vertex. □

In addition to this property, given a minimally separating multicurve γ we can precisely understand the symmetry group $\mathrm{Sym}(\gamma)$. In fact, in [80, Section 4.1 and proof of Theorem 4.2 Claim 2], Mirzakhani also studies multicurves of this type and states that $|\mathrm{Sym}(\gamma)| = k!$ when γ has k components. As this is a crucial point for us, we include a proof.

Lemma 3.6.7. *If γ is a minimally separating multicurve with k components, then $\mathrm{Sym}(\gamma) \simeq \mathfrak{S}_k$, where $\mathfrak{S}_k$ is the symmetric group on $\{1, \ldots, k\}$.*

Proof. Recall the definition from (3.11) of the symmetry group

$$\mathrm{Sym}(\gamma) = \mathrm{Stab}(\gamma) / \bigcap_{i=1}^{k} \mathrm{Stab}(\gamma_i).$$

By construction, $\mathrm{Sym}(\gamma)$ can be identified with a subgroup $\mathfrak{h}$ of $\mathfrak{S}_k$. Indeed,

$\mathrm{Stab}(\gamma)$ acts on the k curves by permutations and the quotient makes this action faithful. To show that $\mathfrak{h} = \mathfrak{S}_k$ it suffices to show that the subgroup $\mathfrak{h}$ contains the transpositions.

This is the case as for any two components γ_1 and γ_2 of γ, we can find an orientation preserving homeomorphism h that permutes γ_1 and γ_2 and leaves the other curves invariant. To see this, we use the existence of the following elementary transformation: Let b_1 and b_2 be two boundary components of a connected surface and a an arc joining the two components. Let Ω be a regular neighbourhood of $a \cup b_1 \cup b_2$. There exists an orientation preserving homeomorphism that exchanges b_1 and b_2 and is equal to the identity outside of Ω.

Now to construct the homeomorphism h, we cut open the surface along the multicurve γ. By Lemma 3.6.6, we obtain two subsurfaces with k boundary components, and such that each of the k curves of γ contribute one boundary component to each subsurface. Let b_1 and b_2 be the two boundary components coming respectively from γ_1 and γ_2 on one of the two subsurfaces. We can connect b_1 and b_2 by an arc a that does not touch the other boundary components. We can therefore find a homeomorphism that swaps b_1 and b_2 and is the identity in a neighbourhood of all the other boundary components. Similarly, we can swap the boundaries coming from γ_1 and γ_2 on the second subsurface. Moreover, since the neighbourhoods that we consider in each of the subsurfaces when swapping the boundary components can be taken to be homeomorphic, we can arrange it so that the homeomorphisms on each subsurface match up in a homeomorphic way across the geodesics γ_1 and γ_2 when we glue back together the boundary components of each curve of γ. By this procedure we have built a homeomorphism h that maps γ_1 to γ_2 and γ_2 to γ_1 while being the identity map in a neighbourhood of the other curves. $\square$

Theorem 3.6.8. *Choosing c and d sufficiently small independently of g and*

$K(g) = O(g^d)$, we have that

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \in \mathcal{M}_g : \begin{array}{l} X \textit{ contains a separating multicurve } \gamma \textit{ with at} \\ \textit{most } K(g) \textit{ disjoint simple closed curve com-} \\ \textit{ponents of total length at most } 4c\log g. \end{array} \right) \to 0,$$

as $g \to \infty$. In fact there exists a universal constant $\delta > 0$ such that this probability is $O(g^{-\frac{1}{2}+\delta(c+d)})$, where the implied constant is independent of c and d.

Proof. Suppose that γ is a separating multicurve satisfying the desired properties on its length and number of curve components. As has been outlined in the lines preceding Lemma 3.6.6, we can extract a minimally separating multicurve from γ satisfying the same hypotheses on the number of components and their lengths. The probability that we are interested in computing is thus bounded by

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \in \mathcal{M}_g : \begin{array}{l} X \text{ contains a minimally separating multic-} \\ \text{urve } \gamma \text{ with at most } K(g) \text{ disjoint simple} \\ \text{closed curve components of total length at} \\ \text{most } 4c\log g. \end{array} \right),$$

which we will now proceed to bound. Let $K(g) = O(g^d)$. Suppose that $N(X)$ is the number of minimally separating multicurves $\gamma = \sum_{i=1}^{k(\gamma)} \gamma_i$ on X for each natural number $k(\gamma) \leq K(g)$ and $\ell(\gamma) \leq 4c\log(g)$. Then the event described above is given by $\{X \in \mathcal{M}_g : N(X) \geq 1\}$. By Markov's inequality, we have

$$\mathbb{P}_g^{\mathrm{WP}}(N(X) \geq 1) \leq \frac{1}{V_g} \int_{\mathcal{M}_g} N(X)\mathrm{d}X.$$

Let us now bound $\int N(X)\mathrm{d}X$ using Mirzakhani's integration formula Theorem 3.5.5 and the volume estimates for Weil-Petersson volume.

Fix $1 \leq k \leq K(g)$ and consider minimally separating γ with $k(\gamma) = k$.

Define the following non-negative symmetric function

$$F(x_1, \ldots, x_k) = \mathbf{1}_{[0,L]}(x_1 + \cdots + x_k), \quad (x_1, \ldots, x_k) \in \mathbb{R}_+^k,$$

where $L = 4c\log(g)$. Given a multicurve $\gamma = \sum_{i=1}^k \gamma_i$ on Σ_g as above, the associated geometric function is obtained by

$$F_\gamma(X) = \sum_{\gamma' \in [\gamma]} \mathbf{1}_{[0,L]}(\ell_{\gamma'_1}(X) + \cdots + \ell_{\gamma'_k}(X)),$$

where the sum runs over all multicurves $\gamma' = \sum_{i=1}^k \gamma'_i$ in the mapping class group orbit of $[\gamma]$ and $\{\gamma'_i : 1 \leq i \leq k\}$ is the set of components of γ'. Then for any $X \in \mathcal{M}_g$, we have

$$N(X) \leq \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} F_\gamma(X),$$

where this inner summation runs over all possible mapping class group orbits of minimally separating multicurves $\gamma = \sum_{i=1}^k \gamma_i$ with $k(\gamma) = k$ and $k = 1, \ldots, K(g)$. We thus obtain the following bound on the probability of interest:

$$\mathbb{P}_g^{\mathrm{WP}}(N(X) \geq 1) \leq \frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} \int_{\mathcal{M}_g} F_\gamma(X) \mathrm{d}X.$$

We may pass this integral over the moduli space to an integral over Euclidean space via the Mirzakhani integral formula provided in Theorem 3.5.5, and hence obtain an upper bound to the above of the form

$$\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} \frac{1}{|\mathrm{Sym}(\gamma)| 2^{M(\gamma)}} \int_{\mathbb{R}_{\geq 0}^k} \mathbf{1}_{[0,L]}(x_1 + \cdots + x_k) x_1 \cdots x_k V_g(\Gamma, \mathbf{x}) \bigwedge_{i=1}^k \mathrm{d}x_i,$$

where $V_g(\Gamma, \mathbf{x})$, $\mathrm{Sym}(\gamma)$ and $M(\gamma)$ are defined as in the lines preceding Theorem 3.5.5 for $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$ and $\Gamma = (\gamma_1, \ldots, \gamma_k)$.

Let us now proceed to estimate the above quantity. By Lemma 3.6.7, we know that the minimal separating property of γ means that $|\mathrm{Sym}(\gamma)| = k!$.

Next, let us address the volume term $V_g(\Gamma, \mathbf{x})$. Using Lemma 3.6.6, the cut surface necessarily has only two components with k boundaries on each component. Thus, this volume term is of the form

$$V_g(\Gamma, \mathbf{x}) = V_{g_1(\gamma),k}(\mathbf{x}^1) V_{g_2(\gamma),k}(\mathbf{x}^2),$$

where $g_1(\gamma) + g_2(\gamma) + k - 1 = g$ and the length vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ are given by the coordinates of $\mathbf{x}$ that correspond to which components of the multicurve form the boundaries of the two subsurfaces. These volumes can be bounded using the volume estimates of Lemma 3.5.1 to obtain

$$V_g(\Gamma, \mathbf{x}) \leq e^{x_1 + \cdots + x_k} V_{g_1(\gamma),k} V_{g_2(\gamma),k}.$$

Thus we obtain that

$$\begin{aligned}
&\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} \frac{1}{|\mathrm{Sym}(\gamma)| 2^{M(\gamma)}} \int_{\mathbb{R}_{\geq 0}^k} \mathbf{1}_{[0,L]}(x_1 + \cdots + x_k) V_g(\Gamma, \mathbf{x}) \mathbf{x} \cdot \mathrm{d}\mathbf{x} \\
&\leq \frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} \frac{1}{k!} V_{g_1(\gamma),k} V_{g_2(\gamma),k} \int_{\mathbb{R}_{\geq 0}^k} \mathbf{1}_{[0,L]}(x_1 + \cdots + x_k) e^{\sum_{i=1}^k x_i} \mathbf{x} \cdot \mathrm{d}\mathbf{x} \\
&\leq \frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} \frac{1}{k!} V_{g_1(\gamma),k} V_{g_2(\gamma),k} e^L \int_{\mathbb{R}_{\geq 0}^k} \mathbf{1}_{[0,L]}(x_1 + \cdots + x_k) \mathbf{x} \cdot \mathrm{d}\mathbf{x} \\
&\leq \frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{\substack{[\gamma] \\ k(\gamma)=k}} \frac{1}{k!} V_{g_1(\gamma),k} V_{g_2(\gamma),k} e^L L^{2k} k^{-k},
\end{aligned}$$

with the factor $L^{2k}k^{-k}$ arising from the fact that the maximum of $x_1 \cdots x_k$ subject to $\sum_{i=1}^{k} x_i = L$ arises when each x_i is equal to Lk^{-1} and the measure of the set $\sum_{i=1}^{k} x_i = L$ is bounded by L^k.

Notice that the sum over $[\gamma]$ with $k(\gamma) = k$ may be re-written as the sum over the pairs $(g_1, g_2) \in \mathbb{Z}^2_{\geq 0}$ satisfying the Euler characteristic criteria $g_1 + g_2 + k - 1 = g$, multiplied by the number of mapping class group orbits of the minimally separating multicurves with k components that separate the surface into a topological decomposition of a genus g_1 and genus g_2 subsurface each with k boundary components. Since two such multicurves are in the same mapping class orbit if and only if their dual multicurve graphs described above are the same, the number of these mapping class group orbits is precisely equal to the number of possible different dual multicurve graphs that can arise from our multicurves of k components. However, as described above each such graph has two vertices, k edges and no self-loops and so there is precisely one orbit for each genus (g_1, g_2) decomposition. Hence, the probability we are interested in is bounded by

$$\frac{1}{V_g} \sum_{k=1}^{K(g)} \frac{e^L L^{2k}}{k! k^k} \sum_{\{(g_1,g_2): g_1+g_2+k-1=g\}} V_{g_1,k} V_{g_2,k}.$$

We now make use of Lemma 3.6.5 for $q = 2$, to deduce that

$$\sum_{\{(g_1,g_2): g_1+g_2+k-1=g\}} V_{g_1,k} V_{g_2,k} = O\left(\frac{V_g D^k \sqrt{k}}{g^{\frac{1}{2}}} \right),$$

with the implied constant being independent of g and k. Up to a constant, we thus have the probability bounded by

$$g^{-\frac{1}{2}} \sum_{k=1}^{K(g)} \frac{e^L L^{2k} D^k \sqrt{k}}{k! k^k}.$$

Using $L = 4c\log(g)$ we can write $e^L = g^{4c}$ and bound $\sqrt{k} \leq \sqrt{K(g)} \lesssim g^{\frac{1}{2}d}$.

Moreover, using Stirling's approximation, $k!$ is bounded up to a constant above and below by $k^{k+\frac{1}{2}}e^{-k}$. In particular, up to bounding by a universal constant, we may replace $k^k k!$ by $(2k)!e^k$ providing an upper bound of the form

$$\begin{aligned} g^{-\frac{1}{2}+4c+\frac{1}{2}d} \sum_{k=1}^{K(g)} \frac{(4ce^{-\frac{1}{2}}D^{\frac{1}{2}}\log(g))^{2k}}{(2k)!} &\leq g^{-\frac{1}{2}+4c+\frac{1}{2}d}\cosh(4ce^{-\frac{1}{2}}D^{\frac{1}{2}}\log(g)) \\ &\leq g^{\delta(c+d)-\frac{1}{2}}, \end{aligned}$$

for some universal constant $\delta > 0$. Taking c and d sufficiently small, we can then insist that this probability tends to zero as $g \to \infty$. □

We now combine Theorem 3.6.8 with Theorem 4.2 of [80] to obtain Theorem 3.1.2.

Proof of Theorem 3.1.2. Let b and c be chosen such that

$$A_g^{b,c} = \left\{ X \in \mathcal{M}_g : \begin{array}{c} X \text{ contains a separating multicurve } \gamma \text{ with at most } K(g) \\ \text{disjoint simple closed curve components of total length} \\ \text{at most } 4c\log g. \end{array} \right\},$$

where $K(g) = O(g^d)$, $d = 2b + \varepsilon$ for any $\varepsilon > 0$ tends to zero as $g \to \infty$. By Theorem 3.6.8 when these constants are suitably chosen, the rate of this is $O(g^{-\frac{1}{2}+\delta(d+c)})$ for some universal constant δ. Moreover, let

$$B_g^b = \{X \in \mathcal{M}_g : \ \mathrm{InjRad}(X) \leq g^{-b}\}.$$

By a result of Mirzakhani [80, Theorem 4.2], we have that

$$\mathbb{P}_g^{\mathrm{WP}}(B_g^b) = O(g^{-2b}).$$

Notice then that

$$(\mathcal{M}_g \setminus A_g^{b,c}) \cap (\mathcal{M}_g \setminus B_g^b) \subseteq \{X \in (\mathcal{M}_g)_{\geq g^{-b}} : N_{c\log g}(X) \leq 1\}.$$

Indeed, suppose X is contained in the left-hand set then by definition of $\mathcal{M}_g \setminus B_g^b$ it is contained in $(\mathcal{M}_g)_{\geq g^{-b}}$. Moreover, if it had $N_{c\log(g)} > 1$ then Lemma 3.6.4 would imply that there would exist a multicurve on X of the form described in the definition of $A_g^{b,c}$ which is a contradiction and thus the inclusion holds. This means that Theorem 3.1.2 follows from a lower bound on the probability of the event on the left-hand side. But this can be determined as follows:

$$\begin{aligned}
\mathbb{P}_g^{\mathrm{WP}}((\mathcal{M}_g \setminus A_g^{b,c}) \cap (\mathcal{M}_g \setminus B_g^b)) &= \mathbb{P}_g^{\mathrm{WP}}(\mathcal{M}_g \setminus (A_g^{b,c} \cup B_g^b)) \\
&\geq 1 - (\mathbb{P}_g^{\mathrm{WP}}(A_g^{b,c}) + \mathbb{P}_g^{\mathrm{WP}}(B_g^b)) \\
&= 1 - O(g^{-\frac{1}{2}+\delta(b+c)} + g^{-2b}),
\end{aligned}$$

by applying Theorem 4.2 of [80] and Theorem 3.6.8 above as required. $\square$

Thus, if one sets $\mathcal{A}_g^{b,c}$ to be the event

$$\{X \in (\mathcal{M}_g)_{\geq g^{-b}} : N_{c\log g}(X) \leq 1\},$$

then combining Theorems 3.1.2, 3.1.3 and Lemma 3.6.1 gives that the bounds of Theorem 3.1.1 hold for any surface in $\mathcal{A}_g^{b,c}$ which has probability tending to 1 as $g \to +\infty$ with the rate given by Theorem 3.1.2.

4 The Tangle-free Hypothesis on Random Hyperbolic Surfaces

Abstract

This article introduces the notion of L-tangle-free compact hyperbolic surfaces, inspired by the identically named property for regular graphs. Random surfaces of genus g, picked with the Weil-Petersson probability measure, are $(a \log g)$-tangle-free for any $a < 1$. This is almost optimal, for any surface is $(4 \log g + O(1))$-tangled. We establish various geometric consequences of the tangle-free hypothesis at a scale L, amongst which the fact that closed geodesics of length $< \frac{L}{4}$ are simple, disjoint and embedded in disjoint hyperbolic cylinders of width $\geq \frac{L}{4}$.

4.1 Introduction

In this article, we introduce the tangle-free hypothesis on compact (connected, oriented) hyperbolic surfaces (without boundary), and explore some of its geometric implications, with a special emphasis on random surfaces, which we show are almost optimally tangle-free.

This work follows several recent articles aimed at adapting results on random regular graphs in both geometry and spectral theory to the setting of random hyperbolic surfaces – see [80, 81, 48, 83, 109, 73] for instance. Though the initial motivation was to provide some useful tools for spectral theory, the results and techniques developed here are purely geometric. Several of our results are significant improvements of useful properties of geodesics on compact hyperbolic surfaces, allowed by the random setting: the length scale at which they apply goes from constant to logarithmic in the genus.

A key innovation of this article is finding an elementary geometric condition which is simultaneously easy to prove for random surfaces, and has far-reaching consequences on their geometry (notably their geodesics) at a large scale. Similar geometric assumptions have been made recently by Mirzakhani and Petri [81, Proposition 4.5] and Gilmore, Le Masson, Sahlsten and Thomas [48]. The use of the tangle-free hypothesis simplifies and improves the result in [48], and generalises one consequence of [81, Proposition 4.5] to a larger scale.

The tangle-free hypothesis for hyperbolic surfaces

Let us first define what we mean by tangle-free and contrast it with existing concepts in the graph theoretic and hyperbolic surface literature. Heuristically speaking, we shall say that a surface is tangle-free if it does not contain embedded pairs of pants or one-holed tori with 'short' boundaries. More precisely, we make the following definition.

Definition 4.1.1. Let X be a compact hyperbolic surface and $L > 0$. Then, X is said to be *L-tangle-free* if all embedded pairs of pants and one-holed tori in X have total boundary length larger than $2L$. Otherwise, X is *L-tangled.*

To be precise, we emphasise that a pair of pants and a one-holed torus are respectively surfaces of signature $(0, 3)$ and $(1, 1)$, and the embedded surfaces we consider have totally geodesic boundary. The total boundary length is defined as the sum of the length of all the boundary geodesics. One should note that we could also have defined the notion of tangle-free using the maximum boundary length (the length of the longest boundary geodesic) and the results of this paper would follow through (up to changes of constants).

It may not be so clear to the reader why we call such a property *tangle-free*. In order to clarify this, we prove that, when a surface is tangled, it contains a non-simple geodesic; that is, a *tangled* geodesic in the literal sense of the word.

Proposition A (Proposition 4.2.1)**.** *Any L-tangled surface contains a self-intersecting geodesic of length smaller than $2L + 2\pi$.*

Tangle-free graphs

One can motivate the study of this geometric property of surfaces through the medium of regular graphs. Indeed, the naming of this property is inspired by a similar notion Bordenave introduced in [22] in order to prove Friedman's theorem [46] regarding the spectral gap of the Laplacian on large regular graphs (we shall come back to this result in more detail at the end of the introduction). A graph $G = (V, E)$ is said to be *L-tangle-free* if, for any vertex v, the ball for the graph distance dist_G

$$\mathcal{B}_L(v) = \{w \in V \,:\, \mathrm{dist}_G(v, w) \leq L\}\,,$$

contains at most one cycle. This definition might seem quite different to the surface definition given above, but we shall prove that balls on tangle-free surfaces contain at most one 'cycle' in the following sense.

Proposition B (Proposition 4.4.4)**.** *If a surface X is L-tangle-free, then for any point $z \in X$, the ball*

$$\mathcal{B}_{\frac{L}{8}}(z) = \left\{w \in X \,:\, \mathrm{dist}_X(z, w) < \frac{L}{8}\right\}$$

is isometric to a ball in the hyperbolic plane or a hyperbolic cylinder.

It is worth noting that in the original proof by Friedman [46], there is also a notion of 'supercritical tangle' in a graph, which are small subgraphs with many cycles. In a sense, pairs of pants or one-holed tori with small total boundary lengths can be seen as analogues of these bad tangles for surfaces.

Admissible values of L

Let us now discuss typical values that L can take in Definition 4.1.1 both for being tangle-free and tangled. Throughout, we shall use the notation $A = O(B)$ to indicate that there is a constant $C > 0$ such that $|A| \leq C|B|$ with C independent of all other variables such as the genus.

It is clear that a surface of injectivity radius r is r-tangle-free, for it has no closed geodesic of length smaller than $2r$. In a deterministic setting, it is hard to say much more than this.

On the other hand, we know that a hyperbolic surface of genus g admits a pants decomposition with all boundary components smaller than the *Bers constant* $\mathcal{B}_g$ – see [32, Chapter 5]. We know that $\mathcal{B}_g \geq \sqrt{6g} - 2$ [32, Theorem 5.1.3], and the best known upper bounds on $\mathcal{B}_g$ are linear in g [33, 91]. All surfaces of genus g are $\frac{3}{2}\mathcal{B}_g$-tangled. This bound however is rather loose, since it follows from cutting *all* of the surface into pairs of pants rather than isolating a single short pair of pants. In light of this, we in fact prove the following, using a method based on Parlier's work [91].

Proposition C (Proposition 4.5.1)**.** *Any hyperbolic surface of genus g is L-tangled for $L = 4 \log g + O(1)$.*

Random graphs and surfaces

How tangle-free can a *typical* surface be? Can L be much larger than the injectivity radius for a large class of surfaces? An instructive method to answer these questions is to consider the setting of random surfaces, and to find an L for which most surfaces are L-tangle-free.

For d-regular graphs with n vertices, sampled with the uniform probability measure $\mathbb{P}_n^{(d)}$, Bordenave proved [24] that for any real number $0 < a < \frac{1}{4}$,

$$\mathbb{P}_n^{(d)}(G \text{ is } (a \log_{d-1}(n))\text{-tangle-free}) \underset{n\to+\infty}{\longrightarrow} 1.$$

This is a key ingredient in Bordenave's proof of Friedman's theorem [22].

In this article, we will consider the Weil-Petersson probability $\mathbb{P}_g^{\mathrm{WP}}$ on the set of closed hyperbolic surfaces of genus g. However, one should note that there exist other non-equivalent random surface models such as that of Brooks and Makover [26] or Magee, Naud and Puder [73]. We introduce the model in detail in Section 4.3, and then prove that, in this setting, random surfaces are tangle-free at a scale $\log g$ with high probability.

Theorem D (Theorem 4.3.2)**.** *For any real number* $0 < a < 1$*,*

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \textit{ is } (a \log g)\textit{-tangle-free}\right) = 1 - O\left(\frac{(\log g)^2}{g^{1-a}}\right).$$

Since any surface of genus g is $(4 \log g + O(1))$-tangled, random surfaces are almost as tangle-free as possible. The scale $\log g$ is a very large scale on a random hyperbolic surface of high genus: by work of Mirzakhani [80] the diameter of such a surface is $\leq 40 \log g$ with high probability. Mirzakhani and Petri [81] also proved that the mean value of its injectivity radius goes to a constant value $\simeq 0.807$ as g approaches infinity, hence proving that random surfaces of high genus have short closed geodesics. These closed geodesics do not bound any pair of pants.

Geometric implications of the tangle-free hypothesis

The L-tangle-free hypothesis has various consequences on the local geometry of the surface at a scale (roughly) L, which we explore in Section 4.4. This will be particularly interesting when L is large; in the case of random surfaces notably, where $L = a \log g$ for $a < 1$. All the results are stated for any L-

tangle-free surface, with a general L and no other assumption, so that they can be directly applied to another setting in which a tangle-free hypothesis is established.

First and foremost, we analyse the embedded cylinders around simple closed geodesics. In a hyperbolic surface with no further geometric assumptions to it, the standard collar theorem [32, Theorem 4.1.1] proves that the collar of width $\operatorname{arcsinh}\left(\sinh(\ell/2)^{-1}\right)$ around a simple closed geodesic of length ℓ is an embedded cylinder; moreover, at this width, disjoint simple closed geodesics have disjoint collars. The width of this deterministic collar is optimal and very satisfying for small ℓ. For larger values of ℓ however, it becomes very poor. Under the tangle-free hypothesis, we are able to obtain significant improvements to the collar theorem that remedy this issue at larger scales.

Theorem E (Theorem 4.4.1)**.** *On a L-tangle-free hyperbolic surface, the collar of width $\frac{L-\ell}{2}$ around a closed geodesic of length $\ell < L$ is isometric to a cylinder.*

This implies that we can find wide collars around geodesics of size $a \log g$, $a < 1$, on random surfaces; as a comparison, the width of the deterministic collar around such a geodesic decreases like $g^{-\frac{a}{2}}$. By a volume argument, Theorem 4.4.1 is optimal up to multiplication of the width by a factor two.

The methodology to prove this result is to examine the topology of an expanding neighbourhood of the geodesic. Since the two simplest hyperbolic subsurfaces (namely the pair of pants and one-holed torus) cannot be encountered up to a scale $\sim L$ due to the tangle-free hypothesis, the neighbourhood remains a cylinder.

An immediate consequence of this improved collar theorem is a bound on the number of intersections of a closed geodesic of length $\ell < L$ and any other geodesic of length ℓ'. We prove in Corollary 4.4.1 that two such geodesics intersect at most $1 + \frac{\ell'}{L-\ell}$ times (and we can remove the 1 if the two geodesics are closed). Therefore, two closed geodesics of length $< \frac{L}{2}$ do not intersect;

Proposition 4.4.2 furthermore states that the collars of width $\frac{L}{2} - \ell$ around two such geodesics are disjoint.

As well as the neighbourhood of geodesics, one can look at the geometric consequences that the tangle-free hypothesis has on the neighbourhood of points. To this end, we explore the set of geodesic loops based at a point on the surface on length scales up to L. As has already been mentioned above in Proposition 4.4.4, which establishes a link between our tangle-free definition and that of graphs, on an L-tangle-free surface, balls of radius $\frac{L}{8}$ are isometric to balls in either the hyperbolic plane or a hyperbolic cylinder. There are several ways to prove this property, some of which are similar to the proof of the improved collar theorem. In order to present different methods, we rather deduce it from the following slightly more general result.

Theorem F (Theorem 4.4.2)**.** *If z is a point on a L-tangle-free surface, and δ_z is the shortest geodesic loop based at z, then any other loop β based at z such that $\ell(\delta_z) + \ell(\beta) < L$ is homotopic to a power of δ_z.*

Another consequence of Theorem 4.4.2 is Corollary 4.4.5, which states that any closed geodesic of length $< L$ on a L-tangle-free surface is simple. Put together, these observations imply the following corollary.

Corollary G. *On a L-tangle-free hyperbolic surface,*

1. *all closed geodesics of length $< L$ are simple;*
2. *all closed geodesics of length $< \frac{L}{2}$ are pairwise disjoint;*
3. *all closed geodesics of length $< \frac{L}{4}$ are embedded in pairwise disjoint cylinders of width $\geq \frac{L}{4}$.*

In the random case, this result is an improvement of the very useful collar theorem II [32, Theorem 4.1.6], which states that all closed geodesics of length $< 2 \operatorname{arcsinh} 1$ on a hyperbolic surface are simple and do not intersect.

Short closed geodesics in random hyperbolic surfaces have been studied by Mirzakhani and Petri [80, 81]. One can deduce from [81, Proposition 4.5] and Markov's inequality that, for any fixed M,

$$1 - \mathbb{P}_g^{\mathrm{WP}}\left(\text{all closed geodesics of length } < M \text{ are simple}\right) \leq \frac{C_M}{g}$$

for a constant $C_M > 0$, when we prove that, for any real number $0 < a < 1$,

$$1 - \mathbb{P}_g^{\mathrm{WP}}\left(\text{all closed geodesics of length } < a \log g \text{ are simple}\right) \leq C \frac{(\log g)^2}{g^{1-a}}$$

for a constant $C > 0$. In order to push the proof in [81] to a scale $\log g$, one would need to use strong properties of the Weil-Petersson volumes and deal with technical estimates, while our approach is quite elementary in both the geometric and probabilistic sense.

As illustrated in Section 4.3, the tools used to study random surfaces in the Weil-Petersson setting require to reduce problems to the study of multicurves. Knowing that all closed geodesics of length $< \frac{a}{2} \log g$ form a multicurve can be useful to the understanding of other properties of random surfaces.

Furthermore, McShane and Parlier proved in [77] that for any $g \geq 2$,

$$\mathbb{P}_g^{\mathrm{WP}}\left(\text{the } \textit{simple} \text{ length spectrum has no multiplicity}\right) = 1,$$

where the simple length spectrum of a surface is the list of all the lengths of its simple closed geodesics. Corollary 4.4.5 then implies the following.

Corollary 4.1.2. *For any $a \in (0,1)$, if $\mathcal{L}(X)$ denotes the length spectrum of X, then*

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathcal{L}(X) \cap [0, a \log g] \textit{ has no multiplicity}\right) = 1 - O\left(\frac{(\log g)^2}{g^{1-a}}\right).$$

This could be surprising since, by the work of Horowitz and Randol, for any compact hyperbolic surface, the length spectrum has unbounded multiplicity [32, Theorem 3.7.1]. However, these high multiplicities are constructed in embedded pairs of pants, and therefore it is natural that their lengths are large for tangle-free surfaces.

Motivations in spectral theory

To conclude this introduction we will outline the connection between the geometry of hyperbolic surfaces and their spectral theory and in particular discuss how the tangle-free hypothesis and its implications on the geometry of surfaces on $\log g$ scales, which is a crucial scale in spectral theory, could be used to tackle some open problems in this area. As promised, let us first return to the relation of the tangle-free hypothesis with spectral graph theory and contrast this with that of surfaces.

Friedman's theorem

Let G be a d-regular graph, and A be its adjacency matrix. We will call eigenvalues of G the eigenvalues of the matrix A. They are linked to the eigenvalues of the Laplacian Δ through the relation $-A + d\,\mathrm{Id} = \Delta$. The value d is always an eigenvalue of G corresponding to constant functions, and $-d$ is an eigenvalue if and only if G is bipartite; both d and $-d$ are referred to as trivial eigenvalues. Friedman's theorem [46], first conjectured by Alon [3], states that for any $\varepsilon > 0$,

$$\mathbb{P}_n^{(d)}\left(\forall\lambda \text{ non-trivial eigenvalue of } G, |\lambda| < 2\sqrt{d-1} + \varepsilon\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

This means that large random regular graphs have an optimal spectral gap, by a result of Alon [88].

Let us compare this to what one may expect of surfaces. We will refer to the spectrum of a compact hyperbolic surface X as meaning the spectrum of the (positive) Laplace-Beltrami operator Δ on X. It is a non-decreasing sequence of eigenvalues $(\lambda_n)_{n\geq 0}$, $\lambda_n \geq 0$. The value $\lambda_0 = 0$ is known as the trivial eigenvalue; it is simple and the corresponding eigenfunctions are the constant functions. The equivalent surface conjecture of the Friedman theorem was formulated by Wright [113], and states that for any small enough $\varepsilon > 0$,

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \geq \frac{1}{4} - \varepsilon\right) \underset{g \to +\infty}{\longrightarrow} 1.$$

Note that this conjecture could concern any reasonable probabilistic setting, and the remarkable properties of the Weil-Petersson model (like Wolpert's magic formula [112] and Mirzakhani's integration formula [78]) make it an excellent candidate. Recently, Magee, Naud and Puder [73] have proved that if X is a surface such that $\lambda_1(X) \geq \frac{1}{4}$ (such a surface exists [62]), then for any $\varepsilon > 0$,

$$\mathbb{P}_n^{\mathrm{RC}}\left(\lambda_1(Y) \geq \frac{3}{16} - \varepsilon\right) \underset{n \to +\infty}{\longrightarrow} 1$$

where Y is sampled uniformly amongst the finite number of degree n covers of X. The conjecture with $\frac{1}{4}$ instead of $\frac{3}{16}$ is still open in this setting too.

Short cycles on graphs and surfaces

In spectral theory, when studying large-scale limits ($n \to +\infty$ for a graph, $g \to +\infty$ for a surface), it is important to know that the small-scale geometry of the object will not affect the spectrum. Often, a simple assumption to avoid this is to assume the injectivity radius to be large.

Unfortunately, random regular graphs have an asymptotically non-zero probability of having a small injectivity radius (see [112]). The same occurs with surfaces taken with the Weil-Petersson probability, by work of Mirzakhani [80]. As a consequence, in both cases, if we want to prove results true with probability approaching 1 in the large-scale limit, one needs to impose weaker and more typical geometric conditions.

For instance, Brooks and Lindenstrauss [29] and Brooks and Le Masson [27] studied eigenfunctions on regular graphs of size $n \to +\infty$, under assumptions on the number of cycles up to a certain length L. This parameter L can always be taken to be the injectivity radius, but in the case of random graphs, it can be increased to be of order $\log n$. In a recent article of Gilmore, Le Masson, Sahlsten and Thomas [48], a similar geometric hypothesis on the number of geodesic loops shorter than a scale L based at each point is made, in order to control the L^p-norms of eigenfunctions of the Laplacian on hyperbolic surfaces. The authors prove it holds for random surfaces of high genus g at a scale $L = c \log g$, but the proof provides no explicit value of the constant $c > 0$.

This limitation could be seen as originating from the methodology used to study the geometry of the surfaces. In essence, the authors prove that the loop condition is implied by a geometric condition, which is typical. This condition however is quite complex, and both the proof of its sufficiency and typicality are rather technical, leaving the local geometry of the random surfaces that are selected to remain quite opaque.

It follows from Corollary 4.4.3 that the constant c in [48] can be taken to be any value $< \frac{1}{4}$. In turn, this improves (and makes precise) the rate of convergence of the probability for which the L^p-norm estimates in [48] hold. This is rather demonstrative of the capabilities of the tangle-free geometric condition allowing for a firm grasp over $\log(g)$ scale geometries for spectral theoretic purposes.

Benjamini-Schramm convergence

The notion of Benjamini-Schramm convergence is another way to study spectral properties of graphs and surfaces in the large-scale limit despite the existence of short cycles. In both cases, there is a general definition of Benjamini-Schramm convergence of a sequence to a limiting object [14, 1, 2], but when the limit is the infinite d-regular tree (for graphs) or the hyperbolic plane (for surfaces), the definition is equivalent to a simpler characterisation. A sequence of hyperbolic surfaces $(X_g)_g$ converges to the hyperbolic plane if and only if

$$\forall R > 0, \quad \frac{\mathrm{Vol}(\{z \in X_g \,:\, \mathrm{injrad}_{X_g}(z) < R\})}{\mathrm{Vol}(X_g)} \underset{g \to +\infty}{\longrightarrow} 0,$$

and the definition for graphs is the same, replacing volumes by cardinalities.

Random graphs and surfaces satisfy this property for an R proportional to $\log n$ and $\log g$ respectively, and this has consequences on their eigenvalues and eigenfunctions (see Anantharaman, Le Masson [7] and Anantharaman [5] for graphs, Le Masson, Sahlsten [67] and Monk [83] for surfaces). The notion of Benjamini-Schramm convergence and the tangle-free hypothesis correspond to assuming the objects have few cycles, but in different ways. The former means that the points which are the base of *at least one* short loop are *scarce* on the surface, while the latter implies that *no* point has *more than one* loop. Both approaches can be useful in different settings.

Outline of the paper

The paper is organised as follows:

- Section 4.2: tangled surfaces have tangled geodesics.
- Section 4.3: random surfaces are $(a \log g)$-tangle-free for any $a < 1$.
- Section 4.4: geometric consequences of the tangle-free hypothesis.
- Section 4.5: any surface of genus g is $(4 \log g + O(1))$-tangled.

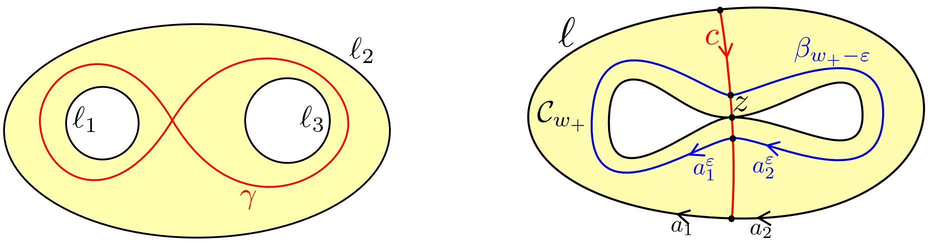
4.2 Tangled surfaces have tangled geodesics

The aim in this section is to prove that being tangled implies having a tangled geodesic - that is to say a *non-simple* closed geodesic of length $\leq 2L + O(1)$.

Proposition 4.2.1. *Let X be a compact hyperbolic surface and $L > 0$. Assume that X is L-tangled. Then, there exists a closed geodesic γ in X of length smaller than $2L + 2\pi$ with one self-intersection.*

The geodesic we construct is what is called a *figure eight.* Any non-simple geodesic on a hyperbolic surface has length greater than $4 \operatorname{arcsinh} 1 \approx 3.52\ldots$, and this result is sharp (see [32, Theorem 4.2.2]).

Proof. It suffices to prove that there is such a geodesic in any pair of pants or one-holed torus of total boundary length smaller than $2L$.



(a) for a pair of pants

(b) for a one-holed torus

Figure 4.1: Construction of a short self-intersecting geodesic.

Let us first consider a hyperbolic pair of pants of boundary lengths ℓ_1, ℓ_2, ℓ_3, such that $\ell_1 + \ell_2 + \ell_3 < 2L$. We construct a closed curve with one self-intersection as represented in Fig. 4.1a. By [32, Formula 4.2.3],

$$\cosh \frac{\ell(\gamma)}{2} = 2 \cosh \frac{\ell_1}{2} \cosh \frac{\ell_3}{2} + \cosh \frac{\ell_2}{2} \leq 3\, e^L.$$

Since $\cosh x \geq \frac{e^x}{2}$, we deduce that the length of γ is smaller than $2L + 2\log 6$.

We use a different proof in the one-holed torus case, because we do not have access to several small geodesics straight away. Let us study a one-holed torus T of boundary length $\ell \leq 2L$. Let $w > 0$, and $\mathcal{C}_w$ be the w-neighborhood of the boundary geodesic

$$\mathcal{C}_w = \{z \in T \,:\, \operatorname{dist}(z, \partial T) < w\}.$$

By the collar theorem [32, Theorem 4.1.1], when w is small enough, $\mathcal{C}_w$ is a half-cylinder with Fermi coordinates (ρ, t), in which the hyperbolic metric is $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \cosh^2\rho\ \mathrm{d}t^2$. This isometry has to break down at some point, because the area of the one-holed torus is 2π, and as long as the isometry holds

$$\mathrm{Vol}(\mathcal{C}_w) = \int_0^\ell \int_0^w \cosh\rho\ \mathrm{d}\rho\ \mathrm{d}t = \ell \sinh w \leq 2\pi. \tag{4.1}$$

We pick w_+ to be the supremum of the widths for which the isometry holds. By continuity, w_+ satisfies inequality (4.1).

The fact that the isometry ceases implies that there is (at least) one self-intersection point z at the boundary of $\mathcal{C}_{w_+}$. By definition, there are two distinct geodesic segments c_1, c_2 of length w_+ from ∂T to z. Furthermore, these segments are orthogonal to the inner boundary $\beta_{w_+} := \partial\mathcal{C}_{w_+} \setminus \partial T$ of the w_+-neighbourhood $\mathcal{C}_{w_+}$. By minimality of w_+, the two tangents of β_{w_+} at z are aligned, and therefore the two segments c_1, c_2 connect to form a geodesic segment c from ∂T to itself.

The regular neighbourhood of ∂T and c is a topological pair of pants, with three boundary components, γ_1, ∂T, γ_2. Neither γ_1 nor γ_2 is contractible because they are freely homotopic to geodesic bigons (c and a portion of ∂T). Then, replacing γ_1 and γ_2 by the closed geodesics $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ in their respective free homotopy classes yields a decomposition of the handle into a pair of pants of boundary components $(\tilde{\gamma}_1, \partial T, \tilde{\gamma}_2)$. Let γ denote the figure-eight geodesic constucted in the pair-of-pants case, which is freely homotopic to $a_1 c a_2^{-1} c$, where a_1 and a_2 are the portions of ∂T delimited by c as represented in Fig. 4.1b. We shall estimate the length of γ.

Let $\varepsilon > 0$. We observe that the portion c_ε of the geodesic segment c outside of $\mathcal{C}_{w_+-\varepsilon}$ is a geodesic segment of length 2ε, connecting two points of $\beta_{w_+-\varepsilon}$. Let a_1^ε, a_2^ε be the two portions of $\beta_{w_+-\varepsilon}$ delimited by c_ε. Then, the loop

$a_1^\varepsilon c_\varepsilon (a_2^\varepsilon)^{-1} c_\varepsilon$ is freely homotopic to $a_1 c a_2^{-1} c$ and hence γ. Its length is equal to

$$\ell(\beta_{w_+ - \varepsilon}) + 4\varepsilon = \ell \cosh(w_+ - \varepsilon) + 4\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \ell \cosh(w_+).$$

By minimality of the geodesic representative in a free homotopy class,

$$\ell(\gamma) \leq \ell \cosh(w_+) = \ell \sqrt{1 + \sinh^2(w_+)} \leq \ell \sqrt{1 + \frac{4\pi^2}{\ell^2}} \leq 2L + 2\pi$$

by equation (4.1), which allows us to conclude. $\square$

4.3 Random surfaces are $(a \log g)$-tangle-free

In this section, we will show that, for any $0 < a < 1$, typical surfaces of genus g are $(a \log g)$-tangle-free. By typical we mean in the probabilistic sense for the Weil-Petersson model of random surfaces. To be precise we shall introduce this model briefly here, a more thorough overview can be found in [59] or [113].

4.3.1 Teichmüller and moduli spaces

For integers g, n such that $2g - 2 + n > 0$, fix a connected and oriented smooth surface $\Sigma_{g,n}$ of genus g and with n numbered boundary components. Let us also fix a length vector $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n_{>0}$. Define the *Teichmüller space* $\mathcal{T}_{g,n}(\ell)$ by

$$\mathcal{T}_{g,n}(\ell) = \left\{ (X, f) : \begin{array}{l} f : \Sigma_{g,n} \to X \text{ diffeomorphism} \\ X \text{ hyperbolic surface} \\ i\text{-th boundary component of length } \ell_i \text{ for } 1 \leq \\ i \leq n \end{array} \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation $(X_1, f_1) \sim (X_2, f_2)$ if and only if there exists an isometry $h : X_1 \to X_2$ such that $f_2 \circ h \circ f_1^{-1} : \Sigma_{g,n} \to \Sigma_{g,n}$ is isotopic to the identity.

The elements of $\mathcal{T}_{g,n}(\ell)$ are surfaces with a *marking*. Many surfaces are isometric, but have a different marking. If one wants to pick a random surface, it is more natural to take it in the *moduli space*

$$\mathcal{M}_{g,n}(\ell) = \left\{ \begin{array}{l} \text{hyperbolic surfaces of genus } g \\ \text{with } n \text{ boundary components} \\ i\text{-th component of length } \ell_i \text{ for } 1 \leq \\ i \leq n \end{array} \right\} \Big/ \{\text{isometry}\}$$

where the quotient is over the set of isometries that preserve the i-th component setwise, for all $i \in \{1, \dots, n\}$. The moduli space can be obtained as a quotient of the Teichmüller space by the action of the mapping class group

$$\mathcal{M}_{g,n}(\ell) = \mathcal{T}_{g,n}(\ell)/\mathrm{MCG}(\Sigma_{g,n}).$$

We recall that $\mathrm{MCG}(\Sigma_{g,n})$ is the group of orientation preserving diffeomorphisms of $\Sigma_{g,n}$ that setwise preserve the boundary components of the surface, up to isotopy, and it acts on the Teichmüller space by precomposition of the marking.

In the case when $n = 0$ (and the surface is compact, with no boundary), we will suppress the mention of n (and the empty vector ℓ), and write Σ_g, $\mathcal{M}_g$, $\mathcal{T}_g$.

4.3.2 The Weil-Petersson probability

The Teichmüller space $\mathcal{T}_{g,n}(\ell)$ possesses a natural symplectic structure, the *Weil-Petersson form* $\omega_{g,n,\ell}^{\mathrm{WP}}$, which is invariant under the action of the mapping class group and therefore descends to the moduli space.

The symplectic form induces a volume form $\mathrm{dVol}_{g,n,\ell}^{\mathrm{WP}} = \frac{1}{N!}(\omega_{g,n,\ell}^{\mathrm{WP}})^{\wedge N}$ for $N = 3g - 3 + n$, called the *Weil-Petersson volume form*. The volume of the moduli space is a finite quantity

$$V_{g,n}(\ell) := \mathrm{Vol}^{\mathrm{WP}}_{g,n,\ell}(\mathcal{M}_{g,n}(\ell)).$$

When $n = 0$ (and the surface is compact, with no boundary), we write $\mathrm{Vol}^{\mathrm{WP}}_g$ and V_g to simplify notations. We will see in the next section why we need to introduce these volumes for surfaces with boundary components, even when we only want to study boundary-free compact surfaces.

We can normalise $\mathrm{Vol}^{\mathrm{WP}}_g$ and obtain the *Weil-Petersson probability measure* $\mathbb{P}^{\mathrm{WP}}_g = \frac{1}{V_g}\mathrm{Vol}^{\mathrm{WP}}_g$ on the moduli space $\mathcal{M}_g$. The Weil-Petersson form can be expressed in Fenchel-Nielsen coordinates thanks to Wolpert's theorem [112]. This geometric expression has deep consequences, and is what ultimately allows for explicit computations in this model.

4.3.3 Mirzakhani's integration formula

In this subsection, we explain how Mirzakhani's integration formula [78] can be used to compute expectations of a certain class of functions known as geometric functions. Knowing how to compute expectations then allows one to estimate the probability of certain events by, for instance, using Markov's inequality $\mathbb{P}(|X| > a) \leq \frac{1}{a}\,\mathbb{E}(|X|)$.

Definition 4.3.1. A *geometric function* is a function $\mathcal{M}_g \to \mathbb{R}$ that can be written as:

$$F^{\Gamma}(X) = \sum_{(\gamma_1,\ldots,\gamma_k)\in\mathcal{O}(\Gamma)} F(\ell_X(\gamma_1), \ldots, \ell_X(\gamma_k)),$$

where:

- $F : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ is a positive measurable function
- Γ is a multi-curve on Σ_g, and $\mathcal{O}(\Gamma)$ is the orbit of Γ under the action by the mapping class group $\mathrm{MCG}(\Sigma_g)$

- for a closed curve γ on Σ_g and $(X, f) \in \mathcal{T}_g$, $\ell_X(\gamma)$ is the length of the unique closed geodesic freely homotopic to the image of γ on X under the marking map f.

Though a fixed term of the sum in the previous definition only really makes sense for an element of the Teichmüller space, the summation over the orbit makes it invariant under the action of the mapping class group, and hence a well-defined function on the moduli space $\mathcal{M}_g$.

The following result is an expression of the integral of any geometric function as an integral over $\mathbb{R}^k_{\geq 0}$. In order to write the formula, we must understand the surface resulting in cutting Σ_g by the curves in Γ. For this, we observe that the cut surface $\Sigma_g \setminus \Gamma$ can be written as the disjoint union $\bigsqcup_{i=1}^q \Sigma_{g_i,n_i}$ of its connected pieces.

The k curves of Γ form $2k$ boundary components of the cut surface. If the multi-curve Γ had lengths $\ell \in \mathbb{R}^k_{\geq 0}$ on X, then these lengths become the boundary lengths of the surface X cut along Γ. Each component Σ_{g_i,n_i} therefore has a length vector $\ell^{(i)} \in \mathbb{R}^{n_i}_{\geq 0}$. We then define

$$V_g(\Gamma, \ell) := \prod_{i=1}^{q} V_{g_i,n_i}(\ell^{(i)}).$$

Mirzakhani's integration formula can then be formulated as follows.

Theorem 4.3.1 ([78])**.** *Given a multi-curve* Γ *and a function* $F : \mathbb{R}^k_{\geq 0} \to \mathbb{R}_+$ *there exists a constant* $0 < C_\Gamma \leq 1$ *dependent only on* Γ *for which*

$$\int_{\mathcal{M}_g} F^\Gamma(X)\, \mathrm{dVol}_g^{\mathrm{WP}}(X) = C_\Gamma \int_{\mathbb{R}^k_{\geq 0}} F(x)\, V_g(\Gamma, \ell)\, \ell_1 \cdots \ell_k \, \mathrm{d}\ell_1 \cdots \mathrm{d}\ell_k.$$

4.3.4 Volume estimates

The previous formula indicates that in order to estimate expectations, we need to understand the asymptotic behaviour of Weil-Petersson volumes. In our proof, we will only use a handful of them, grouped in the following Lemma.

Lemma 4.3.2 (Lemmas 3.2 and 3.3 [80]). *Given $g, n \geq 0$ such that $2g-2+n > 0$,*

1. $\ell_1 \dots \ell_n V_{g,n}(\ell_1, \dots, \ell_n) \leq 2^n \prod_{i=1}^n \sinh\left(\frac{\ell_i}{2}\right) V_{g,n}$,
2. $V_{g,n+2} \leq V_{g+1,n}$,
3. *there exists a constant C independent of g and n such that*
$$V_{g,n} \leq C \frac{V_{g,n+1}}{2g-2+n},$$
4. *there exists a constant C_n independent of g such that for any integers n_1, n_2 satisfying $n_1 + n_2 = n$,*
$$\sum_{g_1+g_2=g} V_{g_1,n_1+1} V_{g_2,n_2+1} \leq C_n \frac{V_{g,n}}{g}.$$

4.3.5 Probabilistic result

We can now state and prove our probabilistic result.

Theorem 4.3.2. *For any real number $0 < a < 1$,*

$$\mathbb{P}_g^{\mathrm{WP}} \left(X \textit{ is } (a \log g)\textit{-tangle-free} \right) = 1 - O\left(\frac{(\log g)^2}{g^{1-a}} \right).$$

Proof. Let us list all the topological types of embedded one-holed tori or pair of pants in a genus g surface (see 4.2):

(i) a curve separating a one-holed torus;

(ii) three curves cutting Σ_g into a pair of pants and a component $\Sigma_{g-2,3}$;

(iii) three curves cutting Σ_g into a pair of pants and two components $\Sigma_{g_1,1}$ and $\Sigma_{g_2,2}$ such that $g_1 + g_2 = g - 1$;

(iv) three curves cutting Σ_g into a pair of pants and three connected components $\Sigma_{g_1,1}$, $\Sigma_{g_2,1}$ and $\Sigma_{g_3,1}$ with $1 \leq g_1 \leq g_2 \leq g_3$ and $g_1 + g_2 + g_3 = g$.

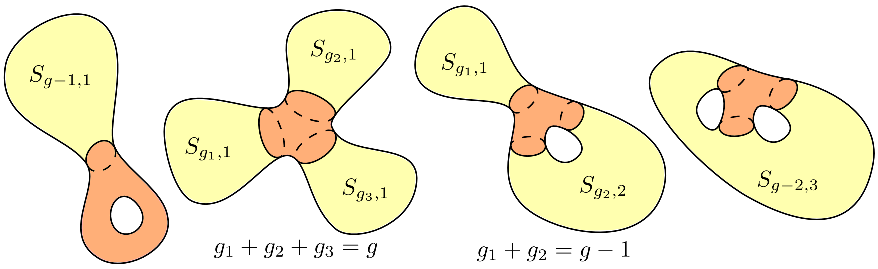


Figure 4.2: The different topological ways to embed a one-holed torus or pair of pants in a surface of genus g.

For any topological situation, we will consider a multicurve α on the base surface Σ_g realising the topological configuration and study the counting function

$$N_L^\alpha(X) = \#\{\beta \in \mathcal{O}(\alpha) \ : \ \ell_X(\beta) \leq 2L\},$$

where the length of a multicurve is defined as the sum of the lengths of its components. Then, the probability of finding a component in the topological situation α of total boundary length $\leq 2L$ can be bounded by Markov's inequality:

$$\mathbb{P}_g^{\mathrm{WP}}\left(N_L^\alpha(X) \geq 1\right) \leq \mathbb{E}_g N_L^\alpha(X).$$

We observe that $N_L^\alpha(X)$ is a geometric function, and its expectation can therefore be computed using Mirzakhani's integration formula (4.3.1). This reduces the problem to estimating integrals with Weil-Petersson volumes, which we will now detail.

In case (i), the integral that appears is

$$\int_0^{2L} V_{1,1}(\ell) V_{g-1,1}(\ell)\, \ell \, \mathrm{d}\ell.$$

From [85], it is known that $V_{1,1}(\ell) = \frac{\ell^2}{24} + \frac{\pi^2}{6}$. Moreover, by Lemma 4.3.2,

$$\ell V_{g-1,1}(\ell) \leq 2e^{\frac{\ell}{2}} V_{g-1,1}.$$

It follows that the probability is smaller than

$$\frac{V_{g-1,1}}{V_g} \int_0^{2L} 2\left(\frac{\ell^2}{24} + \frac{\pi^2}{6}\right) e^{\frac{\ell}{2}}\, \mathrm{d}\ell = O\left(\frac{V_{g-1,1}}{V_g} L^2 e^L\right) = O\left(\frac{(\log g)^2}{g^{1-a}}\right)$$

where the last bound is deduced from Lemma 4.3.2 parts (2) and (3) and taking $L = a \log g$.

In case (ii), the integral that appears is

$$\frac{1}{V_g} \iiint_{0 \leq \ell_1+\ell_2+\ell_3 \leq 2L} V_{0,3}(\ell_1, \ell_2, \ell_3) V_{g-2,3}(\ell_1, \ell_2, \ell_3)\, \ell_1 \ell_2 \ell_3\, \mathrm{d}\ell_1\, \mathrm{d}\ell_2\, \mathrm{d}\ell_3.$$

Due to the fact that $V_{0,3}(\ell_1, \ell_2, \ell_3) = 1$ and by Lemma 4.3.2(1), we need to estimate

$$\frac{V_{g-2,3}}{V_g} \iiint_{0 \leq \ell_1+\ell_2+\ell_3 \leq 2L} \exp\left(\frac{\ell_1+\ell_2+\ell_3}{2}\right) \mathrm{d}\ell_1\, \mathrm{d}\ell_2\, \mathrm{d}\ell_3 = O\left(\frac{(\log g)^2}{g^{1-a}}\right)$$

by Lemma 4.3.2 (2-3).

Let us now bound the sum of all the topological situations of case (iii). By the same manipulations, we obtain that the probability is

$$O\left(\frac{L^2 e^L}{V_g} \sum_{g_1+g_2=g-1} V_{g_1,1} V_{g_2,2}\right) = O\left(\frac{(\log g)^2}{g^{1-a}} \frac{V_{g-1,1}}{V_g}\right) = O\left(\frac{(\log g)^2}{g^{2-a}}\right)$$

by Lemma 4.3.2(4) and then Lemma 4.3.2(2-3).

Finally, in the last case we have to estimate

$$\sum_{\substack{g_1+g_2+g_3=g \\ 1 \leq g_1 \leq g_2 \leq g_3}} V_{g_1,1} V_{g_2,1} V_{g_3,1}$$

$$= \sum_{g_1=1}^{\lfloor \frac{g-2}{3} \rfloor} V_{g_1,1} \sum_{g_2+g_3=g-g_1} V_{g_2,1}V_{g_3,1} \leq C_0 \sum_{g_1=1}^{\lfloor \frac{g-2}{3} \rfloor} \frac{V_{g_1,1}V_{g-g_1,0}}{g-g_1}$$

where C_0 is the constant from Lemma 4.3.2(4). We observe that $g - g_1 \geq \frac{2}{3}g$ and use Lemma 4.3.2(3) to conclude that the probability is

$$O\left(\frac{(\log g)^2}{V_g g^{2-a}} \sum_{g_1=1}^{\lfloor \frac{g-2}{3} \rfloor} V_{g_1,1}V_{g-g_1,1} \right) = O\left(\frac{(\log g)^2}{g^{3-a}} \right)$$

by Lemma 4.3.2(4). □

Remark. In the cases (i), (iii) and (iv), there is a separating geodesic of length $\leq 2a \log g$. Therefore, we could have bounded these probabilities by the probability of having a separating geodesic of length $\leq 2a \log g$, which has been estimated by Mirzakhani in [80, Theorem 4.4]. This approach yields the same end result, but the authors decided to detail the four cases for the sake of self-containment. Furthermore, this more detailed study allows us to see that the most likely cases are cases (i.) and (ii.), and therefore we expect the first length at which the surface is tangled to be obtained by one of these two topological situations.

4.4 Geometry of tangle-free surfaces

The aim of this section is to provide information about geodesics and neighbourhoods of points on tangle-free surfaces. The results will be expressed in terms of an arbitrary L-tangle-free surface X, but can also been seen as result that are true with high probability for $L = a \log g$, $a < 1$ due to Theorem 4.3.2.

4.4.1 An improved collar theorem

Theorem 4.4.1. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Let γ be a simple closed geodesic of length $\ell < L$. Then, for $w := \frac{L-\ell}{2}$, the* *neighbourhood*

$$\mathcal{C}_w(\gamma) = \{z \in X \,:\, \mathrm{dist}(z,\gamma) < w\}$$

is isometric to a cylinder.

The collar theorem [32] is a similar result, with the width $\operatorname{arcsinh}\big(\sinh(\ell/2)^{-1}\big)$.

We recall that, in the random case, for $a < 1$, with high probability, we can take $L = a \log g$. This result therefore is a significant improvement for geodesics of length $b \log g$, $0 < b < a$. We obtain a collar of width $w = \frac{a-b}{2} \log g$, which is expanding with the genus, as opposed to the deterministic collar, of width $\simeq g^{-\frac{b}{2}}$.

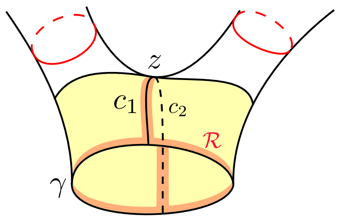
For very short geodesics, the width of this new collar is $\simeq \frac{a}{2} \log g$. It might seem less good than the deterministic collar, which is of width $\simeq -\log(\ell)$. However, by Theorem 4.2 in [80], the injectivity radius of a random surface is greater than $g^{-\frac{a}{2}}$ with probability $1 - O(g^{-a})$. Under this additional probabilistic assumption, the two collars are of similar sizes.

Proof. For small enough w, the neighbourhood $\mathcal{C}_w(\gamma)$ is a cylinder, with two boundary components $\gamma_w^{\pm}$. Let us assume that, for a certain w, the topology of the neighbourhood changes. There are two ways for this to happen (and both can happen simultaneously) – see Fig. 4.3.

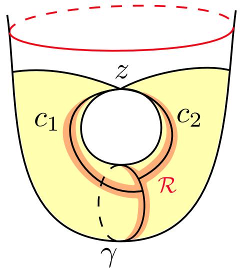
(A) One boundary component, γ_w^+ or γ_w^-, self-intersects.

(B) The two boundary components γ_w^+ and γ_w^- intersect one another.

In both cases, let $z \in X$ denote one intersection point. Since the distance between z and γ is w, there are two distinct geodesic arcs c_1, c_2 of length w, going from z to points of γ, and intersecting γ perpendicularly. Both c_1 and c_2 are orthogonal to the boundaries of the cylinder and the two boundaries are tangent to one another by minimality of the width w. As a consequence, the curve $c = c_1^{-1} c_2$ is a geodesic arc.



(a) one side self-intersects



(b) the two sides intersect one another

Figure 4.3: Illustration of the ways the isometry breaks down when expanding a cylinder around the geodesic γ.

The regular neighbourhood of the curves γ and c has Euler characteristic -1. There are two possible topologies for this neighbourhood.

- If it is a pair of pants, then it has three boundary components. Neither of them is contractible on the surface X. Indeed, one component is freely homotopic to γ, and the two others to c and a portion of γ, which are geodesic bigons. Therefore, when we replace the boundary components of the regular neighbourhood by the closed geodesic in their free homotopy classes, we obtain a pair of pants or a one-holed torus (if two of the boundary components are freely homotopic to one another), of total boundary length smaller than $2\ell + 4w$.

- Otherwise, it is a one-holed torus. Its boundary component is not contractible, because there is no hyperbolic surface of signature $(1, 0)$. Therefore, the closed geodesic in its free homotopy class separates a one-holed torus with boundary length smaller than $2\ell + 4w$ from X.

In both cases, by the tangle-free hypothesis, $2L < 2\ell + 4w$, which allows us to conclude. □

Remark. Let $\mathcal{A}_g \subset \mathcal{M}_g$ be the event "the surface has a simple closed geodesic of length between 1 and 2". By work of Mirzakhani and Petri [81],

$$\mathbb{P}_g^{\mathrm{WP}}(\mathcal{A}_g) \underset{g \to +\infty}{\longrightarrow} 1 - \exp\left(-\int_1^2 \frac{e^t + e^{-t} - 2}{2t} \, \mathrm{d}t\right) > 0,$$

so this event has asymptotically non-zero probability.

Let X be an element of $\mathcal{A}_g$ which is also $(a \log g)$-tangle-free, and let γ be a closed geodesic on X of length $\ell \in [1,2]$. Then, the collar $\mathcal{C}_w(\gamma)$ given by Theorem 4.4.1 has volume

$$\mathrm{Vol}(\mathcal{C}_w(\gamma)) = 2\ell \sinh w \geq 2 \sinh\Big(\frac{a}{2} \log g - 1\Big) \sim g^{\frac{a}{2}} \quad \text{as } g \to +\infty.$$

However, $\mathrm{Vol}(\mathcal{C}_w(\gamma)) \leq \mathrm{Vol}\, X = 2\pi(2g-2)$. This leads to a contradiction for g approaching $+\infty$ as soon as $a > 2$. Hence, for large g, the elements of $\mathcal{A}_g$ are not $(a \log g)$-tangle-free for $a > 2$:

$$\limsup_{g \to +\infty} \mathbb{P}_g^{\mathrm{WP}} \left(X \text{ is } (a \log g)\text{-tangled} \right) \geq \lim_{g \to +\infty} \mathbb{P}_g^{\mathrm{WP}} \left(\mathcal{A}_g \right) > 0.$$

Therefore, for all $a > 2$, random surfaces do *not* have high probability of being $(a \log g)$-tangle-free.

By taking a close to but larger than 1, this same line of reasoning and the fact that we know surfaces to be $(a \log g)$-tangle-free with high probability implies that the improved collar cannot be much larger than $L - \ell$. As a consequence, our result is optimal up to multiplication by 2.

4.4.2 Number of intersections of geodesics

A consequence of this improved collar theorem is a bound on the number of intersections of a short closed geodesic with any other geodesic.

Corollary 4.4.1. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface.*

Let γ be a simple closed geodesic of length $< L$ on X. Then, for any geodesic γ' transverse to γ, the number of intersections $i(\gamma, \gamma')$ between γ and γ' satisfies

$$i(\gamma, \gamma') \leq \frac{\ell(\gamma')}{L - \ell(\gamma)} + 1.$$

In the case where γ' is also closed, then

$$i(\gamma,\gamma') \leq \frac{\ell(\gamma')}{L-\ell(\gamma)}.$$

In particular, if $\ell(\gamma)+\ell(\gamma') < L$, then γ and γ' do not intersect.

Proof. By Theorem 4.4.1, γ is embedded in an open cylinder $\mathcal{C}$ of width $w = \frac{L-\ell(\gamma)}{2}$.

Let us parametrize the geodesic $\gamma' : [0,1] \to X$. The set of times when γ' visits the cylinder can be decomposed as

$$\bigsqcup_{i=1}^{k} (t_i^-, t_i^+), \qquad 0 \leq t_1^- < t_1^+ \leq \ldots \leq t_k^- < t_k^+ \leq 1,$$

as respresented in Fig. 4.4. The restriction c_i of γ' between t_i^- and t_i^+ is a

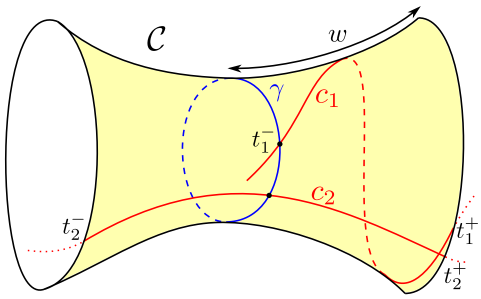


Figure 4.4: Illustration of the proof of Corollary 4.4.1.

geodesic in the cylinder $\mathcal{C}$, transverse to the central geodesic γ. Therefore, if c_i intersects γ, then it does at most once. Let $I \subset \{1,\ldots,k\}$ be the set of i such that c_i intersect γ. We have that $i(\gamma,\gamma') = \#I \leq k$.

We assume that $\#I \geq 2$ (otherwise there is nothing to prove). Any geodesic intersecting the central geodesic transversally travels through the entire cylinder, and is therefore of length greater than $2w$. As a consequence, for any $i \in I$ different from 1 and k, $\ell(c_i) \geq 2w$. Also, if $i = 1$ or k belongs in I, then

$\ell(c_i) \geq w$. This leads to our claim, because

$$(i(\gamma,\gamma')-1)(L-\ell(\gamma)) = (\#I-1)\cdot 2w \leq \sum_{i\in I}\ell(c_i) \leq \ell(\gamma').$$

The case when the curve γ' is closed can be obtained observing that, in this case, $\ell(c_1)$ and $\ell(c_k)$ also are greater than $2w$ (when 1 or k belongs in I). $\square$

Like the collars from the usual collar theorem, the collars of two small enough distinct geodesics are disjoint.

Proposition 4.4.2. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Let γ, γ' be two distinct simple closed geodesics such that $\ell(\gamma) + \ell(\gamma') < L$. Then, the distance between γ and γ' is greater than $L - \ell(\gamma) - \ell(\gamma')$.*

In particular, if $\ell(\gamma), \ell(\gamma') < \frac{L}{2}$, then the collars of width $\frac{L}{2} - \ell(\gamma)$ around γ and $\frac{L}{2} - \ell(\gamma')$ around γ' are two disjoint embedded cylinders.

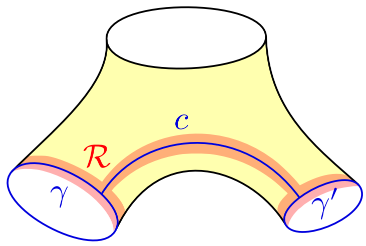


Figure 4.5: Illustration of the proof of Proposition 4.4.2

Proof. We already know, owed to Corollary 4.4.1, that γ and γ' do not intersect. Let c be a length-minimising curve with one endpoint on γ and the other on γ' (see Fig. 4.5). Then, by minimality, c is simple and only intersects γ and γ' at is endpoints. The regular neighbourhood $\mathcal{R}$ of γ, γ' and c is a topological pair of pants of total boundary length less than $2(\ell(\gamma) + \ell(\gamma') + \ell(c))$. Since γ and γ' are non-contractible and not freely homotopic to one another, the third boundary component is not contractible and $\mathcal{R}$ corresponds to an embedded pair of pants or one-holed torus on X. By the tangle-free hypothesis,

$\ell(\gamma) + \ell(\gamma') + \ell(c) \geq L$, and therefore the distance between γ and γ' is greater than $L - \ell(\gamma) - \ell(\gamma')$. This implies our claim. □

4.4.3 Short loops based at a point

Let us now study short loops based at a point on a tangle-free surface.

Theorem 4.4.2. *Let $L > 0$, and X be an L-tangle-free hyperbolic surface. Let $z \in X$, and let δ_z be a shortest geodesic loop based at z. If β is a (not necessarily geodesic) loop based at z, such that $\ell(\beta) + \ell(\delta_z) < L$ then β is homotopic with fixed endpoints to a power of δ_z.*

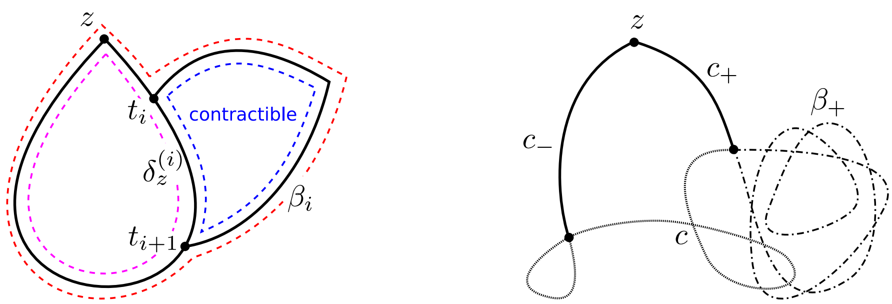
The result is empty if the injectivity radius of the point z is greater than $\frac{L}{2}$. The "shortest geodesic loop" δ_z is not necessarily unique. It will be as soon as the injectivity radius at z is smaller than $\frac{L}{4}$. More precisely, we directly deduce from Theorem 4.4.2 the following corollary, which was used in [48] for random surfaces (with a length $L = a \log g$, but the value of a was not explicit). Note the similarity of this result to the classical Margulis lemma [94]. In particular, we obtain an explicit constant for the Margulis lemma in the case of tangle-free surfaces in the same way that the classical collar theorem provides.

Corollary 4.4.3. *Let $L > 0$, and $X = \Gamma \backslash \mathbb{H}$ be an L-tangle-free hyperbolic surface. Then, for any $z \in \mathbb{H}$, the set $\{T \in \Gamma \ : \ \mathrm{dist}_{\mathbb{H}}(z, T \cdot z) < \frac{L}{2}\}$ is:*

- *reduced to the identity element (when the injectivity radius at z is $\geq \frac{L}{4}$),*
- *or included in the subgroup $\langle T_0 \rangle$ generated by the element $T_0 \in \Gamma$ corresponding to the shortest geodesic loop through z.*

We recall that any compact hyperbolic surface is isometric to a quotient of the hyperbolic plane $\mathbb{H}$ by a Fuchsian co-compact group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ – see [64] for more details.

We could prove Theorem 4.4.2 using the same method as we used for Theorem 4.4.1 and Corollary 4.4.1, expanding a cylinder around δ_z. However, our initial proof used a different method, which we decided to present here, in order to expose different ways to use the tangle-free hypothesis.



(a) Case $k = 0$. (b) Case $k > 0$.

Figure 4.6: Illustrations of the proof of Theorem 4.4.2.

Proof of Theorem 4.4.2. By replacing β by a new curve in its homotopy class, we can assume that β has a finite number of self-intersections, and of intersections with δ_z, while still satisfying the length condition.

We now prove this result by induction on the number of self-intersections $k \geq 0$ of β. We start with the base case of $k = 0$ so that β is simple. We parametrise $\beta : [0, 1] \to X$. Let $0 = t_0 < t_1 < \ldots < t_I = 1$ be the times when β meets δ_z.

Let $0 \leq i < I$, and β_i be the restriction of β to $[t_i, t_{i+1}]$ – see Fig. 4.6a. Then, the regular neighbourhood $\mathcal{R}$ of δ_z and β_i has Euler characteristic -1, and total boundary length $\leq 2(\ell(\delta_z) + \ell(\beta_i)) < 2L$. If $\mathcal{R}$ is a topological one-holed torus, then by the tangle-free hypothesis, its boundary component is contractible, which is impossible for there is no hyperbolic surface of signature $(1, 0)$.

Therefore, $\mathcal{R}$ is a topological pair of pants. By the tangle-free hypothesis, one of its boundary components is contractible. It can not be the component corresponding to δ_z, so it is another one. Hence, β_i is homotopic with fixed endpoints to a portion $\delta_z^{(i)}$ of δ_z.

As a consequence, $\beta = \beta_0 \ldots \beta_{I-1}$ is homotopic with fixed endpoints to the product

$$c = \delta_z^{(0)} \delta_z^{(1)} \dots \delta_z^{(I-1)}.$$

c goes from z to z following only portions of δ_z. Therefore, c is homotopic with fixed endpoints to a power δ_z^j of δ_z.

We now move forward to the case $k > 0$. We assume the result to hold for any smaller k. The idea is to find a way to cut β into smaller loops on which to apply the induction hypothesis; the construction is represented in Fig. 4.6b.

Let $\ell = \ell(\beta)$. We pick a length parametrisation of $\beta : \mathbb{R}/\ell\mathbb{Z} \to X$ such that $\beta(0) = z$. We look for the first intersection point of β, starting a 0, but looking in both directions:

$$\ell_+ = \min\{t \geq 0 \,:\, \exists s \in (t, \ell) \text{ such that } \beta(s) = \beta(t)\}$$
$$\ell_- = \min\{t \geq 0 \,:\, \exists s \in (t, \ell) \text{ such that } \beta(-s) = \beta(-t)\}.$$

Up to a change of orientation of β, we can assume that $\ell_+ \leq \ell_-$. Then, we set

$$t = \max\{s \in (\ell_+, \ell) \,:\, \beta(s) = \beta(\ell_+)\}$$

to be the last time at which β visits $\beta(\ell_+)$, so that the restriction of β to $[\ell_+, t]$ is a loop β_+. The curve has no self-intersection between $\ell - \ell_-$ and ℓ, so $t \leq \ell - \ell_-$. Then, if we denote by c_+, c and c_- the respective restrictions of β to $[0, \ell_+]$, $[t, \ell - \ell_-]$ and $[\ell - \ell_-, \ell]$, we can write $\beta = c_+ \, \beta_+ \, c \, c_-$, which is homotopic with fixed endpoints to $(c_+ \, \beta_+ \, c_+^{-1})\,(c_+ \, c \, c_-)$.

Let us apply the induction hypothesis to the two loops $c_+ \, \beta_+ \, c_+^{-1}$ and $c_+ \, c \, c_-$. It will follow that they, and hence β, are homotopic with fixed endpoints to a power of δ_z.

β_+ is a sub-loop of β. As a consequence, $c_+ \, c \, c_-$ has less self-intersections than β, and hence strictly less than k. Furthermore, it is shorter, so it satisfies the length hypothesis $\ell(c_+ \, c \, c_-) + \ell(\delta_z) < L$. So we can apply the induction hypothesis.

c_+ is simple and does not intersect β_+ (except at its endpoint). As a consequence, we can find a curve b homotopic to $c_+ \, \beta_+ \, c_+^{-1}$ with as many self-intersections as β_+. β_+ is a strict sub-loop of β, so this intersection number is strictly smaller than k. The length of b can be taken as close as desired to that of $c_+ \, \beta_+ \, c_+^{-1}$. Moreover,

$$\ell(c_+ \, \beta_+ \, c_+^{-1}) = 2\ell_+ + \ell(\beta_+) \leq \ell_+ + \ell_- + \ell(\beta_+) \leq \ell(\beta)$$

so b can be chosen to satisfy the length hypothesis $\ell(\delta_z) + \ell(b) < L$, and we can apply the induction hypothesis to it. □

4.4.4 Neighbourhood of a point and graph definition

Now that we know about short loops based at a point, we can understand the geometry (and topology) of balls on a tangle-free surface.

Proposition 4.4.4. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. For a point z in X, let $\mathcal{B}_{\frac{L}{8}}(z) := \left\{ w \in X \, : \, \mathrm{dist}_X(z, w) < \frac{L}{8} \right\}$. Then, $\mathcal{B}_{\frac{L}{8}}(z)$ is isometric to a ball in either the hyperbolic plane (whenever the injectivity radius at z is $\geq \frac{L}{8}$) or a hyperbolic cylinder.*

In the second case, since the injectivity radius at z is greater than $\frac{L}{8}$, the ball $\mathcal{B}_{\frac{L}{8}}(z)$ is not contractible on X; it is therefore homeomorphic to a cylinder (see Fig. 4.7).

In a sense, this corollary proves that our notion of tangle-free implies the natural translation of the notion of tangle-free for graphs. Indeed, the ball $\mathcal{B}_{\frac{L}{8}}(z)$ has either no non-contractible geodesic loop, or only one (and its iterates). We could have picked Proposition 4.4.4 to be a definition for tangle-free, but we consider the pair of pants definition to be both convenient to use and natural in the context of hyperbolic geometry and the Weil-Petersson model.

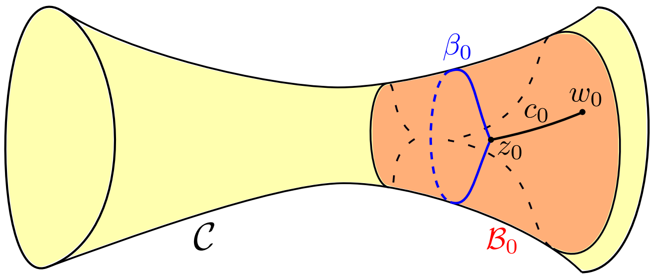


Figure 4.7: Illustration of the proof of Proposition 4.4.4 in the cylinder $\mathcal{C}$. Neighbourhoods of points of small injectivity radius on a tangle-free surface are isometric to balls in cylinders, like $\mathcal{B}_0$.

Proof. In order to prove this result, we will work in the universal cover $\mathbb{H}$ of X. Let us write $X = \Gamma \backslash \mathbb{H}$, for a co-compact Fuchsian group Γ.

Let z be a point on X of injectivity radius smaller than $\frac{L}{8}$ (otherwise, the conclusion is immediate). Then, the shortest geodesic loop β based at z satisfies $\ell(\beta) < \frac{L}{4}$.

Let $\tilde{z} \in \mathbb{H}$ be a lift of z, $\tilde{\beta}$ be a lift of β starting at $\tilde{z}$, and $\tilde{\mathcal{B}}$ be the ball of radius $\frac{L}{8}$ around $\tilde{z}$ in $\mathbb{H}$. Let $T_\beta \in \Gamma$ be the covering transformation corresponding to β. The quotient $\mathcal{C} = \mathbb{H} \backslash \langle T_\beta \rangle$ is a hyperbolic cylinder. The ball $\tilde{\mathcal{B}}$ is projected on a ball $\mathcal{B}_0$ on $\mathcal{C}$. Let us prove that the projection from $\mathcal{B}_0$ on $\mathcal{C}$ to $\mathcal{B}$ on X is an isometry.

In order to do so, we shall establish that for any $\tilde{w} \in \tilde{\mathcal{B}}$, the set of transformations $T \in \Gamma$ such that $T \cdot \tilde{w} \in \tilde{\mathcal{B}}$ is included in $\langle T_\beta \rangle$. Since any two points in $\tilde{\mathcal{B}}$ are at a distance at most $\frac{L}{4} < \frac{L}{2}$, this will follow from proving

$$\Gamma_L(\tilde{w}) := \left\{ T \in \Gamma \ : \ \mathrm{dist}_{\mathbb{H}}(\tilde{w}, T \cdot \tilde{w}) < \frac{L}{2} \right\} \subset \langle T_\beta \rangle.$$

Let c be the shortest path from $\tilde{w}$ to $\tilde{z}$. The path $c\,\tilde{\beta}\,(T_\beta \circ c^{-1})$ is a path from $\tilde{w}$ to $T_\beta \cdot \tilde{w}$. Its length is $2\ell(c) + \ell(\beta) < 2 \times \frac{L}{8} + \frac{L}{4} = \frac{L}{2}$. As a consequence, T_β belongs in $\Gamma_L(\tilde{w})$. Then, $\Gamma_L(\tilde{w})$ is not reduced to $\{\mathrm{id}\}$. By Corollary 4.4.3, it is included in a cyclic subgroup $\langle T_0 \rangle$. T_β hence is a power of T_0, but T_β is primitive. Therefore, $T_\beta = T_0^{\pm 1}$, and the conclusion follows. $\square$

4.4.5 Short geodesics are simple

Corollary 4.4.5. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Any primitive closed geodesic on X of length $< L$ is simple.*

This consequence of Theorem 4.4.2 can also be deduced from the fact that the shortest non-simple primitive closed geodesic on a compact hyperbolic surface is a figure eight geodesic [32, Theorem 4.2.4], which is embedded in a pair of pants or one-holed torus.

Proof. Let us assume by contradiction that γ is not simple; we can then pick an intersection point z. This allows us to write γ as the product of two geodesic loops γ_1, γ_2 based at z. Since $\ell(\gamma_1) + \ell(\gamma_2) < L$, one of them is $< L/2$. Up to a change of notation, we take it to be γ_1.

Let δ_z be the shortest geodesic loop based at z. By definition, $\ell(\delta_z) \leq \ell(\gamma_1)$. So γ_1 and γ_2 both satisfy the length hypothesis of Theorem 4.4.2:

$$\ell(\gamma_1) + \ell(\delta_z) \leq 2\ell(\gamma_1) < L$$

$$\ell(\gamma_2) + \ell(\delta_z) \leq \ell(\gamma) < L.$$

Therefore, they are both homotopic with fixed endpoints to powers of δ_z, which implies γ is too. So γ is freely homotopic to a power j of the simple closed geodesic γ_0 in the free homotopy class of δ_z. By uniqueness, $\gamma = \gamma_0^j$. γ is primitive, so $j = 0$ or 1. But γ is not contractible (so $j \neq 0$) and not simple (so $j \neq 1$): we reach a contradiction, which allows us to conclude. $\square$

Remark. Put together, Corollary 4.4.5 and 4.4.1 imply that all primitive closed geodesics of length $< \frac{L}{2}$ are simple and disjoint. Any such family of curves has cardinality at most $2g - 2$. But we know that the number of primitive closed geodesics of length $< \frac{L}{2}$ on a fixed surface is equivalent to $\frac{2}{L} e^{\frac{L}{2}}$ as $L \to +\infty$

[58, 32]. This can be seen as another indicator that, if X is L-tangle-free of large genus, then we expect L to be at most logarithmic in g.

4.5 Any surface of genus g is $(4 \log g + O(1))$-tangled

We recall that any surface is L-tangled for $L = \frac{3}{2}\mathcal{B}_g$, the Bers constant, because it can be entirely decomposed in pairs of pants of maximal boundary length smaller than $\mathcal{B}_g$. The best known estimates on the Bers constant $\mathcal{B}_g$ are linear in the genus g [33, 91], which is pretty far off the $c \log g$ we obtained for random surfaces. This is not a surprise, because in order to prove that a surface is tangled, we only need to find one embedded pair of pants or one-holed torus. In Buser and Parlier's estimates on $\mathcal{B}_g$ [32, 91], the pair of pants decomposition is constructed by successively exhibiting short curves on the surface; the first ones are of length $\simeq \log g$, but as the construction goes on, and we find $2g-2$ curves to entirely cut the surface, a linear factor appears.

In our case, we only need to stop the construction as soon as we manage to separate a pair of pants. Following Parlier's approach in [91] to bound the Bers constant, we prove the following.

Proposition 4.5.1. *There exists a constant $C > 0$ such that, for any $g \geq 2$, any compact hyperbolic surface of genus g is X is L-tangled for $L = 4 \log g + C$.*

This goes to prove that random hyperbolic surfaces are almost optimally tangle-free, despite the possibility of having a small injectivity radius.

The proof relies on the following two Lemmas, which are all used by Parlier [91]. Lemma 4.5.2, due to Bavard [13], allows us to find a small geodesic loop on our surface.

Lemma 4.5.2. *Let X be a hyperbolic surface of genus g. For any $z \in X$, the* *length of the shortest geodesic loop through z is smaller than*

$$2\operatorname{arccosh}\left(\frac{1}{2\sin\frac{\pi}{12g-6}}\right) = 2\log g + O(1).$$

Some problems will arise in the proof if the geodesic loop we obtain using this result is too small. These difficulties can be solved by assuming a lower bound on the injectivity radius of the surface; for instance, for random surfaces, with high probability, one can assume that the injectivity radius is bounded below by $g^{-\varepsilon}$ for a $\varepsilon > 0$ [80]. However, such an assumption makes the final inequality weaker.

Another way to fix this issue, used in [91], is to expand all the small geodesics, and by this process obtain a new surface, with an injectivity radius bounded below, and in which the lengths of all the curves are longer. For our purposes, we only need to expand one curve. This is achieved by the following Lemma.

Lemma 4.5.3 (Theorem 3.2 in [90])**.** *Let $\Sigma_{g,n}$ be a base surface with $n > 0$ boundary components. Let $(X, f) \in \mathcal{T}_{g,n}(\ell_1, \dots, \ell_n)$ and $\varepsilon_1, \dots, \varepsilon_n \geq 0$. Then, there exists a marked hyperbolic surface $(\tilde{X}, \tilde{f})$ in $\mathcal{T}_{g,n}(\ell_1+\varepsilon_1, \dots, \ell_n+\varepsilon_n)$ such that, for any closed curve c on the base surface $\Sigma_{g,n}$, $\ell_X(c) \leq \ell_{\tilde{X}}(c)$.*

We are now able to prove the result.

Proof. Let γ be the systole of X which is necessarily simple. We cut the surface X along this curve, and obtain a (possibly disconnected) hyperbolic surface X_{cut} with two boundary components. By the extension Lemma (applied to both components separately if need be), there exists a surface X_{cut}^+ such that:

- the boundary components β_1, β_2 in X_{cut}^+ are of length $1 \leq \ell \leq 2\log g + O(1)$.

- for any closed curve c not intersecting γ, $\ell_{X_{\mathrm{cut}}}(c) \leq \ell_{X_{\mathrm{cut}}^+}(c)$.

We shall find a pair of pants in X^+_{cut}, and use the relationship between lengths in X and X^+_{cut} to conclude.

For $w > 0$, let us consider the w-neighbourhood of one component β_1 of the boundary of X^+_{cut}

$$\mathcal{C}_w(\beta_1) = \{z \in X^+_{\mathrm{cut}} \, : \, \mathrm{dist}(z, \beta_1) < w\}.$$

For small enough w, $\mathcal{C}_w(\beta_1)$ is a half-cylinder. However, there is a w at which this isometry stops. This w can be bounded by a volume argument: as long as $\mathcal{C}_w$ is a half-cylinder,

$$\mathrm{Vol}(\mathcal{C}_w(\beta_1)) = \ell \sinh w \leq \mathrm{Vol}\, X = 2\pi(2g-2).$$

However, $\ell \geq 1$. It follows that $w \leq \log g + O(1)$.

There are two reasons for this neighbourhood to cease being isometric to a cylinder.

- The half-cylinder self-intersects inside the surface (see Fig. 4.3a). Then, one can construct an embedded pair of pants on X^+_{cut}, of total boundary length $\leq 2\ell + 4w$. This pair of pants will also be one on X, with shorter boundary components.

- The half-cylinder reaches the boundary of X^+_{cut}. It can only do so by intersecting the component β_2. Then, one can construct an embedded pair of pants on X^+_{cut} of boundaries shorter than ℓ, ℓ, and $2\ell + 2w$, which corresponds to a one-holed torus on X, of boundary shorter than $2\ell + 2w$ (see Fig. 4.3b, but expanding only a half-cylinder).

We can conclude that the surface X is L-tangled, for $L = \ell + 2w \leq 4 \log g + O(1)$. $\square$

5 Delocalisation of Eigenfunctions on Large Genus Random Surfaces

Abstract

We prove that eigenfunctions of the Laplacian on a compact hyperbolic surface delocalise in terms of a geometric parameter dependent upon the number of short closed geodesics on the surface. In particular, we show that an L^2 normalised eigenfunction restricted to a measurable subset of the surface has squared L^2-norm $\varepsilon > 0$, only if the set has a relatively large size – exponential in the geometric parameter. For random surfaces with respect to the Weil-Petersson probability measure, we then show, with high probability as $g \to \infty$, that the size of the set must be at least the genus of the surface to some power dependent upon the eigenvalue and ε.

5.1 Introduction

5.1.1 Background

The study of the Laplacian operator $\Delta = -\operatorname{div}\operatorname{grad}$ has been undertaken from a multitude of different perspectives. When considered as an operator on function spaces of Riemannian manifolds, a commonplace theme has been to study the connection of properties of the eigenfunctions with respect to their eigenvalue. For example, in a quantum chaotic setting, that is, where the underlying dynamics of the geodesic flow are chaotic, there is great interest in the behaviour of the probability measures $|\psi|^2 \mathrm{dVol}_M$. Here, ψ is an L^2-normalised eigenfunction of the Laplacian and dVol_M is the standard volume measure on M. In particular, if the manifold M is compact, then one can consider an orthonormal basis of $L^2(M)$ consisting of Laplacian eigenfunctions $\{\psi_j\}_{j\geq 0}$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \ldots \to \infty$. One may then consider weak-* limits of the measures $|\psi_j|^2 \mathrm{dVol}_M$ as $j \to \infty$. An overarching conjecture by Rudnick and Sarnak [96], called the Quantum Unique Ergodicity conjecture, states that when the manifold has negative sectional curvature, these measures converge to the volume measure on the space. Essentially, this is asking whether the eigenfunctions become fully delocalised in the eigenvalue aspect. In this limit, there have been several angles of approach to demonstrating delocalisation; for example through computing L^∞-norm bounds of eigenfunctions, and studying the entropy of the limit measures arising from weak-* limits of lifts of the eigenfunction measures (microlocal lifts) to the unit tangent bundle (see [4, 8, 30, 61]). Recently, in the setting of compact hyperbolic surfaces, Dyatlov and Jin [39] have also made a breakthrough showing that the weak-* limits of the microlocal lifts must have full support, that is, they assign a positive measure to any open subset of the unit tangent bundle.

Rather than the eigenvalue aspect, in this article we shall consider a delocalisation result of the eigenfunctions in the large *spatial* aspect on compact hyperbolic surfaces. This allows one to understand how the eigenfunctions are affected by a large volume geometry (or equivalently, a large genus by the Gauss-Bonnet Theorem). Such a perspective is commonplace in the regular graph literature, since the spectrum of the Laplacian is bounded. Moreover, the spatial aspect has been the subject of several recent results on surfaces in part due to their spectral and geometric similarities to regular graphs. In the arithmetic surface setting, it is a natural limit to study due to its connection with the level aspect. In this setting, some delocalisation results have been obtained by Saha [99] and Hu and Saha [56] in terms of the sup norms of the eigenfunctions of the form

$$\|\psi_\lambda\|_\infty \lesssim_\lambda g^{-\alpha}\|\psi_\lambda\|_2,$$

for some exponent $\alpha > 0$, and g the genus of the surface. The notion of delocalisation that we shall explore here will seek to understand if eigenfunctions can have partial, or conversely, near full concentration, on certain subsets of a compact hyperbolic surface.

Before stating our results precisely, it will be insightful to discuss some recent work that has connected Laplacian eigenfunctions to the surface geometry, as a way to highlight the different geometric influences that have so far been explored. In the work of Le Masson and Sahlsten [67], a spatial analogue of quantum ergodicity for compact hyperbolic surfaces was developed via a Benjamini-Schramm type of convergence. This result is analogous to similar work on regular graphs by Anantharaman and Le Masson [7]. A sequence of compact hyperbolic surfaces $(X_n)_{n\geq 0}$ Benjamini-Schramm converges to the hyperbolic plane if for all $R > 0$,

$$\frac{\mathrm{Vol}(z \in X_n : \mathrm{InjRad}_{X_n}(z) < R)}{\mathrm{Vol}(X_n)} \to 0,$$

as $n \to \infty$ (here $\mathrm{InjRad}_{X_n}(z)$ is the injectivity radius of the surface X_n at the point $z \in X_n$). Geometrically, this means that the proportion of points on the surface with small injectivity radius, or equivalently at least one short geodesic loop based at that point, is small in the limit. For a more general notion of Benjamini-Schramm convergence, see the articles [1, 2]. Benjamini-Schramm convergence can be seen as an assumption on the *global* geometry of the surfaces as it requires that the geometry of the surface around *most* points is 'well-behaved'. It turns out that this global geometric assumption is typical for a fixed surface at appropriately chosen scales of R, dependent upon the surface volume/genus. More precisely, Monk [83] shows that for each $g \geq 2$, there exists a subset $\mathcal{M}'_g$ of the moduli space of compact genus g hyperbolic surfaces such that, for any surface $X \in \mathcal{M}'_g$ one has

$$\frac{\mathrm{Vol}(z \in X : \mathrm{InjRad}_X(z) < \frac{1}{6}\log(g))}{\mathrm{Vol}(X)} = O\left(g^{-\frac{1}{3}}\right).$$

When one considers the Weil-Petersson probability measure on the moduli space of fixed genus (we will discuss this random model in more detail later), the probability of $\mathcal{M}'_g$ tends to one as $g \to \infty$. Thus, selecting a sequence of surfaces $(X_g)_g$ where g is the genus of X_g, and $X_g \in \mathcal{M}'_g$ for each g, one obtains Benjamini-Schramm convergence of this sequence, and the probability of $\mathcal{M}'_g$ approaches one as $g \to \infty$. Using this condition, Monk is able to obtain information about the structure of the Laplacian spectrum for random surfaces of large genus.

This Benjamini-Schramm assumption on surfaces can be contrasted with a *local* geometric assumption upon the surface geometry, that has been exploited by Gilmore, Le Masson, Sahlsten and Thomas in [48]. The focus of their work was on the L^p norms of the eigenfunctions of the Laplacian, rather than the spectrum, and how they are influenced by a large surface genus. There the authors require a strong control over the local topology around *every* point on the surface, not just control around a large proportion of points as is offered by the Benjamini-Schramm condition. Specifically, they require that every point on the surface is the base of only a small number of 'short' primitive geodesic loops. By comparison, the Benjamini-Schramm condition roughly states that the proportion of points that are the base of at *least* one 'short' geodesic loop is small compared to the surface volume.

The reason why this control over all points on the surface is beneficial, is highlighted when using the Selberg pre-trace formula to infer properties about Laplacian eigenfunctions. Indeed, understanding the behaviour of an eigenfunction at a certain point with this formula requires one to look at all the geodesic loops on the surface based at that point. On the other hand, being a global property, the Benjamini-Schramm condition seems more suited to understanding properties of the *spectrum* of the Laplacian. This is because one can use the Selberg trace formula (the integral of the pre-trace formula) to link the spectrum to an integral over the surface of a function evaluated at lengths of geodesic loops on the surface. Due to the presence of the integral, one only requires that the geodesic loops are well-behaved at *most* points on the surface.

In this article, we are dealing with properties of the eigenfunctions of the Laplacian, and again require strong control over the local topology of all points on the surface. For this reason, we will utilise the geometric condition for surfaces that was introduced by Gilmore, Le Masson, Sahlsten and Thomas, and this is written precisely in the statement of equation (5.1) below. It turns out that the length scale at which one can understand this local topology of points corresponds greatly to the strength of the results for the eigenfunctions. Indeed, this scale corresponds to the parameter $R(X)$ in equation (5.1) below, and from Theorem 5.1.3, the larger that this can be taken, the more we can infer about eigenfunction concentration. Here, we will consider estimates for this scale for both deterministic surfaces, and those chosen with high probability as the genus of the surface tends to infinity, with respect to the Weil-Petersson random model. To aid in the understanding of how large the length scale can be, it is beneficial for us to utilise another geometric property, from which one can directly infer the geometric loop properties at every point. Indeed, this is the perspective taken by Monk and Thomas in [84] where the *tangle-free parameter* of a surface is introduced, leading to more precise length scales. We introduce this parameter now.

Definition 5.1.1. Given $L > 0$, a compact hyperbolic surface X is said to L-tangle-free if every embedded pair of pants and one-holed torus in X has total boundary length at least $2L$.

Recall that a pair of pants is a hyperbolic surface of genus zero with three simple closed geodesic boundaries, and a one-holed torus is a genus 1 hyperbolic surface with a single simple closed geodesic boundary. When we consider total boundary length, we will mean the sum of the lengths of these geodesic boundaries on the subsurfaces.

Although stated in terms of pants and one-holed tori (the fundamental building blocks of a hyperbolic surface), the tangle-free parameter L of a surface provides understanding on the local topology of the surface around all points, as is required here. Indeed, this is highlighted in the following theorem.

Theorem 5.1.2 ([84, Theorem 9])**.** *Suppose that X is an L-tangle-free surface, and let $z \in X$ with δ_z a geodesic loop based at z of shortest length, $\ell(\delta_z)$. Then, any (not necessarily geodesic) loop β based at z, whose length $\ell(\beta)$ satisfies*

$$\ell(\beta) + \ell(\delta_z) < L,$$

is homotopic with fixed endpoints to a power of δ_z.

In other words, if the injectivity radius of an L-tangle-free surface at a point is less than $\frac{L}{2}$, then the shortest geodesic loop δ_z based at that point is unique. Furthermore, any other geodesic loop based at that point with length less than $\frac{L}{2}$, is homotopic with endpoints fixed at z to a power of δ_z. This means that the topology of the $\frac{L}{2}$-neighbourhood of any point on such a surface is well understood.

In the next subsection, we will state precisely how this length scale corresponds to the required local geometric condition for this article. Of course, understanding this correspondence is only useful if one can obtain estimates on how large the parameter L can be. Deterministically, notice that every surface is L-tangle-free for L at least $\mathrm{InjRad}(X)$, the injectivity radius of the surface. This is because the total boundary length of any pair of pants and one-holed tori embedded in the surface is at least $6\mathrm{InjRad}(X)$, and $2\mathrm{InjRad}(X)$ respectively. For a surface chosen at random from the moduli space of genus g with respect to the Weil-Petersson model, one may take $L = c\log(g)$ for any $0 < c < 1$, with probability tending to 1 as $g \to \infty$. Further details of this will be discussed in Subsection 5.1.3.

5.1.2 Deterministic Delocalisation

Let us now state precisely the delocalisation result that we prove in this article. The type of delocalisation that we will examine here answers the following: suppose an eigenfunction carries some L^2 contribution on a subset of the surface, what information can be deduced about such a subset? Inspired by analogous results obtained for regular graphs in [29, 30, 47], we address how large such a subset can be in terms of the genus of the surface. Recall that, in this setting, the genus is an equivalent parameter to the volume by the Gauss-Bonnet Theorem.

In a near fully delocalised case, an eigenfunction would assign a value of the order $\frac{1}{\sqrt{g}}$ across the whole manifold X (due to the L^2 normalisation), where g is the genus of X. Thus, on a measurable subset $E \subseteq X$, one should expect to see the L^2 norm of the eigenfunction restricted to this set, to be of an order proportional to the size of the set itself. In other words, if E were a subset such that $\|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon$, then one would expect a bound of the form

$$\mathrm{Vol}(E) \geq C\varepsilon g,$$

for some constant C, independent of the genus g and the L^2 contribution ε (when g is considered large enough).

What is obtained in this article, is a drop in the exponent of the genus, dependent upon the eigenvalue of the eigenfunction, and the contribution ε. This result is obtained for surfaces chosen with respect to the Weil-Petersson model with probability tending to one as $g \to \infty$. To achieve a result of this form, we begin with a lower bound on the volume of E holding for all surfaces. This lower bound can be understood in terms of the L-tangle-free parameter above, or more generally, the parameter $R(X)$ associated with $(R(X), C(X))$-admissibility introduced below. Then, we use probabilistic estimates for these parameters to obtain bounds in terms of the genus. We will discuss the concept of $(R(X), C(X))$-admissibility now, and contrast it to a similar parameter used for regular graphs.

Geometric Parameter

For regular graphs, Brooks and Lindenstrauss [29] prove that if a graph Laplacian eigenfunction has some L^2 contribution on a subset of the vertices, then this subset is bounded below in terms of the size of the graph and the L^2 contribution on the subset. The starting point for this result is the introduction of a geometric parameter that provides a length scale at which there are few distinct, non-backtracking walks between any two vertices in the graph shorter than this length. In particular, this can be deduced from bounds on the number of cycles based at a point in the graph, whose length are controlled by a similar length scale. This is analogous to the control offered by the tangle-free parameter discussed above on the geodesic loops based at any point on the surface. In fact, the exact formulation of the geometric property that we require here is a combinatorial bound on the number of geodesic paths between points on the surface. And, as is the case with graphs, this can be inferred from similar combinatorial bounds on the number of geodesic loops based at points.

More precisely, for $R, C > 0$ we say that a compact hyperbolic surface $X = \Gamma\backslash\mathbb{H}$ is (R, C)-admissible if for all $\delta > 0$ there exists $C_0(\delta) > 0$ (independent of R, C and X) such that for any $z, w \in \mathbb{H}$ one has

$$|\{\gamma \in \Gamma : d(z, \gamma w) \leq r\}| \leq C C_0(\delta) e^{\delta r}, \quad \text{for all } r \leq R. \tag{5.1}$$

Of course, one can always find a pair (R, C) for which X is (R, C)-admissible: take $R = c\mathrm{InjRad}(X)$ for any $c < 1$, $C = 1$ and $C_0(\delta) = 2$. Indeed, for $r \leq R$ there can be at most two elements in the set on the left-hand side otherwise one would obtain a geodesic loop on the surface of length shorter than $\mathrm{InjRad}(X)$. The crucial point is that the parameter R represents the length scale up to which we can understand the local geometry about every point on the surface (this is highlighted in Lemma 5.3.1). This means, it will be the controlling parameter for the lower bound on the volume of a set E (see the statement of Theorem 5.1.3 for the precise relation). The strength of the theorem thus relies on one being able to take R as large as possible. In fact, we will wish for the parameter R to grow in terms of genus of the surface. In the Weil-Petersson model, surfaces can have small injectivity radius with positive probability (see also Theorem 5.1.5), and so the idea is to ensure that we can go past the injectivity radius scale for typical surfaces.

Pushing past the injectivity radius scale for R can possibly require the combinatorial bound of equation (5.1) to have some dependence upon the surface itself. This means that the constant C may also depend upon a geometric feature of the surface X such as its injectivity radius. Due to these dependencies and our interest in understanding the influence of the geometry on the eigenfunctions, we shall instead often write that a surface is $(R(X), C(X))$-admissible to emphasise and keep track of these surface dependent parameters. If one is not so careful, this can cause some problems when observing how the constant $C(X)$ manifests itself in the lower bound on the volume of the set E in Theorem 5.1.3. Thus, one must ensure that $C(X)$ is well understood, so that it will not ruin the obtained bounds. Using the tangle-free parameter allows us to find a pair $(R(X), C(X))$ that can be studied probabilistically, and that result in lower bounds on the volume of the set E in terms of the genus. Indeed, we shall show in Lemma 5.3.1 that if X is an L-tangle-free surface, then it is $(\frac{L}{4}, \min\{1, \mathrm{InjRad}(X)\}^{-1})$-admissible. For these parameters, the constant $C_0(\delta)$ can be stated explicitly as in Lemma 5.3.1, but its precise value is unimportant to our discussion here. The idea then is to show that with probability tending to one as $g \to \infty$ a surface is $c\log(g)$-tangle-free and has injectivity radius at least $g^{-\varepsilon}$ for any $0 < c < 1$ and $\varepsilon > 0$ (see Subsection 5.1.3). This will allow us to pass from estimates on delocalisation in terms of an admissibility pair (R, C) to to estimates in terms of the surface genus.

The deterministic result that we obtain is split into two components. First, there is the case of tempered eigenfunctions that have eigenvalues in $[\frac{1}{4}, \infty)$. These are dealt with by using a similar approach to Brooks and Lindenstrauss [29] and Ganguly and Srivastava [47] on regular graphs, through what can be seen as a smoothed cosine wave propagation operator. The untempered eigenfunctions, whose eigenvalues are in $(0, \frac{1}{4})$, are analysed through a rescaled ball averaging operator, and we can actually obtain a stronger delocalisation result in this case.

Theorem 5.1.3. *Let $0 < \varepsilon < 1$ be given, and suppose that X is an $(R(X), C(X))$-admissible compact hyperbolic surface. Suppose that ψ_λ is an L^2-normalised* *eigenfunction of the Laplacian on X with eigenvalue λ, and suppose that $E \subseteq X$ is a measurable set for which*

$$\|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon.$$

Then there exists a constant $R_0 > 0$ dependent only upon ε and λ, such that if $R(X) > R_0$, then we have the following bounds.

(1) If $\lambda \geq \frac{1}{4}$, there exists a universal constant $A > 0$ (independent of all other parameters and the surface), and a constant $d(\lambda) > 0$ for which

$$\mathrm{Vol}(E) \geq \frac{A\varepsilon}{C(X)} e^{d(\lambda)\varepsilon R(X)} \min\{1, \mathrm{InjRad}(X)^2\},$$

(2) If $\lambda = \frac{1}{4} - \sigma$ for some $0 < \sigma \leq \frac{1}{4}$, there exists a universal constant $A > 0$ (independent of all other parameters and the surface), such that

$$\mathrm{Vol}(E) \geq \frac{A\varepsilon}{C(X)} e^{(\frac{1}{4}+\frac{1}{2}\sqrt{\sigma})R(X)}.$$

The constant $d(\lambda)$ above is made explicit later (see Theorem 5.3.5). Theorem 5.1.3 in particular shows that the eigenfunctions can not be large on a small set if (for instance) the L-tangle-free parameter of the surface is large compared to the eigenvalue (taking $R(X) = \frac{L}{4}$).

5.1.3 Random Surface Delocalisation

Let us now discuss how one can use Theorem 5.1.3 to obtain a probabilistic result in terms of the genus/volume of the surface as desired. For this, we shall first describe the construction of the Weil-Petersson random surface model that we shall employ; a more detailed account can be found in [59, 79, 80]. Note that other *distinct* random surface constructions could also be considered, such as a random triangulations model by Brooks and Makover [26], and a random cover model by Magee, Naud and Puder [73]. It would be interesting to see if similar results to those presented here could be realised in these models.

Fix a genus $g \geq 2$ and let $\mathcal{T}_g$ denote the Teichmüller space of marked genus g closed Riemann surfaces up to marking equivalence. Then, there is a $(6g-6)$-dimensional real-analytic structure on $\mathcal{T}_g$ which carries a symplectic form ω_{WP} called the *Weil-Petersson form.* One obtains a volume form on $\mathcal{T}_g$ by taking a $(3g-3)$-fold wedge product of ω_{WP} and normalising by $(3g-3)!$. In addition to this volume structure, there is a natural group acting on $\mathcal{T}_g$ called the *mapping class group*, denoted by MCG_g, which acts by changing the marking on a point in $\mathcal{T}_g$. The *moduli space* of genus g is then defined as the quotient by this action:

$$\mathcal{M}_g = \mathcal{T}_g/\mathrm{MCG}_g.$$

This space can be thought of as the collection of hyperbolic metrics that can be endowed on a genus g surface, identified up to isometry. An important feature of the Weil-Petersson volume form defined on $\mathcal{T}_g$ is that it is invariant under the action of MCG_g, and so it descends naturally to the moduli space. With respect to this measure, the moduli space has finite volume (see [32] for an upper bound, and [82] for more specific asymptotics of this volume for large genus). This allows one to define a probability measure on $\mathcal{M}_g$ called the *Weil-Petersson probability measure*, and calculate probabilities in the natural way:

$$\mathbb{P}_g^{\mathrm{WP}}(A) = \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} \mathbf{1}_A(X) \mathrm{d}X,$$

where $\mathrm{d}X$ is used to denote the volume form. Commonly, one takes A to be a collection of surfaces satisfying some desired geometric property. By using integration tools and volume estimates obtained by Mirzakhani [78, 79, 80], one is able to obtain upper bounds for these probabilities as functions of the genus, and determine events that hold with high probability as $g \to \infty$.

Recall that the probabilistic result that we require is estimating the probability of a collection of surfaces that are (R, C)-admissible for suitable R and C. In fact, from Lemma 5.3.1 we will see that an L-tangle-free surface is $(\frac{L}{4}, \min\{1, \mathrm{InjRad}(X)\}^{-1})$-admissible. We then have the following probabilistic estimate for L from Monk and Thomas in [84].

Theorem 5.1.4 ([84, Theorem 4])**.** *For any $0 < c < 1$, one has*

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \in \mathcal{M}_g : X \text{ is } c\log(g)\text{-tangle-free}\right) = 1 - O\left(\frac{(c\log(g))^2}{g^{1-c}}\right),$$

as $g \to \infty$.

For the parameter $\min\{1, \mathrm{InjRad}(X)\}^{-1}$, it suffices to estimate the injectivity radius. The following result of Mirzakhani is sufficient for our purposes.

Theorem 5.1.5 ([80, Theorem 4.20])**.** *For any $a > 0$,*

$$\mathbb{P}_g^{\mathrm{WP}}(X : \mathrm{InjRad}(X) \geq g^{-a}) = 1 - O(g^{-2a}),$$

as $g \to \infty$.

Thus, it will suffice for us to choose surfaces that are both $c\log(g)$-tangle-free and that have injectivity radius at least g^{-a} for some $0 < c < 1$ and $a > 0$ since this set will have probability tending to one as $g \to \infty$, and any surface X in this set will be $(R(X), C(X))$-admissible with appropriate values of $R(X) = c\log(g)$ and $C(X) = g^a$. Hence using Theorem 5.1.3, we obtain the following random result.

Theorem 5.1.6. *Let $\varepsilon > 0$ be given, and suppose that X is a compact hyperbolic surface with genus g chosen randomly according to the Weil-Petersson* *probability model. Suppose further that λ is an eigenvalue of the Laplacian on X, and ψ_λ is an L^2-normalised eigenfunction of the Laplacian with eigenvalue λ. Let $E \subseteq X$ be a measurable set for which*

$$\|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon.$$

Then, if $\lambda \geq \frac{1}{4}$, there exists a universal constant $A > 0$ (independent of the surface and all other parameters) such that for any $0 < c < \frac{1}{4}$, and $a > 0$, one obtains

$$\mathrm{Vol}(E) \geq A\varepsilon g^{c\varepsilon d(\lambda)-a},$$

with $d(\lambda)$ as in Theorem 5.1.3. If $\lambda = \frac{1}{4} - \sigma$ for some $0 < \sigma < \frac{1}{4}$, then there exists a universal constant $A > 0$ (independent of the surface and all other parameters) such that for any $0 < c < \frac{1}{4}$ and $a > 0$, one obtains

$$\mathrm{Vol}(E) \geq A\varepsilon g^{c(\frac{1}{4}+\frac{1}{2}\sqrt{\sigma})-a}.$$

Both bounds hold with probability

$$1 - O\left(\frac{\log(g)^2}{g^{1-4c}} + g^{-2a}\right),$$

as $g \to \infty$.

Remark. As noted, the exponent of the genus in the above result is governed exclusively by finding a collection of surfaces with probability tending to 1 as $g \to \infty$ that are also (R, C)-admissible for suitable R and C. For our result, these followed from probabilistic estimates of the tangle-free parameter L. To improve the exponent using this method, one would need to show that typical surfaces can have a tangle-free parameter of the size $A\log(g)$ for A

large. However, in Monk and Thomas [84], it is shown that no surface is more than $(4\log(g)+O(1))$-tangle-free, which would not be sufficient for this. Thus, any significant improvement to the exponent would require a new approach to estimating R for a typical surface.

5.2 Harmonic Analysis for Hyperbolic Surfaces

We begin by defining our main object of study, hyperbolic surfaces, as well as outlining necessary tools from harmonic analysis that are used to obtain our results. One can find further details on these topics in Katok [64], Bergeron [15] and Iwaniec [60].

The *hyperbolic upper half-plane* will be a sufficient model of hyperbolic space for our purposes. This is defined by

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \colon y > 0\},$$

and is equipped with the Riemannian metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2},$$

which induces the standard Riemannian volume form

$$\mathrm{d}\mu = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}.$$

The set of orientation preserving isometries of $\mathbb{H}$ with this metric are the Möbius transformations given by

$$z \mapsto \frac{az + b}{cz + d},$$

for some $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. They can be identified with the group

$\mathrm{PSL}(2,\mathbb{R})$ with the natural associated group action. Using this, one can make the identification $\mathbb{H} = \mathrm{PSL}(2,\mathbb{R})/\mathrm{SO}(2)$.

A convenient definition for a hyperbolic surface is then obtained through this group action. Indeed, consider a discrete subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ that acts freely upon $\mathbb{H}$. A *hyperbolic surface* is a manifold quotient $X = \Gamma\backslash\mathbb{H}$. That is, the surface consists of points on $\mathbb{H}$ identified up to orbits of isometries in Γ. The Riemannian metric and volume measure are induced upon the surface in a natural manner. To each such subgroup Γ (and hence to each surface), one may determine (non-uniquely) a fundamental domain in $\mathbb{H}$. Functions defined on the surface can be identified with Γ-invariant functions upon $\mathbb{H}$, or functions on such a fundamental domain. We will deal in this article exclusively with the case when X is compact.

The harmonic analysis tools that are required to show our result are given by invariant integral operators and the Selberg transform. Such operators are constructed from radial functions. These are bounded, even and measurable functions $k : (-\infty, \infty) \to \mathbb{C}$. They give rise to a function, which we also denote by the same symbol $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$, through the correspondence

$$k(z, w) = k(d(z, w)),$$

where $d(z, w)$ is the hyperbolic distance between $z, w \in \mathbb{H}$. This function has the important property that it is invariant under the diagonal action of $\mathrm{PSL}(2,\mathbb{R})$. That is, for any $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ and $z, w \in \mathbb{H}$ one has

$$k(\gamma z, \gamma w) = k(z, w).$$

From this, one then *formally* defines a function $k_\Gamma : X \times X \to \mathbb{C}$ called an *automorphic kernel* by

$$k_\Gamma(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w),$$

where we have defined k_Γ as a Γ-periodic function on $\mathbb{H}$. For this sum to converge, one requires an appropriate decay condition on k; for instance

$$|k(\rho)| = O(e^{-\rho(1+\delta)}),$$

for some $\delta > 0$ would suffice, and we assume such a condition from now on. We can then define an invariant integral operator T_k on functions on X through the formula

$$(T_k f)(z) = \int_D k_\Gamma(z,w) f(w) \mathrm{d}\mu(w),$$

where D is a fundamental domain for X. The importance of operators defined in this manner is their connection to the Laplacian operator which we recall is defined in coordinates on $\mathbb{H}$ as

$$\Delta = -\mathrm{div}\,\mathrm{grad} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

This operator commutes with isometries on $\mathbb{H}$ and so naturally passes to an operator on the hyperbolic surface X. Since X is compact, the Laplacian has a discrete spectrum contained in $[0,\infty)$, with the 0-eigenspace being simple and consisting of the constant functions. In addition, there exists an orthonormal basis $\{\psi_j\}_{j=0}^{\infty}$ of Laplacian eigenfunctions for $L^2(X)$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$.

An important observation is that any eigenfunction of the Laplacian is also an eigenfunction of an invariant integral operator T_k on the surface. The eigenvalue of such an eigenfunction for T_k can be determined by taking a *Selberg transform* of the initial radial kernel. This is defined to be the Fourier transform

$$\mathcal{S}(k)(r) = h(r) = \int_{-\infty}^{+\infty} e^{iru} g(u) \mathrm{d}u,$$

of the function

$$g(u) = \sqrt{2} \int_{|u|}^{+\infty} \frac{k(\rho) \sinh \rho}{\sqrt{\cosh \rho - \cosh u}} \mathrm{d}\rho.$$

The spectrum is then provided from the following result.

Theorem 5.2.1 ([15, Theorem 3.8])**.** *Let $X = \Gamma \backslash \mathbb{H}$ be a hyperbolic surface and $k \colon [0, \infty) \to \mathbb{C}$ a radial kernel. Suppose that ψ_λ is an eigenfunction of the Laplacian with eigenvalue $\lambda = s_\lambda^2 + \frac{1}{4}$ for $s_\lambda \in \mathbb{C}$. Then ψ_λ is an eigenfunction of the convolution operator T_k with invariant kernel k and*

$$(T_k \psi_\lambda)(z) = \int_X k_\Gamma(d(z, w)) \psi_\lambda(w) \, \mathrm{d}\mu(w) = h(s_\lambda) \psi_\lambda(z),$$

where $h(s_\lambda) = \mathcal{S}(k)(s_\lambda)$.

One refers to s_λ in the above result as the *spectral parameter* associated to λ. Through this result, and the Selberg transform, one can also reconstruct an invariant kernel operator with a specified spectrum. Indeed, given a suitable function h one can take an inverse Selberg transform to obtain a radial kernel k through the formulae

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isu} h(s) \mathrm{d}s,$$

and then

$$k(\rho) = -\frac{1}{\sqrt{2}\pi}\int_{\rho}^{+\infty}\frac{g'(u)}{\sqrt{\cosh u - \cosh\rho}}\mathrm{d}u.$$

5.3 Delocalisation of Tempered Eigenfunctions on Large Genus Surfaces

We start with the deterministic version of our result, and thus consider a fixed compact hyperbolic surface $X = \Gamma\backslash\mathbb{H}$. Let $D \subseteq \mathbb{H}$ be a fundamental domain of X, and $E \subseteq X$ a measurable set. Suppose that X is $(R(X), C(X))$-admissible for some $R, C > 0$. Suppose also that $\{\psi_j\}_{j=0}^{\infty}$ is an orthonormal basis for $L^2(X)$ of Laplacian eigenfunctions with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leq \ldots \to \infty$. It is clear that in the case of the constant eigenfunctions corresponding to λ_0 that the delocalisation result holds, and so we will fix an eigenvalue $\lambda = \lambda_j$ for some $j \geq 1$. In particular, in this section we will further assume that the eigenfunction is tempered so that $\lambda \geq \frac{1}{4}$. Let $s_\lambda \in [0, \infty)$ be the spectral parameter associated with λ through the equation $s_\lambda^2 + \frac{1}{4} = \lambda$.

5.3.1 Outline of the proof

The connection between the eigenfunction, the geodesic loop parameter $R(X)$ from an admissible pair $(R(X), C(X))$ for X, and the volume of the set E is unified in the construction of a propagation operator. To exhibit this we utilise the following methodology:

1. We consider a family of operators that are to be seen as a smoothed cosine wave kernel, and recall how these operators act upon Laplacian eigenfunctions using results of [48]. [Lemma 5.3.2]

2. By selecting certain members of this family of operators and weighting them appropriately, we construct another family of operators now

specialised to a certain fixed eigenvalue λ, as well as some secondary parameters that will later depend on the parameter ε. We then determine the operator norm of these operators. [Lemma 5.3.3]

3. The eigenfunctions of the Laplacian are also eigenfunctions of the constructed family of operators, and so we study their eigenvalues under this operator family. This is done by showing that they can be written in terms of Fejér kernels of certain orders, and so we obtain bounds on the eigenvalues using properties of these kernels. [Lemma 5.3.4]

4. Through studying the spectral decomposition of the restricted eigenfunction $\psi_\lambda \mathbf{1}_E$ over an orthonormal basis, we can relate a lower bound on the volume of E to the previously obtained bounds on the eigenvalues and operator norms of the constructed family of operators. [Theorem 5.3.5]

To begin, let us explain how an L-tangle-free surface is $(\frac{L}{4}, \min\{1, \mathrm{InjRad}(X)^{-1}\})$-admissible. This will allow one to contextualise the results in terms of L, and allow for Theorem 5.1.6 to be deduced immediately from Theorems 5.1.3, 5.1.4 and 5.1.5 .

Lemma 5.3.1. *Suppose that X is an L-tangle-free compact hyperbolic surface. Then, for any $\delta > 0$, there exists a constant $C_0(\delta) > 0$ such that for all $z, w \in \mathbb{H}$, one has*

$$|\{\gamma \in \Gamma : d(z, \gamma w) \leq r\}| \leq \frac{C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)\}} e^{\delta r}, \quad \textit{for all } r \leq \frac{L}{4}.$$

In other words, X is $(\frac{L}{4}, \min\{1, \mathrm{InjRad}(X)^{-1}\})$-admissible.

Proof. It is clear from Theorem 5.1.2 that for any $r \leq \frac{L}{2}$, and any $z \in \mathbb{H}$, there is at most one non-identity primitive $\gamma \in \Gamma$ that is in the set

$$\{\gamma \in \Gamma : d(z, \gamma z) \leq r\}. \tag{5.2}$$

Any element in Γ is the power of a primitive element as X is compact. Moreover, if $\gamma_1 \in \Gamma$ is equal to γ_0^n for some primitive element $\gamma_0 \in \Gamma$, then

$$d(z, \gamma_0 z) \leq d(z, \gamma_1 z).$$

This means that if γ_1 is in this set, then the primitive element γ_0 is also. Combining these observations, the only elements in this set are γ^n for some powers $n \in \mathbb{Z}$, and $\gamma \in \Gamma$ a single primitive element.

To determine an upper bound on the number of elements, we use the fact that the distance $d(z, \gamma^n z)$ is at least n times the translation distance of γ (it would be precisely equal if z were on the axis of γ). By definition, the translation distance is bounded below by twice the injectivity radius of the surface. Considering the identity and both the positive and negative powers of γ, we see the maximal number of elements in the set (5.2) is

$$1 + \left\lfloor \frac{r}{\mathrm{InjRad}(X)} \right\rfloor, \tag{5.3}$$

for any $z \in \mathbb{H}$. A bound on the cardinality of the set (5.2) provides a bound on the cardinality of the set

$$\left\{ \gamma \in \Gamma : d(z, \gamma w) \leq \frac{r}{2} \right\}, \tag{5.4}$$

for $z, w \in \mathbb{H}$. Indeed, suppose there were at least

$$m = 2 + \left\lfloor \frac{r}{\mathrm{InjRad}(X)} \right\rfloor$$

non-identity elements in the set (5.4), labelled γ_i for $1 \leq i \leq m$. For each $2 \leq i \leq m$ we have

$$d(\gamma_1 w, (\gamma_i \gamma_1^{-1})(\gamma_1 w)) \leq d(\gamma_1 w, z) + d(z, \gamma_i w) \leq r,$$

by the triangle inequality, and the fact that the γ_i are in (5.4). As the γ_i are distinct, $\gamma_i\gamma_1^{-1}$ is not the identity for any $i = 2, \ldots, m$. This means that there are at least $m-1$ non-identity elements in the set

$$\{\gamma \in \Gamma : d(\gamma_1 w, \gamma\gamma_1 w) \leq r\},$$

contradicting the previous bound on the cardinality in (5.3). Thus, when including the identity, this means there are at most m elements in the set stated in equation (5.4). Notice that

$$m \leq 2 + \left\lfloor \frac{r}{\min\{1, \mathrm{InjRad}(X)\}} \right\rfloor.$$

We wish to find a constant $C_0(\delta) > 0$ so that

$$m \leq \frac{C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)\}} e^{\delta r},$$

for all $\delta > 0$. We first bound $2 + r$. If $r < 1$, then trivially, this is bounded by $3 \leq 3e^{\delta r}$. If $r \geq 1$ then we can observe that

$$2 + r \leq 3r \leq 3e^{\frac{1}{\delta}} e^{\delta r}.$$

Indeed,

$$3e^{\frac{1}{\delta}} e^{\delta r} \geq 3\left(1 + \frac{1}{\delta}\right)(1 + \delta r) \geq 3r.$$

Hence given $\delta > 0$, we set $C_0(\delta) = 3e^{\frac{1}{\delta}}$. Then,

$$\begin{aligned} \left|\left\{\gamma \in \Gamma : d(z, \gamma w) \leq \frac{r}{2}\right\}\right| = m &\leq 2 + \left\lfloor \frac{r}{\min\{1, \mathrm{InjRad}(X)\}} \right\rfloor \\ &\leq \frac{2 + r}{\min\{1, \mathrm{InjRad}(X)\}} \end{aligned}$$

$$\leq \frac{C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)\}} e^{\delta r},$$

providing the required bound. $\square$

5.3.2 Construction of a Family of Propagation Operators

In this subsection we take the first step of defining an appropriate family of operators. These are largely based on their similarity to wave propagation operators, and are defined through the inverse Selberg transform. Indeed, define

$$h_t(r) = \frac{\cos(rt)}{\cosh(\frac{\pi r}{2})},$$

for appropriate values of $r \in \mathbb{C}$ and $t \geq 0$. Denote by $k_t(\rho)$ the radial kernel obtained via the inverse Selberg transform of h_t. This defines an integral operator P_t on functions of the hyperbolic plane via

$$P_t f(z) = \int_{\mathbb{H}} k_t(d(z,w)) f(w) \mathrm{d}\mu(w).$$

The construction of this operator is similar to that used in Iwaniec and Sarnak [61] when computing sup norm bounds for Laplacian eigenfunctions on arithmetic surfaces. Indeed, in their article they construct a propagation operator whose kernel is based on the Fourier transform of h_t. The exact kernel k_t defined above has been studied by Brooks and Lindenstrauss in [30], and also by Gilmore, Le Masson, Sahlsten and Thomas in [48], and several important facts about the associated operator P_t will be utilised here.

Through use of the automorphic kernel one may consider P_t as an operator on functions of the surface X. That is, we consider P_t on such functions acting by

$$P_t f(z) = \int_D \sum_{\gamma \in \Gamma} k_t(d(z, \gamma w)) f(w) \mathrm{d}\mu(w),$$

with D a fundamental domain of X as before. Let Π denote the projection operator to the subspace orthogonal to constants defined by

$$\Pi f(z) = f(z) - \frac{1}{\mathrm{Vol}(X)} \int_D f(w) \mathrm{d}\mu(w).$$

Then [48] shows that the operators $P_t \Pi$ are bounded linear operators from $L^q(X) \to L^p(X)$ for $1 \leq q \leq 2 \leq p \leq \infty$ conjugate indices, when t is not too large. In fact, an explicit upper bound is obtained on the operator norm. Here it will suffice to consider only the $L^1(X) \to L^\infty(X)$ norm estimates, and we replicate the statement of these bounds for the reader's convenience.

Lemma 5.3.2 ([48, Lemma 3.3, Theorem 4.3])**.** *Suppose that X is an $(R(X), C(X)$-admissible compact hyperbolic surface with $R(X) > 1$. Then for $t \leq \frac{1}{4}(R(X) - 1)$ and any $\delta > 0$, there exists a constant $C_0(\delta) > 0$ dependent only upon δ such that*

$$\|P_t \Pi\|_{L^1(X) \to L^\infty(X)} \leq \frac{C(X) C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)^2\}} e^{-(\frac{1}{2} - \delta)t},$$

The proof of this result relies on estimating the expression

$$\sum_{\gamma \in \Gamma} |k_t(d(z, \gamma w))|,$$

for $z, w \in D$, which arises from the automorphic kernel of $P_t \Pi$. Outside of a ball of radius $4t$, Brooks and Lindenstrauss show that the kernel k_t satisfies some strong exponential decay. Inside the ball of radius $4t$, we are considering points $z, w \in D$ for which $d(z, \gamma w) \leq R(X)$. The number of these terms is bounded by a sub-exponential growth from the $(R(X), C(X))$-admissibility which is off-set by a slight exponential decay of the kernel. Crucially, this is where control over the geodesics between all points on the surface with lengths up to the scale $R(X)$ is utilised, and results in an exponential decay for the operator norm of $P_t\Pi$.

Armed with this upper bound for the operator norm of $P_t\Pi$, we wish to construct a new operator that is specialised to the eigenvalue λ, and two auxiliary parameters that will later depend upon ε. This is done by taking a certain linear combination of members of the $P_t\Pi$ family for select values of t at which the above bounds hold. The choice of t is delicate. On one hand, we need to take enough operators in the linear combination so that the operator has an appropriate spectral action on Laplacian eigenfunctions. On the other hand, taking too many operators in the linear combination will inflate the operator norm too much. To this end, we follow the approach of [47] used for regular graphs which refines and improves upon the original techniques and bounds obtained in [29, 30].

Recall that the Fejér kernel of order N is defined by

$$F_N(s) = \frac{1}{N}\frac{\sin^2(Ns/2)}{\sin^2 s/2} = 1 + \sum_{j=1}^{N} \frac{N-j}{N/2}\cos(js).$$

By dividing the summation in the right hand side by certain hyperbolic cosines, one recovers a summation of functions similar to the h_t defined above. We will exploit this observation to understand the spectral action of a certain linear combination of $P_t\Pi$ as a function of the Fejér kernel. To this end, for positive integers N and r, define

$$W_{\lambda,r,N} = \sum_{j=1}^{N} \frac{N-j}{N}(\cos(rs_\lambda j)+1)P_{jr}\Pi.$$

With control over the values of N and r, we can utilise the upper bound on the operator norm of $P_t\Pi$ to see that this is a bounded operator from $L^1(X) \to L^\infty(X)$, and obtain explicit bounds on the operator norm.

Lemma 5.3.3. *Suppose that $\lambda \geq \frac{1}{4}$ is an eigenvalue of the Laplacian on an $(R(X), C(X))$-admissible compact hyperbolic surface X for some $R(X) > 1$. Given positive integers N and r satisfying $Nr \leq \frac{1}{4}(R(X)-1)$, and any $\delta < 0.01$, the operator $W_{\lambda,r,N} : L^1(X) \to L^\infty(X)$ is a bounded linear operator with norm*

$$\|W_{\lambda,r,N}\|_{L^1(X)\to L^\infty(X)} \leq \frac{C(X)A(\delta)}{\min\{1, \mathrm{InjRad}(X)^2\}} e^{-(\frac{1}{2}-\delta)r},$$

for some constant $A(\delta) > 0$ dependent only upon δ.

Proof. From the conditions on N and r, we have $jr \leq \frac{1}{4}(R(X)-1)$ for each $j = 1, \dots, N$. Utilising Lemma 5.3.2 we obtain for any $\delta > 0$

$$\begin{aligned}
\|W_{\lambda,r,N}\|_{L^1(X)\to L^\infty(X)} &\leq \sum_{j=1}^{N} \left| \frac{N-j}{N} (\cos(rs_\lambda j) + 1) \right| \|P_{jr}\Pi\|_{L^1(X)\to L^\infty(X)} \\
&\leq \frac{2C(X)C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)^2\}} \sum_{j=1}^{N} e^{-(\frac{1}{2}-\delta)jr} \\
&\leq \frac{2C(X)C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)^2\}} \left(e^{-(\frac{1}{2}-\delta)r} + \sum_{j=2}^{\infty} e^{-(\frac{1}{2}-\delta)jr} \right) \\
&= \frac{2C(X)C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)^2\}} \left(e^{-(\frac{1}{2}-\delta)r} + \frac{e^{-(\frac{1}{2}-\delta)r}}{e^{(\frac{1}{2}-\delta)r} - 1} \right).
\end{aligned}$$

For $\delta < 0.01$ we have $2\sinh\left(\frac{1}{2}-\delta\right) = e^{\left(\frac{1}{2}-\delta\right)} - e^{-\left(\frac{1}{2}-\delta\right)} \geq 1$. Under this condition, since $r \geq 1$, we obtain

$$e^{-\left(\frac{1}{2}-\delta\right)r}\left(1 + e^{\left(\frac{1}{2}-\delta\right)}\right) \leq e^{-\left(\frac{1}{2}-\delta\right)}\left(1 + e^{\left(\frac{1}{2}-\delta\right)}\right) \leq e^{\left(\frac{1}{2}-\delta\right)}.$$

This is equivalent to

$$e^{-\left(\frac{1}{2}-\delta\right)r} \le e^{\left(\frac{1}{2}-\delta\right)} e^{-\left(\frac{1}{2}-\delta\right)r} \left(e^{\left(\frac{1}{2}-\delta\right)r} - 1\right).$$

Plugging this estimate into the bounds for the operator norm above then gives

$$\|W_{\lambda,r,N}\|_{L^1(X)\to L^\infty(X)} \le \frac{2C(X)C_0(\delta)}{\min\{1, \mathrm{InjRad}(X)^2\}} \left(1 + e^{\left(\frac{1}{2}-\delta\right)}\right) e^{-\left(\frac{1}{2}-\delta\right)r}.$$

Setting $A(\delta) = 2C_0(\delta)\left(1 + e^{\left(\frac{1}{2}-\delta\right)}\right)$ then gives the result. □

5.3.3 Determining the Spectral Action and Proof of Theorem 5.1.3 for Tempered Eigenfunctions

We now analyse the spectrum of the operator $W_{\lambda,r,N}$ defined above. We do this by testing it against the orthonormal basis of Laplacian eigenfunctions considered at the start of this section. We remark that more accurate estimates on the $W_{\lambda,r,N}$ eigenvalue of an eigenfunction with Laplacian eigenvalue μ close to λ may be possible to obtain, but they are not needed for our purposes here.

Lemma 5.3.4. *Suppose that $\lambda \ge \frac{1}{4}$ and $\mu \in [0, \infty)$ are eigenvalues of the Laplacian on an $(R(X), C(X))$-admissible compact hyperbolic surface X with $R(X) > 1$. Fix positive integers N and r satisfying $Nr \le \frac{1}{4}(R(X) - 1)$. If ψ_μ is an eigenfunction of the Laplacian on X with eigenvalue μ, then ψ_μ is an eigenfunction of the operator $W_{\lambda,r,N}$, and the following bounds on its eigenvalue hold.*

1. *If $\mu \ge \frac{1}{4}$, then the eigenvalue of ψ_μ under the action of $W_{\lambda,r,N}$ is at least -1.*

2. *If $\mu \in [0, \frac{1}{4})$, then the eigenvalue of ψ_μ under the action of $W_{\lambda,r,N}$ is at least 0.*

3. *The eigenvalue of ψ_λ under $W_{\lambda,r,N}$ is at least $\frac{N-4}{4\cosh(s_\lambda \pi/2)}$.*

Proof. The fact that ψ_μ is an eigenfunction of $W_{\lambda,r,N}$ is immediate from the construction of the operator as a linear combination of $P_t\Pi$ for various values of t. To analyse the eigenvalue of ψ_μ, we will rewrite it as a function of Fejér kernels. If $\mu = 0$, then it is obvious from the definition of Π that the eigenvalue is zero, so assume that $\mu > 0$. Then,

$$W_{\lambda,r,N}\psi_\mu = \sum_{j=1}^{N} \frac{N-j}{N}(\cos(rs_\lambda j)+1)\frac{\cos(rs_\mu j)}{\cosh\left(\frac{s_\mu \pi}{2}\right)}\psi_\mu.$$

For small eigenvalues $\mu \in (0, \frac{1}{4})$, it is easy to see that the summation is non-negative. Indeed, s_μ will be purely imaginary and lie in $(0, \frac{1}{2})i$, so that $\frac{s_\mu \pi}{2} \in (0, \frac{\pi}{4})i$ and each term in the summation is non-negative. To deal with values of μ at least $\frac{1}{4}$, we rewrite the above eigenvalue by splitting the summation. Notice that

$$\begin{aligned}
&\frac{1}{\cosh\left(\frac{s_\mu \pi}{2}\right)} \sum_{j=1}^{N} \frac{N-j}{N} \cos(rs_\lambda j)\cos(rs_\mu j) \\
&\quad = \frac{1}{2\cosh\left(\frac{s_\mu \pi}{2}\right)} \sum_{j=1}^{N} \frac{N-j}{N}(\cos(jr(s_\lambda + s_\mu)) + \cos(jr(s_\lambda - s_\mu))) \\
&\quad = \frac{1}{4\cosh\left(\frac{s_\mu \pi}{2}\right)} \left(1 + 2\sum_{j=1}^{N} \frac{N-j}{N}\cos(jr(s_\lambda + s_\mu)) \right.\\
&\qquad\qquad \left. +1 + 2\sum_{j=1}^{N} \frac{N-j}{N}\cos(jr(s_\lambda - s_\mu)) - 2\right) \\
&\quad = \frac{1}{4\cosh\left(\frac{s_\mu \pi}{2}\right)}(F_N(r(s_\lambda + s_\mu)) + F_N(r(s_\lambda - s_\mu)) - 2).
\end{aligned}$$

Similarly, we have

$$\frac{1}{\cosh\left(\frac{s_\mu \pi}{2}\right)} \sum_{j=1}^{N} \frac{N-j}{N}\cos(rs_\mu j) = \frac{1}{2\cosh\left(\frac{s_\mu \pi}{2}\right)}(F_N(rs_\mu) - 1).$$

The eigenvalue can then be analysed by using properties of the Fejér kernel. Indeed, we have that $F_N(s) \geq 0$ from the sine representation of the Fejér kernel for all $s \in \mathbb{R}$. Thus, the eigenvalue is bounded below by

$$\frac{1}{4\cosh\left(\frac{s_\mu \pi}{2}\right)}(0+0-2)+\frac{1}{2\cosh\left(\frac{s_\mu \pi}{2}\right)}(0-1)=-\frac{1}{\cosh\left(\frac{s_\mu \pi}{2}\right)} \geq -1.$$

For $\mu = \lambda$ we note that $F_N(0) = N$ so that a lower bound is given by

$$\frac{1}{4\cosh\left(\frac{s_\lambda \pi}{2}\right)}(N+0-2)+\frac{1}{2\cosh\left(\frac{s_\lambda \pi}{2}\right)}(0-1)=\frac{N-4}{4\cosh\left(\frac{s_\lambda \pi}{2}\right)},$$

as required. $\square$

Understanding the bounds on the spectrum of $W_{\lambda,r,N}$ allows one to obtain inequalities involving the matrix coefficients of certain functions under the operator. This is crucial since we will examine the action of $W_{\lambda,r,N}$ upon $\psi_\lambda \mathbf{1}_E$ via a decomposition over an orthonormal basis of eigenfunctions for $L^2(X)$. In fact, by manipulation of norms, we will see that the lower bounds on eigenvalues from Lemma 5.3.4, along with the upper bound for the operator norm in Lemma 5.3.3, will be sufficient to obtain a lower bound on the set volume.

Theorem 5.3.5. *Fix $0 < \varepsilon < 1$ and suppose that X is an $(R(X), C(X))$-admissible compact hyperbolic surface. Let ψ_λ be an L^2-normalised eigenfunction of the Laplacian on X with eigenvalue $\lambda \geq \frac{1}{4}$, and suppose that $E \subseteq X$ is a measurable set for which*

$$\|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon.$$

Then if $R(X) \geq 64\varepsilon^{-1}\cosh(\frac{s_\lambda \pi}{2})$, there exists a universal constant $A > 0$ (independent of all other parameters and the surface X) for which

$$\operatorname{Vol}(E) \geq \frac{A\varepsilon \min\{1, \operatorname{InjRad}(X)^2\}}{C(X)} e^{d(\lambda)\varepsilon R(X)},$$

where $d(\lambda)$ can be taken to be

$$d(\lambda) = \frac{1}{256 \cosh\left(\frac{s_\lambda \pi}{2}\right)}.$$

Proof. Set the parameters r and N as follows:

$$\begin{aligned} N &= \left\lfloor 8\varepsilon^{-1} \cosh\left(\frac{s_\lambda \pi}{2}\right)\right\rfloor, \\ r &= \left\lceil \frac{1}{8} N^{-1} R \right\rceil. \end{aligned}$$

Since ψ_λ is L^2-normalised, the parameter ε is bounded above by 1. Thus $N \geq 1$, and both N and r are positive integers. Additionally,

$$rN \leq \frac{1}{8} R + N \leq \frac{1}{4} R,$$

when $R \geq 64\varepsilon^{-1} \cosh\left(\frac{s_\lambda \pi}{2}\right)$. Thus, the parameters N and r satisfy the hypotheses of Lemmas 5.3.3 and 5.3.4. Fix $\delta > 0$ to be $0.005 < 0.01$ as required for the operator norm bounds from Lemma 5.3.3.

By use of the Hölder and Cauchy-Schwarz inequalities,

$$\begin{aligned} |\langle W_{\lambda,r,N}(\psi_\lambda \mathbf{1}_E), \psi_\lambda \mathbf{1}_E\rangle| &\leq \|W_{\lambda,r,N}(\psi_\lambda \mathbf{1}_E)\overline{\psi_\lambda \mathbf{1}_E}\|_1 \\ &\leq \|W_{\lambda,r,N}(\psi_\lambda \mathbf{1}_E)\|_\infty \|\psi_\lambda \mathbf{1}_E\|_1 \\ &\leq \|W_{\lambda,r,N}\|_{L^1 \to L^\infty} \|\psi_\lambda \mathbf{1}_E\|_1^2 \\ &\leq \|W_{\lambda,r,N}\|_{L^1 \to L^\infty} \operatorname{Vol}(E) \|\psi_\lambda \mathbf{1}_E\|_2^2 \\ &= \|W_{\lambda,r,N}\|_{L^1 \to L^\infty} \operatorname{Vol}(E)\varepsilon. \end{aligned}$$

Applying the operator norm bound of Lemma 5.3.3 then gives

$$|\langle W_{\lambda,r,N}(\psi_\lambda \mathbf{1}_E), \psi_\lambda \mathbf{1}_E\rangle| \leq \frac{C(X)A(0.005)}{\min\{1, \operatorname{InjRad}(X)^2\}} e^{-(\frac{1}{2} - 0.005)r} \operatorname{Vol}(E)\varepsilon.$$

We now seek a lower bound on this same inner product. We do this by considering the action of the operator $W_{\lambda,r,N}$ on the spectral decomposition of $\psi_\lambda \mathbf{1}_E$ over the orthonormal basis. Indeed, write

$$\psi_\lambda \mathbf{1}_E = \langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle \psi_\lambda + f_{\text{temp}} + f_{\text{untemp}},$$

where f_{temp} corresponds to the tempered part of the spectrum with the term corresponding to ψ_λ removed, and f_{untemp} corresponds to the untempered part of the spectrum. From Lemma 5.3.4, we known the action of $W_{\lambda,r,N}$ on each Laplacian eigenfunctions and hence also on f_{temp} and f_{untemp} from orthogonality of the eigenfunctions and thus,

$$\begin{aligned}
\langle W_{\lambda,r,N}(\langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle \psi_\lambda), \langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle \psi_\lambda \rangle &\geq \varepsilon^{-1} \|\psi_\lambda\|_2^2 |\langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle|^2 \\
&= \varepsilon^{-1} |\langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle|^2, \\
\langle W_{\lambda,r,N}(f_{\text{temp}}), f_{\text{temp}} \rangle &\geq -\|f_{\text{temp}}\|_2^2, \\
\langle W_{\lambda,r,N}(f_{\text{untemp}}), f_{\text{untemp}} \rangle &\geq 0.
\end{aligned}$$

Hence by one again using orthogonality of the eigenfunctions, we see that

$$\langle W_{\lambda,r,N}(\psi_\lambda \mathbf{1}_E), \psi_\lambda \mathbf{1}_E \rangle \geq \varepsilon^{-1} |\langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle|^2 - \|f_{\text{temp}}\|_2^2. \tag{5.5}$$

Now, notice that

$$|\langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle| = \|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon,$$

and also by an application of Pythagoras' theorem that

$$\begin{aligned}
\|f_{\text{temp}}\|_2^2 &\leq \|\psi_\lambda \mathbf{1}_E\|_2^2 - |\langle \psi_\lambda \mathbf{1}_E, \psi_\lambda \rangle|^2 \\
&= \|\psi_\lambda \mathbf{1}_E\|_2^2 (1 - \|\psi_\lambda \mathbf{1}_E\|_2^2)
\end{aligned}$$

$$= \|\psi_\lambda \mathbf{1}_E\|_2^2(1-\varepsilon).$$

Putting these into the lower bound of equation (5.5) gives

$$\langle W_{\lambda,r,N}(\psi_\lambda \mathbf{1}_E), \psi_\lambda \mathbf{1}_E\rangle \geq \|\psi_\lambda \mathbf{1}_E\|_2^2(1-(1-\varepsilon)) = \varepsilon^2.$$

Combining the upper and lower bounds on the inner product then provides

$$\mathrm{Vol}(E) \geq \frac{A\varepsilon \min\{1, \mathrm{InjRad}(X)^2\}}{C(X)} e^{(\frac{1}{2}-0.005)r},$$

where $A = \frac{1}{A(0.005)}$. We now compute using the assigned values of r and N that

$$\left(\frac{1}{2} - 0.005\right) r \geq \frac{1}{32} N^{-1} R \geq \frac{\varepsilon R}{256 \cosh\left(\frac{s_\lambda \pi}{2}\right)}.$$

This concludes the proof with

$$d(\lambda) = \frac{1}{256 \cosh\left(\frac{s_\lambda \pi}{2}\right)}.$$

□

5.4 Delocalisation of Untempered Eigenfunctions on Large Surfaces

We now turn to studying the eigenfunctions corresponding to small eigenvalues. As before, let $X = \Gamma\backslash\mathbb{H}$ be an $(R(X), C(X))$-admissible compact hyperbolic surface with associated fundamental domain $D \subseteq \mathbb{H}$, and let $E \subseteq X$ be a measurable subset. We will suppose this time that ψ_λ is an eigenfunction of the Laplacian with eigenvalue $\lambda = \frac{1}{4} - \sigma$ for some $0 < \sigma \leq \frac{1}{4}$. This in particular means that the spectral parameter s_λ for the eigenvalue is $\sqrt{\sigma} i$.

The methodology for bounding the volume of E will follow the same steps as in the tempered case. In fact, one can obtain an identical lower bound on the volume by using the work of the previous section, along with the operator

$$W_{\lambda,r,N} := W_{\frac{1}{4},r,N},$$

for $\lambda \in (0, \frac{1}{4})$. We instead opt here to use a different operator which allows us to obtain a stronger delocalisation result, by removing the ε dependence from the exponent of the volume lower bound.

The operator we use will be a rescaled ball-averaging operator on the surface. The kernel of this operator is given by

$$k_{t,\lambda}(\rho) = \frac{\mathbf{1}_{\{\rho \leq t\}}(\rho)}{\cosh(t)^{\frac{1}{2}(1+\sqrt{\sigma})}}.$$

In the usual way, we obtain an operator acting on functions on the surface through the following formula:

$$B_{t,\lambda} f(z) = \frac{1}{\cosh(t)^{\frac{1}{2}(1+\sqrt{\sigma})}} \int_D \sum_{\gamma \in \Gamma} \mathbf{1}_{\{d(z,\gamma w) \leq t\}}(w) f(w) \mathrm{d}\mu(w).$$

The $L^1(X) \to L^\infty(X)$ operator norm of $B_{t,\lambda}$ is then bounded by

$$\sup_{z,w \in D} \frac{1}{\cosh(t)^{\frac{1}{2}(1+\sqrt{\sigma})}} \sum_{\gamma \in \Gamma} |\mathbf{1}_{\{d(z,\gamma w) \leq t\}}(w)|.$$

Suppose that $C_0(\delta) > 0$ is a $\delta > 0$ dependent constant associated to the surface X via the $(R(X), C(X))$-admissibility. For $t \leq R(X)$, and fixed $z, w \in D$, the number of terms in the summand is bounded by $C(X)C_0(\delta)e^{\delta t}$. The $L^1(X) \to L^\infty(X)$ operator norm of $B_{t,\lambda}$ for $t \leq R(X)$ is thus bounded by

$$\|B_{t,\lambda}\|_{L^1(X) \to L^\infty(X)} \leq C(X)C_0(\delta)e^{\frac{1}{2}(2\delta - 1 - \sqrt{\sigma})t}, \tag{5.6}$$

for any $\delta > 0$. This bound serves the same function as Lemma 5.3.3 from the previous section. The second key ingredient required is an analogue of Lemma 5.3.4 to understand the spectral action of $B_{t,\lambda}$. Recall that the spectral action of operators defined through a kernel in this way, is obtained from the Selberg transform of the kernel. This can be computed as follows.

Lemma 5.4.1. *The Selberg transform of the function $k_{t,\lambda}(\rho)$ is given by*

$$h_{t,\lambda}(r) = \frac{4\sqrt{2}}{\cosh(t)^{\frac{1}{2}\sqrt{\sigma}}} \int_0^t \cos(ru)\sqrt{1 - \frac{\cosh(u)}{\cosh(t)}}\mathrm{d}u.$$

Proof. We use the formulae quoted in Section 5.2 to determine the Selberg transform. Firstly notice that

$$\begin{aligned} g(u) &= \frac{\sqrt{2}}{\cosh(t)^{\frac{1}{2}(1+\sqrt{\sigma})}} \int_{|u|}^t \frac{\sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}}\mathrm{d}\rho \mathbf{1}_{\{|u|\leq t\}}(u) \\ &= \frac{2\sqrt{2}}{\cosh(t)^{\frac{1}{2}(1+\sqrt{\sigma})}} \sqrt{\cosh(t) - \cosh(u)}\mathbf{1}_{\{|u|\leq t\}}(u). \end{aligned}$$

Thus, one obtains

$$\begin{aligned} h_{t,\lambda}(r) &= \frac{2\sqrt{2}}{\cosh(t)^{\frac{1}{2}\sqrt{\sigma}}} \int_{-t}^t e^{iru}\sqrt{1 - \frac{\cosh(u)}{\cosh(t)}}\mathrm{d}u \\ &= \frac{4\sqrt{2}}{\cosh(t)^{\frac{1}{2}\sqrt{\sigma}}} \int_0^t \cos(ru)\sqrt{1 - \frac{\cosh(u)}{\cosh(t)}}\mathrm{d}u. \end{aligned}$$

□

From now on, we will work with the operator $B_{t,\lambda}$ for $t = R(X)$. To obtain the desired bounds on the spectral action, we will require the following lemma that we isolate for readability. The result is a purely technical calculation, and so the reader who wishes to follow the main line of argument for the volume bounds will be at no loss by skipping over the proof.

Lemma 5.4.2. *Suppose that $a \in (\sqrt{\sigma}, \frac{1}{2})$ for some $\sigma > 0$, then for all $R \geq 2$,*

$$\int_0^R \cosh(au)\sqrt{1 - \frac{\cosh(u)}{\cosh(R)}}\mathrm{d}u \geq \frac{1}{3}\sinh(\sqrt{\sigma}R).$$

Proof. Since $\cosh(au)$ is an increasing function in u, the integrand is non-negative and the expression under the square root is contained in $[0, 1]$, we may bound the integral as follows:

$$\begin{aligned}
&\int_0^R \cosh(au)\sqrt{1 - \frac{\cosh(u)}{\cosh(R)}}\mathrm{d}u \\
&\qquad \geq \int_0^R \cosh(\sqrt{\sigma}u)\left(1 - \frac{\cosh(u)}{\cosh(R)}\right)\mathrm{d}u \\
&\qquad = \frac{\sinh(R\sqrt{\sigma})}{\sqrt{\sigma}} - \frac{1}{2}\left(\frac{\sinh((\sqrt{\sigma}+1)R)}{(\sqrt{\sigma}+1)\cosh(R)} + \frac{\sinh((1-\sqrt{\sigma})R)}{(1-\sqrt{\sigma})\cosh(R)}\right).
\end{aligned}$$

This expression is then equal to

$$\frac{\begin{array}{l} 2\sinh(R\sqrt{\sigma})\cosh(R)(1-\sigma) \\ \qquad\qquad - \sinh((\sqrt{\sigma}+1)R)(\sqrt{\sigma}-\sigma) - \sinh((1-\sqrt{\sigma})R)(\sigma+\sqrt{\sigma}) \end{array}}{2\sqrt{\sigma}(\sqrt{\sigma}+1)(1-\sqrt{\sigma})\cosh(R)}. \tag{5.7}$$

Since $\sqrt{\sigma} \leq \frac{1}{2}$, the denominator is bounded above by $\frac{3}{2}\cosh(R)$, and so we seek a lower bound on the numerator. Using angle sum formulae for the hyperbolic functions we see that

$$\begin{aligned}
-\sqrt{\sigma}(\sinh((\sqrt{\sigma}+1)R) + \sinh((1-\sqrt{\sigma})R)) &= -2\sqrt{\sigma}\sinh(R)\cosh(R\sqrt{\sigma}) \\
\sigma(\sinh((\sqrt{\sigma}+1)R) - \sinh((1-\sqrt{\sigma})R)) &= 2\sigma\sinh(R\sqrt{\sigma})\cosh(R).
\end{aligned}$$

The numerator of equation (5.7) is thus equal to

$$2\sinh(R\sqrt{\sigma})\cosh(R) - 2\sqrt{\sigma}\sinh(R)\cosh(R\sqrt{\sigma}).$$

Which again, using angle sum formulae for the hyperbolic functions, reduces to

$$(2-2\sqrt{\sigma})\sinh(R\sqrt{\sigma})\cosh(R) - 2\sqrt{\sigma}\sinh((1-\sqrt{\sigma})R).$$

Using this lower bound on the numerator, and the upper bound on the denominator shown above, we see that (5.7) is bounded below by

$$\frac{2}{3}(2-2\sqrt{\sigma})\sinh(R\sqrt{\sigma}) - \frac{4\sqrt{\sigma}}{3}\frac{\sinh((1-\sqrt{\sigma})R)}{\cosh(R)}.$$

Claim. For $R \geq 2$,

$$\frac{4\sqrt{\sigma}}{3}\frac{\sinh((1-\sqrt{\sigma})R)}{\cosh(R)} \leq \frac{1}{3}(2-2\sqrt{\sigma})\sinh(R\sqrt{\sigma}).$$

If true, this shows that the integral is bounded below by

$$\frac{1}{3}(2-2\sqrt{\sigma})\sinh(R\sqrt{\sigma}) \geq \frac{1}{3}\sinh(R\sqrt{\sigma}),$$

whenever $R \geq 2$, and thus the result follows.

Proof of Claim. By using an angle sum formula expansion of $\sinh((1-\sqrt{\sigma})R)$, and rearranging the inequality, we see that it suffices to show that

$$1 \leq \left(\frac{2+2\sqrt{\sigma}}{4\sqrt{\sigma}}\right)\tanh(R\sqrt{\sigma})\coth(R).$$

For fixed $R \geq 2$, consider the function

$$x \mapsto \left(\frac{2+2x}{4x}\right)\tanh(Rx),$$

defined for $x \in (0, \frac{1}{2}]$. By differentiating, one can see that this function has a single stationary point in the interval $(0, \frac{1}{2}]$ for $R \geq 2$. Moreover, the function is strictly increasing in a neighbourhood of zero, and strictly decreasing in a neighbourhood of $\frac{1}{2}$. Thus, the stationary point is a local maxima. The values taken by this function in the domain $(0, \frac{1}{2}]$ are then bounded below by the values taken at $x = \frac{1}{2}$ and as $x \to 0^+$; these are $\frac{3}{2}\tanh(\frac{1}{2}R)$ and $\frac{1}{2}R$ respectively. In either case, we can conclude that

$$\left(\frac{2+2\sqrt{\sigma}}{4\sqrt{\sigma}}\right)\tanh(R\sqrt{\sigma})\coth(R) \geq 1,$$

where $R \geq 2$, as required. $\square$

Lemma 5.4.3. *Suppose that $\lambda = \frac{1}{4} - \sigma$ for some $0 < \sigma \leq \frac{1}{4}$ and $\mu \in [0, \infty)$ are eigenvalues of the Laplacian on an $(R(X), C(X))$-admissible compact hyperbolic surface X. Let ψ_μ be an eigenfunction of the Laplacian on X with eigenvalue μ. Then, if $R(X) \geq \frac{2}{\sqrt{\sigma}}\log(2 + 2\varepsilon^{-1})$, the following bounds hold.*

1. *If $\mu \geq \frac{1}{4}$, then the eigenvalue of ψ_μ under the action of $B_{R,\lambda}$ is at least -1.*

2. *If $\mu \in [0, \frac{1}{4})$, then the eigenvalue of ψ_μ under the action of $B_{R,\lambda}$ is at least 0.*

3. *The eigenvalue of ψ_λ under $B_{R,\lambda}$ is at least ε^{-1}.*

Proof. First, suppose that $\mu \geq \frac{1}{4}$. Then,

$$\begin{aligned} h_{R,\lambda}(s_\mu) &\geq -\frac{4\sqrt{2}}{\cosh(R)^{\frac{1}{2}\sqrt{\sigma}}}\int_0^R \sqrt{1 - \frac{\cosh(u)}{\cosh(R)}}\mathrm{d}u \\ &\geq -\frac{4\sqrt{2}R}{\cosh(R)^{\frac{1}{2}\sqrt{\sigma}}}, \end{aligned}$$

For R sufficiently large, this is bounded below by -1. The case when $\mu \in [0, \frac{1}{4})$ is trivial since the integrand is non-negative from the spectral parameter s_μ being purely imaginary.

For the spectral action on ψ_λ, write $\lambda = \frac{1}{4} - a_\lambda^2$, so that the spectral parameter of λ is $s_\lambda = a_\lambda i$. Then by defintion,

$$h_{R,\lambda}(s_\lambda) = \frac{4\sqrt{2}}{\cosh(R)^{\frac{1}{2}\sqrt{\sigma}}} \int_0^R \cosh(a_\lambda u)\sqrt{1 - \frac{\cosh(u)}{\cosh(R)}}\mathrm{d}u.$$

By assumption on λ, it follows that $a_\lambda \in (\sqrt{\sigma}, \frac{1}{2})$. Lemma 5.4.2 then shows that

$$h_{R,\lambda}(s_\lambda) \geq \frac{4\sqrt{2}}{3} \frac{\sinh(R\sqrt{\sigma})}{\cosh(R)^{\frac{1}{2}\sqrt{\sigma}}},$$

whenever $R \geq 2$. This expression is subsequently bounded below by $\frac{1}{2}e^{\frac{1}{2}\sqrt{\sigma}R} - 1$ which is at least ε^{-1} whenever $R \geq \frac{2}{\sqrt{\sigma}} \log(2 + 2\varepsilon^{-1})$. $\square$

We now combine the upper bound (5.6) with Lemma 5.4.3 to obtain the desired delocalisation result for small eigenvalues.

Theorem 5.4.4. *Fix $0 < \varepsilon < 1$, and suppose that X is an $(R(X), C(X))$-admissible compact hyperbolic surface. Suppose that $\lambda = \frac{1}{4} - \sigma$ for some $0 < \sigma \leq \frac{1}{4}$ is an eigenvalue of the Laplacian on X, and ψ_λ is an L^2-normalised eigenfunction with eigenvalue λ. Let $E \subseteq X$ be a measurable set for which*

$$\|\psi_\lambda \mathbf{1}_E\|_2^2 = \varepsilon.$$

Then if $R(X) \geq R_0(\lambda, \varepsilon)$ for some constant $R_0(\lambda, \varepsilon)$ dependent only upon λ and ε, there exists a universal constant $A > 0$ (independent of all other parameters and the surface), such that,

$$\mathrm{Vol}(E) \geq \frac{A\varepsilon}{C(X)} e^{(\frac{1}{4} + \frac{1}{2}\sqrt{\sigma})R}.$$

Proof. Suppose that $R \geq R_0(\lambda, \varepsilon)$ given by Lemma 5.4.3, so that the bounds for the spectral action of $B_{R,\lambda}$ hold. As in the proof of Theorem 5.3.5, we can use Lemma 5.4.3 to see that

$$\varepsilon^2 \leq \|B_{R,\lambda}\|_{L^1(X)\to L^\infty(X)}\varepsilon \mathrm{Vol}(E).$$

The operator norm of $B_{R,\lambda}$ is controlled as in equation (5.6) by

$$\|B_{R,\lambda}\|_{L^1(X)\to L^\infty(X)} \leq C(X)C_0(\delta)e^{\frac{1}{2}(2\delta-1-\sqrt{\sigma})R},$$

for any $\delta > 0$. Set $\delta = \frac{1}{4}$, then we obtain the lower bound

$$\mathrm{Vol}(E) \geq \frac{C\varepsilon}{C(X)}e^{(\frac{1}{4}+\frac{1}{2}\sqrt{\sigma})R},$$

where $C = \frac{1}{C_0(\frac{1}{4})}$. □

Combining Theorems 5.3.5 and 5.4.4 then gives the deterministic result Theorem 5.1.3. Theorem 5.1.6 is then obtained by using Theorem 5.1.4 to probabilistically set $R(X) = c\log(g)$ and $C(X) = \frac{1}{\min\{1,\mathrm{InjRad}(X)\}}$ and control the injectivity radius in Theorem 5.1.3.

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