Intermediate dimension of images of sequences under fractional Brownian motion

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Abstract

We show that the almost sure θ-intermediate dimension of the image of the set $F_p = \{0, 1, \frac{1}{2^p}, \frac{1}{3^p}, \ldots\}$ under index-h fractional Brownian motion is $\frac{\theta}{ph+\theta}$, a value that is smaller than that given by directly applying the Hölder bound for fractional Brownian motion. In particular this establishes the box-counting dimension of these images.

1 Introduction

Intermediate dimensions were introduced in [10] to interpolate between the Hausdorff dimension and box-counting dimensions of sets where these differ, see the recent surveys [9, 11] for surveys on intermediate dimensions and dimension interpolation. The lower and upper intermediate dimensions, $\underline{\dim}_\theta E$ and $\overline{\dim}_\theta E$ of a set $E \subseteq \mathbb{R}^n$ depend on a parameter $\theta \in [0, 1]$, with $\underline{\dim}_0 E = \overline{\dim}_0 E = \dim_{\mathrm{H}} E$ and $\underline{\dim}_1 E = \underline{\dim}_{\mathrm{B}} E$ and $\overline{\dim}_1 E = \overline{\dim}_{\mathrm{B}} E$, where $\dim_{\mathrm{H}}, \underline{\dim}_{\mathrm{B}}$ and $\overline{\dim}_{\mathrm{B}}$ denote Hausdorff, and lower and upper box-counting dimensions, respectively. Various properties of intermediate dimensions are established in [1, 10] with the intermediate dimensions reflecting the range of diameters of sets needed to get coverings that are efficient for estimating dimensions. In particular $\underline{\dim}_\theta E$ and $\overline{\dim}_\theta E$ are monotonically increasing in $\theta \in [0, 1]$, are continuous except perhaps at $\theta = 0$, and are invariant under bi-Lipschitz mappings. Intermediate dimensions have been calculated for many sets which have differing Hausdorff and box-counting dimensions, including for sets of the form F_p given in (1.5) below in [10], as well as for attractors of infinitely generated conformal iterated function systems [2], spirals [6], countable families of concentric spheres [16] and topologists' sine curves [16], with non-trivial bounds obtained for self-affine carpets [10, 13].

Specifically, for $E \subseteq \mathbb{R}^n$ and $0 \leq \theta \leq 1$, the *lower intermediate dimension* of E may be defined as

$$\underline{\dim}_\theta E = \inf \big\{ s \geq 0 : \text{ for all } \epsilon > 0 \text{ and all } 0 < r_0 < 1, \text{ there exists } 0 < r \leq r_0 \qquad (1.1)$$
$$\text{and a cover } \{U_i\} \text{ of } E \text{ such that } r^{1/\theta} \leq |U_i| \leq r \text{ and } \textstyle\sum |U_i|^s \leq \epsilon \big\}$$

and the corresponding *upper intermediate dimension* by

$$\overline{\dim}_\theta E = \inf\big\{s \geq 0 : \text{ for all } \epsilon > 0 \text{ there exists } 0 < r_0 < 1 \text{ such that for all } 0 < r \leq r_0, \tag{1.2}$$
$$\text{there is a cover } \{U_i\} \text{ of } E \text{ such that } r^{1/\theta} \leq |U_i| \leq r \text{ and } \textstyle\sum |U_i|^s \leq \epsilon\big\},$$

where $|U|$ denotes the diameter of a set $U \subseteq \mathbb{R}^n$. When $\theta = 0$ (1.1) and (1.2) reduce to Hausdorff dimension, since there are no lower bounds on the diameters of covering sets. When $\theta = 1$ all covering sets are forced to have the same diameter and we recover the lower and upper box-counting dimensions.

It is convenient to work with equivalent definitions of these intermediate dimensions in terms of the exponential behaviour of sums over covers. For $E \subseteq \mathbb{R}^n$ bounded and non-empty, $\theta \in (0, 1]$, $r > 0$ and $s \in [0, n]$, define

$$S^s_{r,\theta}(E) := \inf\Big\{\sum_i |U_i|^s : \{U_i\}_i \text{ is a cover of } E \text{ such that } r \leq |U_i| \leq r^\theta \text{ for all } i\Big\}. \tag{1.3}$$

It is immediate that

$$\underline{\dim}_\theta E = \text{ the unique } s \in [0, n] \text{ such that } \liminf_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r} = 0 \tag{1.4}$$

and

$$\overline{\dim}_\theta E = \text{ the unique } s \in [0, n] \text{ such that } \limsup_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r} = 0.$$

For $p > 0$ let

$$F_p \;=\; \Big\{0, \frac{1}{1^p}, \frac{1}{2^p}, \ldots, \frac{1}{3^p} \cdots\Big\}. \tag{1.5}$$

It is shown in [10] that, for all $\theta \in [0, 1]$,

$$\dim_\theta F_p \;=\; \frac{\theta}{p+\theta}.$$

Index-h fractional Brownian motion ($0 < h < 1$) is the stochastic process $B_h : \mathbb{R}^{\geq 0} \to \mathbb{R}$ such that, almost surely, B_h is continuous with $B_h(0) = 0$, and the increments $B_h(x) - B_h(y)$ are stationary and Gaussian with mean 0 and variance $|x - y|^{2h}$, see, for example, [7, 8, 12, 14, 15]. There is a considerable literature on the dimensions of images of sets under stochastic processes, see [12] for Hausdorff dimensions, and [17] where the packing dimensions of images are expressed in terms of dimension profiles.

Here we investigate the almost sure intermediate dimensions of $B_h(F_p)$, the image of F_p under index-h fractional Brownian motion. By a simple estimate on the intermediate dimensions of Hölder images of sets, or see [1, Section 4], since B_h has almost sure Hölder exponent $h - \epsilon$ for all $\epsilon > 0$,

$$\dim_\theta B_h(F_p) \;\leq\; \frac{1}{h}\dim_\theta F_p \;=\; \frac{\theta}{h(p+\theta)};$$

however the actual value is smaller than this.

Theorem 1.1. *Let $B_h : \mathbb{R} \to \mathbb{R}$ be index-h fractional Brownian motion. Then almost surely, for all $\theta \in [0,1]$,*

$$\dim_\theta B_h(F_p) \;=\; \frac{\theta}{ph+\theta}, \tag{1.6}$$

and in particular

$$\dim_B B_h(F_p) \;=\; \frac{1}{ph+1}.$$

We obtain the upper bound for $\dim_\theta B_h(F_p)$ by, for each r, covering the part of $B_h(F_p)$ near 0 by abutting intervals of lengths r^θ and the remaining points individually by intervals of length r. The lower bound uses a potential theoretic method, estimating an energy of the image under B_h of the measure given by equal point masses on the points of F_p between $1/(2M-1)^p$ and $1/M^p$.

'Intermediate dimension profiles' were introduced in [5] to develop a general theory of intermediate dimensions including their behaviour under projections and this was developed for random images in [4]. By [4, Theorem 3.4]

$$\dim_\theta B_h(E) \;=\; \frac{1}{h}\dim_\theta^h E$$

where $\dim_\theta^s E$ is the θ-dimension profile of a general compact $E \subseteq \mathbb{R}$, but this does not give an explicit value of $\dim_\theta B_h(F_p)$. Whilst this might be found by an awkward calculation of dimension profiles, our proof here of (1.6) is self-contained.

2 Proofs

We recall from [5] the energy kernels $\widetilde{\phi}^s_{r,\theta}$ on $\mathbb{R}^m$ defined for $0 < r < 1, \theta \in [0,1]$ and $0 < s \le m$ by

$$\widetilde{\phi}^s_{r,\theta}(x) = \begin{cases} 1 & |x| < r \\ \left(\dfrac{r}{|x|}\right)^s & r \le |x| < r^\theta \\ 0 & r^\theta \le |x| \end{cases} \tag{2.1}$$

(here we only need the case of $m = 1$).

The proof of the lower bound for (1.6) uses the following three lemmas. The first is a slight variant of [5, Lemma 4.3] that relates the covering sums to energies with respect to the kernel $\widetilde{\phi}^s_{r,\theta}$. We write $\mathcal{M}(F)$ for the set of Borel probability measures supported by$F$.

Lemma 2.1. *Let $F \subseteq \mathbb{R}^m$ be compact, $\theta \in (0,1]$, $0 < r < 1$ and $0 \le s \le m$ and let $\mu \in \mathcal{M}(F)$. Then*

$$S^s_{r,\theta}(F) \;\ge\; r^s\left[\iint \widetilde{\phi}^s_{r,\theta}(x-y)d\mu(x)d\mu(y)\right]^{-1} \tag{2.2}$$

Proof. Since $\widetilde{\phi}^s_{r,\theta}$ is lower semicontinuous, by standard potential theory there is an equilibrium measure μ_0 for which $\iint \widetilde{\phi}^s_{r,\theta}(x-x)d\mu(x)d\mu(y)$ attains its minimum, say,

$$\iint \widetilde{\phi}^s_{r,\theta}(x-y)d\mu_0(x)d\mu_0(y) = \gamma.$$

Moreover,

$$\int \widetilde{\phi}^s_{r,\theta}(x-y)d\mu_0(y) \ \geq \ \gamma$$

for all $x \in F$, with equality if $x \in F_0$ for a set $F_0 \subseteq F$ with $\mu_0(F_0) = 1$.

If $r \leq \delta < r^\theta$ and $x \in F_0$ then using (2.1),

$$\gamma = \int \widetilde{\phi}^s_{r,\theta}(x-y)d\mu(y) \geq \int \Big(\frac{r}{\delta}\Big)^s 1_{B(0,\delta)}(x-y)d\mu(y) \geq \Big(\frac{r}{\delta}\Big)^s \mu(B(x,\delta)). \tag{2.3}$$

Let $\{U_i\}_i$ be a finite cover of F by sets of diameters $r \leq |U_i| < r^\theta$ and write $\mathcal{I} = \{i : U_i \cap F_0 \neq \emptyset\}$, so for each $i \in \mathcal{I}$ we may choose $x_i \in U_i \cap F_0$ such that $U_i \subseteq B(x_i, |U_i|)$. Then

$$1 = \mu(F_0) \leq \sum_{i\in\mathcal{I}} \mu(U_i) \leq \sum_{i\in\mathcal{I}} \mu(B(x_i,|U_i|)) \leq r^{-s}\gamma \sum_{i\in\mathcal{I}} |U_i|^s$$

by (2.3), so

$$\sum_i |U_i|^s \geq r^s\gamma^{-1};$$

taking the infimum over all such covers gives (2.2). (Note that considering covers with $r \leq |U_i| < r^\theta$ makes no difference in the definition (1.3).) □

We next bound the expectation of $\widetilde{\phi}^s_{r,\theta}$ evaluated on increments of fractional Brownian motion in terms of another kernel:

$$\psi^s_{r,\theta}(x) \ = \ \min\Big\{1, \frac{r^{\theta(1-s)+s}}{|x|^h}\Big\}. \tag{2.4}$$

Lemma 2.2. *Let $B_h : \mathbb{R} \to \mathbb{R}$ be index-h fractional Brownian motion. Then there is a constant c depending only on s such that for $0 < r < 1$, $0 < \theta \leq 1$ and $0 < s < 1$,*

$$\mathbb{E}\big(\widetilde{\phi}^s_{r,\theta}(B_h(x) - B_h(y))\big) \ \leq \ c\ \psi^s_{r,\theta}(x-y).$$

Proof. Since $B_h(x) - B_h(y)$ has Gaussian density with mean 0 and variance $|x-y|^{2h}$,

$$\mathbb{E}\big(\widetilde{\phi}^s_{r,\theta}(B_h(x) - B_h(y))\big) = \frac{1}{\sqrt{2\pi}}\frac{1}{|x-y|^h}\int_{-\infty}^{\infty} \widetilde{\phi}^s_{r,\theta}(t)\exp\Big(\frac{-t^2}{2|x-y|^{2h}}\Big)dt.$$

This is bounded above by 1 since $\widetilde{\phi}^s_{r,\theta}(t) \leq 1$ and the Gaussian has integral 1. With $c_1 = 2/\sqrt{2\pi}$,

$$\begin{aligned}
\mathbb{E}\big(\widetilde{\phi}^s_{r,\theta}(B_h(x) - B_h(y))\big) &= \frac{c_1}{|x-y|^h}\int_0^\infty \widetilde{\phi}^s_{r,\theta}(t)\exp\Big(\frac{-t^2}{2|x-y|^{2h}}\Big)dt \\
&= \frac{c_1}{|x-y|^h}\Big[\int_0^r \exp\Big(\frac{-t^2}{2|x-y|^{2h}}\Big)dt + \int_r^{r^\theta} \frac{r^s}{t^s}\exp\Big(\frac{-t^2}{2|x-y|^{2h}}\Big)dt\Big] \\
&\leq \frac{c_1}{|x-y|^h}\Big[\int_0^r dt + \int_r^{r^\theta}\frac{r^s}{t^s}dt\Big] \\
&\leq \frac{c_1}{|x-y|^h}\Big[r + \frac{1}{1-s}r^{s+\theta(1-s)}dt\Big] \\
&\leq c_2\frac{r^{s+\theta(1-s)}}{|x-y|^h}
\end{aligned}$$

since $r < 1$ and $s + \theta(1-s) \leq 1$. □

The next lemma estimates the energy of a measure on F_p under the kernel $\psi^s_{r,\theta}$.

Lemma 2.3. *Given $\theta, s \in (0,1]$ and $h \in (0,1)$ there is a number $c > 0$ such that for all $0 < r < 1$ there is a measure $\mu_r \in \mathcal{M}(F_p)$ such that*

$$E(\mu_r) := \int\!\!\int \psi^s_{r,\theta}(x-y)d\mu_r(x)d\mu_r(y) \ \leq\ c\, r^{(\theta(1-s)+s)/(1+ph)}.$$

Proof. Given r we will find $M \equiv M(r) \in \mathbb{N}$ such that the measure μ_r formed by placing a point mass of $1/M$ on each of the M points of

$$F_p^M \ = \ \left\{\frac{1}{(2M-1)^p}, \frac{1}{(2M-2)^p}, \ldots, \frac{1}{M^p}\right\} \ \subseteq \ F_p$$

satisfies the conclusion. Note that if $M \leq k \leq 2M-1$, then

$$\frac{p}{(2M)^{p+1}} \ \leq \ \frac{p}{(k+1)^{p+1}} \ < \ \left|\frac{1}{k^p} - \frac{1}{(k+1)^p}\right| \ < \ \frac{p}{k^{p+1}} \ \leq \ \frac{p}{M^{p+1}},$$

by a mean value theorem estimate. In particular, if g is the minimum gap length between any pair of points of F_p^M, then $g \geq p/(2M)^{p+1}$ so any interval of length $R \geq g$ intersects at most $aM^{p+1}R$ points of F_p^M, where $a = 2^{p+2}/p$. We estimate the energy

$$E(\mu_r) \ = \ \int\!\!\int \psi^s_{r,\theta}(x-y)d\mu_r(x)d\mu_r(y) \ = \ \frac{1}{M^2}\sum_{M\leq i,j\leq 2M-1}\psi^s_{r,\theta}(x_i - x_j)$$

where for convenience we write $x_i = 1/i^p$. Let m be the greatest integer such that $2^m g \leq M^{-p}$. Using (2.4),

$$\begin{aligned}
M^2E(\mu_r) \ &= \ \sum_{x_i=x_j}\psi^s_{r,\theta}(x_i-x_j) \ + \sum_{g\leq|x_i-x_j|\leq 1/M^p}\psi^s_{r,\theta}(x_i-x_j)\\
&= \ M \ + \sum_{g\leq|x_i-x_j|\leq 1/M^p}\frac{r^{\theta(1-s)+s}}{|x_i-x_j|^h}\\
&\leq \ M \ + \sum_{k=0}^{m}\ \sum_{2^k g\leq|x_i-x_j|\leq 2^{k+1}g}\frac{r^{\theta(1-s)+s}}{|x_i-x_j|^h}\\
&\leq \ M \ + \ r^{\theta(1-s)+s}\sum_{k=0}^{m}(2^k g)^{-h}MaM^{p+1}2^{k+1}g\\
&\leq \ M \ + \ c_1M^{p+2}r^{\theta(1-s)+s}g^{1-h}\,2^{m(1-h)}\\
&\leq \ M \ + \ c_1M^{p+2}r^{\theta(1-s)+s}g^{1-h}(M^{-p}g^{-1})^{(1-h)}\\
&= \ M \ + c_1M^{2+ph}r^{\theta(1-s)+s}
\end{aligned}$$

where c_1 is a constant depending only on h and we have taken the dominant term of the geometric sum. Thus

$$\begin{aligned}
E(\mu_r) \ &\leq \ M^{-1} \ + \ c_1M^{ph}r^{\theta(1-s)+s}\\
&= \ (2+c_12^{ph})r^{(\theta(1-s)+s)/(1+ph)}
\end{aligned}$$

on setting $M = \lceil r^{-(\theta(1-s)+s)/(1+ph)}\rceil \leq 2r^{-(\theta(1-s)+s)/(1+ph)}$ as $r<1$. □

Proof of Theorem 1.1. The conclusion is clear when $\theta = 0$, as $B_h(F_p)$ is countable so has Hausdorff dimension 0, so assume $\theta \in (0, 1]$.

Upper bound: Let $0 < \epsilon < h$. Index-h fractional Brownian motion satisfies

$$|B_h(x)| \ \leq \ Kx^{h-\epsilon} \qquad (0 \leq x \leq 1)$$

almost surely for some $K < \infty$ see, for example, [3].

Let $0 < r < 1$ and M be an integer and take a cover of $B_h(F_p)$ by intervals $\{U_i\}$ with $r \leq |U_i| \leq r^\theta$ by covering each point $B_h(1/k^p)$ $(1 \leq k \leq M)$ by an interval of length r and covering $B_h\big([0, 1/M^p]\big) \subseteq [-KM^{-p(h-\epsilon)}, KM^{-p(h-\epsilon)}]$ by abutting intervals of length r^θ. Then

$$\sum_i |U_i|^s \ \leq \ Mr^s + \left[\frac{2KM^{-p(h-\epsilon)}}{r^\theta} + 1\right](r^\theta)^s \ = \ Mr^s + 2Kr^{\theta(s-1)}M^{-p(h-\epsilon)} + r^{s\theta}.$$

Setting $M = \lceil r^{(\theta(s-1)-s)/(1+p(h-\epsilon))}\rceil \geq 2$ gives

$$\sum_i |U_i|^s \ \leq \ 2(1+K)r^{(s(p(h-\epsilon)+\theta)-\theta)/(1+p(h-\epsilon))} + r^{s\theta} \to 0$$

as $r \to 0$ provided that $s > \theta/(p(h-\epsilon)+\theta)$. Taking ϵ arbitrarily small we conclude that

$$\overline{\dim}_\theta B_h(F_p) \leq \frac{\theta}{ph+\theta} \tag{2.5}$$

almost surely, by the definition (1.2) of $\overline{\dim}_\theta$.

Lower bound: From Lemmas 2.2 and 2.3 and using Fubini's theorem, there is a number c independent of r such that for all $0 < r < 1$ there is a measure μ_r on F_p such that

$$\mathbb{E}\bigg(\iint \widetilde{\phi}^s_{r,\theta}\big(B_h(x) - B_h(y)\big)d\mu_r(x)d\mu_r(y)\bigg) \ \leq c\ r^{(\theta(1-s)+s)/(1+ph)}.$$

For $\epsilon > 0$, setting $r = 2^{-k}, k \in \mathbb{N}$, and summing,

$$\mathbb{E}\bigg(\sum_{k=1}^\infty 2^{k[(\theta(1-s)+s)/(1+ph)-\epsilon]} \iint \widetilde{\phi}^s_{2^{-k},\theta}\big(B_h(x) - B_h(y)\big)d\mu_{2^{-k}}(x)d\mu_{2^{-k}}(y)\bigg) \ \leq c\sum_{k=1}^\infty 2^{-k\epsilon} \ < \infty.$$

Hence, almost surely there exists a random $K < \infty$ such that

$$\iint \widetilde{\phi}^s_{r,\theta}\big(B_h(x) - B_h(y)\big)d\mu_r(x)d\mu_r(y) \ \leq \ Kr^{(\theta(1-s)+s)/(1+ph)-\epsilon}$$

for $r = 2^{-k}$ for all $k \in \mathbb{N}$ and thus for all $0 < r < 1$ with a modified K, noting that the two sides change only by a bounded ratio on replacing r by 2^{-k} for the least k such that $2^{-k} \leq r$. Writing $\widetilde{\mu}_r$ for the image measure of μ_r under B_h, so $\widetilde{\mu}_r$ is supported by $B_h(F_p)$ (in the notation of Lemma 2.3 $\widetilde{\mu}_r$ consists of a mass of $1/M$ on each point $B_r(x_i)$), this becomes

$$\iint \widetilde{\phi}^s_{r,\theta}(u-v)d\widetilde{\mu}_r(u)d\widetilde{\mu}_r(v) \ \leq \ Kr^{(\theta(1-s)+s)/(1+ph)-\epsilon}.$$

Thus, by Lemma 2.1, almost surely there is a $K < \infty$ such that for all $0 < r < 1$,

$$S^s_{r,\theta}(B_h(F_p)) \;\geq\; K^{-1} r^{s-(\theta(1-s)+s)/(1+ph)+\epsilon}$$

so

$$\liminf_{r\to 0} \frac{\log S^s_{r,\theta}(B_h(F_p))}{-\log r} \;\geq\; -s + \frac{\theta(1-s)+s}{(1+ph)} - \epsilon \;=\; \frac{\theta(1-s)-sph}{(1+ph)} - \epsilon$$

and this remains true almost surely on setting $\epsilon = 0$. Hence $\liminf_{r\to 0} \log S^s_{r,\theta}(B_h(F_p))/-\log r \geq 0$ if $\theta(1-s)+sph = 0$, that is if $s = \theta/(ph+\theta)$, so by (1.4)

$$\underline{\dim}_\theta B_h(F_p) \;\geq\; \frac{\theta}{ph+\theta}. \tag{2.6}$$

Combined with (2.5) this gives (1.6) almost surely for each $\theta \in [0,1]$. Thus, almost surely, (1.6) holds for all rational $\theta \in [0,1]$ simultaneously, and so, since intermediate dimensions are continuous for $\theta \in (0,1]$, for all $\theta \in [0,1]$ simultaneously. □

Finally we remark that a similar approach can be used to find or estimate the intermediate and box-counting dimensions of fractional Brownian images of sets defined by other sequences tending to 0. For example, let $f : [1,\infty) \to \mathbb{R}^+$ be a decreasing function and let $F = \{0, f(1), f(2), \ldots\}$. Then if $f(x) = O(x^{-p})$ the upper bound argument gives

$$\overline{\dim}_\theta B_h(F) \leq \frac{\theta}{ph+\theta} \tag{2.7}$$

almost surely. If we also assume that f is differentiable with non-increasing absolute derivative $|f'|$ then F has 'decreasing gaps', that is $f(k) - f(k+1)$ is non-increasing, and if

$$\frac{f(x)^{1-h}}{|f'(2x)|} = O(x^{1+ph})$$

a similar energy argument gives that almost surely, for all $\theta \in [0,1]$,

$$\underline{\dim}_\theta B_h(F) \;\geq\; \frac{\theta}{ph+\theta}. \tag{2.8}$$

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