



Citation for published version:

Kiselev, AV & Ryzhov, V 2022 'Mathematical Heritage of Sergey Naboko: Functional Models of Non-Self-Adjoint Operators'.

Publication date:
2022

[Link to publication](#)

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Mathematical Heritage of Sergey Naboko: Functional Models of Non-Self-Adjoint Operators

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April 6, 2022

1 Dilation theory for dissipative operators

Functional model construction for a contractive linear operator T acting on a Hilbert space K is a well developed domain of the operator theory. Since pioneering works by B. Sz.-Nagy, C. Foiaş [40], P. D. Lax, R. S. Phillips [14], L. de Branges, J. Rovnyak [4, 5], and M. Livšic [15], this research area attracted many specialists in operator theory, complex analysis, system control, gaussian processes and other disciplines. Multiple studies culminated in the development of a comprehensive theory complemented by various applications, see [9, 10, 28, 29, 31] and references therein.

The underlying idea of functional model is the fundamental theorem of B. Sz.-Nagy and C. Foias stating that for a dissipative operator L under the assumption $\mathbb{C}_- \subset \rho(L)$ (dissipative operators satisfying this condition are called *maximal*), there exists a selfadjoint dilation of L , which is a selfadjoint operator \mathcal{L} on a wider space $\mathcal{H} \supset K$ such that

$$(L - zI)^{-1} = P_K(\mathcal{L} - zI)^{-1}|_K, \quad z \in \mathbb{C}_-, \quad (1)$$

where P_K is an orthogonal projection from \mathcal{H} onto K .

In applications, such a dilation \mathcal{L} should be *minimal*; it should not contain any reducing self-adjoint parts not related to the operator L . Mathematically the minimality condition is expressed as the equality

$$\text{clos} \bigvee_{z \notin \mathbb{R}} (\mathcal{L} - zI)^{-1} |_K = \mathcal{H},$$

where \mathcal{H} is the dilation space $\mathcal{H} \supset K$. Construction of a dilation satisfying this condition is a non-trivial task successfully solved for contractions by Sz.-Nagy and Foiaş [40] with the help of Neumark's theorem [27], and by B. Pavov [34, 35] for two important cases of dissipative operators arising in mathematical physics and successfully extended later to a general setting (more on this in the following sections).

The functional model theory of non-selfadjoint operators studies operators L which have no non-trivial reducing selfadjoint parts. Such operators are called *completely non-selfadjoint* or, using a less accurate term, *simple*. In what follows, all non-selfadjoint operators are assumed closed, densely defined and simple, with regular points in both lower and upper half-planes.

1.1 Additive perturbations

Let $A = A^*$ be a selfadjoint unbounded operator on a Hilbert space K and V a bounded (for simplicity) non-negative operator $V = V^* = \alpha^2/2 \geq 0$, where $\alpha = (2V)^{1/2}$. Let $L = A + \frac{i}{2}\alpha^2$. The operators A and $V = \alpha^2/2$ are the real and imaginary parts of L defined on $\text{dom}(L) = \text{dom}(A)$.

Following Pavlov, denote $E = \text{clos ran } \alpha$ and define the dilation space as the direct sum of K and the equivalents of incoming and outgoing channels of the Lax-Phillips scattering theory, see [14], $\mathcal{D}_\pm = L^2(\mathbb{R}_\pm, E)$,

$$\mathcal{H} = \mathcal{D}_- \oplus K \oplus \mathcal{D}_+. \quad (2)$$

Elements of \mathcal{H} are represented as three-component vectors (v_-, u, v_+) with $v_\pm \in \mathcal{D}_\pm$ and $u \in K$. The action of \mathcal{L} on the channels \mathcal{D}_\pm is defined by $\mathcal{L} : (v_-, 0, v_+) \mapsto (iv'_-, 0, iv'_+)$. The self-adjointness of $\mathcal{L} = \mathcal{L}^*$ and the requirement (1) lead to the form of dilation \mathcal{L} suggested in [34],

$$\mathcal{L} \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} = \begin{pmatrix} v_- \\ Au + \frac{\alpha}{2} [v_+(0) + v_-(0)] \\ v_+ \end{pmatrix}, \quad (3)$$

defined on the domain

$$\text{dom}(\mathcal{L}) = \{(v_-, u, v_+) \in \mathcal{H} \mid v_\pm \in W_2^1(\mathbb{R}_\pm, E), u \in \text{dom}(A), v_+(0) - v_-(0) = i\alpha u\}$$

The ‘‘boundary condition’’ $v_+(0) - v_-(0) = i\alpha u$ can be interpreted as a coupling between the incoming and outgoing channels \mathcal{D}_\pm , realised by the imaginary part of L acting on E . The characteristic function of L is the contractive operator-valued function defined by the formula

$$S(z) = I_E + i\alpha(L^* - zI)^{-1}\alpha : E \rightarrow E, \quad z \in \mathbb{C}_+. \quad (4)$$

Owing to the general theory [40], the operator L is unitary equivalent to its model in the spectral representation of \mathcal{L} in accordance with (1).

Due to the operator version of Fatou’s theorem [40], non-tangential boundary values of the function S exist in the strong operator topology almost everywhere on the real line. Denote $S = S(k) = \text{s-lim}_{\varepsilon \downarrow 0} S(k + i\varepsilon)$, a. e. $k \in \mathbb{R}$. Similarly, let $S^* = S^*(k) := \text{s-lim}_{\varepsilon \downarrow 0} [S(k + i\varepsilon)]^*$, which exists for almost all $k \in \mathbb{R}$. The symmetric form of the functional model is obtained by factorisation and completion of the dense linear set of vector-valued functions from the space $L^2(E) \oplus L^2(E)$ with respect to the norm

$$\left\| \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}^2 := \int_{\mathbb{R}} \left\langle \begin{pmatrix} I & S^* \\ S & I \end{pmatrix} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\rangle_{E \oplus E} dk \quad (5)$$

Note that the elements of \mathcal{H} are not individual functions from $L^2(E) \oplus L^2(E)$, but rather equivalence classes formed after factorization over elements with zero \mathcal{H} -norm, followed by completion [29, 30]. It is easily seen that for each $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}$ the expressions $g_+ := S\tilde{g} + g$ and $g_- := \tilde{g} + S^*g$ are in fact usual square summable vector-functions from $L_2(E)$.

The space $\mathcal{H} = L_2\left(\begin{smallmatrix} I & S^* \\ S & I \end{smallmatrix}\right)$ with the norm defined by (5) turns out to be the spectral representation space of the self-adjoint dilation \mathcal{L} of the operator L . Henceforth we will denote the corresponding unitary mapping of \mathcal{H} onto \mathcal{H} by Φ . It means that the operator of multiplication by the independent variable acting on \mathcal{H} , i.e., the operator $f(k) \mapsto kf(k)$, is unitary equivalent to the dilation \mathcal{L} . Hence, for $z \in \mathbb{C} \setminus \mathbb{R}$, the mapping $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \mapsto (k - z)^{-1} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}$, where $\begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}$ is

unitary equivalent to the resolvent of \mathcal{L} and therefore L is mapped to its functional model (with the symbol \simeq denoting unitary equivalence),

$$(L - zI)^{-1} \simeq P_{\mathcal{K}}(k - z)^{-1}|_{\mathcal{K}}, \quad z \in \mathbb{C}_- \quad (6)$$

The incoming and outgoing subspaces of the dilation space \mathcal{H} admit the form

$$\mathcal{D}_+ := \begin{pmatrix} H_2^+(E) \\ 0 \end{pmatrix}, \quad \mathcal{D}_- := \begin{pmatrix} 0 \\ H_2^-(E) \end{pmatrix}, \quad \mathcal{K} := \mathcal{H} \ominus [\mathcal{D}_+ \oplus \mathcal{D}_-]$$

where $H_2^\pm(E)$ are the Hardy classes of E -valued vector-functions analytic in \mathbb{C}_\pm and $\mathcal{D}_\pm = \Phi \mathcal{D}_\pm$. As usual [36], the functions from vector-valued Hardy classes $H_2^\pm(E)$ are identified with their boundary values existing almost everywhere on the real line. They form two complementary mutually orthogonal subspaces, so that $L_2(E) = H_2^+(E) \oplus H_2^-(E)$.

The image \mathcal{K} of K under the spectral mapping Φ of the dilation space \mathcal{H} to \mathcal{H} is the subspace

$$\mathcal{K} = \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H} \mid \tilde{g} + S^*g \in H_2^-(E), S\tilde{g} + g \in H_2^+(E) \right\}$$

The orthogonal projection $P_{\mathcal{K}}$ from \mathcal{H} onto \mathcal{K} is defined by formula (7). Note that the following definition has to be understood on the dense set of functions from $L^2(E) \oplus L^2(E)$ in \mathcal{H} .

$$P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+(\tilde{g} + S^*g) \\ g - P_-(S\tilde{g} + g) \end{pmatrix}, \quad \tilde{g} \in L_2(E), g \in L_2(E), \quad (7)$$

where P_\pm are the orthogonal projections from L_2 onto the Hardy classes H_2^\pm .

2 Naboko's functional model for a family of additive perturbations

The model approach to the analysis of dissipative operators outlined above relies exclusively on the knowledge of a characteristic function of a dissipative completely non-selfadjoint operator L . The properties of the operator are expressed in terms of its characteristic function, i. e., in the language of analytic operator-valued functions theory. This represents the true value of the functional model approach: all the abstract results obtained using model techniques become immediately available, once the characteristic function of the operator is known.

Successful applications of the functional model approach for contractions and dissipative operators have inspired the search for models of non-dissipative operators. The attempts to follow the blueprints of Sz.-Nagy-Foias and Lax-Phillips meet serious challenges rooted in the absence of a self-adjoint dilation for such operators.

The breakthrough came in the late seventies with the publication of papers [16, 17] and especially [18] by S. Naboko, who found a way to represent a non-dissipative operator in a model space of a suitably chosen dissipative one. Apart from the model construction, his works largely contributed to the development of various areas in the non-self-adjoint operator theory. In contrast to the earlier results, his model representation does not rely on the uniqueness (up to a unitary equivalence) of the characteristic function of a completely non-selfadjoint operator. Based on the dilation (3), the paper [18] provides an isometry between the dilation space (2) and the model space (5) in an explicit form. This explicitness plays a crucial rôle in passage to the model representation for non-dissipative operators using nothing more than Hilbert resolvent identities. All

the building blocks of the method are clearly presented in terms of the original problem, which is especially appealing from the applications' perspective. We next give a brief overview of the key ideas presented in [16, 17, 18].

2.1 Isometric map between the dilation and model spaces

Consider a non-self-adjoint operator

$$L = A + iV \quad (8)$$

acting in the Hilbert space K , where $A = A^*$ and $V = V^*$ is A -bounded with the relative bound less than 1. The domains of A and L coincide and the operator L is closed. Note that V can be written in the form $V = \frac{\alpha J \alpha}{2}$ with $\alpha = \sqrt{2|V|}$, $J := \text{sign } V : E \rightarrow E$ defined according to the functional calculus of self-adjoint operators. Like in (4), $E := \text{clos ran } \alpha$. The characteristic function of L admits the form (see, e.g., [39])

$$\Theta(z) = I_E + iJ\alpha(L^* - zI)^{-1}\alpha : E \rightarrow E, \quad z \in \rho(L^*). \quad (9)$$

Alongside with L introduce the operator L^\parallel on the same domain $\text{dom}(L^\parallel) = \text{dom}(L)$ as follows:

$$L^\parallel := A + i|V| = A + i\frac{\alpha^2}{2}. \quad (10)$$

The operator L^\parallel is precisely the dissipative operator of the preceding Section. The work [18] contains the model construction, the definition of the isometry $\Phi : \mathcal{H} \rightarrow \mathcal{K}$ from (a dense set in) the dilation space (2) to the model space (5) of L^\parallel , which is a preliminary step towards the model for its additive perturbations of the form (8). Note that the characteristic function S of L^\parallel is given by the expression (4) where L is replaced by L^\parallel :

$$S(z) = I_E + i\alpha(L^{\parallel*} - zI)^{-1}\alpha, \quad z \in \rho(L^{\parallel*}), \quad L^{\parallel*} := (L^\parallel)^*. \quad (11)$$

The argument of [16] shows that the characteristic functions of L and L^\parallel are related via the Potapov-Ginzburg operator linear-fractional transformation, or PG-transform [3]. This fact is essentially geometric. It relates contractions on Kreĭn spaces (i. e., the spaces with an indefinite metric defined by the involution $J = J^* = J^{-1}$) to contractions on Hilbert spaces. The PG-transform is invertible and the following assertion pointed out in [16] holds.

Proposition 2.1. *The characteristic function (9) of $L = A + iV$ is J -contractive on its domain and the PG-transform maps Θ to the contractive characteristic function of $L^\parallel = A + i|V|$ defined by (11), as follows:*

$$\Theta \mapsto S = -(\chi^+ - \Theta\chi^-)^{-1}(\chi^- - \Theta\chi^+), \quad S \mapsto \Theta = (\chi^- + \chi^+S)(\chi^+ + \chi^-S)^{-1}, \quad (12)$$

where $\chi^\pm = \frac{1}{2}(I_E \pm J)$ are orthogonal projections onto the subspaces χ^+E (χ^-E , respectively).

It appears somewhat unexpected that two operator-valued functions connected by formulae (12) can be explicitly written down in terms of their ‘‘main operators’’ L and L^\parallel . This relationship between the characteristic functions of L and L^\parallel goes in fact much deeper, see [2, 3]. In particular, the self-adjoint dilation of L^\parallel and the J -self-adjoint dilation of L are also related via a suitably

adjusted version of the PG-transform. Similar statements hold for the corresponding linear systems or “generating operators” of the functions Θ and S , see [2, 3]. This fact is crucial for the construction of a model of a general closed, densely defined non-self-adjoint operator, see [38].

Assume as usual that the operator L^\parallel is completely non-self-adjoint, and let \mathcal{L} be the minimal self-adjoint dilation of L^\parallel of the form (3).

Theorem 2.2 ([18], Theorem 2). *There exists a mapping Φ from the dilation space \mathcal{H} onto Pavlov’s model space \mathcal{H} defined by (5) with the following properties.*

1. Φ is isometric.
2. $\tilde{g} + S^*g = \mathcal{F}_+h$, $S\tilde{g} + g = \mathcal{F}_-h$, where $(\tilde{g}) = \Phi h$, $h \in \mathcal{H}$
3. $\Phi \circ (\mathcal{L} - zI)^{-1} = (k - z)^{-1} \circ \Phi$, $z \in \mathbb{C} \setminus \mathbb{R}$
4. $\Phi\mathcal{H} = \mathcal{H}$, $\Phi\mathcal{D}_\pm = \mathcal{D}_\pm$, $\Phi K = \mathcal{K}$
5. $\Phi \circ (\mathcal{L} - zI)^{-1} = (k - z)^{-1} \circ \Phi$, $z \in \mathbb{C} \setminus \mathbb{R}$.

Here the bounded maps $\mathcal{F}_\pm : \mathcal{H} \rightarrow L^2(\mathbb{R}, E)$ are defined by the formulae

$$\begin{aligned}\mathcal{F}_+ : h &\mapsto -\frac{1}{\sqrt{2\pi}} \alpha(L^\parallel - k + i0)^{-1}u + S^*(k)\hat{v}_-(k) + \hat{v}_+(k), \\ \mathcal{F}_- : h &\mapsto -\frac{1}{\sqrt{2\pi}} \alpha(L^{-\parallel} - k - i0)^{-1}u + \hat{v}_-(k) + S(k)\hat{v}_+(k),\end{aligned}$$

where $h = (v_-, u, v_+) \in \mathcal{H}$ and \hat{v}_\pm are the Fourier transforms of $v_\pm \in L^2(\mathbb{R}_\pm, E)$.

2.2 Model representation of additive perturbations

Theorem 2.2 opens a possibility of expressing a larger class of perturbations of A in the model space \mathcal{H} . Namely, consider operators in K of the form

$$L^\varkappa = A + \frac{\alpha\varkappa\alpha}{2}, \quad \text{dom}(L^\varkappa) = \text{dom}(A), \quad (13)$$

where \varkappa is a bounded operator in E . The family $\{L^\varkappa \mid \varkappa : E \rightarrow E\}$ includes A for $\varkappa = 0$, the dissipative operator L^\parallel for $\varkappa = iI_E$, its adjoint $L^{-\parallel}$ for $\varkappa = -iI_E$, as well as self-adjoint and non-self-adjoint operators corresponding to other values of the “parameter” \varkappa . In particular, the non-dissipative operator $L = A + iV = A + i\frac{\alpha J \alpha}{2}$ of (8) is recovered by putting $\varkappa = iJ$. Representations of the resolvent $(L^\varkappa - zI)^{-1}$, $z \in \rho(L^\varkappa)$ in the model space \mathcal{H} are obtained using the properties of \mathcal{F}_\pm given in Theorem 2.2 and resolvent identities for $(L^\parallel - zI)^{-1}$, $(L^{-\parallel} - zI)^{-1}$, and $(L^\varkappa - zI)^{-1}$. The key component of the proofs is the representation of $\mathcal{F}_\pm(L^\varkappa - zI)^{-1}u$ in terms of $\mathcal{F}_\pm u$ for $u \in K$. For instance, it can be shown that there exist two analytic operator-functions $\Theta'_\varkappa, \Theta_\varkappa : E \rightarrow E$, bounded in $\mathbb{C}_-, \mathbb{C}_+$ respectively, such that for $z_0 \in \rho(L^\varkappa)$, $\text{Im } z_0 < 0$, and all $u \in K$

$$\begin{aligned}\mathcal{F}_+(L^\varkappa - z_0I)^{-1}u &= \frac{1}{k - z_0}(\mathcal{F}_+u)(k - i0) - \frac{1}{k - z_0}\Theta'_\varkappa(k - i0)[\Theta'_\varkappa(z_0)]^{-1}(\mathcal{F}_+u)(z_0) \\ \mathcal{F}_-(L^\varkappa - z_0I)^{-1}u &= \frac{1}{k - z_0}(\mathcal{F}_-u)(k + i0) - \frac{1}{k - z_0}\Theta_\varkappa(k + i0)[\Theta'_\varkappa(z_0)]^{-1}(\mathcal{F}_+u)(z_0)\end{aligned} \quad (14)$$

Here $\mathcal{F}_\pm u \in H_2^\mp(E)$ since $u \in K$ and $(\mathcal{F}_+ u)(z_0) = (\tilde{g} + S^*g)(z_0)$ is the analytic continuation of the function $(\tilde{g} + S^*g)$ to the point z_0 in the lower half-plane. The possibility to express $\mathcal{F}_\pm(L^\mathcal{Z} - z_0 I)^{-1}u$ using the spectral mappings \mathcal{F}_\pm applied to $u \in K$ found on the right hand side of (14) is the key ingredient of calculations leading to the main theorem.

Theorem 2.3 (Model Theorem, [18]). *If $z_0 \in \mathbb{C}_- \cap \rho(L^\mathcal{Z})$ and $(\tilde{g}) \in \mathcal{H}$, then*

$$\Phi(L^\mathcal{Z} - z_0 I)^{-1} \Phi^* \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_{\mathcal{K}} \frac{1}{k - z_0} \begin{pmatrix} \tilde{g} \\ g - \frac{1+i\mathcal{Z}}{2} [\Theta'_\mathcal{Z}(z_0)]^{-1} (\tilde{g} + S^*g)(z_0) \end{pmatrix}$$

If $z_0 \in \mathbb{C}_+ \cap \rho(L^\mathcal{Z})$ and $(\tilde{g}) \in K$, then

$$\Phi(L^\mathcal{Z} - z_0 I)^{-1} \Phi^* \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_{\mathcal{K}} \frac{1}{k - z_0} \begin{pmatrix} \tilde{g} - \frac{1-i\mathcal{Z}}{2} [\Theta_\mathcal{Z}(z_0)]^{-1} (S\tilde{g} + g)(z_0) \\ g \end{pmatrix}$$

2.3 Smooth vectors and the absolutely continuous subspace

In [18, 20] Sergey Naboko introduced absolutely continuous subspaces of the family $L^\mathcal{Z}$. He always admired Mark Kreĭn, and in particular liked to quote him as saying: “the major instruments of self-adjoint spectral analysis arise from the Hilbert space geometry, whereas in the non-self-adjoint setup the modern complex analysis has to take the role of the main tool”. It is therefore not surprising that his definition of spectral subspaces is formulated in the language of complex analysis.

In the functional model space \mathcal{H} consider two subspaces $\mathcal{N}_\pm^\mathcal{Z}$ defined as follows:

$$\mathcal{N}_\pm^\mathcal{Z} := \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H} : P_\pm (\chi_\mathcal{Z}^+ (\tilde{g} + S^*g) + \chi_\mathcal{Z}^- (S\tilde{g} + g)) = 0 \right\}, \text{ where } \chi_\mathcal{Z}^\pm := \frac{I \pm i\mathcal{Z}}{2}.$$

These subspaces are then characterised in terms of the resolvent of the operator $L^\mathcal{Z}$. This, again, can be seen as a consequence of a much more general argument (see, e.g., [38, 37]). Consider the counterparts of $\mathcal{N}_\pm^\mathcal{Z}$ in the original Hilbert space K :

$$\tilde{N}_\pm^\mathcal{Z} := \Phi^* P_{\mathcal{K}} \mathcal{N}_\pm^\mathcal{Z}, \quad N_\pm^\mathcal{Z} := \text{clos } \tilde{N}_\pm^\mathcal{Z}.$$

Now introduce the set $\tilde{N}_e^\mathcal{Z} := \tilde{N}_+^\mathcal{Z} \cap \tilde{N}_-^\mathcal{Z}$ of so-called *smooth vectors* and its closure $N_e^\mathcal{Z}(L^\mathcal{Z}) := \text{clos}(\tilde{N}_e^\mathcal{Z})$.

The next assertion has been always singled out by S. Naboko in his lectures on functional models as “the main result of the whole lecture course”. In particular, it motivates the term “the set of smooth vectors” used for $\tilde{N}_e^\mathcal{Z}$ and opens up a possibility to construct a rich functional calculus of the absolutely continuous “part” of the operator, leading in particular to the scattering theory (see details in the next Section).

Theorem 2.4. *The sets $\tilde{N}_\pm^\mathcal{Z}$ are described as follows:*

$$\tilde{N}_\pm^\mathcal{Z} = \{u \in \mathcal{H} : \alpha(L^\mathcal{Z} - zI)^{-1}u \in H_\pm^2(E)\}.$$

Moreover, for the functional model image of $\tilde{N}_e^\mathcal{Z}$ the following representation holds:

$$\Phi \tilde{N}_e^\mathcal{Z} = \left\{ P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H} : \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H} \text{ satisfies } \Phi(L^\mathcal{Z} - zI)^{-1} \Phi^* P_{\mathcal{K}} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_{\mathcal{K}} \frac{1}{\cdot - z} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \quad \forall z \in \mathbb{C}_- \cup \mathbb{C}_+ \right\}. \quad (15)$$

The above Theorem together with Theorem 2.5 motivated generalising the notion of the absolutely continuous subspace $\mathcal{H}_{\text{ac}}(L^\varkappa)$ to the case of non-self-adjoint operators L^\varkappa by identifying it with the set N_e^\varkappa .

Definition 2.1. For a non-self-adjoint L^\varkappa the absolutely continuous subspace $\mathcal{H}_{\text{ac}}(L^\varkappa)$ is defined by the formula $\mathcal{H}_{\text{ac}}(L^\varkappa) = N_e^\varkappa(L^\varkappa)$.

In the case of a self-adjoint operator L^\varkappa , $\mathcal{H}_{\text{ac}}(L^\varkappa)$ is to be understood in the sense of the classical definition of the absolutely continuous subspace of a self-adjoint operator.

Theorem 2.5. Assume that $\varkappa = \varkappa^*$ and let $\alpha(L^\varkappa - zI)^{-1}$ be a Hilbert-Schmidt operator for at least one point $z \in \rho(L^\varkappa)$. Then the definition $\mathcal{H}_{\text{ac}}(L^\varkappa) = N_e^\varkappa$ is equivalent to the classical definition of the absolutely continuous subspace of a self-adjoint operator, i.e., $N_e^\varkappa = \mathcal{H}_{\text{ac}}(L^\varkappa)$.

Remark 1. Alternative conditions, which are even less restrictive in general, that guarantee the validity of the assertion of Theorem 2.5 were obtained in [20]. The absolutely continuous subspace of a non-self-adjoint operator also admits different definitions [37], which in generic case can be not equivalent to the one given above. This question is treated in full details by Romanov in [41].

2.4 Scattering theory

The intrinsic relationship between the scattering theory and the theory of dilations and functional models is due to [14]. The fact that the characteristic function of an arbitrary dissipative operator L can be realised as the scattering matrix of its dilation \mathcal{L} was observed by Adamyan and Arov in [1]. This fact, as was reiterated by Sergey on many occasions, together with Birman's seminal works on the mathematical scattering theory, motivated his work on the construction of wave and scattering operators in the functional model representation. With the introduction of smooth vector sets which are dense in absolutely continuous subspaces of operators L^\varkappa , it was natural to define (see [16, 18]) the action of exponential groups $\exp iL^\varkappa t$ in \mathcal{H} as multiplication by $\exp(ikt)$ on the smooth vectors.

In view of the classical definition of the wave operator of a pair of self-adjoint operators,

$$W_\pm(L^0, L^\varkappa) := \text{s-lim}_{t \rightarrow \pm\infty} e^{iL^0 t} e^{-iL^\varkappa t} P_{\text{ac}}^\varkappa,$$

where P_{ac}^\varkappa is the projection onto the absolutely continuous subspace of L^\varkappa , he observed that, at least formally, for $\Phi^* P_{\mathcal{X}} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \tilde{N}_e^\varkappa$ one has

$$W_-(L^0, L^\varkappa) \Phi^* P_{\mathcal{X}} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \Phi^* P_{\mathcal{X}} \begin{pmatrix} -(I + S)^{-1}(I + S^*)g \\ g \end{pmatrix}, \quad (16)$$

and similar formulae hold for $W_+(L^0, L^\varkappa)$, $W_\pm(L^\varkappa, L^0)$.

The need to attribute rigorous meaning to the right hand side of the latter equality, and thus to prove the existence and completeness of wave operators, motivated Sergey to investigate the boundary behaviour of operator-valued R -functions, see [19, 20] and references therein. This research has since found numerous applications in as seemingly unrelated areas as, say, the theory of Anderson localisation of stochastic differential operators. In the scattering theory (see [20]) it has allowed him to prove the classical Kreĭn–Birman–Kuroda theorem, the invariance principle and their non-self-adjoint generalisations by following the approach sketched above. It is worth mentioning that the latter effectively blends together non-stationary, stationary and smooth formulations of the self-adjoint scattering theory.

2.5 Singular spectrum of non-self-adjoint operators

A major thrust of Sergey's research was towards the analysis of singular spectral subspaces of non-self-adjoint operators. In the present section, we mention some of his results obtained in this direction. The notation throughout is as in Sections 2.3 and 2.2, with \varkappa set to be equal to iJ with an involution J (see Section 2.1). To simplify the notation, we therefore consistently drop the corresponding superscripts, as in $L = L^\varkappa$. It is further assumed throughout that the non-real spectrum of L is countable, with finite multiplicity. This latter condition holds in particular when the perturbation V is in trace class, which we will assume satisfied (similar results under less restrictive conditions are also available).

The singular subspace of L is defined as follows: $N_i(L) := H \ominus N_e(L^*)$. For the operator L^* , it is set by $N_i(L^*) := H \ominus N_e(L)$. These definitions prove to be consistent with the classical one for self-adjoint operators due to the characterisation

$$N_i(L) = \{u \in K : \langle ((L - t - i\varepsilon)^{-1} - (L - t + i\varepsilon)^{-1})u, v \rangle \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ for all } v \in K\}.$$

Define

$$\Theta_1(z) = \chi^- + S(z)\chi^+, \quad \Theta_2(z) = \chi^+ + S(z)\chi^-, \quad \Theta'_1(z) = \chi^- + S^*(\bar{z})\chi^+, \quad \Theta'_2(z) = \chi^+ + S^*(\bar{z})\chi^-, \quad (17)$$

so that for the characteristic function $\Theta(z)$ one has (cf. (12))

$$\Theta(z) = \Theta'_1(\bar{z})(\Theta'_2)^{-1}(\bar{z}), \quad z \in \mathbb{C}_+; \quad \Theta(z) = \Theta'_2(\bar{z})(\Theta'_1)^{-1}(\bar{z}), \quad z \in \mathbb{C}_-.$$

Set

$$\tilde{N}_+^i(L) = \Phi^* P_{\mathcal{K}} \left(\begin{array}{c} H_2^-(E) \ominus \Theta'_1 H_2^-(E) \\ 0 \end{array} \right), \quad \tilde{N}_-^i(L) = \Phi^* P_{\mathcal{K}} \left(\begin{array}{c} 0 \\ H_2^+(E) \ominus \Theta_2 H_2^+(E) \end{array} \right)$$

for the operator L and similarly

$$\tilde{N}_+^i(L^*) = \Phi^* P_{\mathcal{K}} \left(\begin{array}{c} H_2^-(E) \ominus \Theta'_2 H_2^-(E) \\ 0 \end{array} \right), \quad \tilde{N}_-^i(L^*) = \Phi^* P_{\mathcal{K}} \left(\begin{array}{c} 0 \\ H_2^+(E) \ominus \Theta_1 H_2^+(E) \end{array} \right)$$

for the operator L^* . The respective closures of these sets $N_+^i(L), N_-^i(L), N_+^i(L^*)$ and $N_-^i(L^*)$ are introduced in [21]. These subspaces are invariant with respect to the resolvents of $(L - z)^{-1}, (L^* - z)^{-1}$. It is shown that $N_\pm^i(L)$ can be seen as spectral for L , representing the parts of the singular spectrum pertaining to the (closed) upper and lower half-planes, respectively. In particular, eigenvectors and root vectors of the operator L , corresponding to $z \in \mathbb{C}_+$ ($z \in \mathbb{C}_-$), belong to $\tilde{N}_+^i(L)$ ($\tilde{N}_-^i(L)$, respectively). The paper [21] discusses the conditions of separability of spectral subspaces under the additional condition

$$\sup_{\text{Im } z > 0} \max\{\|\chi^+ S(z)\chi^-\|, \|\chi^- S(z)\chi^+\| \} < 1, \quad (18)$$

which guarantees that the ‘‘interaction’’ of the positive and negative ‘‘parts’’ of the perturbation V is ‘‘small’’. This is to say that it restricts the class of operators considered to those which are not too far from an orthogonal sum of a dissipative and an anti-dissipative ($\text{Im } L \leq 0$) operators.

In particular, [21] provides non-restrictive additional conditions such that

$$N_i(L) \cap N_e(L) = \{0\}, \quad N_i(L) \vee N_e(L) = K$$

and sharp estimates for the angle between $N_i(L)$ and $N_e(L)$. What's more,

$$N_-^i(L) \cap N_+^i(L) = \{0\}, \quad N_-^i(L) \vee N_+^i(L) = N_i(L)$$

with an explicit estimate for the angle between $N_-^i(L)$ and $N_+^i(L)$. Further, $L|_{N_+^i}$ ($L|_{N_-^i}$) is similar to a dissipative (anti-dissipative, respectively) operator with purely singular spectrum.

Dropping the separability condition (18) makes the spectral analysis of L much more involved. The corresponding problems were posed by S. Naboko in [22]. Most of them are still awaiting resolution, including the problem of a general spectral resolution of identity for a non-self-adjoint operator of the class considered here, but some were successfully tackled in [23] by S. Naboko and his student V. Veselov as well as in subsequent papers of V. Veselov. In particular, the named paper concerns with an in-depth study of the spectral subspace $N_0^i(L)$, introduced in [22]. The main result is formulated for $V \in \mathfrak{S}_1$ as follows:

$$\det \Theta(z) = \det \Theta_{L|_{N_+(L) \vee N_-(L)}}(z), \quad N_+(L) \vee N_-(L) = N_-^i(L) \vee N_e(L) \vee N_+^i(L),$$

generalising the corresponding result of Gohberg and Kreĭn. It shows that the determinant of the characteristic function of L contains no information on the spectral subspace

$$N_0^i(L) := K \ominus \{N_-(L^*) \vee N_+(L^*)\} \subset N_i(L), \quad (19)$$

i.e., $\det \Theta_{L|_{N_0^i(L)}}(z) \equiv 1$. Here in notation of Section 2.3 $N_{\pm}(L^*) = N_{\pm}^{-iJ}$.

The subspace N_0^i is precisely the ‘‘additional’’ spectral subspace corresponding to the real part of the spectrum of L (in particular, it contains the eigenvectors and root vectors corresponding to real values of the spectral parameter), the analytic structure of which has no parallels in the case of dissipative operators. In a nutshell, it appears due to the interaction of the ‘‘incoming’’ and ‘‘outgoing’’ energy channels in the non-conservative system modelled by L .

The rôle of N_0^i for the spectral analysis of non-dissipative operators is further revealed by the following assertion:

$$N_i(L) \cap N_e(L) \subset N_0^i(L),$$

i.e., *if* the absolutely continuous and singular subspaces intersect, the intersection *must* lie in N_0^i . It is therefore the presence of N_0^i that ensures that $N_e(L^*) \vee N_i(L^*) \neq K$, which prevents a spectral decomposition for the operator L^* .

Sergey had mentioned to us, that he had seven to eight papers worth of further material on the functional model and spectral analysis of non-dissipative operators. Unfortunately, he had never published these results.

2.6 A functional model based on the Strauss characteristic function

In contrast to the model theory for contractions associated with the names of Sz.-Nagy-Foiaş and de Branges-Rovnyak, the models of unbounded non-selfadjoint operators are usually concerned with ‘‘concrete’’ operators arising in applications. In particular, the functional model for non-self-adjoint additive perturbations discussed above was motivated by the spectral analysis of the Schrödinger operator with a complex potential, see, e.g., [34, 24, 21, 20]. In fact, Sergey Naboko had reiterated to us on a number of occasions, that his primary concern was the spectral theory of the Schrödinger operator, rather than the development of abstract mathematical concepts: the functional model in

his view was simply the tool of choice in this area. More precisely, the Schrödinger operator $-\Delta + p(x) + iq(x)$ in $L_2(\mathbb{R}^3)$, where $p(x), q(x)$ are real-valued bounded functions of $x \in \mathbb{R}^3$, can be written in the form (8) with the operator α defined as $\alpha : f \mapsto |q(x)|^{1/2}f$, where $f \in L_2(\mathbb{R}^3)$. It is important to note that all the building blocks of the model construction are explicitly given in terms of the problem at hand. Indeed, both the characteristic function $S(z)$ and the “spectral maps” \mathcal{F}_\pm are expressed via the non-real part of the complex potential (and the operator itself). The true nature of the problem’s “non-selfadjointness”, i.e., the non-triviality of the imaginary part of the potential, is thus faithfully preserved in the model representation.

The same observation is valid for other model constructions of non-selfadjoint operators available in the literature, see, e.g., [8] in the present volume for the case of non-self-adjoint extensions of symmetric operators. Therefore it becomes increasingly important to express the non-selfadjointness of the problem not in abstract terms (as it is commonly done in the operator theory), but rather in terms of the concrete operator present in the problem statement.

The standard way to calculate the characteristic function of a non-self-adjoint operator is based on the definition given by A. Strauss in [39]. For a dissipative operator it reads as follows

Definition 2.2 ([39]). *Let L be a closed maximal densely defined dissipative operator on a Hilbert space K . The characteristic function of L is a bounded operator-valued analytic function $S(z) : E \rightarrow E_*$, $z \in \rho(L^*)$, such that*

$$S(z)\Gamma f = \Gamma_*(L^* - zI)^{-1}(L - zI)f, \quad f \in \text{dom}(L),$$

where the boundary operators Γ, Γ_* are defined for $u, v \in \text{dom}(L)$, $u', v' \in \text{dom}(L^*)$ by the equalities

$$(Au, v) - (u, Av) = i(\Gamma u, \Gamma v)_E, \quad (u', A^*v') - (A^*u', v') = i(\Gamma_*u', \Gamma_*v')_{E_*}$$

and $E := \text{clos ran}(\Gamma)$, $E_* := \text{clos ran}(\Gamma_*)$ are Hilbert spaces.

According to this definition, the concrete form of the characteristic function of L depends on the choice of boundary operators Γ, Γ_* . It is easy to see that for any Hilbert space isometries $\pi : E \rightarrow E'$, $\pi_* : E_* \rightarrow E'_*$, the maps $\pi\Gamma$ and $\pi_*\Gamma_*$ are also boundary operators with the corresponding characteristic function $\pi_*S(z)\pi^* : E' \rightarrow E'_*$. In applications, a suitable definition of the boundary operators is determined according to the problem statement itself. For example, the operator α of (10) (the root cause of the operator’s non-selfadjointness) admits the rôle of both Γ and Γ_* . Convenient boundary operators appear “naturally” in the analysis of non-self-adjoint extensions of symmetric operators as well. Once the triple $\{\Gamma, \Gamma_*, S(z)\}$ is explicitly defined, the construction of the functional model follows the blueprint of S. Naboko [18].

A further important contribution is contained in the two recent papers [6, 7] by B.M. Brown, M. Marletta, S. Naboko, and I. Wood. The authors offer a model construction carried out in the abstract setting of Strauss’ boundary operators Γ, Γ_* , resorting to no specific realisation of them. This work therefore makes all the steps of the model construction explicit, regardless of any particular form of the characteristic function, the latter to be set based on the requirements imposed by a concrete application at hand. In particular, this makes it possible to construct a functional model in the case where both the differential expression itself and the boundary conditions are non-self-adjoint, which in our view is especially relevant for topical problems of materials science.

2.7 Applications of the functional model technique

Here we list some notable applications of the functional model technique, in which Sergey Naboko was involved, in addition to his work on the spectral analysis of non-self-adjoint Schrödinger operators mentioned earlier, see, e.g., [21] and references therein.

1. In [33, 32] Sergey, together with Yu. Kuperin and R. Romanov, studied the non-self-adjoint single-velocity Boltzmann transport operator. Using the functional model techniques, the absolute continuity of this operator's continuous spectrum was proved; the similarity problem of the absolutely continuous “part” of the operator to a self-adjoint one was fully settled, and the existence of a spectral singularity at zero ascertained for a singular set of multiplication coefficients.

2. In [25, 26], together with R. Romanov, Sergey Naboko analysed the impact of spectral singularities on the asymptotic behaviour of the group of exponentials, generated by a maximal dissipative operator L . It was shown that this asymptotics allows one to recover the orders and locations of spectral singularities in the case, where their number is finite and they are of a finite power order.

3. In [11, 12], for a non-dissipative trace class perturbation L of a self-adjoint operator on K such that $N_0^i(L)$ coincides with the Hilbert space K , a generalisation of the Caley identity was obtained in the following form: there exists an outer in the upper half-plane \mathbb{C}_+ uniformly bounded scalar analytic function $\gamma(\lambda)$ such that $w - \lim_{\varepsilon \downarrow 0} \gamma(L + i\varepsilon) = 0$. A generalisation of this result was further obtained to the case of relative trace class perturbations.

4. In [13], the so-called matrix model was introduced and studied in some detail, i.e., a rank two non-dissipative additive perturbation L in K of a self-adjoint operator under the assumption that $K = N_0^i(L)$. This model represents the simplest possible case of a non-dissipative operator which exhibits the properties not found in any dissipative one; despite its seeming simplicity, it already includes the main analytic obstacles found in the general case. It has to be noted that this model was the favourite sandbox of Sergey; unfortunately, many results obtained by him, up to and including a von Neumann type estimate in BMO classes for functions of the operator L , have never been published.

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