NORM-RESOLVENT CONVERGENCE FOR NEUMANN LAPLACIANS ON MANIFOLDS THINNING TO GRAPHS

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ABSTRACT. Norm-resolvent convergence with order-sharp error estimate is established for Neumann Laplacians on thin domains in $\mathbb{R}^2$ and $\mathbb{R}^3$, converging to metric graphs in the limit of vanishing thickness parameter in the resonant case.

1. Introduction

In [15, 20, 21], see also references therein, Neumann Laplacians A_ε on thin manifolds, converging to metric graphs G, were studied, see, e.g., Fig. 1. The named works attacked the question of spectral convergence of such PDEs to the spectrum of a graph Laplacian with some matching conditions at the graph vertices. Denoting the set of edges e of the limiting graph G by E, each $e = [0, l_e]$ is associated with the Hilbert space $L^2(e)$. Accordingly, $L^2(G) := \oplus_e L^2(e)$. For each e, the graph Laplacian A_G is generated by the differential expression $-u''$, $u \in H^2(e)$ (see [4] for details).

It was proved, that the spectra of A_ε converge within any compact $K \in \mathbb{C}$ in the sense of Hausdorff to the spectrum of a graph Laplacian A_G, where the matching conditions at the vertices might be either of:

- Kirchhoff, or standard, if the volumes of vertex subdomains are decaying faster, than the volumes of edge subdomains;
- Resonant, described in terms of $\delta-$type matching conditions with coupling constants proportionate to the spectral parameter z, in the case when the volumes of vertex subdomains are of the same order as the volumes of the edge ones;
- Finally, the limiting graph Laplacian is completely decoupled (i.e., the Dirichlet condition is imposed at every vertex) if the vertex subdomains vanish slower than the edge ones.

In the present paper, we are dealing with the most interesting resonant case. We show that the Neumann Laplacians A_ε in fact converge in norm-resolvent sense to an ODE acting in the Hilbert space $L^2(G) \oplus \mathbb{C}^N$, where N is the number of vertices. The operator to which it converges is in fact the one pointed out in [20] as the self-adjoint operator which spectrum coincides with the Hausdorff limit of spectra for the family A_ε. We also obtain a sharp error bound, which in the planar case is $O(\varepsilon/|\log \varepsilon|)$ and in the case of $\mathbb{R}^3$ is $O(\varepsilon)$, where in both cases ε is the radius of the edge domain section.

This result easily implies the Hausdorff spectral convergence, at the same time yielding a sharp estimate on its rate. Moreover, it paves the way to the consideration of higher frequency regimes, i.e., of the setup where the spectral parameter is no longer constrained to

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a compact (but is still constrained by some power of the small parameter ε). This argument will appear elsewhere. In view of better clarity of the paper, here we restrict ourselves to the case where (with not a huge loss of generality) the edge subdomains are assumed to be straight and uniformly thin, whereas the vertex subdomains are smooth with the exception of the points where they meet the edges.

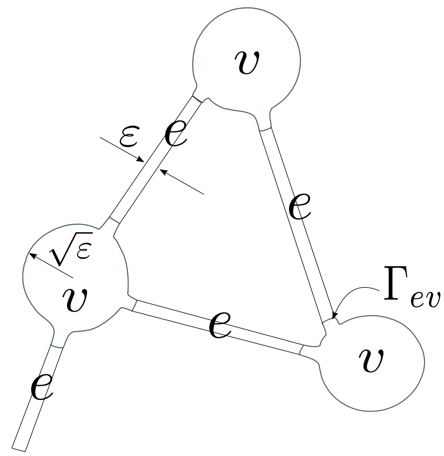

FIGURE 1. Thin structure: an example.

2. Problem setup and preliminaries

For simplicity, we will only treat the planar case of dimension 2 here. The corresponding setup for higher dimensions is introduced likewise.

For the limiting graph (see Fig. 1), the following notation will be used: the graph G will be identified with the set of edges E, each individual edge denoted by $e \in E$ and treated as an interval $[0, l_e]$. The set of vertices V of the graph is a collection of individual vertices $v \in V$, treated as the sets of edge endpoints meeting at v. The graph G is assumed oriented throughout.

Passing over to the setup pertaining to the Neumann Laplacian on a thin graph-convergent structure, let a connected domain Q be the union of the vertex part Q_V and the edge part Q_E, where Q_E will be assumed to be a finite collection of $\varepsilon-$thin rectangular boxes, $Q_E = \cup_e Q_e$. For each e, the domain Q_e is assumed to be, up to a linear change of variables, defined by

$$Q_e = \{x \in \mathbb{R}^2 : x_1 \in (0, l_e), x_2 \in (0, \varepsilon)\}.$$

It is further assumed that $Q_V = \cup_v Q_v$, where each of disjoint domains Q_v is assumed simply connected with piecewise smooth boundary ∂Q_v. This boundary is assumed to be decomposed as $\partial Q_v = \Gamma_v \cup \Gamma_v^\varepsilon$, with Γ_v^ε further decomposable into a union of straight segments, $\Gamma_v^\varepsilon = \cup \Gamma_{ve}$. Here the union is taken over all edge domains Q_e which are connected to Q_v, $\overline{Q_e} \cap \overline{Q_v} = \Gamma_{ve}$. In what follows, we will refer to the segments Γ_{ve} as *contact plates*. Since operators of the Zaremba (or mixed) boundary value problem will be used below, we further require that the contact plates Γ_{ve} meet Γ_v at angles strictly less than π, see [6, 7] for the precise formulation. We will further assume that the curves Γ_v are smooth.

Moreover, the total number of vertex domains will be denoted by N, and each Q_v will be represented as a linear shift of a scaled fixed (i.e., $\varepsilon-$independent) domain Q_v^0, that is, that for each v one has $Q_v = \varepsilon^{1/2} Q_v^0 + b_v$, where b_v is a vector in $\mathbb{R}^2$. We remark that this guarantees that we are in the "resonant" case of [15, 20], i.e., that the volumes of the contact plates are proportionate to the volumes of vertex domains.

Note that Q, Q_v and Q_e are all assumed to be $\varepsilon-$dependent, so that $|Q| \to 0$ as $\varepsilon \to 0$. We elected to omit this dependence in notation for the sake of convenience.

On the domain Q we consider a family of self-adjoint operators A_ε defined by the differential expression $-\Delta$ subject to Neumann boundary conditions.

Precisely, we will deal with the resolvent $(A_\varepsilon - z)^{-1}$ of a self-adjoint on $L^2(\mathbb{R}^d)$ operator A_ε, defined by means of its coercive sesquilinear form. We will always assume that $z \in \mathbb{C}$ is separated from the spectrum of the original operator family, more precisely, we assume that $z \in K_\sigma$, where

$$K_\sigma := \big\{ z \in \mathbb{C} |\ z \in K \text{ a compact set in } \mathbb{C},\ \operatorname{dist}(z, \mathbb{R}) \geq \sigma > 0 \big\}.$$

After we have established the operator-norm asymptotics of $(A_\varepsilon - z)^{-1}$ for $z \in K_\sigma$, the result is extended by analyticity to a compact set K_σ^{ext}, the distance of which to the spectrum of the leading order of the asymptotics is bounded below by σ.

Similar to, e.g., [16] in a related area of critical-contrast homogenisation and facilitated by the abstract framework of [26], instead of the form-based definition of the operators A_ε we will consider them as operators of transmission problems, see [27] and references therein, relative to the *internal boundary* $\Gamma := \cup_{e,v} \Gamma_{ev}$

The transmission problem introduced above is then formulated as, given a $f \in L^2(Q)$, finding a $u \in L^2(Q)$ such that it solves (in the variational, or weak, formulation) the boundary value problem

$$\begin{cases} -\Delta u(x) - z u(x) = f(x), & x \in Q_V \text{ and } x \in Q_E \\ u_v(x) = u_e(x), \quad \frac{\partial u_v}{\partial n} - \frac{\partial u_e}{\partial n} = 0 & \text{on } \Gamma_{ev}, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial Q. \end{cases} \tag{1}$$

Here $u_v := u|_{Q_v}$, $u_e := u|_{Q_e}$ for all admissible e and v, and $\partial/\partial n$ represents the exterior normal on ∂Q and the "edge-inward" normal (i.e., directed from Q_v to Q_e) on any of contact plates Γ_{ev}. By a classical argument the weak solution of the above problem is shown to be equal to $(A_\varepsilon - z)^{-1} f$.

It of course remains to be seen that the linear operator of the transmission problem (1) defined via Ryzhov's technique of [26], which we briefly recall below, is precisely the same operator A_ε; the proof of this fact follows easily from [26] combined with the main estimate of [27].

Following [26] (cf. [3],[8] and references therein for alternative approaches) which is in fact based on the ideology of the classical Birman-Kreĭn-Višik theory (see [5, 19, 30]), the linear operator of the transmission BVP is introduced as follows. Let $\mathcal{H} := L^2(\Gamma) = \oplus_{e,v} L^2(\Gamma_{ev})$, and consider the harmonic lift operators Π_V and Π_E defined on $\phi \in \mathcal{H}$ via

$$\Pi_V \phi := u_\phi, \begin{cases} \Delta u_\phi = 0, \quad u_\phi \in L^2(Q_V) \\ u_\phi|_\Gamma = \phi \end{cases} \quad \text{and } \Pi_E \phi := u_\phi, \begin{cases} \Delta u_\phi = 0, \quad u_\phi \in L^2(Q_E) \\ u_\phi|_\Gamma = \phi \end{cases} \tag{2}$$

subject to Neumann boundary conditions on ∂Q.

These operators are first defined on $\phi \in C^2(\Gamma)$, in which case the corresponding solutions u_ϕ can be seen as classical. The results of [6] allow to extend both harmonic lifts to bounded (in fact, compact) operators on $\mathcal{H}$, in which case u_ϕ are to be treated as distributional solutions of the respective BVPs.

The solution operator $\Pi : \mathcal{H} \mapsto L^2(Q) = L^2(Q_V) \oplus L^2(Q_E)$ is defined as follows:

$$\Pi\phi := \Pi_V\phi \oplus \Pi_E\phi.$$

Consider the self-adjoint operator family A_0 (note that we have elected to drop the subscript ε for notational convenience) to be the Dirichlet decoupling of the operator family A_ε, i.e., the operator of the boundary value problem on both Q_V and Q_E, where the Dirichlet boundary conditions are imposed on Γ in conjunction with Neumann boundary conditions on ∂Q. The operator A_0 is generated by the same differential expression as A_ε. Clearly, one has $A_0 = A_0^V \oplus A_0^E$ relative to the orthogonal decomposition $L^2(Q) = L^2(Q_V) \oplus L^2(Q_E)$; all three operators A_0, A_0^V and A_0^E are self-adjoint and positive-definite. Moreover, by [6, 32] there exists a bounded A_0^{-1}.

The intersection of $\operatorname{dom} A_0$ with $\operatorname{ran}\Pi$ is clearly trivial, see [26].

Denoting $\tilde{\Gamma}_0^{V(E)}$ the left inverses of $\Pi_{V(E)}$, respectively, one introduces the trace operator $\Gamma_0^{V(E)}$ as the null extension of $\tilde{\Gamma}_0^{V(E)}$ to the domain $\operatorname{dom} A_0^{V(E)} \dotplus \operatorname{ran}\Pi_{V(E)}$. In the same way we introduce the operator $\tilde{\Gamma}_0$ and its null extension Γ_0 to the domain $\operatorname{dom} A_0 \dotplus \operatorname{ran}\Pi$.

The solution operators S_z^V, S_z^E of BVPs

$$\begin{cases} -\Delta u_\phi - z u_\phi = 0, & u_\phi \in \operatorname{dom} A_0^V \dotplus \operatorname{ran}\Pi_V \\ \Gamma_0^V u_\phi = \phi \end{cases} \quad \text{and}$$

$$\begin{cases} -\Delta u_\phi - z u_\phi = 0, & u_\phi \in \operatorname{dom} A_0^E \dotplus \operatorname{ran}\Pi_E \\ \Gamma_0^E u_\phi = \phi \end{cases}$$

are defined as linear mappings from ϕ to u_ϕ, respectively. These operators are bounded from $L_2(\Gamma)$ to $L_2(Q_V)$ and $L_2(Q_E)$, respectively, and admit the following representations:

$$S_z^E = (1 - z(A_0^E)^{-1})^{-1}\Pi_E, \quad S_z^V = (1 - z(A_0^V)^{-1})^{-1}\Pi_V. \tag{3}$$

The solution operator S_z from $L^2(\Gamma)$ to $L^2(Q_V) \oplus L^2(Q_E)$ is defined as $S_z = S_z^V \oplus S_z^E$; it admits the representation $S_z = (1 - z(A_0)^{-1})^{-1}\Pi$ and is bounded.

Having introduced orthogonal projections P_V and P_E from $L^2(Q)$ onto $L^2(Q_V)$ and $L^2(Q_E)$, respectively, one has the obvious identities

$$S_z^V = P_V S_z, \quad S_z^E = P_E S_z, \quad \text{and } \Pi_V = P_V\Pi, \quad \Pi_E = P_E\Pi. \tag{4}$$

Fix self-adjoint (and, in general, unbounded) operators Λ^E, Λ^V defined on domains $\operatorname{dom}\Lambda^E$, $\operatorname{dom}\Lambda^V \subset L^2(\Gamma)$ (in what follows these operators will be chosen as Dirichlet-to-Neumann maps of Zaremba problems on Q_E and Q_V, respectively, and well-defined on $H^1(\Gamma)$, where $H^1(\Gamma)$ is the standard Sobolev space pertaining to the internal boundary Γ). Still following [26], we define the "second boundary operators" Γ_1^E and Γ_1^V to be linear operators on the domains

$$\operatorname{dom}\Gamma_1^E := \operatorname{dom} A_0^E \dotplus \Pi_E \operatorname{dom}\Lambda^E, \quad \operatorname{dom}\Gamma_1^V := \operatorname{dom} A_0^V \dotplus \Pi_V \operatorname{dom}\Lambda^V. \tag{5}$$

The action of $\Gamma_1^{E(V)}$ is set by:

$$\Gamma_1^E : \ (A_0^E)^{-1} f \dotplus \Pi_E\phi \mapsto \Pi_E^* f + \Lambda^E\phi, \quad \Gamma_1^V : \ (A_0^V)^{-1} f \dotplus \Pi_V\phi \mapsto \Pi_V^* f + \Lambda^V\phi \tag{6}$$

for all $f \in L^2(Q_E)$, $\phi \in \operatorname{dom}\Lambda^E$ and $f \in L^2(Q_V)$, $\phi \in \operatorname{dom}\Lambda^V$, respectively.

Alongside $\Gamma_1^{E(V)}$, introduce a self-adjoint Λ on $\operatorname{dom}\Lambda \subset \mathcal{H}$ and the boundary operator Γ_1 on the domain

$$\operatorname{dom}\Gamma_1 := \operatorname{dom} A_0^{-1} \dotplus \Pi \operatorname{dom}\Lambda;$$
$$\Gamma_1 : \; A_0^{-1} f \dotplus \Pi\phi \mapsto \Pi^* f + \Lambda\phi \quad \forall\, f \in L^2(Q) \text{ and } \phi \in \operatorname{dom}\Lambda.$$

We remark that the operators $\Gamma_1, \Gamma_1^{E(V)}$ thus defined are assumed to be neither closed nor indeed closable.

In our setup, we make the following concrete choice of the operators $\Lambda^{E(V)}$: these operators in what follows are the Dirichlet-to-Neumann maps pertaining to the components Q_E and Q_V, respectively. Precisely, for the problem

$$\begin{aligned} \Delta u_\phi = 0; &\quad u_\phi \in L^2(Q_E) \\ u_\phi|_\Gamma = \phi; &\quad \partial_n u_\phi|_{\partial Q} = 0 \end{aligned}$$

Λ^E maps the boundary values ϕ of u_ϕ to the negative traces of its normal derivative[1] $\partial_n u_\phi|_\Gamma$, where $n = -n_E$ is as above the "edge-inward" normal. This operator is well-defined by its sesquilinear form as a self-adjoint operator on $L^2(\Gamma)$ (see, e.g., [40, 39]); one has $H^1(\Gamma) \subset \operatorname{dom}\Lambda^E$ by [6].

On the vertex part Q_V one considers the problem

$$\begin{aligned} \Delta u_\phi = 0; &\quad u_\phi \in L^2(Q_V) \\ u_\phi|_\Gamma = \phi; &\quad \partial_n u_\phi|_{\partial Q} = 0, \end{aligned} \tag{7}$$

and defines Λ^V as the operator mapping the boundary values ϕ of u_ϕ to the negative traces of its normal derivative $-\partial_n u_\phi|_\Gamma$, where $n = n_V$ is the "edge-inward" normal. The self-adjointness of Λ^V on $\operatorname{dom}\Lambda^V \supset H^1(\Gamma)$ follows by an unchanged argument.

Finally we introduce the operator Λ which on $\phi := H^1(\Gamma)$ is the sum $\Lambda\phi = \Lambda^V\phi + \Lambda^E\phi$. It is also a self-adjoint operator on $\operatorname{dom}\Lambda \supset H^1(\Gamma)$. This can be ascertained either by the argument of [27], in which case it is defined as the inverse of a compact self-adjoint operator on $L^2(\Gamma) \ominus \{c\mathbf{1}\}$, extended to $\{c\mathbf{1}\}$ by zero, or, alternatively, from its definition by a closed sesquilinear form.

The choice of $\Lambda^{E(V)}$ made above allows us to consider Γ_1 on the domain $\operatorname{dom} A_0 \dotplus \Pi \operatorname{dom}\Lambda$. One then writes [26] the second Green identity in the following form:

$$\langle Au, v\rangle_{L^2(Q)} - \langle u, Av\rangle_{L^2(Q)} = \langle \Gamma_1 u, \Gamma_0 v\rangle_{L^2(\Gamma)} - \langle \Gamma_0 u, \Gamma_1 v\rangle_{L^2(\Gamma)} \tag{8}$$

for all $u, v \in \operatorname{dom}\Gamma_1 = \operatorname{dom} A_0 \dotplus \Pi \operatorname{dom}\Lambda$, where the operator A is the null extension (see [25]) of the operator A_0 onto $\operatorname{dom}\Gamma_1$. Thus the triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is closely related to a boundary quasi-triple of [3] (see also [2]) for the transmission problem considered; cf. [8] for an alternative approach.

The calculation of Π^* in [26] shows that $\Pi^* = \Gamma_1 A_0^{-1}$ and therefore Γ_1 as introduced above acts as follows:

$$\Gamma_1 : \; u = P_E u + P_V u \mapsto \partial_n P_e u|_\Gamma - \partial_n P_V u|_\Gamma,$$

where P_E and P_V are the orthogonal projections of $L^2(Q)$ onto $L^2(Q_E)$, $L^2(Q_V)$, respectively. The transmission problem at hand therefore (at least, formally so far) corresponds to the matching condition $\Gamma_1 u = 0$.

[1]This definition is inspired by [26]. Note that the operator thus defined is negative the classical Dirichlet to Neumann map DN of e.g. [16].

Definition 2.1 ([26]). *The operator-valued function $M(z)$ defined on the domain* $\operatorname{dom}\Lambda$ *for* $z \in \rho(A_0)$ *(and in particular, for $z \in K_\sigma$) by the formula*

$$M(z)\phi = \Gamma_1 S_z \phi = \Gamma_1 (1 - zA_0^{-1})^{-1}\Pi\phi \tag{9}$$

is called the M-function of the problem (1)*.*

The following result of [26] summarises the properties of the $M-$function which we will need in what follows.

Proposition 2.2 ([26], Theorem 3.3.)**.** *1. One has the following representation:*

$$M(z) = \Lambda + z\Pi^*(1 - zA_0^{-1})^{-1}\Pi, \quad z \in \rho(A_0). \tag{10}$$

2. $M(z)$ is an analytic operator-function with values in the set of closed operators in $L_2(\Gamma)$ densely defined on the z-independent domain $\operatorname{dom}\Lambda$*.*

3. For $z, \zeta \in \rho(A_0)$ the operator $M(z) - M(\zeta)$ is bounded and

$$M(z) - M(\zeta) = (z - \zeta)S_{\bar{z}}^* S_\zeta$$

In particular, $\Im M(z) = (\Im z)S_{\bar{z}}^* S_{\bar{z}}$ and $(M(z))^* = M(\bar{z})$.

4. For $u_z \in \ker(A - zI) \cap \{\operatorname{dom} A_0 \dotplus \Pi \operatorname{dom}\Lambda\}$ the formula holds:

$$M(z)\Gamma_0 u_z = \Gamma_1 u_z.$$

Alongside $M(z)$, we define $M_V(z)$ and $M_E(z)$ pertaining to the vertex Q_V and edge Q_E parts of the domain Q by the formulae

$$\begin{aligned} M_V(z)\phi &= \Gamma_1^V S_z^V \phi = \Gamma_1^V (1 - z(A_0^V)^{-1})^{-1}\Pi_V \phi, \\ M_E(z)\phi &= \Gamma_1^E S_z^E \phi = \Gamma_1^E (1 - z(A_0^E)^{-1})^{-1}\Pi_E \phi. \end{aligned} \tag{11}$$

The value of the fact that on $\phi \in H^1(\Gamma)$ one has $M(z)\phi = M_E\phi + M_V\phi$ is clear: in contrast to A_ε which cannot be additively decomposed into "independent parts" pertaining to the vertex and edge parts of the medium Q owing to the transmission interface conditions on the internal boundary Γ of the two, the $M-$function turns out to be in fact additive (cf., e.g., [13], where this additivity was observed and exploited in an independent, but closely related, setting of scattering). In what follows we will observe that the resolvent $(A_\varepsilon - z)^{-1}$ can be expressed in terms of $M(z)$ via a version of the celebrated Kreĭn formula, thus reducing the asymptotic analysis of the resolvent to that of the corresponding $M-$function (cf., e.g., [1, 3] for alternative approaches to derivation of a Kreĭn formula in our setting).

Alongside the transmission problem (1), the boundary conditions of which can be now (so far, formally) represented as $u \in \operatorname{dom} A_0 \dotplus \Pi L^2(\Gamma)$, $\Gamma_1 u = 0$, in what follows we will require a wider class of problems of this type. This class is formally given by the transmission conditions of the type

$$u \in \operatorname{dom} A \equiv \operatorname{dom} A_0 \dotplus \Pi L^2(\Gamma), \quad \beta_0\Gamma_0 u + \beta_1\Gamma_1 u = 0,$$

where β_1 is a bounded operator on $L^2(\Gamma)$ and β_0 is a linear operator defined on the domain $\operatorname{dom}\beta_0 \supset \operatorname{dom}\Lambda$.

In general, the operator $\beta_0\Gamma_0 + \beta_1\Gamma_1$ is not defined on the domain $\operatorname{dom} A$. This problem is being taken care of by the following assumption, which will in fact be satisfied throughout:

$$\beta_0 + \beta_1\Lambda \text{ defined on } \operatorname{dom}\Lambda \text{ is closable in } \mathcal{H}.$$

We remark that by Proposition 2.2 the operators $\overline{\beta_0+\beta_1 M(z)}$ are closable for all $z\in\rho(A_0)$, and the domains of their closures coincide with $\operatorname{dom}\overline{\beta_0+\beta_1\Lambda}$.

For any $f\in H$ and any $\phi\in\operatorname{dom}\Lambda$, the equality

$$(\beta_0\Gamma_0+\beta_1\Gamma_1)(A_0^{-1}f+\Pi\phi)=\beta_1\Pi^* f+(\beta_0+\beta_1\Lambda)\phi$$

shows that the operator $\beta_0\Gamma_0+\beta_1\Gamma_1$ is correctly defined on $A_0^{-1}H\dotplus\Pi\operatorname{dom}\Lambda\subset\operatorname{dom}A$. Denoting $\mathcal{B}:=\overline{\beta_0+\beta_1\Lambda}$ with the domain $\operatorname{dom}\mathcal{B}\supset\operatorname{dom}\Lambda$, one checks that $H_{\mathcal{B}}:=A_0^{-1}H\dotplus\Pi\operatorname{dom}\mathcal{B}$ is a Hilbert space w.r.t. the norm

$$\|u\|^2_{\mathcal{B}}:=\|f\|^2_H+\|\phi\|^2_{\mathcal{H}}+\|\mathcal{B}\phi\|^2_{\mathcal{H}}\quad\text{for}\quad u=A_0^{-1}f+\Pi\phi.$$

It is then proved [26, Lemma 4.1] that $\beta_0\Gamma_0+\beta_1\Gamma_1$ extends to a bounded operator from $H_{\mathcal{B}}$ to $\mathcal{H}$, for which extension the same notation $\beta_0\Gamma_0+\beta_1\Gamma_1$ is preserved for the sake of convenience.

We will make use of the following version of the celebrated Kreĭn formula:

Proposition 2.3 ([26],Theorem 5.1). *Let $z\in\rho(A_0)$ be such that the operator $\overline{\beta_0+\beta_1 M(z)}$ defined on $\operatorname{dom}\mathcal{B}$ is boundedly invertible. Then*

$$R_{\beta_0,\beta_1}(z):=(A_0-z)^{-1}+S_zQ_{\beta_0,\beta_1}(z)S^*_{\bar z},\ \textit{where}\ Q_{\beta_0,\beta_1}:=-(\overline{\beta_0+\beta_1M(z)})^{-1}\beta_1\qquad(12)$$

is the resolvent of a closed densely defined operator A_{β_0,β_1} with the domain

$$\operatorname{dom}A_{\beta_0,\beta_1}=\{u\in H_{\mathcal{B}}|(\beta_0\Gamma_0+\beta_1\Gamma_1)u=0\}=\ker(\beta_0\Gamma_0+\beta_1\Gamma_1).$$

In particular, the (self-adjoint) operator of the transmission problem (1), which corresponds to the choice $\beta_0=0,\beta_1=I$, admits the following characterisation in terms of its resolvent:

$$R_{0,I}(z)=(A_0-z)^{-1}-S_zM^{-1}(z)S^*_{\bar z}.\qquad(13)$$

In this case, one clearly has $H_{\mathcal{B}}=A_0^{-1}H\dotplus\Pi\operatorname{dom}\Lambda$ and $\operatorname{dom}A_{0,I}=\{u\in H_{\mathcal{B}}|\Gamma_1 u=0\}$ which, together with the discussion at the very beginning of this section, yields $A_{0,I}=A_\varepsilon$.

We remark that the operators β_0 and β_1 above can be assumed $z-$dependent, as this change does not impact the corresponding proofs of [26]. In this case however, the corresponding operator-function $R_{\beta_0,\beta_1}(z)$ is shown to be a resolvent of a $z-$dependent operator family. Within the self-adjoint setup of the present paper, $R_{\beta_0,\beta_1}(z)$ will be guaranteed to represent a generalised resolvent in the sense of [23, 24, 28].

3. Auxiliary estimates

In the present Section, we prove a number of auxiliary statements which we need to prove our main result.

We start with the analysis of the operators Π_V and S^V_z introduced in Section 2. First we note that each of these operators admits a decomposition into an orthogonal sum over N vertex domains $\{Q_v\}$ of Q.

It suffices therefore to consider a single vertex domain Q_v (we recall for readers' convenience that the volume of this domain is assumed to be decaying with $\varepsilon\to0$). Its boundary ∂Q_v contains a disjoint set of straight segments belonging to the internal boundary Γ, which are, in line with what has been said above, are denoted as Γ_{ev}; the union of the latter is Γ^ε_v.

The decoupled operator A_0 has $L^2(Q_v)$ as its invariant subspace. We will denote by $A_0^{(v)}$ its self-adjoint restriction, $A_0^{(v)}:=A_0|_{L^2(Q_v)}$. By construction, the operator $A_0^{(v)}$ is the Laplacian with the so-called Zaremba, or Neumann-Dirichlet mixed, boundary conditions [31, 17]. It is subject to Dirichlet boundary conditions on Γ_v^ε and to Neumann boundary conditions on its complement Γ_v. Clearly this operator is boundedly invertible; moreover, one has the following

Proposition 3.1 (see [32, 33])**.** *The following estimate holds:*

$$\|(A_0^{(v)})^{-1}\| \leq C\frac{|Q_v|}{|\log\varepsilon|}.$$

Remark 3.2. *We remark that the latter Proposition in fact holds under much more general conditions than imposed by us. Namely, the domain Q_v is only required to be Lipschitz and no conditions whatsoever are imposed on the geometry of the set Γ_v^ε.*

Next, we turn our attention to the solution operator $S_z^{(v)} := S_z^V|_{\Gamma_v^\varepsilon}$ and the corresponding harmonic lift $\Pi^{(v)} := \Pi_V|_{\Gamma_v^\varepsilon}$. The two are clearly related by the formula

$$S_z^{(v)} = (1 - z(A_0^{(v)})^{-1})^{-1}\Pi^{(v)}.$$

In order to bound the norm of $\Pi^{(v)}$, we first consider the corresponding Zaremba problem on Q_v^0. We follow by relating the norm of the corresponding Poisson operator to the least Steklov eigenvalue of the bi-Laplacian, following the blueprint of [36], based in turn on Fichera's principle of duality, see [37]. Since the boundary of Q_v^0 is non-smooth, in doing so we follow the generalisations developed in [34, 38], with obvious modifications required in passing from Dirichlet to Zaremba setup. The estimate for the said Steklov eigenvalue is then taken from the norm of the compact embedding of $H^2(Q_v^0)$ to the traces of normal derivatives on the contact plates, see e.g. [35]. Rescaling back to Q_v, we obtain

Lemma 3.3. *The following estimate holds:*

$$\|\Pi^{(v)}\| \leq C \quad \textit{independent of } \varepsilon.$$

By Proposition 3.1, this yields the following estimate for the solution operator $S_z^{(j)}$:

$$S_z^{(j)} = \left(1 + O\left(\frac{\varepsilon}{|\log\varepsilon|}\right)\right)\Pi^{(v)} = \Pi^{(v)} + O\left(\frac{\varepsilon}{|\log\varepsilon|}\right),$$

where the error bounds are understood in the uniform operator norm topology.

Our next step is the analysis of the "part" of the Dirichlet-to-Neumann map Λ^V pertaining to the vertex domain Q_v. We will denote by Λ_v^V its self-adjoint restriction, $\Lambda_v^V := \Lambda^V|_{L^2(\Gamma_v^\varepsilon)}$.

Firstly, we note that the spectrum of Λ_v^V (which can be termed as the Steklov spectrum of the sloshing problem pertaining to $A_0^{(v)}$, see [22]) is discrete and accumulates to negative infinity. The point $\lambda_1 = 0$ is the least (by absolute value) Steklov eigenvalue with $\psi_v = |\Gamma_v^\varepsilon|^{-1/2}\mathbf{1}|_{\Gamma_v^\varepsilon}$ being the corresponding eigenvector. For the second eigenvalue λ_2 one has by scaling, see, e.g., [40] and references therein:

Lemma 3.4.

$$|\lambda_2| \geq C\frac{1}{\varepsilon}.$$

Introduce the N-dimensional orthogonal projection

$$\mathcal{P} := \sum_v \langle \cdot, \psi_v \rangle \psi_v,$$

define $\mathcal{P}_\perp := 1 - \mathcal{P}$ and consider the operator $\mathcal{P}_\perp M(z) \mathcal{P}_\perp$. Since obviously $\mathcal{P} \operatorname{dom} \Lambda \subset \operatorname{dom} \Lambda$, this is well-defined and by a straightforward estimate for the sesquilinear forms one has, taking into account (10) applied to $M_V(z)$, Proposition 3.1, and Lemmata 3.3, 3.4

Lemma 3.5.

$$\|(\mathcal{P}_\perp M(z) \mathcal{P}_\perp)^{-1}\| \leq C\varepsilon, z \in K_\sigma,$$

where the operator $\mathcal{P}_\perp M(z) \mathcal{P}_\perp$ is considered as a linear (unbounded) operator in $\mathcal{PH}$.

4. Norm-resolvent asymptotics

We will make use of the Kreĭn formula (13) to obtain a norm-resolvent asymptotics of the family A_ε. In so doing, we will compute the asymptotics of $M^{-1}(z)$ based on a Schur-Frobenius type inversion formula, having first rewritten $M(z)$ as a two by two operator matrix relative to the orthogonal decomposition of the Hilbert space $\mathcal{H} = \mathcal{PH} \oplus \mathcal{P}_\perp \mathcal{H}$. In the study of operator matrices, we rely upon the material of [29] and references therein.

The operator $M(z)$ admits the block matrix representation,

$$M(z) = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix} \text{ with } \mathbb{A}, \mathbb{B}, \mathbb{E} \text{ bounded.}$$

For the inversion of $M(z)$ we then use the Schur-Frobenius inversion formula [29, Theorem 2.3.3]

$$\overline{\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}}^{-1} = \begin{pmatrix} \mathbb{A}^{-1} + \overline{\mathbb{A}^{-1}\mathbb{B}\mathcal{S}}^{-1}\mathbb{E}\mathbb{A}^{-1} & -\overline{\mathbb{A}^{-1}\mathbb{B}\mathcal{S}}^{-1} \\ -\overline{\mathcal{S}}^{-1}\mathbb{E}\mathbb{A}^{-1} & \overline{\mathcal{S}}^{-1} \end{pmatrix} \text{ with } \mathcal{S} := \mathbb{D} - \mathbb{E}\mathbb{A}^{-1}\mathbb{B}. \tag{14}$$

Note that by Proposition 2.2, one has $\Im M(z) = (\Im z) S_{\bar{z}}^* S_{\bar{z}}$. Moreover, since $S_z = (1 - zA_0^{-1})^{-1}\Pi$, one has

$$S_{\bar{z}}^* S_{\bar{z}} = \Pi^*(1 - zA_0^{-1})^{-1}(1 - \bar{z}A_0^{-1})^{-1}\Pi,$$

and therefore, for some constants $c_1, c_2 > 0$,

$$\langle S_{\bar{z}}^* S_{\bar{z}} \mathcal{P}\phi, \mathcal{P}\phi \rangle_{\mathcal{H}} = \left\|(1 - \bar{z}A_0^{-1})^{-1}\Pi\mathcal{P}\phi\right\|^2 \geq c_1\|\Pi\mathcal{P}\phi\|^2 \geq c_1\|\Pi_V\mathcal{P}\phi\|^2 \geq c_2\|\mathcal{P}\phi\|_{\mathcal{H}}^2$$

for all $\phi \in \mathcal{H}, z \in K_\sigma$, where we have used the fact that the operator A_0 is bounded below by a positive constant. It follows that $\mathbb{A}^{-1} = (\mathcal{P}M(z)\mathcal{P})^{-1}$ is boundedly invertible.

Proceeding exactly as in [10] based on the estimate provided by Lemma 3.5 which reads

$$\|\mathbb{D}^{-1}\| \leq C\varepsilon,$$

use $\mathcal{S}^{-1} = (I - \mathbb{D}^{-1}\mathbb{E}\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbb{D}^{-1}$ to obtain $\mathcal{S}^{-1} = O(\varepsilon)$.

Returning to (14), one gets

$$M(z)^{-1} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(\varepsilon) \tag{15}$$

with a uniform estimate for the remainder term. Comparing our result with (12) of Proposition 2.3 with $\beta_0 := \mathcal{P}_\perp$ and $\beta_1 := \mathcal{P}$, one arrives at the following

Theorem 4.1. *For $z \in K_\sigma$ one has the estimate*

$$\|(A_\varepsilon - z)^{-1} - (A_{\beta_0,\beta_1} - z)^{-1}\| \le C\varepsilon$$

for a universal constant C and $\beta_0 = \mathcal{P}_\perp$, $\beta_1 = \mathcal{P}$, where the operator A_{β_0,β_1} is defined in Proposition 2.3.

Proof. The proof is identical to that of [10, Theorem 3.1], we include it here for the sake of completeness.

For the resolvent $(A_\varepsilon - z)^{-1}$ the formula (13) is applicable, in which for $M(z)^{-1}$ we use (15). As for the resolvent $\big(A_{\mathcal{P}_\perp,\mathcal{P}} - z\big)^{-1}$, Proposition 2.3 with $\beta_0 = \mathcal{P}_\perp$, $\beta_1 = \mathcal{P}$ is clearly applicable. Moreover, for this choice of β_0, β_1, the operator

$$Q_{\mathcal{P}_\perp,\mathcal{P}}(z) = -\big(\overline{\mathcal{P}_\perp + \mathcal{P}M(z)}\big)^{-1}\mathcal{P}$$

in (12) is easily computable (*e.g.*, by the Schur-Frobenius inversion formula of [29], see (14))[2], yielding

$$Q_{\mathcal{P}_\perp,\mathcal{P}}(z) = -\mathcal{P}\big(\mathcal{P}M(z)\mathcal{P}\big)^{-1}\mathcal{P}, \tag{16}$$

and the claim follows. $\square$

The estimate of Theorem 4.1 already establishes norm-resolvent convergence of the family A_ε to an operator which is by (16) a relative finite-dimensional perturbation of the decoupled operator A_0. It is however possible to obtain a further simplification of this answer, relating the leading order asymptotic term to a self-adjoint operator on the limiting metric graph. This procedure follows the blueprint of our paper [10]. In what follows, we briefly outline this argument.

Note first that $(A_0 - z)^{-1} = (A_0^V - z)^{-1} \oplus (A_0^E - z)^{-1}$ is easily analysed. Indeed, by Proposition 3.1 one has $(A_0^V - z)^{-1} = O(\varepsilon/|\log\varepsilon|)$, whereas $(A_0^E - z)^{-1}$ by the separation of variables converges to the Dirichlet Laplacian on the space

$$H_G := \oplus_v L^2([0, l_e] \times \mathbf{1}_\varepsilon),$$

where $\mathbf{1}_e := \varepsilon^{-1/2}\mathbf{1}$ is the constant normalised function in the variable perpendicular to the direction of the edge e. The operator $(A_0 - z)^{-1}$ is therefore $O(\varepsilon/|\log\varepsilon|)$-close, uniformly in $z \in K_\sigma$, to the operator unitary equivalent to the resolvent of A_0^G, where A_0^G is the Dirichlet-decoupled graph Laplacian pertaining to the graph G. The finite-dimensional second summand on the right hand side of (16) should therefore represent the matching conditions at the vertices of the limiting graph G. In order to see this, one passes over to the generalised resolvent $R_\varepsilon(z) := P_E(A_\varepsilon - z)^{-1}P_E$, which is shown to admit the following asymptotics.

Theorem 4.2. *The operator family $R_\varepsilon(z)$ admits the following asymptotics in the operator-norm topology for $z \in K_\sigma$:*

$$R_\varepsilon(z) - R_{\text{eff}}(z) = O(\varepsilon),$$

where $R_{\text{eff}}(z)$ is the solution operator for the following spectral BVP on the edge domain Q_E:

$$\begin{aligned} -\Delta u - zu = f, \quad & f \in L^2(Q_E), \\ \beta_0(z)\Gamma_0^E u + \beta_1\Gamma_1^E u = 0, & \end{aligned} \tag{17}$$

[2] We remark that $\mathcal{P}_\perp + \mathcal{P}M(z)$ is triangular ($\mathbb{A} = \mathcal{P}M(z)\mathcal{P}$, $\mathbb{B} = \mathcal{P}M(z)\mathcal{P}_\perp$, $\mathbb{E} = 0$, $\mathbb{D} = I$ in (14)) with respect to the decomposition $\mathcal{H} = \mathcal{P}\mathcal{H} \oplus \mathcal{P}_\perp\mathcal{H}$.

with $\beta_0(z) = \mathcal{P}_\perp - \mathcal{P}B(z)\mathcal{P}$, $B(z) := -M_V(z)$ *and* $\beta_1 = \mathcal{P}$.

The boundary condition in (17) can be written in the more conventional form

$$\mathcal{P}_\perp u|_\Gamma = 0, \quad \mathcal{P}\partial_n u = \mathcal{P}B(z)\mathcal{P}u\big|_\Gamma.$$

Equivalently,

$$R_\varepsilon(z) - \left(A^E_{\mathcal{P}_\perp - \mathcal{P}B(z)\mathcal{P},\mathcal{P}} - z\right)^{-1} = O(\varepsilon),$$

where $A^E_{\mathcal{P}_\perp - \mathcal{P}B(z)\mathcal{P},\mathcal{P}}$, *for any fixed* z, *is the operator in* $L^2(Q_E)$ *defined by Proposition 2.3 relative to the triple* $(\mathcal{H}, \Pi_E, \Lambda^E)$, *where the term "triple" is understood in the sense of* [26]. *This operator is maximal anti-dissipative for* $z \in \mathbb{C}_+$ *and maximal dissipative for* $z \in \mathbb{C}_-$, *see* [28].

The *proof* of the theorem follows immediately from Theorem 4.1, see [10, Theorem 3.6] together with the observation that

$$\mathcal{P}M(z)\mathcal{P} = \mathcal{P}M_E(z)\mathcal{P} + \mathcal{P}M_V(z)\mathcal{P}.$$

The next step of our argument is to introduce the truncated[3] (reduced) boundary space $\breve{\mathcal{H}}$ in order to make all the ingredients of our setup finite-dimensional.

We put $\breve{\mathcal{H}} := \mathcal{P}\mathcal{H}$ (note, that in our setup $\breve{\mathcal{H}}$ is N-dimensional). Introduce the truncated Poisson operator on $\breve{\mathcal{H}}$ by $\breve{\Pi}_E := \Pi_E|_{\breve{\mathcal{H}}}$ and the truncated Dirichlet-to-Neumann map $\breve{\Lambda}^E := \mathcal{P}\Lambda^E|_{\breve{\mathcal{H}}}$. Then

Proposition 4.3 ([10], Theorem 3.7)**.** *1. The formula*

$$R_{\text{eff}}(z) = \left(A_0^E - z\right)^{-1} - \breve{S}_z^E\left(\breve{M}_E(z) - \mathcal{P}B(z)\mathcal{P}\right)^{-1}(\breve{S}_{\bar{z}}^E)^* \tag{18}$$

holds, where $\breve{S}_z^E$ *is the solution operator of the problem*

$$-\Delta u_\phi - z u_\phi = 0, \quad u_\phi \in \operatorname{dom} A_0^E \dotplus \operatorname{ran} \breve{\Pi}_E,$$
$$\Gamma_0^E u_\phi = \phi, \quad \phi \in \breve{\mathcal{H}},$$

and $\breve{M}_E$ *is the* M*-operator defined in accordance with* (9), (11) *relative to the triple* $(\breve{\mathcal{H}}, \breve{\Pi}_E, \breve{\Lambda}^E)$.

2. The "effective" generalised resolvent $R_{\text{eff}}(z)$ *is represented as the generalised resolvent of the problem*

$$\begin{gathered}-\Delta u - zu = f, \quad f \in L^2(Q_E), \quad u \in \operatorname{dom} A_0^E \dotplus \operatorname{ran} \breve{\Pi}_E,\\ \mathcal{P}\partial_n u\big|_\Gamma = \mathcal{P}B(z)\mathcal{P}u\big|_\Gamma.\end{gathered} \tag{19}$$

3. The triple $(\breve{\mathcal{H}}, \breve{\Gamma}_0^E, \breve{\Gamma}_1^E)$ *is the classical boundary triple* [18, 14] *for the operator* $A_{\max}$ *defined by the differential expression* $-\Delta$ *on the domain* $\operatorname{dom} A_{\max} = \operatorname{dom} A_0^E \dotplus \operatorname{ran} \breve{\Pi}_E$. *Here* $\breve{\Gamma}_0^E$ *and* $\breve{\Gamma}_1^E$ *are defined on* $\operatorname{dom} A_{\max}$ *as the operator of the boundary trace on* Γ *and* $\mathcal{P}\partial_n u$, *respectively.*

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[3]In what follows we consistently supply the (finite-dimensional) "truncated" spaces and operators pertaining to them by the breve overscript.

We now consider the operator $\mathcal{P}B(z)\mathcal{P}$ in (18); since $B=-M_V$ by definition, we invoke the estimates derived in Section 3 to obtain

$$\mathcal{P}B\mathcal{P}=-\mathcal{P}\Lambda^V\mathcal{P}-z\mathcal{P}\Pi_V^*\Pi_V\mathcal{P}+O(\varepsilon/|\log\varepsilon|)=-z\breve{\Pi}_V^*\breve{\Pi}_V+O(\varepsilon/|\log\varepsilon|),$$

with a uniform estimate for the remainder term. Here the truncated Poisson operator $\breve{\Pi}_V$ is introduced as $\breve{\Pi}_V:=\Pi_V|_{\breve{\mathcal{H}}}$ relative to the same truncated boundary space as above, $\breve{\mathcal{H}}=\mathcal{P}\mathcal{H}$. As a result, we obtain

$$R_{\text{eff}}(z)-R_{\text{hom}}(z)=O(\varepsilon/|\log\varepsilon|),$$

with

$$R_{\text{hom}}(z):=(A_0^E-z)^{-1}-\breve{S}_z^E\big(\breve{M}_E(z)+z\breve{\Pi}_V^*\breve{\Pi}_V\big)^{-1}(\breve{S}_{\bar z}^E)^*. \tag{20}$$

By a classical result of [28] (see also [23, 24]), the operator $R_{\text{hom}}(z)$ is a generalised resolvent, so it defines a z-dependent family of closed densely defined operators in $L^2(Q_E)$, which are maximal anti-dissipative for $z\in\mathbb{C}_+$ and maximal dissipative for $z\in\mathbb{C}_-$. Writing the resolvent $(A_\varepsilon-z)^{-1}$ in the matrix form relative to the orthogonal decomposition $L^2(Q)=P_EL^2(Q)\oplus P_VL^2(Q)=L^2(Q_E)\oplus L^2(Q_V)$ then yields the following

Theorem 4.4. *The resolvent $(A_\varepsilon-z)^{-1}$ admits the following asymptotics in the uniform operator-norm topology:*

$$\big(A_\varepsilon-z\big)^{-1}=\mathcal{R}_{\text{hom}}(z)+O(\varepsilon/|\log\varepsilon|),$$

where the operator $\mathcal{R}_{\text{hom}}(z)$ has the following representation relative to the decomposition $L^2(Q_E)\oplus L^2(Q_V)$:

$$\mathcal{R}_{\text{hom}}(z)=\begin{pmatrix} R_{\text{hom}}(z) & \Big(\mathfrak{K}_{\bar z}\big[R_{\text{hom}}(\bar z)-(A_0^E-\bar z)^{-1}\big]\Big)^*\breve{\Pi}_V^* \\ \breve{\Pi}_V\mathfrak{K}_z\big[R_{\text{hom}}(z)-(A_0^E-z)^{-1}\big] & \breve{\Pi}_V\mathfrak{K}_z\Big(\mathfrak{K}_{\bar z}\big[R_{\text{hom}}(\bar z)-(A_0^E-\bar z)^{-1}\big]\Big)^*\breve{\Pi}_V^*\end{pmatrix}. \tag{21}$$

Here $\mathfrak{K}_z:=\Gamma_0^E|_{\mathfrak{N}_z}$ with $\mathfrak{N}_z:=\operatorname{ran}S_z^E\mathcal{P}$, $z\in\mathbb{C}_\pm$, and the generalised resolvent $R_{\text{hom}}(z)$ is defined by (20).

The above theorem provides us with the simplest possible leading-order term of the asymptotic expansion for $(A_\varepsilon-z)^{-1}$. However, it is not yet obvious whether this leading-order term $\mathcal{R}_{\text{hom}}(z)$ is a resolvent of some self-adjoint operator in the space $L^2(Q_E)\oplus\breve{\Pi}_V\breve{\mathcal{H}}\subset L^2(Q)$. It turns out that answer to this question is positive, which is proved by the following explicit construction.

Put $L^2(G):=\oplus_eL^2[0,l_e]$, $H^2(G):=\oplus_eH^2[0,l_e]$. For any $u\in H^2(G)$ introduce the notation u_{ev} for the limit of $u_e(x):=u|_e(x)$ at the vertex v. Let $H_{\text{hom}}=L^2(G)\oplus\mathbb{C}^N$, and set

$$\operatorname{dom}\mathcal{A}_{\text{hom}}=\Big\{(u,\beta)^\top\in H_{\text{hom}}:\ u\in H^2(G),\ u_v:=u_{ev}=u_{ve'}\text{ for any }v$$
$$\text{and }e,e'\text{ incident to }v,\ \text{and }\beta=\kappa u_V\Big\}, \tag{22}$$

where u_V is the N-dimensional vector of $\{u_v\}_{v\in V}$ and κ is the diagonal matrix,

$$\kappa=\operatorname{diag}\{|Q_v^0|^{1/2}\}.$$

The action of the operator is set by

$$\mathcal{A}_{\text{hom}}\begin{pmatrix}u\\ \beta\end{pmatrix}=\begin{pmatrix}-u''\\ -\kappa^{-1}\partial_n u|_V\end{pmatrix},\qquad \begin{pmatrix}u\\ \beta\end{pmatrix}\in\operatorname{dom}\mathcal{A}_{\text{hom}},\tag{23}$$

where $\partial_n u|_V$ is the $N-$dimensional vector of $\{\sum_{e\in v}\partial_n u_e|_v\}_{v\in V}$, i.e., the vector, each element of which is represented by the sum of edge-inward normal derivatives of the function u over all the edges incident to the vertex v.

The main result of the present work, which follows by an explicit computation of the resolvent of (22)–(23) followed by the comparison of the latter with (21), is as follows.

Theorem 4.5. *The resolvent $(A_\varepsilon - z)^{-1}$ admits the following estimate in the uniform operator norm topology, uniform in $z\in K_\sigma$:*

$$\big(A_\varepsilon - z\big)^{-1} - \Theta\big(\mathcal{A}_{\text{hom}} - z\big)^{-1}\Theta^* = O(\varepsilon/|\log\varepsilon|),\tag{24}$$

where Θ is a partial isometry from H_{hom} onto $L^2(Q)$, acting as follows:

- *for every edge $e\in G$, $e=[0,l_e]$, it embeds $u\in H^2(e)$ into $L^2(Q_e)$ as $u(x)\times\varepsilon^{-1/2}\mathbf{1}(y)$;*
- *for every vertex $v\in G$, it embeds the value u_v, i.e., the common value of $u\in H^2(G)$ at the vertex v, into $L^2(Q_v)$ as $\varepsilon^{-1/2}u_v\mathbf{1}$.*

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