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THE THEORY OF HIGH FREQUENCY GRAVITATIONAL RADIATION, AND ITS APPLICATION TO COSMOLOGY

by

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TO

MY MOTHER AND

FATHER

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Summary.

In Chapter 1 a brief introduction to the theory of gravitational radiation in general relativity is presented, and an outline of the variety of different methods that have been used to study it is given.

In the second chapter, a single theoretical approach, upon which to base the subsequent treatment, is chosen. This approach, developed by R. A. Isaacson (1968a,b), involves obtaining approximate gravitational wave solutions to the vacuum Einstein equations by supposing that the radiation is of high frequency. The work of Isaacson is reviewed to show how the high frequency approximation leads to a tensor representation of the gravitational field energy.

In Chapter 3, the Isaacson theory is extended, by the present writer, so that it may be applied to situations in which gravitational radiation is present in a matter filled manifold. The work of J. Madore, who has also considered gravitational radiation in a material fluid, is discussed to show that his results may be found, as are Isaacson's, as special cases within the proposed general formalism. Provided that certain assumptions are made, the wave energy in matter is shown to be of the same form as that found in vacuum.

It is in Chapter 4 that the cosmological applications of the general formalism of Chapter 3 are first considered. Radiation travelling through a cosmological 'background' space-time is examined, with the intention of discovering

how this background geometry interacts with the radiation. A Friedmann line element, with the space curvature constant $\kappa = 0$, is used to represent the cosmological background upon which the radiation propagates. Employing this example I am able to show that test particles, located in a plane perpendicular to the direction of propagation of a monodirectional gravitational wave, experience accelerations due to the wave superimposed upon the effects of the model's cosmological expansion. Further, the theory for constructing an isotropic gravitational radiation field is developed. It is shown that the energy tensor of such a field may be represented by a perfect fluid energy tensor, with an equation of state in which the radiation pressure is one-third of the energy density.

In Chapter 5, the manner in which the radiation modifies the cosmological background is considered. The work is motivated by a model of the Universe, containing matter and gravitational radiation, proposed by Isaacson and Winicour (1972, 1973). They assumed that matter was converted into an isotropic field of gravitational radiation. However, as is shown in Chapter 5, the model can lead to a negative mass density. This difficulty is overcome by bringing the conversion to an end at some pre-assigned instant in cosmic time. This ensures that the energy distribution in the model remains physically acceptable throughout the model's development. The cosmological equations for the model are solved, by numerical methods where necessary, and a number of examples of the resulting universes are given in diagrammatic and tabular form.

Finally, in an appendix, the possibility that gravitational radiation is generated during the 'fireball' era of the Universe is briefly considered.

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Chapter 1.

Introduction.

The problem of obtaining a theoretical description of a radiative gravitational field is one that was first considered by Albert Einstein in 1916. Over half a century later the problem still remains one of the most interesting posed by the general theory of relativity, since in that time it has still evaded a satisfactory solution. The apparently elusive nature of the answer to the problem has however inspired and motivated many researchers to explore a variety of different theoretical avenues in search of ways to resolve the difficulties. Work on gravitational waves may be divided into several groups, each reflecting a different approach to the problem.

The first of these seeks to give a rigourous definition of the conditions which the space-time metric must satisfy in order to describe a wave-like field. Pirani (1957) produced one of the most important studies in this group when he recognized that the procedure of classifying electromagnetic fields, by an examination of the eigenvectors of the field's energy tensor, could be carried over into gravitational theory. An observer is envisaged who moves in an electromagnetic field in such a way that he measures no net energy flow due to the field (in Pirani's original terminology the observer is said to be 'following the field'). The four velocity of such an observer may be shown to be an eigenvector of the field's energy tensor. In this scheme a null electromagnetic field is identified by the fact that its energy tensor has no timelike eigenvectors (Synge, 1956), and hence an observer following the field would need to attain the fundamental velocity. The eigenvectors of the gravitational field were defined by Pirani with the aid of Petrov's algebraic classification (Petrov, 1954, 1969) of empty space-time Riemann-Christoffel tensors into canonical types. Of the five algebraically distinct types of gravity field, only three, generally denoted by II, III and N, do not possess timelike eigenvectors. By analogy with the electromagnetic case, Pirani introduced the criterion that gravitational waves exist in a region of space-time only if the Riemann-Christoffel tensor there is of Petrov type II, III or N. Other works in this group, that contain algebraic or geometrical criteria for the existance of wave-like gravity fields, have been presented by Bel (1959) and Lichnerowicz (1960).

A second group is composed of investigations containing exact solutions of Einstein's field equations which the authors have interpreted as gravitational waves. Some examples of these may be found in the studies of Einstein and Rosen (1937), Bondi, Pirani and Robinson (1959) and Robinson and Trautman (1960). These solutions are regarded as representations of waves with cylindrical, plane and spherical symmetry respectively, and have been shown to fall into the type II category of the Petrov classification.

In a third group of works, gravitational waves are treated by approximation methods. This approach was used by Einstein (1916) when he employed the linearized equations of gravity to show that, in certain coordinate systems, their solutions exhibited obvious wave-like characteristics. However, due to the inherent nonlinearity of the equations of general relativity, this result was treated with some caution. It was not until 1938 that a method of approximation was devised which took account of this nonlinearity (Einstein, Infeld and Hoffmann, 1938). This work however did not lend weight to the idea that gravitational waves had any physical significance, since it contained no indication that freely gravitating particles produced gravitational emission. Indeed there was a good deal of controversy about the physical existence of radiative fields until the late 1950's and this situation was not alleviated by the fact that the exact solutions known up to that time described free wave fields, uncoupled from sources. New light was cast upon this question when Bonnor (1959) developed a new approximation technique also capable of examining some of the nonlinear aspects of the field equations. Applying this to the problem of gravitational waves generated by a specified source system, he was able to show that the energy of the radiating field, as calculated from the energy pseudotensor of the linear theory, corresponded to the amount of mass lost by the source during the emission process. Bonnor's method was subsequently developed and applied to problems of astrophysical interest by, for example, Bonnor and Rotenberg (1966). Many such studies of waves by approximation methods have been made. Instances of these may be found in Price and Thorne (1969) and Infeld and Michalska-Trautman (1969).

Investigations in a fourth category deal with the gravitational emission of elementary particles. This method of attack, which is neither fully in the domain of the quantum theorist ~~not~~ nor the relativist, has proved to be a very difficult one. Although a great deal of work has been attempted (Vladimirov, 1964; Carmeli, 1967; Isham, Salam and Strathdee, 1974), fruitful results will perhaps only be forthcoming when the extensive problems of reconciling quantum theory and gravitation have been resolved.

A fifth group comprises works devoted to the experimental study of gravitational radiation. A comprehensive account of the theory and practice of wave detection may be found in Weber (1961). The relevant details of the present day observational climate in this field is set out in the text of this thesis.

Finally, in a sixth category, the authors arrive at a definition of gravitational radiation by first establishing a definition of the gravitational field energy. Examples of this approach may be found in the works of Brill (1959) and Isaacson (1968a,b).

The study of gravitational waves that is to follow is broadly based upon the technique outlined in the investigations of category six, although the approximation methods of group three have a bearing, since approximate solutions of Einstein's field equations will be considered. In this brief introduction I have given only the barest summary of the efforts of some sixty years of work. It is hoped however that it has been sufficient to indicate the diversity and ingenuity of the ideas employed in the attempt to understand this interesting physical and theoretical problem.

Chapter 2.

A review of the work of Isaacson.

2.1 Discussion of approximation techniques.

A number of different approaches to the problem of gravitational radiation have been mentioned in Chapter 1. The approach adopted here was first fully developed by R. A. Isaacson (1968a,b). The technique is one of finding approximate solutions to Einstein's equations, beginning with the assumption that the metric of space-time takes the form

$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + \varepsilon \overset{(1)}{g}_{\mu\nu} , \quad 0 < \varepsilon << 1. \tag{2.101}$$

Here $\varepsilon \overset{(1)}{g}_{\mu\nu}$ is regarded as a small perturbation superimposed upon $\overset{(0)}{g}_{\mu\nu}$. The field equations are then expanded in terms of powers of the smallness parameter ε. Approximate solutions to the resulting system of equations are then found which may be interpreted as representing gravitational radiation.

The similarity between this procedure and earlier approximation schemes is immediately apparent, and in light of this, perhaps it would be prudent to ask what significant improvements in the theory did Isaacson propose. The essential difference lies in the choice of $\overset{(0)}{g}_{\mu\nu}$ in equation (2.101). Prior to 1968, analyses of approximate wave-like fields dealt, in the zeroeth approximation, with the flat space-time metric. That is, the components of $\overset{(0)}{g}_{\mu\nu}$ were identified with those of the Minkowski metric, and the gravitational field entered the scheme only as a small correction to flat space. This type of procedure is severely limited in its domain of applicability.

An important step towards overcoming some of the shortcomings of linearized theory, was taken by Isaacson when he proposed that the metric $\overset{(0)}{g}_{\mu\nu}$ be represented by the metric of an arbitrary, curved space-time. His work was the first entirely devoted to an investigation of approximate wave-like fields in general relativity, employing a completely general choice of $\overset{(0)}{g}_{\mu\nu}$.

A review of Isaacson's work is undertaken here, since most of what is to follow will find its origins there.

Treating gravitational waves in a curved space-time, we will discover a linear wave equation which will describe how the curved background interacts with, and modifies the 'high frequency' radiation field. Then, using higher order terms afforded us by the non-linear framework adopted by Isaacson, it will be possible to show how the wave acts back upon the geometry of the curved space-time. If it is assumed that energy transportation occurs in radiative gravity fields, then the radiation itself should be expected to act as a source in creating some part of the curvature of the space-time through which it propagates. The energy of the field in this case, like all other forms of energy, will have an effective gravitational mass. Why it is necessary to suppose that the radiation is of 'high frequency' will be explained later, as well as what is meant by this expression.

It is assumed that the total metric, $g_{\mu\nu}$, takes the form given by equation (2.101), and this will be written as

$$g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon h_{\mu\nu}, \quad 0 < \varepsilon << 1, \qquad (2.102)$$

where $\gamma_{\mu\nu}$ is a slowly varying function of space-time, and $h_{\mu\nu}$

is a relatively rapidly changing perturbation linearly superimposed upon $\gamma_{\mu\nu}$. Following Isaacson, it will be supposed that $h_{\mu\nu}$ is identified with a high frequency gravitational radiation field, and the $\gamma_{\mu\nu}$ with the curved "background" space-time through which it propagates.

2.2 Definition of high frequency radiation.

To formalize what has been said about the relative rates of change of $\gamma_{\mu\nu}$ and $h_{\mu\nu}$, the concept of 'characteristic length' is introduced. Let $(x^\alpha) \equiv (x^1, x^2, x^3, x^4)$ be a dimensionless coordinate system, where x^i, i=1,2,3 are spacial coordinates, and x^4, a time coordinate. Thus, in this system the components of the total metric, $g_{\mu\nu}$, will be of zero physical dimensions. The theory will be developed employing a dimensionless analysis unless it is specifically stated otherwise. In particular, units are chosen such that G=c=1, where G is the Newtonian gravitational constant and c is the velocity of light.

Now, if a frequency Ω is attributed to the radiation then it is possible to define a dimensionless wavelength, $\lambda \approx \Omega^{-1}$. Since the radiation will vary significantly over this distance, λ is chosen to be the characteristic length of the $h_{\mu\nu}$. This is expressed as

$$\left|\frac{\partial h_{\mu\nu}}{\partial x^\alpha}\right| \approx \lambda^{-1}|h_{\mu\nu}|. \tag{2.201}$$

The characteristic length L, over which the background changes appreciably, may be defined in the following manner. Provided attention is restricted to the non-zero components of the background, let

$$\left|\frac{\partial \gamma_{\mu\nu}}{\partial x^\alpha}\right| \approx (L^\alpha_{\mu\nu})^{-1}|\gamma_{\mu\nu}|.$$

Then

$$L = \min\left(L^{\alpha}_{\mu\nu} \right), \tag{2.202}$$

where L , $L^{\alpha}_{\mu\nu}$ are pure numbers and where no summation is implied by repeated indices.

For those derivatives of $\gamma_{\mu\nu}$ which are non-zero, we may write for simplicity

$$\left| \frac{\partial \gamma_{\mu\nu}}{\partial x^{\alpha}} \right| \approx L^{-1} \left| \gamma_{\mu\nu} \right|. \tag{2.203}$$

By consideration of the conditions (2.201) and (2.203) gravitational radiation is said to be of high frequency provided that

$$L >> \lambda. \tag{2.204}$$

It is of interest to note that if $\gamma_{\mu\nu} \equiv \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the metric of Minkowski, then the condition for high frequency radiation is fulfilled for any finite characteristic length λ. Thus, the results of the linearized theory should be expected to emerge from Isaacson's more general high frequency approximation theory.

Henceforth, only radiation satisfying the definition of high frequency will be considered. The limitations of the validity of this assumption will be briefly discussed later.

In Isaacson's treatment, the parameter ε is defined as follows. Let $\varepsilon = \lambda / L$ and choose L such that $L \approx 1$. Then, indeed, for high frequency radiation $\varepsilon << 1$, and it can be seen that the magnitude of the wavelength λ is of the order ε . Further, since the frequency Ω will now have an ε dependence of the form $\Omega \propto \varepsilon^{-1}$, the limiting process given by $\varepsilon \rightarrow 0$ will be called the 'high frequency limit'.

2.3 Notations and conventions.

The geometry of space-time will be represented by a four-dimensional Riemannian manifold, with fundamental form

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu ,$$

where the signature of the total metric $g_{\mu\nu}$ is -2.

Greek indices will be regarded as taking the values 1,2, 3,4, latin indices as taking the the values 1,2,3, and the convention of summing over repeated indices is adopted unless it is specifically stated otherwise.

Partial derivatives will be denoted thus

$$\frac{\partial \psi}{\partial x^\alpha} \equiv \psi_{,\alpha} \, ,$$

and Christoffel symbols of the second kind for a metric tensor $b_{\mu\nu}$ will be defined by

$$\Gamma^{\lambda}_{\alpha\beta} \equiv \tfrac{1}{2} b^{\lambda\sigma}\left(b_{\sigma\alpha,\beta} + b_{\sigma\beta,\alpha} - b_{\alpha\beta,\sigma}\right).$$

Covariant derivatives with respect to the background metric $\gamma_{\mu\nu}$ will be indicated by a semicolon. For example

$$\psi_{\mu\nu;\alpha} = \psi_{\mu\nu,\alpha} - \Gamma^{\sigma}_{\mu\alpha}(\gamma_{\rho\tau})\psi_{\sigma\nu} - \Gamma^{\sigma}_{\nu\alpha}(\gamma_{\rho\tau})\psi_{\mu\sigma}.$$

All indices will be raised or lowered by the use of the background metric tensor, and the conventions adopted for the Riemann-Christoffel and Ricci tensors will be

$$v_{\sigma;\mu\nu} - v_{\sigma;\nu\mu} = R^{\rho}{}_{\sigma\mu\nu}(\gamma_{\alpha\beta})\, v_\rho ,$$

where v_σ is an arbitrary covariant vector, and

$$R_{\mu\nu} = \gamma^{\alpha\beta} R_{\alpha\mu\nu\beta}, \quad R = \gamma^{\alpha\beta} R_{\alpha\beta}.$$

Finally, the notation $f(x) = O(\varepsilon^n)$ should be read 'f is of the order of magnitude ε^n'. If $f(x) = O(\varepsilon^n)$, then this will be taken to mean that there exists a constant

$P > 0$ such that

$$|f(x)| < P\varepsilon^n, \quad \text{as } \varepsilon \to 0.$$

2.4 Constructing an approximation scheme in vacuum.

Before continuing this review of the work of Isaacson, no attempt should be made to evade the critism that the splitting of a metric into parts, in the manner displayed in equation (2.102), is in general not allowable. This absence of a superposition principle in general relativity is, of course, a consequence of the notorious non-linearity of the field equations. The form of the metric (2.102), first postulated by Brill and Hartle (1964), should rather be regarded as an initial assumption the adequacy of which is to be tested in the particular situation under consideration.

The definition of high frequency gravitational radiation employed by Isaacson, the basis of his work, is as follows.

A metric is said to contain a high frequency wave if and only if there exist a family of coordinate systems in which the total metric takes the form

$$g_{\mu\nu} = \gamma_{\mu\nu}(x) + \varepsilon h_{\mu\nu}(x, \varepsilon), \quad 0 < \varepsilon << 1, \tag{2.401}$$

where

$$\begin{aligned} \gamma_{\mu\nu} &= O(1), & h_{\mu\nu} &= O(1), \\ \gamma_{\mu\nu,\alpha} &= O(1), & h_{\mu\nu,\alpha} &= O(\varepsilon^{-1}), \\ \gamma_{\mu\nu,\alpha\beta} &= O(1), & h_{\mu\nu,\alpha\beta} &= O(\varepsilon^{-2}). \end{aligned} \tag{2.402}$$

Following Isaacson, we will use this definition to derive approximate vacuum field equations, and to demonstrate that the results thus obtained may indeed be interpreted as gravitational radiation.

In vacuum the Einstein field equations may be written

$$R_{\mu\nu} = 0. \tag{2.403}$$

If the metric tensor (2.401) is used, the total Ricci tensor may be expressed as a power series in $\mathcal{E}$ given by

$$\begin{aligned} R_{\mu\nu}(g_{\alpha\beta}) = {} & R^{(0)}_{\mu\nu} + \mathcal{E} R^{(1)}_{\mu\nu} \\ & + \mathcal{E}^2 R^{(2)}_{\mu\nu} + \mathcal{E}^3 R^{(3+)}_{\mu\nu}, \end{aligned} \tag{2.404}$$

where $R^{(3+)}_{\mu\nu}$ is a remainder term.

After some calculation, it is found that

$$R^{(0)}_{\mu\nu} = R_{\mu\nu}(\gamma_{\alpha\beta}), \tag{2.405}$$

$$R^{(1)}_{\mu\nu} = \Box^{\sigma}_{\mu\sigma;\nu} - \Box^{\sigma}_{\mu\nu;\sigma}, \tag{2.406}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = {} & (h^{\sigma}{}_{\alpha} \Box^{\alpha}_{\mu\nu})_{;\sigma} - (h^{\sigma}{}_{\alpha} \Box^{\alpha}_{\mu\sigma})_{;\nu} \\ & + \Box^{\tau}_{\mu\sigma} \Box^{\sigma}_{\nu\tau} - \Box^{\tau}_{\mu\nu} \Box^{\sigma}_{\tau\sigma}, \end{aligned} \tag{2.407}$$

where

$$\Box^{\lambda}_{\mu\nu} \equiv \frac{1}{2}(h^{\lambda}{}_{\mu;\nu} + h^{\lambda}{}_{\nu;\mu} - h_{\mu\nu;}{}^{\lambda}). \tag{2.408}$$

Thus the vacuum field equations (2.403) become

$$R^{(0)}_{\mu\nu} + \mathcal{E} R^{(1)}_{\mu\nu} + \mathcal{E}^2 R^{(2)}_{\mu\nu} + \mathcal{E}^3 R^{(3+)}_{\mu\nu} = 0. \tag{2.409}$$

The order of magnitude of each term in this expansion is now assessed by appealing to equations (2.405), (2.406) and (2.407), and to the definition (2.402). Since $R^{(0)}_{\mu\nu}$ is a function of the background metric only, it is argued that

$$R^{(0)}_{\mu\nu} = O(1). \tag{2.410}$$

The expression $R^{(1)}_{\mu\nu}$, on the other hand, is a function of terms typically of the form $h_{\mu\nu,\alpha\beta}$ and thus it is supposed that

$$\varepsilon R^{(1)}_{\mu\nu} = O(\varepsilon^{-1}). \tag{2.411}$$

A similar argument applied to the remaining terms in the expansion (2.409) will give

$$\varepsilon^2 R^{(2)}_{\mu\nu} = O(1) \,, \quad \varepsilon^3 R^{(3+)}_{\mu\nu} = O(\varepsilon). \tag{2.412}$$

The estimates given here led Isaacson to propose that the vacuum equations may be decomposed in the following manner. To first order, equations (2.409) become

$$R^{(1)}_{\mu\nu} = 0 \,, \tag{2.413}$$

and, to second order

$$R^{(0)}_{\mu\nu} = -\varepsilon^2 R^{(2)}_{\mu\nu} \,. \tag{2.414}$$

Henceforth, equations (2.413) and (2.414) will be regarded as describing the propagation, in vacuum, of high frequency gravitational radiation through a curved background space-time.

An account of the results obtained by an analysis of these equations will follow. Before we do this however, certain preliminaries still require attention.

2.5 Coordinate conditions.

Consider the question; to what extent are the wave 'potentials', $h_{\mu\nu}$, uniquely determined?

It is easily seen from equations (2.406), (2.407) that the vacuum equations, given by equations (2.413), (2.414), are very cumbersome and complex expressions. It is not difficult to foresee the formidable challenge they pose if an analysis of them is to be made. Some assistance can be obtained here by noting, as is well known, that the Einstein equations in themselves fail to uniquely determine the metric tensor $g_{\mu\nu}$. This allows some freedom to impose restrictions on the coordinate system which act to eliminate the ambiguity in the determinacy of $g_{\mu\nu}$.

Thus, if such restrictions are to be chosen, it would be natural to make the choice with a view to reducing the labour involved in studying the vacuum equations.

Mathematical convenience however is not the only criterion that must be considered when deciding upon suitable conditions. To illustrate another, it is of use to briefly review the equations of linearized general relativity. If a perturbed metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} ,$$

is assumed, where $\eta_{\mu\nu}$ is the flat space metric of Minkowski, and powers of ε greater than one are neglected, it is found that the vacuum Einstein equations become

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{,\alpha\beta} = 0 , \qquad (2.501)$$

provided that the conditions

$$h^{\mu\nu}{}_{,\nu} = 0 , \quad h = \eta^{\alpha\beta} h_{\alpha\beta} = 0, \qquad (2.502)$$

are satisfied. Although the foundations upon which this weak field treatment is built severely limits its applicability, it would never-the-less be desirable to adopt a correspondence principle between the linear theory and the theory of Isaacson. In the limit when the general background metric $\gamma_{\mu\nu} \rightarrow \eta_{\mu\nu}$, the wave equations (2.413) will be required to reduce to the 'flat space' equations (2.501). To demonstrate the consequences of this principle, the explicit form of equations (2.413) will now be derived.

Substitution of equations (2.408) into equations (2.406) will give

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\gamma^{\sigma\rho}(h_{\sigma\rho;\mu\nu} + h_{\mu\nu;\sigma\rho} - h_{\rho\mu;\nu\sigma} - h_{\rho\nu;\mu\sigma}) , \tag{2.503}$$

so that the first order vacuum equations (2.413) become .

$$\gamma^{\sigma\rho}(h_{\sigma\rho;\mu\nu} + h_{\mu\nu;\sigma\rho} - h_{\rho\mu;\nu\sigma} - h_{\rho\nu;\mu\sigma}) = 0. \tag{2.504}$$

Now since

$$h_{\mu\nu;\tau\rho} - h_{\mu\nu;\rho\tau} = R^{(0)}_{\mu\sigma\rho\tau}h^{\sigma}{}_{\nu} + R^{(0)}_{\nu\sigma\rho\tau}h_{\mu}{}^{\sigma} , \tag{2.505}$$

the wave equations (2.504) may be written

$$\begin{aligned} & h_{\mu\nu;}{}^{\rho}{}_{\rho} + h^{\rho}{}_{\rho;\mu\nu} - h_{\rho\mu;}{}^{\rho}{}_{\nu} \\ & - h_{\rho\nu;}{}^{\rho}{}_{\mu} + h^{\sigma}{}_{\mu}R^{(0)}_{\nu\sigma} + h^{\sigma}{}_{\nu}R^{(0)}_{\mu\sigma} \\ & - 2R^{(0)}_{\rho\mu\nu\sigma}h^{\rho\sigma} = 0 , \end{aligned} \tag{2.506}$$

where

$$h_{\alpha\beta;}{}^{\sigma} \equiv \gamma^{\sigma\rho}h_{\alpha\beta;\rho} .$$

If the covariant forms of conditions (2.502) are adopted, that is

$$h^{\mu\nu}{}_{;\nu} = 0 , \tag{2.507a}$$

$$h = \gamma^{\alpha\beta}h_{\alpha\beta} = 0 , \tag{2.507b}$$

as the new coordinate conditions, then indeed (2.506) reduces to (2.501) in the limit $\gamma_{\mu\nu} \rightarrow \eta_{\mu\nu}$. The coordinate system in which these conditions are imposed is sometimes referred to as the 'transverse-traceless' (TT) gauge, for reasons which will become evident later.

If the coordinate conditions (2.507) are now imposed upon equations (2.506), we observe that

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(h_{\mu\nu;}{}^{\rho}{}_{\rho} + h^{\sigma}{}_{\mu} R^{(0)}_{\nu\sigma} + h^{\sigma}{}_{\nu} R^{(0)}_{\mu\sigma} - 2R^{(0)}_{\rho\mu\nu\sigma} h^{\rho\sigma} \right). \qquad (2.508)$$

Isaacson has pointed out that the conditions (2.507) and the equations (2.506) are consistent with one another only in cases where the background geometry is of constant curvature. However, he has also demonstrated that the scheme may be applied to cases other than this, since the terms comprising the inconsistency are a factor $\mathcal{E}^2$ smaller than the dominant terms in the wave equations (2.506).

2.6 The 'WKB' assumption.

In his treatment of the radiation problem, Isaacson found it convenient, again for reasons of mathematical simplicity, to analyse his approximate equations making a particular choice of the functions $h_{\mu\nu}$. A most useful and natural way of deciding upon a choice of $h_{\mu\nu}$ is to appeal a second time to the linearized vacuum Einstein equations, given by equations (2.501), (2.502). Clearly, if $\bar{\xi}_{\alpha}$ is a constant null vector then these equations possess a solution of the form

$$h_{\mu\nu} = \bar{A}_{\mu\nu}\, e^{i\bar{\xi}_{\alpha} x^{\alpha}}, \qquad (2.601)$$

where the $\bar{A}_{\mu\nu}$ are constants.

In the more general case, when the background geometry is curved, this type of solution may be expected to be reasonable only within a small, locally Euclidean region of space-time. To account for the expected deviations from the linear solutions (2.601), it is assumed that the $h_{\mu\nu}$ are of the form

$$h_{\mu\nu} = A_{\mu\nu}\, e^{i\phi}, \qquad (2.602)$$

where now the $A_{\mu\nu}$ are functions of space-time, and where the 'phase' ϕ has a functional dependence not necessarily of the kind displayed in solutions (2.601). Actually, only the real part of (2.602) is to be used. If a correspondence principle is supposed between (2.602) and (2.601) in the limit $\gamma_{\mu\nu} \rightarrow \eta_{\mu\nu}$, then this, and the initial definition (2.402), will restrict (2.602) such that the $A_{\mu\nu}$ should be expected to be slowly changing functions of space-time, and ϕ to be a relatively rapidly varying function of space-time.

The test solution (2.602) will be referred to as the WKB assumption after its chief proponents in quantum mechanics, Wentzel, Kramers and Brillouin (Wentzel, 1926; Kramers, 1926; Brillouin, 1926).

With the preliminaries over, it is now possible to consider the results obtained by Isaacson.

2.7 The wave equation.

In this section the properties of the linear equations (2.413) are investigated, employing the TT gauge conditions (2.507) and the WKB assumption (2.602).

The components of the vector normal to the surfaces of constant phase will be denoted thus

$$\xi_\alpha = \phi_{,\alpha} \,. \tag{2.701}$$

Since the surfaces, ϕ = constant, define the 'wave-fronts' of the radiation, the ray vector ξ_α is locally tangential to the path through space-time along which it propagates.

To ensure compatibility between the WKB assumption and the definition (2.402), the $A_{\mu\nu}$ and the phase ϕ must be restricted in the following manner.

$$A_{\mu\nu} = O(1), \quad \xi_\mu = O(\varepsilon^{-1}),$$
$$A_{\mu\nu,\alpha} = O(1), \quad \xi_{\mu,\alpha} = O(1). \qquad (2.702)$$

The assessment of the order of magnitude of $\xi_{\mu,\alpha}$ is not, however, uniquely determined by the above considerations since equation (2.602) would still be consistent with the definition (2.402) in the event that

$$\xi_{\mu,\alpha} = O(\varepsilon^{-1}).$$

This difficulty may be overcome by noting that ξ_μ is unchanging in a small, locally flat region of space-time. This is the reason why it is asserted that the ray vector should vary only slowly in equations (2.702).

From equation (2.508), the wave equations (2.413), become

$$h_{\mu\nu;}{}^{\rho}{}_{\rho} + h^{\sigma}{}_{\mu} R^{(0)}_{\nu\sigma} + h^{\sigma}{}_{\nu} R^{(0)}_{\mu\sigma} \qquad (2.703)$$
$$- 2R^{(0)}_{\rho\mu\nu\sigma} h^{\rho\sigma} = 0,$$

for arbitrary $h_{\mu\nu}$. If the WKB form of $h_{\mu\nu}$ is adopted, then equation (2.703) becomes an equation in $A_{\mu\nu}$ and ϕ given by

$$\{-\xi_\rho \xi^\rho A_{\mu\nu} + A_{\mu\nu;}{}^{\rho}{}_{\rho} + A^{\sigma}{}_{\mu} R^{(0)}_{\nu\sigma}$$
$$+ A^{\sigma}{}_{\nu} R^{(0)}_{\mu\sigma} - 2R^{(0)}_{\rho\mu\nu\sigma} A^{\rho\sigma}\}$$
$$+ i\{2\xi_\rho A_{\mu\nu;}{}^{\rho} + \xi^{\rho}{}_{;\rho} A_{\mu\nu}\} = 0.$$

This equation is assumed to be valid as a complex equation. Thus, it is decomposed into real and imaginary parts so that

$$\xi_\rho \xi^\rho A_{\mu\nu} = A_{\mu\nu;}{}^{\rho}{}_{\rho} + A^{\sigma}{}_{\mu} R^{(0)}_{\nu\sigma} \qquad (2.704)$$
$$+ A^{\sigma}{}_{\nu} R^{(0)}_{\mu\sigma} - 2R^{(0)}_{\rho\mu\nu\sigma} A^{\rho\sigma},$$

$$2\xi_\rho A_{\mu\nu;}{}^{\rho} + \xi^{\rho}{}_{;\rho} A_{\mu\nu} = 0. \tag{2.705}$$

It is observed that the RHS of equation (2.704) is smaller than the LHS by a factor ε^2, so to a good degree of approximation

$$\xi_\rho \xi^\rho = 0. \tag{2.706}$$

Thus, in the high frequency limit, the ray vectors of the radiation are null. To show that the radiation propagates along null geodesics, it is admissable to introduce a family of curves in space-time which have the ray vectors as tangents. That is

$$\frac{dx^\mu}{da} = \xi^\mu, \tag{2.707}$$

where a is a non-zero scalar parameter varying along the curve. Since the ξ_α are null vectors, the solution curves $x^\mu(a)$ may be identified with null geodesics of the background geometry. If equation (2.706) is now covariantly differentiated, it is found that

$$\xi_{\alpha;\beta}\xi^\beta = 0, \tag{2.708}$$

or equivalently

$$\frac{\delta \xi_\alpha}{\delta a} = 0,$$

since ξ_α is a gradient.

Here $\delta/\delta a$ is the absolute derivative. Thus, the rays undergo parallel transport tangentially along the null geodesics.

If equation (2.705) is considered, then following Isaacson we introduce a tensor 'polarization' field $q_{\mu\nu}$ given by

$$A_{\mu\nu} = \mathcal{A} q_{\mu\nu}. \tag{2.709}$$

The tensor $q_{\mu\nu}$ will be required to satisfy $q_{\mu\nu} q^{\mu\nu} = 1$ so that the 'amplitude' $\mathcal{A}$ is a scalar function of space-time given by

$$\mathcal{A} = (A_{\mu\nu} A^{\mu\nu})^{\frac{1}{2}}.$$

Then equation (2.705) becomes

$$(\mathcal{A}_{,\rho} \xi^{\rho} + \tfrac{1}{2} \mathcal{A} \xi^{\rho}{}_{;\rho}) q_{\mu\nu} + \mathcal{A} q_{\mu\nu;\rho} \xi^{\rho} = 0. \qquad (2.710)$$

If this equation is multiplied by $q^{\mu\nu}$ then

$$\mathcal{A}_{,\rho} \xi^{\rho} + \tfrac{1}{2} \mathcal{A} \xi^{\rho}{}_{;\rho} + \tfrac{1}{2} \mathcal{A} (q_{\mu\nu} q^{\mu\nu})_{;\rho} \xi^{\rho} = 0,$$

so that

$$\mathcal{A}_{,\rho} \xi^{\rho} + \tfrac{1}{2} \mathcal{A} \xi^{\rho}{}_{;\rho} = 0, \qquad (2.711)$$

or equivalently

$$(\mathcal{A}^2 \xi^{\rho})_{;\rho} = 0. \qquad (2.712)$$

A comparison of equations (2.710) and (2.711) implies

$$q_{\mu\nu;\rho} \xi^{\rho} = 0, \qquad (2.713)$$

so that

$$\frac{\delta q_{\mu\nu}}{\delta a} = 0.$$

Thus, $q_{\mu\nu}$ also undergoes parallel transport along the null geodesics of the background.

Finally, if the WKB form of $h_{\mu\nu}$ is substituted into the TT coordinate conditions, equation (2.507a) becomes to first order

$$\xi^{\beta} A_{\alpha\beta} = 0, \qquad (2.714)$$

which displays the transverse character of the radiation induced by this choice of coordinate system. The equation (2.507b) gives

$$\gamma^{\alpha\beta} A_{\alpha\beta} = 0. \tag{2.715}$$

If the initial conditions

$$\xi^{\beta} q_{\alpha\beta} = 0 , \qquad \gamma^{\alpha\beta} q_{\alpha\beta} = 0,$$

are imposed at any event E in the radiation field, equations (2.708) and (2.713) will ensure that these conditions are satisfied everywhere along the null geodesic passing through E. Thus consistency with conditions (2.714) and (2.715) is guaranteed.

In summary, it may be concluded that in the first approximation 'WKB' radiation is defined locally by the equations

$$\xi_{\rho} \xi^{\rho} = 0 , \qquad (A^{2} \xi^{\rho})_{;\rho} = 0 , \qquad \frac{\delta q^{\mu\nu}}{\delta a} = 0 , \tag{2.716}$$

provided that the conditions

$$q_{\mu\nu} q^{\mu\nu} = 1 , \qquad \xi^{\beta} q_{\alpha\beta} = 0, \qquad \gamma^{\alpha\beta} q_{\alpha\beta} = 0, \tag{2.717}$$

are satisfied.

2.8 The energy tensor for gravitational radiation.

In this section, an analysis of equations (2.414) is undertaken employing the TT coordinate conditions to demonstrate the existence of a tensor representation of gravitational wave energy, in the high frequency approximation.

The basis of the Isaacson scheme is the vacuum Einstein equations, given by equations (2.403). However, since the background is assumed to be curved, the existence of an effective energy distribution, $E_{\mu\nu}^{\text{eff}}$, must be postulated to account for the creation of this curvature even though the space-time is by definition devoid of material sources. Following Isaacson, we express this as

$$R^{(0)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} R^{(0)} = -8\pi E_{\mu\nu}^{\text{eff}}. \qquad (2.801)$$

In investigating the non-linear aspects of the scheme, Isaacson interpreted equation (2.414) as showing how the radiation itself "acts as a source for the curvature of the background". Substitution of equation (2.414) into equation (2.801), gives the following expression for $E_{\mu\nu}^{\text{eff}}$,

$$E_{\mu\nu}^{\text{eff}} = (\varepsilon^2/8\pi)\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} R^{(2)}\right). \qquad (2.802)$$

This demonstrates how the geometric fluctuations of the field due to the radiation may be regarded as an effective energy tensor, which appears on the RHS of the 'Einstein equations', (2.801). Thus the energy cuving the background is derived from the radiation, so that $E_{\mu\nu}^{\text{eff}}$ may be described as the local energy tensor for gravitational radiation.

From equations (2.407) and (2.408) it is found that

$$\begin{aligned} R^{(2)}_{\mu\nu} = -\frac{1}{2}\Big\{ & \frac{1}{2} h^{\rho\sigma}{}_{;\nu} h_{\rho\sigma;\mu} \\ & + h^{\rho\sigma}(h_{\sigma\rho;\mu\nu} + h_{\mu\nu;\sigma\rho} - h_{\sigma\mu;\nu\rho} - h_{\sigma\nu;\mu\rho}) \qquad (2.803) \\ & + h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}(h_{\sigma\mu;\rho} - h_{\rho\mu;\sigma}) \\ & - (h^{\rho\sigma}{}_{;\rho} - \frac{1}{2} h_{;}{}^{\sigma})(h_{\sigma\mu;\nu} + h_{\sigma\nu;\mu} - h_{\mu\nu;\sigma})\Big\}. \end{aligned}$$

If equation (2.803) is substituted into equations (2.802), and $E_{\mu\nu}^{\text{eff}}$ is written as

$$E_{\mu\nu}^{\text{eff}} = (c^2/16\pi)(U_{\mu\nu} + W_{\mu\nu}{}^{\rho}{}_{;\rho}), \qquad (2.804)$$

then

$$U_{\mu\nu} = \tfrac{1}{2}h^{\rho\sigma}{}_{;\mu}h_{\rho\sigma;\nu} - h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}(h_{\sigma\mu;\rho} - h_{\rho\mu;\sigma})$$
$$-\tfrac{1}{2}h_{;}{}^{\sigma}(h_{\sigma\mu;\nu} + h_{\sigma\nu;\mu} - h_{\mu\nu;\sigma}) \qquad (2.805)$$
$$+\tfrac{1}{2}\delta_{\mu\nu}\{\tfrac{1}{2}h^{\rho\sigma}{}_{;}{}^{\alpha}h_{\rho\sigma;\alpha} - h^{\alpha\sigma}{}_{;}{}^{\rho}h_{\alpha\rho;\sigma}$$
$$+ h_{;}{}^{\sigma}(h_{\sigma\alpha;}{}^{\alpha} - \tfrac{1}{2}h_{;\sigma})\},$$

and

$$W_{\mu\nu}{}^{\rho} = \delta_{\nu}{}^{\rho}h^{\alpha\sigma}h_{\alpha\sigma;\mu} + h^{\rho\sigma}(h_{\mu\nu;\sigma}$$
$$- h_{\sigma\mu;\nu} - h_{\sigma\nu;\mu}) \qquad (2.806)$$
$$+ \delta_{\mu\nu}\{h^{\rho\sigma}(h_{\sigma\alpha;}{}^{\alpha} - \tfrac{1}{2}h_{;\sigma}) - \tfrac{1}{2}h_{\alpha\sigma}h^{\alpha\sigma}{}_{;}{}^{\rho}\}.$$

Clearly the explicit form of $E_{\mu\nu}^{\text{eff}}$ will be very cumbersome, even if the coordinate conditions, (2.507), are imposed. However, it will now be shown that equation (2.802) can provide a simple and elegant expression for the energy of gravitational radiation, provided that certain assumptions are made.

Of great importance in the treatment is Isaacson's assumption that a meaningful result is obtainable only if a space-time average of equation (2.802) is considered. The underlying idea here is that only the macroscopic properties of the wave, and not the fine details of the radiative oscillations, are considered of importance in assessing the gravitational field energy. This is analogous to the situation in electromagnetics where a similar averaging process is carried out when the energy and momentum of electromagnetic radiation is determined. Indeed, the Maxwell equations themselves are obtainable by taking the average of the microscopic field equations of electromagnetism over innumerable, non-static atomic fields. The details of the microscopic structure are, however, irrelevant to problems involving bulk matter on the laboratory scale.

Let $B_{\mu\nu}$ be an arbitrary tensor field, and denote the average of $B_{\mu\nu}$ at the point P(x) by $\langle B_{\mu\nu}(x) \rangle$. It is required that this average should possess contributions from all points P'(x') in a neighbourhood of P(x). However, since tensors at different points have different transformation properties, an average cannot be constructed by simply integrating $B_{\mu\nu}$ over a neighbourhood of P(x). Indeed parallel propagators, $g_{\mu}^{\cdot\alpha'}(x,x')$, are introduced in order that the tensor field at P'(x'), denoted by $B_{\mu'\nu'}$, may be transported along a geodesic to P(x). If we perform this operation for all points P'(x') belonging to a neighbourhood of P(x), in effect contributions are gathered at one point thus making an integral average possible. The propagators, $g_{\mu}^{\cdot\alpha'}$, which transform as vectors at either P(x) or P'(x'), are defined by $V_{\mu} = g_{\mu}^{\cdot\alpha'} V_{\alpha'}$ where, if $V_{\alpha'}$ is a vector at P'(x'), then V_{μ} is the result of parallel transport along a geodesic from P'(x') to P(x).

The average itself is defined by

$$\langle B_{\mu\nu}(x) \rangle = \int_{\mathcal{M}} g_{\mu}^{\cdot\alpha'}(x,x')\, g_{\nu}^{\cdot\beta'}(x,x') \times B_{\alpha'\beta'}(x')\, \Theta(x,x')\, d^4x' , \tag{2.807}$$

where the subscript $\mathcal{M}$ indicates that the integral is evaluated over all space-time, and where Θ is a scalar function which satisfies

$$\int_M \Theta(x,x')\, d^4x' = 1 , \quad \Theta \geqslant 0 . \tag{2.808}$$

The extent of the neighbourhood of P(x) from which contributions to the average are obtained is governed by the function Θ. Denote by $\mathcal{D}$, a region of space-time containing P(x), such that $\mathcal{D}$ is sufficiently large to contain several wavelengths of the radiation field, and yet small enough to ensure that the background geometry remains approximately flat. The function Θ is defined such that, as P'(x') approaches the boundary of $\mathcal{D}$, Θ falls smoothly to zero and remains so for all P'(x') not belonging to $\mathcal{D}$. Within $\mathcal{D}$, Θ and its derivatives are assumed to be of order unity, and in addition a function S_α may be introduced such that

$$\Theta_{,\alpha} = S_\alpha \Theta . \tag{2.809}$$

With an averaging technique suitably defined, we consider again equation (2.804). The averaged energy tensor is denoted thus

$$\begin{aligned} E_{\mu\nu} &\equiv \langle E_{\mu\nu}{}^{\text{eff}} \rangle \\ &= (\mathcal{E}^2/16\pi)\langle U_{\mu\nu} + W_{\mu\nu}{}^{\rho}{}_{;\rho} \rangle . \end{aligned} \tag{2.810}$$

To assist in the task of averaging this equation, Isaacson introduced three rules which are stated, and proved, here.

(i). The term $\mathcal{E}^2\langle W_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle$, in equation (2.810), is smaller than $\mathcal{E}^2 W_{\mu\nu}{}^{\rho}{}_{;\rho}$ by a factor $\mathcal{E}$. In particular, $\mathcal{E}^2\langle W_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle = O(\mathcal{E})$.

(ii). To first order in $\mathcal{E}$

$$\mathcal{E}^2\langle h_\nu{}^{\sigma}{}_{;}{}^{\rho} h_{\rho\mu;\sigma}\rangle = -\mathcal{E}^2\langle h_\nu{}^{\sigma}{}_{;}{}^{\rho}{}_{\sigma} h_{\rho\mu}\rangle .$$

The factor $\mathcal{E}^2$ is introduced for convenience due to the presence of such a factor outside the averaging brackets in equation (2.810).

(iii). Covariant derivatives of $h_{\mu\nu}$ in equation (2.810) may be commuted. That is

$$\mathcal{E}^2 h_{\mu\nu;\rho\sigma} = \mathcal{E}^2 h_{\mu\nu;\sigma\rho}.$$

<u>Proof of (i)</u>.

First, it is required to show that

$$\mathcal{E}^2 \langle W_{\mu\nu}{}^{\rho}{}_{;\rho} \rangle = O(\mathcal{E}).$$

By definition

$$\begin{aligned}\mathcal{E}^2\langle W_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle &= \mathcal{E}^2\int_M g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'}{}_{;\rho'}\,\Theta\, d^4x' \\ &= \mathcal{E}^2\int_M (g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'}\Theta)_{;\rho'}\, d^4x' \\ &- \mathcal{E}^2\int_M \{ g_{\mu}{}^{\cdot\alpha'}{}_{;\rho'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'}\Theta \\ &+ g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'}{}_{;\rho'} W_{\alpha'\beta'}{}^{\rho'}\Theta + g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'}\Theta_{,\rho'}\} d^4x'.\end{aligned} \tag{2.811}$$

The first of these terms may be transformed to a surface integral, which will be zero due to the behaviour of Θ. To assess the order of the remaining terms we consider the modulus of the final integral in equation (2.811).

$$\begin{aligned}|I_{\mu\nu}| &\equiv \mathcal{E}^2 \left|\int_M g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'}\Theta_{,\rho'}\, d^4x'\right| \\ &\leqslant \mathcal{E}^2\int_M |g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'}\Theta_{,\rho'}|\, d^4x' \\ &= \mathcal{E}^2\int_M |g_{\mu}{}^{\cdot\alpha'} g_{\nu}{}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'} S_{\rho'}|\,\Theta\, d^4x',\end{aligned} \tag{2.812}$$

by equation (2.809). Since the propagators, $g_{\mu}{}^{\cdot\alpha'}$, are dependent only upon the background geometry, $g_{\mu}{}^{\cdot\alpha'}$ and $g_{\mu}{}^{\cdot\alpha'}{}_{;\rho}$ may be supposed to be of order unity. Thus, by inspection of conditions (2.402)

$$\varepsilon^2 (g_{\mu}^{\cdot\alpha'} g_{\nu}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'} S_{\rho'}) = O(\varepsilon)$$

So, as $\varepsilon \to 0$, there exists some constant $Q > 0$, such that

$$\varepsilon^2 | g_{\mu}^{\cdot\alpha'} g_{\nu}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'} S_{\rho'} | < Q\varepsilon$$

Hence, continuing from relations (2.812), we find that

$$|I_{\mu\nu}| < \int_M Q\varepsilon \,\Theta\, d^4x' = Q\varepsilon, \text{ as } \varepsilon \to 0.$$

Therefore, by definition, it may be concluded that

$$\varepsilon^2 \int_M g_{\mu}^{\cdot\alpha'} g_{\nu}^{\cdot\beta'} W_{\alpha'\beta'}{}^{\rho'} \Theta_{,\rho'} d^4x' = O(\varepsilon). \qquad (2.813)$$

A similar argument applied to the remaining terms in equation (2.811) leads to the result

$$\varepsilon^2 \langle W_{\mu\nu}{}^{\rho}{}_{;\rho} \rangle = O(\varepsilon).$$

An assessment of the order of $\varepsilon^2 W_{\mu\nu}{}^{\rho}{}_{;\rho}$ must now be made.

From equation (2.806), $W_{\mu\nu}{}^{\rho}{}_{;\rho}$ is seen to consist of terms typically of the form $h_{\mu\nu} h_{\alpha\beta,\rho\sigma}$ or $h_{\mu\nu,\rho} h_{\alpha\beta,\sigma}$. Therefore, by inspection of the definition (2.402), $W_{\mu\nu}{}^{\rho}{}_{;\rho}$ will be of order ε^{-2}, so that

$$\varepsilon^2 W_{\mu\nu}{}^{\rho}{}_{;\rho} = O(1).$$

Thus, a comparison of the orders of magnitude of $\varepsilon^2 W_{\mu\nu}{}^{\rho}{}_{;\rho}$ and $\varepsilon^2 \langle W_{\mu\nu}{}^{\rho}{}_{;\rho} \rangle$ provides the desired result.

Proof of (ii).

Define

$$\varepsilon^2 D_{\mu\nu}{}^{\sigma}{}_{;\sigma} \equiv \varepsilon^2 (h_{\nu}{}^{\sigma}{}_{;}{}^{\rho} h_{\rho\mu})_{;\sigma}$$

Taking the average, and expanding the RHS of this equation we find

$$\varepsilon^2\langle D_{\mu\nu}{}^{\sigma}{}_{;\sigma}\rangle = \varepsilon^2\langle h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}{}_{\sigma}h_{\rho\mu} + h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}h_{\rho\mu;\sigma}\rangle. \tag{2.814}$$

By a similar argument to that contained in the proof of (i), it is found that

$$\varepsilon^2\langle D_{\mu\nu}{}^{\sigma}{}_{;\sigma}\rangle = O(\varepsilon),$$

whereas

$$\varepsilon^2\langle h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}{}_{\sigma}h_{\rho\mu} + h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}h_{\rho\mu;\sigma}\rangle = O(1).$$

Thus, equation (2.814) may be written

$$\varepsilon^2\langle h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}h_{\rho\mu;\sigma}\rangle = -\varepsilon^2\langle h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}{}_{\sigma}h_{\rho\mu}\rangle + O(\varepsilon),$$

so that, indeed, to first order

$$\varepsilon^2\langle h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}h_{\rho\mu;\sigma}\rangle = -\varepsilon^2\langle h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}{}_{\sigma}h_{\rho\mu}\rangle.$$

It is stressed that both sides of this equation are of the order of magnitude unity.

Proof of (iii).

Here equation (2.505) is reconsidered.

$$h_{\mu\nu;\tau\rho} - h_{\mu\nu;\rho\tau} = R^{(0)}_{\mu\sigma\rho\tau}h^{\sigma}{}_{\nu} + R^{(0)}_{\nu\sigma\rho\tau}h_{\mu}{}^{\sigma}.$$

If the definition (2.402) is used, it is possible to assess the orders of magnitude of the terms in this expression. It is found that

$$\varepsilon^2(h_{\mu\nu;\tau\rho} - h_{\mu\nu;\rho\tau}) = O(1),$$

whereas

$$\varepsilon^2(R^{(0)}_{\mu\sigma\rho\tau}h^{\sigma}{}_{\nu} + R^{(0)}_{\nu\sigma\rho\tau}h_{\mu}{}^{\sigma}) = O(\varepsilon^2).$$

Thus, equation (2.505) may be written

$$\varepsilon^2 h_{\mu\nu;\tau\rho} = \varepsilon^2 h_{\mu\nu;\rho\tau} + O(\varepsilon^2),$$

which gives the required result, that is

$$\varepsilon^2 h_{\mu\nu;\tau\rho} = \varepsilon^2 h_{\mu\nu;\rho\tau}.$$

It is noted that both sides of this equation also are of unit magnitude.

Corollary to Rule (i).

With reference to subsection 3.3(III), it may be shown that

$$\varepsilon^2 \langle (\gamma_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta;}{}^{\rho})_{;\rho} \rangle = O(\varepsilon).$$

Proof of Corollary.

This statement follows if $W_{\mu\nu}{}^{\rho}$ is replaced by

$$\gamma_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta;}{}^{\rho},$$

in the proof of Rule (i).

These rules may now be used to simplify equation (2.810). From rule (i), it is observed that

$$(\varepsilon^2/16\pi)\langle U_{\mu\nu}\rangle = O(1),\ (\varepsilon^2/16\pi)\langle W_{\mu\nu}{}^{\rho}{}_{;\rho}\rangle = O(\varepsilon),$$

so that equation (2.810) becomes

$$E_{\mu\nu} = (\varepsilon^2/16\pi)\langle U_{\mu\nu}\rangle + O(\varepsilon). \qquad (2.815)$$

If the TT coordinate conditions are used, then equation (2.805) implies that,

$$\begin{aligned} E_{\mu\nu} = (\varepsilon^2/16\pi)\langle & \tfrac{1}{2} h^{\rho\sigma}{}_{;\mu} h_{\rho\sigma;\nu} \\ & - h_{\nu}{}^{\sigma}{}_{;}{}^{\rho}(h_{\sigma\mu;\rho} - h_{\rho\mu;\sigma}) \\ & + \tfrac{1}{2}\gamma_{\mu\nu}\{\tfrac{1}{2} h^{\rho\sigma}{}_{;}{}^{\alpha} h_{\rho\sigma;\alpha} - h^{\alpha\sigma}{}_{;}{}^{\rho} h_{\alpha\rho;\sigma}\}\rangle. \end{aligned} \qquad (2.816)$$

An examination of this equation shows that only the first term on the RHS can evade manipulation by the use of rule (ii). Indeed, it can be shown that this term becomes the only survivor when rules (i), (ii), (iii), and the coordinate conditions are further applied to equation (2.816). Thus, Isaacson's result for the stress energy of gravitational radiation is found,

$$E_{\mu\nu} = (\varepsilon^2/32\pi)\langle h^{\rho\sigma}{}_{;\mu} h_{\rho\sigma;\nu}\rangle + O(\varepsilon). \qquad (2.817)$$

It is, of course, by virtue of the averaging process that it is possible to reduce the equation (2.804) for $E_{\mu\nu}^{eff}$, to this remarkably elegant form. This, however, is not the only benefit to be derived from this treatment of $E_{\mu\nu}^{eff}$. If we recall the expression for the effective field energy, given by equation (2.804), and consider its behaviour under a general infinitesimal coordinate transformation

$$x^\alpha \to \bar{x}^\alpha = x^\alpha + \varepsilon Z^\alpha, \qquad (2.818)$$

then to first order, $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to \bar{h}_{\mu\nu} = h_{\mu\nu} - Z_{\mu;\nu} - Z_{\nu;\mu},$$

so that the effective field energy can be shown to transform as

$$E_{\mu\nu}^{eff} \to \bar{E}_{\mu\nu}^{eff} = E_{\mu\nu}^{eff} + H_{\mu\nu}{}^{\rho}{}_{;\rho} + O(\varepsilon).$$

On averaging this equation, Isaacson has shown that the divergence term, $H_{\mu\nu}{}^{\rho}{}_{;\rho}$, diminishes in magnitude so we find that $E_{\mu\nu} \equiv \langle E_{\mu\nu}^{eff}\rangle$ transforms as

$$E_{\mu\nu} \to \bar{E}_{\mu\nu} = E_{\mu\nu} + O(\varepsilon). \qquad (2.819)$$

Thus, a very important result is obtained. The averaged gravitational radiation energy tensor is invariant, in the high frequency limit, under the general gauge transformation (2.818), and is therefore given by equation (2.817) for all choices of coordinates.

It is of interest to investigate equation (2.817) assuming the WKB form of $h_{\mu\nu}$. It will be supposed that $h_{\mu\nu} = \mathrm{Real}\{\mathcal{A} q_{\mu\nu} e^{i\phi}\}$, and that equations (2.716), (2.717) are satisfied. Then, to first order

$$E_{\mu\nu} = (\mathcal{E}^2/32\pi)\langle \mathcal{A}^2 \xi_\mu \xi_\nu \sin^2\phi \rangle. \qquad (2.820)$$

By inspection of the definition (2.807), to evaluate this average a knowledge of the propagators, $g_{\mu}^{\ \cdot\alpha'}$, of the background geometry is required as well as a suitable choice of the function Θ, satisfying equations (2.808). Since a general choice of the background metric $\gamma_{\mu\nu}$ is being considered, it is not possible to obtain explicit expressions for the $g_{\mu}^{\ \cdot\alpha'}$. Instead, the averaging integral is modified to find a procedure which will provide an alternative averaging technique of practical use in evaluating the WKB form of $E_{\mu\nu}$.

We recall the rigourous statement of the average given by equation (2.807). Since the function Θ is zero for all points P'(x') not belonging to neighbourhood $\mathcal{D}$ of P(x), a reasonable alternative integral can be constructed by eliminating Θ from the integrand of (2.807) and integrating over $\mathcal{D}$, instead of over the whole manifold, $\mathcal{M}$. Also, as $\mathcal{D}$ is defined to be a locally Euclidean region with respect to the background geometry, the general propagators in the integral over $\mathcal{M}$ may be replaced by the propagators of the Minkowski metric in the integral over $\mathcal{D}$. Since a vector $V_{\mu'}$ at P'(x') is unaltered by the operation of parallel transport to P(x) in the flat region $\mathcal{D}$,

$$g_{\mu}^{\cdot\alpha'}(\eta_{\sigma\rho}) = \delta_{\mu'}^{\ \alpha'},$$

because

$$V_{\mu} = g_{\mu}^{\cdot\alpha'} V_{\alpha'} = \delta_{\mu'}^{\ \alpha'} V_{\alpha'} = V_{\mu'},$$

where $\delta_{\mu'}^{\ \alpha'}$ is the Kronecker delta at P'(x'). Thus, equation (2.820) may be written as

$$E_{\mu\nu} = (\mathcal{E}^2/32\pi) \int_{\mathcal{D}} \delta_{\mu'}^{\ \alpha'} \delta_{\nu'}^{\ \beta'} \mathcal{A}^2(x') \xi_{\alpha'} \xi_{\beta'} \sin^2\phi(x')\, d^4x'$$

$$= (\mathcal{E}^2/32\pi) \int_{\mathcal{D}} \mathcal{A}^2(x') \xi_{\mu'} \xi_{\nu'} \sin^2\phi(x')\, d^4x'.$$

By the definition (2.702), the amplitude and the wave vectors of the radiation can be seen to change at a rate corresponding to the rate of change of the background. Hence, as a further consequence of the Euclidean nature of $\mathcal{D}$, it may be inferred that the product $\mathcal{A}^2(x')\xi_{\mu'}\xi_{\nu'}$ is unchanging over the region of integration. This gives

$$E_{\mu\nu} = (\mathcal{E}^2/32\pi) \mathcal{A}^2 \xi_{\mu} \xi_{\nu} \langle \sin^2\phi \rangle,$$

which leads finally to Isaacson's expression

$$E_{\mu\nu} = (\mathcal{E}^2 \mathcal{A}^2/64\pi) \xi_{\mu} \xi_{\nu}, \qquad (2.821)$$

for the stress energy of monochromatic, WKB radiation in the high frequency limit. It is also of interest to note that the WKB form of $E_{\mu\nu}$ satisfies

$$E^{\mu\nu}{}_{;\nu} = (\mathcal{E}^2/64\pi)\{(\mathcal{A}^2\xi^{\nu})_{;\nu}\xi^{\mu} + \mathcal{A}^2\xi^{\nu}\xi^{\mu}{}_{;\nu}\} = 0, \qquad (2.822)$$

by equations (2.708) and (2.712).

The result (2.821), that the form of the energy tensor for gravitational waves is the same as the form for electromagnetic null fields, has also been found by MacCallum and Taub

(1973) using a different method. Their rederivation of Isaacson's result was obtained by employing a WKB form for $h_{\mu\nu}$, and using the "averaged Lagrangian" technique developed by Whitham (1971). The primary difference between this and the Isaacson treatment is that (2.821) is arrived at by averaging a scalar density, therefore overcoming some of the above mentioned problems of tensor averaging.

In summary; it has been found that, provided the wavelength of gravitational radiation is short compared with the characteristic length of the background geometry, localisation of the energy carried by the radiation is possible. Isaacson has convincingly demonstrated that such 'high frequency' waves possess an energy tensor $E_{\mu\nu}$, satisfying the local conservation laws (2.822), which is second rank, symmetric and gauge invariant. Further, the inclusion of some of the inherent non-linearity of general relativity is an additional attractive feature of Isaacson's treatment. This allows us to display the manner in which the wave field energy contributes to the curvature of the background space-time through which the wave journeys. Thus, the 'Isaacson tensor' $E_{\mu\nu}$ overcomes many of the difficulties which plague the pseudotensorial representations of field energy associated with the linear theory of gravity.

2.9 Validity of approximation.

The scheme which has been reviewed in this chapter is valid for all magnitudes of gravitational field strength. Its applicability is, however, restricted in that it may only be used in situations where the condition for high frequency, (2.204), is well satisfied. The severity of this restriction is not great since radiation emitted by a material source will satisfy (2.204) in the region, a sufficient distance from the source, usually refered to as the "wave zone".

Chapter 3.

A general formalism for the treatment of high frequency gravitational radiation, employing the WKB approximation.

3.1 Motivation.

In this Chapter, the scheme developed by Isaacson, for finding approximate solutions to Einstein's field equations, will again be the subject of attention. From the review of the previous Chapter, it is known that the solutions thus obtained may be interpreted as describing the propagation of high frequency gravitational radiation through a curved 'background' space-time.

The tensor representation of the energy of gravitational radiation discovered by Isaacson has since been recognised as being a considerable improvement upon the inadequate descriptions of field energy provided by the linear theory. Consequently, it has attracted the attention of many writers since its advent, with contributions being made in either developing the technique further (Choquet-Bruhat, 1969; Madore, 1972, 1973; MacCallum and Taub, 1973; Legros and Madore, 1974), or in applying it to astrophysical problems (Price and Thorne, 1969; Rees, 1971; Jackson, 1972; Madore, 1974). In particular, J. Madore may be regarded as having produced some powerful physical results employing a WKB analysis of gravitational radiation, and these will be mentioned later.

The motivation for the work in this Chapter is to devise a formalism extending Isaacson's vacuum scheme to one capable of describing radiation in the presence of matter. It is hoped to do this in such a way as to render the presently published results of Isaacson and Madore as special cases within the general formalism. During the discussion, it will become apparent that the treatments of these two authors contain mathematical differences and the significance of these will be investigated.

3.2 The approximation scheme in matter.

In this Section, a set of field equations capable of describing the propagation of high frequency gravitational radiation within a matter filled background is developed. Einstein's equations for a space-time manifold containing a distribution of material energy may be written

$$\mathcal{G}_{\mu\nu} = -8\pi T_{\mu\nu} , \tag{3.201}$$

where $T_{\mu\nu}$ is a tensor describing the matter content of the manifold, and where

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu}(g_{\alpha\beta}) - \tfrac{1}{2} g_{\mu\nu} R(g_{\alpha\beta}). \tag{3.202}$$

Observe that

$$R(g_{\alpha\beta}) = g^{\mu\nu} R_{\mu\nu}(g_{\alpha\beta}),$$

and suppose, for the moment, that the total metric $g_{\mu\nu}$ is arbitrary.

As is well known, the set of equations (3.201) is interpreted as relating the gravitational field, represented by the geometry of a Riemannian manifold, to the material sources $T_{\mu\nu}$ creating the field. It is important to make the apparently trivial observation that the components of the tensor $T_{\mu\nu}$ contain only the energetic contributions of the material sources, and that contributions due to the gravitational field energy itself do not enter into the RHS of

Einstein's equations.

The effect of introducing gravitational radiation into the system will be to impose restrictions upon the form of the total metric, $g_{\mu\nu}$. To describe the gravitational radiation content of the manifold, a definition similar to that of Section 2.4 is adopted. A metric is said to contain high frequency gravitational radiation if and only if there exists a family of coordinate systems in which the total metric of the matter filled region takes the form of equation (2.401). That is

$$g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon h_{\mu\nu}, \quad 0 < \varepsilon << 1.$$

In addition, it is suppose d that

$$\gamma_{\mu\nu} = O(1) \ , \ h_{\mu\nu} = O(1) , \tag{3.203}$$

where the $\gamma_{\mu\nu}$ again represent the background, and where the $h_{\mu\nu}$ are identified with the 'wave potentials' of the gravitational radiation. In the vacuum treatment of Chapter 2, it was discovered that the effective energy of the radiation field was responsible for the curvature of the background. In this case, when matter is also present, it should be borne in mind that $\gamma_{\mu\nu}$ will be generated by both wave and material energy sources.

The complete definition of Section 2.4, involving detailed estimates of the orders of magnitude of the derivatives of the $\gamma_{\mu\nu}$ and the $h_{\mu\nu}$, is not employed. However, some indication of the rates at which the $\gamma_{\mu\nu}$ and the $h_{\mu\nu}$ change relative to one another is necessary since it was concluded, in Section 2.8, that localization of the wave energy is only possible if the radiation is of high frequency. It is for this reason that the definition of high frequency, as stated in Section 2.2, is a common feature of the works of both Isaacson and Madore. Thus, instead of imposing the totality of the conditions (2.402), it will only be required that the radiation conform to the weaker restrictions of equations (2.201), (2.203) and (2.204), when developing the general scheme.

It is not claimed that the following formulation is unique. The aim is to produce a single theoretical framework which encompasses the known results of Isaacson and Madore. Assumptions are made, and the scheme is developed along lines which provide the desired results by the simplest means.

If the metric (2.401) is substituted into equation (3.202), it is found that

$$\begin{aligned}\mathcal{G}_{\mu\nu} = \sum_{i=0}^{2}\Big\{ & \varepsilon^{i}\big(R^{(i)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(i)}\big) \\ & - \frac{\varepsilon}{2}^{i+1}\Delta^{(i)}_{\mu\nu} + \frac{\varepsilon}{2}^{i+2} h_{\mu\nu}h^{\alpha\beta}R^{(i)}_{\alpha\beta}\Big\} \\ & + \Big[\varepsilon^{3}\big(R^{(3+)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(3+)}\big) - \frac{\varepsilon}{2}^{4}\Delta^{(3+)}_{\mu\nu} + \frac{\varepsilon}{2}^{5} h_{\mu\nu}h^{\alpha\beta}R^{(3+)}_{\alpha\beta}\Big],\end{aligned} \tag{3.204}$$

where the $R^{(i)}_{\mu\nu}$ terms are defined by equations (2.405) to (2.408), and where

$$\begin{aligned} R^{(i)} &\equiv \gamma^{\alpha\beta}R^{(i)}_{\alpha\beta}, \\ \Delta^{(i)}_{\mu\nu} &\equiv h_{\mu\nu}R^{(i)} - \gamma_{\mu\nu}h^{\alpha\beta}R^{(i)}_{\alpha\beta},\end{aligned} \tag{3.205}$$

for i=0,1,2,3+. Hence equation (3.204) becomes

$$\begin{aligned}\mathcal{G}_{\mu\nu} = {} & \mathcal{G}^{(0)}_{\mu\nu} + \varepsilon\,\mathcal{G}^{(1)}_{\mu\nu} \\ & + \varepsilon^{2}\mathcal{G}^{(2)}_{\mu\nu} + \varepsilon^{3}\mathcal{G}^{(3+)}_{\mu\nu},\end{aligned} \tag{3.206}$$

where

$$\mathcal{G}^{(0)}_{\mu\nu} = R^{(0)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(0)} , \tag{3.207}$$

$$\mathcal{G}^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(1)} - \tfrac{1}{2}\Delta^{(0)}_{\mu\nu} , \tag{3.208}$$

$$\mathcal{G}^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)} - \tfrac{1}{2}\Delta^{(1)}_{\mu\nu} \tag{3.209}$$
$$+ \tfrac{1}{2}h_{\mu\nu}h^{\alpha\beta}R^{(0)}_{\alpha\beta} ,$$

$$\mathcal{G}^{(3+)}_{\mu\nu} = R^{(3+)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(3+)} - \tfrac{1}{2}\Delta^{(2)}_{\mu\nu}$$
$$- \frac{\varepsilon}{2}\Delta^{(3+)}_{\mu\nu} + \frac{1}{2}h_{\mu\nu}h^{\alpha\beta}\big(R^{(1)}_{\alpha\beta} \tag{3.210}$$
$$+ \varepsilon R^{(2)}_{\alpha\beta} + \varepsilon^2 R^{(3+)}_{\alpha\beta} \big).$$

Now we consider the matter content of the model. In what follows, matter will be envisaged as a general fluid characterized by a velocity 4-vector u_μ , a density ρ , a pressure p , a temperature T , and variables η , ζ and χ representing the shear viscosity, the bulk viscosity and the thermal conductivity respectively. An explicit statement of this tensor appears in the next Section.

In the presence of the radiation, it is assumed that all of the quantities describing the fluid will be perturbed from their 'background' values in such a way that

$$u_\mu = u^{(0)}_\mu + \varepsilon u^{(1)}_\mu , \quad \rho = \rho^{(0)} + \varepsilon\rho^{(1)}, \tag{3.211}$$

and correspondingly for p , T , η , ζ and χ . Since the radiative perturbation to the metric is small, of the order of ε , the fluid perturbations are also expected to be small although this will be discussed in greater detail later.

If the perturbed variables are now substituted into the functional form of $T_{\mu\nu}$, a power series in $\mathcal{E}$ will arise,

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \mathcal{E}\, T^{(1)}_{\mu\nu} + \mathcal{E}^2 T^{(2)}_{\mu\nu} + \mathcal{E}^3 T^{(3+)}_{\mu\nu}, \qquad (3.212)$$

where $T^{(3+)}_{\mu\nu}$ is a remainder term. Here the particular forms of the $T^{(i)}_{\mu\nu}$ will be dependent upon the choice of $T_{\mu\nu}$.

Thus, using equations (3.206) and (3.212), Einstein's equations become

$$\mathcal{G}^{(0)}_{\mu\nu} + \mathcal{E}\mathcal{G}^{(1)}_{\mu\nu} + \mathcal{E}^2\mathcal{G}^{(2)}_{\mu\nu} + \mathcal{E}^3\mathcal{G}^{(3+)}_{\mu\nu} = -8\pi\left(T^{(0)}_{\mu\nu} + \mathcal{E}\, T^{(1)}_{\mu\nu} + \mathcal{E}^2 T^{(2)}_{\mu\nu} + \mathcal{E}^3 T^{(3+)}_{\mu\nu}\right). \qquad (3.213)$$

As will be seen, these equations provide an adequate machinery with which to describe the propagation of radiation in matter. Attention is now turned to the task of generating an appropriate set of approximate field equations.

Consideration of the vacuum scheme of Isaacson suggests that the presence of gravitational radiation in a region of space-time will give rise to an effective energy density there, which may be attributed to the radiation. Thus in the case now under consideration, there are two energy sources, that of the radiation field and that of the fluid, which act in creating the background curvature.

To accomodate this hypothesis, in matter it is assumed that

$$\mathcal{G}^{(0)}_{\mu\nu} = -8\pi\left(T^{(0)}_{\mu\nu} + E_{\mu\nu}^{\;\mathrm{Eff}}\right), \qquad (3.214)$$

an equation analogous to Isaacson's vacuum equation (2.801).

As before, $E_{\mu\nu}^{eff}$ denotes the effective energy tensor of the radiation field.

As emphasised previously, the equations of Einstein have a source term $T_{\mu\nu}$ which contains all contributions to the energy of space-time, other than that of the gravity field itself.

Since we have again postulated the existence of a gravity field energy $E_{\mu\nu}^{eff}$, the field equations should be expressible so as to exhibit this energy amongst the material energy terms on the RHS of equation (3.213). Insight into how to do this is provided by a reconsideration of the methods employed in the vacuum case. The field energy $E_{\mu\nu}^{eff}$ is looked for amongst the geometrical terms in the expansion on the LHS of equations (3.213). If this energy is to be displayed on the RHS of equations (3.213), then mathematically this amounts to a trivial shifting of terms from one side of the field equations to the other. However, from a physical point of view this simple operation will contribute significantly in obtaining a sensible scheme to describe radiation in this approximation.

As a further step towards the realization of a suitable formalism, another assumption is made. It is supposed that the wave energy resides in the $\overset{(2)}{G}_{\mu\nu}$ term of the field equations. In particular that

$$\begin{aligned} E_{\mu\nu}^{eff} &= (\varepsilon^2/8\pi)\overset{(2)}{G}_{\mu\nu} \\ &= (\varepsilon^2/8\pi)\Big\{\overset{(2)}{R}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)} \\ &\quad - \tfrac{1}{2}\overset{(1)}{\Delta}_{\mu\nu} + \tfrac{1}{2}h_{\mu\nu}h^{\alpha\beta}\overset{(0)}{R}_{\alpha\beta}\Big\}, \end{aligned} \qquad (3.215)$$

from equation (3.209). The validity of this important assumption will be a subject for discussion later. By a comparison of equation (3.215) with its vacuum analogue, equation (2.802), it is immediately apparent that the definition of $E_{\mu\nu}{}^{eff}$ in matter differs from the corresponding definition in vacuo. Despite this modification, however, equation (3.215) will be shown to provide a final averaged energy tensor for high frequency radiation in matter, which has a form identical to the energy tensor derived in the vacuum treatment.

Having identified the energy of the gravitational radiation with an aspect of the geometry of the matter filled manifold, equations (3.213) are rewritten in the following manner

$$\begin{aligned} \mathcal{G}^{(0)}_{\mu\nu} + \varepsilon\mathcal{G}^{(1)}_{\mu\nu} + \varepsilon^3\mathcal{G}^{(3+)}_{\mu\nu} &= -8\pi\left(T^{(0)}_{\mu\nu} + \varepsilon T^{(1)}_{\mu\nu} + \varepsilon^2 T^{(2)}_{\mu\nu}\right. \\ &\qquad \left. + \varepsilon^3 T^{(3+)}_{\mu\nu}\right) - \varepsilon^2\mathcal{G}^{(2)}_{\mu\nu} . \end{aligned}$$

Then, by equation (3.215), this becomes

$$\begin{aligned} \mathcal{G}^{(0)}_{\mu\nu} + \varepsilon\mathcal{G}^{(1)}_{\mu\nu} + \varepsilon^3\mathcal{G}^{(3+)}_{\mu\nu} &= -8\pi\left(T^{(0)}_{\mu\nu} + \varepsilon T^{(1)}_{\mu\nu} + \varepsilon^2 T^{(2)}_{\mu\nu}\right. \\ &\qquad \left. + \varepsilon^3 T^{(3+)}_{\mu\nu}\right) - 8\pi E_{\mu\nu}{}^{eff} , \end{aligned}$$

which provides a modification of the field equations in which both material and the field energy sources are clearly exhibited on the RHS. Finally, the substitution of equations (3.214) into the above equation gives

$$\varepsilon \mathcal{G}^{(1)}_{\mu\nu} + \varepsilon^3 \mathcal{G}^{(3+)}_{\mu\nu} = -8\pi\left(\varepsilon T^{(1)}_{\mu\nu} + \varepsilon^2 T^{(2)}_{\mu\nu} + \varepsilon^3 T^{(3+)}_{\mu\nu}\right). \tag{3.216}$$

Thus, an approximate solution of Einstein's equations, representing gravitational radiation travelling through a material medium, is obtained by simultaneously solving the equations (3.214) and (3.216),

$$\mathcal{G}^{(0)}_{\mu\nu} = -8\pi\left(T^{(0)}_{\mu\nu} + E^{\text{eff}}_{\mu\nu}\right),$$

$$\varepsilon \mathcal{G}^{(1)}_{\mu\nu} + \varepsilon^3 \mathcal{G}^{(3+)}_{\mu\nu} = -8\pi\left(\varepsilon T^{(1)}_{\mu\nu} + \varepsilon^2 T^{(2)}_{\mu\nu} + \varepsilon^3 T^{(3+)}_{\mu\nu}\right),$$

where the effective energy tensor of the high frequency field is given by equation (3.215),

$$E^{\text{eff}}_{\mu\nu} = (\varepsilon^2/8\pi)\, \mathcal{G}^{(2)}_{\mu\nu}.$$

These equations, which are proposed by the present writer, are generalisations of the vacuum field equations (2.801), (2.413) and (2.802) respectively.

The above field equations, which will sometimes be referred to as the 'general formalism', have been obtained with the aid of much guidance provided by the Isaacson vacuum treatment. One criterion that may be employed to measure the adequacy of this general formalism is to check to see that it is, at least, consistent with the published results of Isaacson and Madore. This will now be attempted. In addition, an examination of the Isaacson method when applied to manifolds possessing a non-zero matter content will be undertaken. It will be interesting to discover to what extent the material medium will have a modifying influence upon the original results of Isaacson.

In the course of the discussion, we will endeavour to justify the assumptions made in developing the equations (3.214), (3.215) and (3.216).

3.3 The general formalism and the 'Isaacson approach'.

The approach employed by Isaacson is based upon the definition of high frequency radiation stated at the opening of Section 2.4, which demands that the order of magnitude of the quantities involved, and their derivatives, be specified a priori. This is expressed by the array of equations (2.402). The definition of high frequency given by equations (2.201), (2.203) and (2.204), as well as the equations (3.203), are automatically satisfied by the conditions (2.402), since $\mathcal{E} << 1$.

Other important aspects of the Isaacson method, for example the WKB form of $h_{\mu\nu}$ and the coordinate conditions employed, are features which are shared with the Madore method. Therefore, so as not to detract from the fundamental importance of the equations (2.402) to the Isaacson approach, these other characteristics will be introduced later when necessary.

Equation (3.216) is now considered in both the vacuum and matter cases, using equations (2.402) as a basis for the analysis.

(I). Equation (3.216) in vacuum - the Isaacson method.

In the absence of matter, equation (3.216) immediately reduces to

$$\varepsilon \overset{(1)}{G}_{\mu\nu} + \varepsilon^3 \overset{(3+)}{G}_{\mu\nu} = 0 . \tag{3.301}$$

Since $\overset{(1)}{G}_{\mu\nu}$ is a function of terms the largest of which are of the form $h_{\mu\nu,\alpha\beta}$, the equations (2.402) imply that

$$\varepsilon \overset{(1)}{G}_{\mu\nu} = O(\varepsilon^{-1}). \tag{3.302}$$

Similarly, $\overset{(3+)}{G}_{\mu\nu}$ can be shown to be a function of terms the largest of which are typically of the form $h_{\mu\nu}h_{\rho\tau,\alpha}h_{\lambda\sigma,\beta}$. Thus, it is asserted that

$$\varepsilon^3 \overset{(3+)}{G}_{\mu\nu} = O(\varepsilon) . \tag{3.303}$$

To a good degree of approximation, equation (3.301) may therefore be written

$$\overset{(1)}{G}_{\mu\nu} = 0 .$$

By an inspection of equation (3.208), it can be seen that the appropriate treatment of equation (3.216) of the general formalism gives rise to the original vacuum results of Isaacson (1968a).

(II). Equation (3.216) in matter - the Isaacson method.

It has been emphasised throughout that Isaacson did not attempt to discuss radiation in the presence of material energy in his pioneering papers of 1968. The possibility that Isaacson's method can be extended to deal with radiation in the presence of matter, and thus be applied to cosmology, will now be examined.

Consider again the equation (3.216)

$$\varepsilon \overset{(1)}{G}_{\mu\nu} + \varepsilon^3 \overset{(3+)}{G}_{\mu\nu} = -8\pi\left(\varepsilon \overset{(1)}{T}_{\mu\nu} + \varepsilon^2 \overset{(2)}{T}_{\mu\nu} + \varepsilon^3 \overset{(3+)}{T}_{\mu\nu}\right).$$

In the presence of matter, the estimates (3.302) and (3.303) remain valid, so that the $\varepsilon \mathcal{G}^{(1)}_{\mu\nu}$ term will again be dominant on the LHS. On the RHS, it would appear that there are no terms larger than order ε, so that at first sight the $\varepsilon \mathcal{G}^{(1)}_{\mu\nu}$ term would seem to be larger than all the other terms in the equation by at least a factor ε^{-2}. This implies that equation (3.216) could be written to good accuracy as

$$\mathcal{G}^{(1)}_{\mu\nu} = 0 . \tag{3.304}$$

This procedure, however, does not conform to the spirit of the Isaacson approach in that the term $T^{(1)}_{\mu\nu}$ for example may inherently possess an order of magnitude other than unity. There exists the possibility, which must be considered, that the order of $T^{(1)}_{\mu\nu}$ could become large enough to equal the order of $\mathcal{G}^{(1)}_{\mu\nu}$ during the passage of the gravitational waves. In this event, equation (3.304) would certainly not be the case. A careful analysis is required to resolve this problem.

Firstly, it will be assumed that the 'potential' of the radiation is given by the WKB form, as introduced in Section 2.6. That is

$$h_{\mu\nu} = \mathcal{A} q_{\mu\nu} e^{i\phi} ,$$

where $\mathcal{A}$, $q_{\mu\nu}$ and ϕ represent the amplitude, the polarization and the phase of the radiation respectively. Since the background metric, and its derivatives, are stated in equations (2.402) as having unit magnitude it is further supposed that the unperturbed values of the fluid variables satisfy the following conditions,

$$u_\mu^{(0)},\ \rho^{(0)},\ \ldots\ldots \leqslant O(1), \tag{3.305}$$

$$\frac{\partial u_\mu^{(0)}}{\partial\phi} = \frac{\partial\rho^{(0)}}{\partial\phi} = \ldots\ldots = 0 . \tag{3.306}$$

Since the WKB form of $h_{\mu\nu}$ has been adopted, it is reasonable to suppose that the passage of the radiation will induce fluid perturbations of the form

$$u_\mu^{(1)} = \hat{u}_\mu^{(1)} e^{i\phi}, \quad \rho^{(1)} = \hat{\rho}^{(1)} e^{i\phi}, \tag{3.307}$$

and similarly for $p^{(1)}$, $T^{(1)}$, $\eta^{(1)}$, $\zeta^{(1)}$ and $\chi^{(1)}$, where

$$\hat{u}_\mu^{(1)},\ \hat{\rho}^{(1)},\ \ldots\ldots \leqslant O(1).$$

This type of hypothesis, which is common in hydrodynamical treatments of waves in fluids, is not of course totally satisfactory since any full treatment of the problem must involve a solution of the equations of fluid motion. However, if it is accepted for the time being, a reasonable evaluation of the radiation's effect upon the fluid is possible.

Two cases are briefly considered. Firstly, when the material distribution may be described by a perfect fluid energy tensor, given by

$$T_{\mu\nu} = w u_\mu u_\nu - g_{\mu\nu} p, \tag{3.308}$$

$$w = p + \rho,$$

and secondly, when it may be described by a general dissipative fluid energy tensor, given by

$$\tilde{T}_{\mu\nu} = w u_\mu u_\nu - g_{\mu\nu} p + \eta \Sigma_{\mu\nu} \tag{3.309}$$

$$+ \zeta \Pi_{\mu\nu}(D_\alpha u^\alpha) + \chi(u_\mu \Psi_\nu + u_\nu \Psi_\mu),$$

where

$$\Pi_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu ,$$

$$\Phi_\mu = \Pi^\alpha{}_\mu T_{,\alpha} - T u^\nu (D_\nu u_\mu) ,$$

$$\Sigma_{\mu\nu} = \Pi^\alpha{}_\mu \Pi^\beta{}_\nu (D_\alpha u_\beta + D_\beta u_\alpha) - \frac{2}{3} \Pi_{\mu\nu} (D_\alpha u^\alpha).$$

Here, the variables u_μ, ρ, p, T, η, ζ and χ are as defined in Section 3.2, and the D_μ denotes the covariant derivative with respect to $g_{\mu\nu}$. For an account of the theory of imperfect fluids see, for example, Weinberg (1972).

Consider the passage of the radiation through the perfect fluid distribution. When the hypothesis characterized by equations (3.307) is accepted the following estimates arise,

$$u^{(1)}_\mu \leqslant O(1) , \quad u^{(1)}_{\mu,\alpha} \leqslant O(\varepsilon^{-1}), \qquad (3.310)$$

$$\rho^{(1)} \leqslant O(1) , \quad \rho^{(1)}_{,\alpha} \leqslant O(\varepsilon^{-1}),$$

and similarly for $p^{(1)}$, $T^{(1)}$, $\eta^{(1)}$, $\zeta^{(1)}$ and $\chi^{(1)}$, since $\xi_\mu \equiv \phi_{,\mu} = O(\varepsilon^{-1})$ in the Isaacson scheme. When equations (3.211), (3.307) are combined and the result introduced into the equation (3.308), particular functional forms for $T^{(1)}_{\mu\nu}$, $T^{(2)}_{\mu\nu}$ and $T^{(3+)}_{\mu\nu}$ are obtained. This straight-forward calculation shows that none of these quantities contain derivatives of the form displayed in equations (3.310).

Thus, in this rough approximation, it is possible to conclude that

$$T^{(1)}_{\mu\nu} \leq O(1), \quad T^{(2)}_{\mu\nu} \leq O(1), \qquad (3.311)$$
$$T^{(3+)}_{\mu\nu} \leq O(1).$$

Now consider the radiation incident upon the dissipative fluid. A similar procedure applied to equation (3.309) provides lengthy expressions for the $\widetilde{T}^{(1)}_{\mu\nu}$, $\widetilde{T}^{(2)}_{\mu\nu}$ and $\widetilde{T}^{(3+)}_{\mu\nu}$, which for the sake of brevity are not reproduced here. It is sufficient to indicate the result. For a dissipative fluid it is found that

$$\widetilde{T}^{(1)}_{\mu\nu} \leq O(\varepsilon^{-1}), \quad \widetilde{T}^{(2)}_{\mu\nu} \leq O(\varepsilon^{-1}), \qquad (3.312)$$
$$\widetilde{T}^{(3+)}_{\mu\nu} \leq O(\varepsilon^{-1}),$$

since the $\widetilde{T}^{(i)}_{\mu\nu}$, i= 1,2,3+, are comprised of terms linear in derivatives of the kind contained in equations (3.310).

Before any conclusions are drawn about the form of equation (3.216) treated in matter, it is desirable to justify the equations (3.311) and (3.312) by a rigourous solution of the approximate equations of fluid motion. This may be done as follows for the perfect fluid case.

Consider gravitational radiation travelling through a space-time manifold in which the material distribution is described by equation (3.308). As before, it is supposed that the fluid variables are perturbed in the presence of the radiation in such a way that

$$\left.\begin{aligned} u_\mu &= u^{(0)}_\mu + \varepsilon u^{(1)}_\mu, \quad \rho = \rho^{(0)} + \varepsilon \rho^{(1)}, \\ p &= p^{(0)} + \varepsilon p^{(1)}, \quad w = w^{(0)} + \varepsilon w^{(1)}, \end{aligned}\right\} \qquad (3.313)$$

where $w^{(0)} = p^{(0)} + \rho^{(0)}$ and $w^{(1)} = p^{(1)} + \rho^{(1)}$.

Substitution of equations (3.313) into equation (3.308) gives

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \varepsilon T^{(1)}_{\mu\nu} + \varepsilon^2 T^{(2)}_{\mu\nu} + \varepsilon^3 T^{(3+)}_{\mu\nu} ,$$

where

$$T^{(0)}_{\mu\nu} = w^{(0)} u^{(0)}_{\mu} u^{(0)}_{\nu} - \gamma_{\mu\nu} p^{(0)} , \qquad (3.314a)$$

$$T^{(1)}_{\mu\nu} = w^{(1)} u^{(0)}_{\mu} u^{(0)}_{\nu} + w^{(0)} u^{(0)}_{\mu} u^{(1)}_{\nu} \qquad (3.314b)$$

$$+ w^{(0)} u^{(1)}_{\mu} u^{(0)}_{\nu} - p^{(1)} \gamma_{\mu\nu} - p^{(0)} h_{\mu\nu} ,$$

$$T^{(2)}_{\mu\nu} = w^{(0)} u^{(1)}_{\mu} u^{(1)}_{\nu} + w^{(1)} u^{(0)}_{\mu} u^{(1)}_{\nu} \qquad (3.314c)$$

$$+ w^{(1)} u^{(1)}_{\mu} u^{(0)}_{\nu} - p^{(1)} h_{\mu\nu} ,$$

$$T^{(3+)}_{\mu\nu} = w^{(1)} u^{(1)}_{\mu} u^{(1)}_{\nu} , \qquad (3.314d)$$

and where the conditions (3.305) and (3.306) are valid.

The equations governing the flow of the fluid are

$$D_{\mu} T^{\mu\nu} = 0 , \qquad (3.315)$$

and

$$T^{(0)\mu\nu}{}_{;\mu} = 0 , \qquad (3.316)$$

where D_{μ} represents the covariant derivative with respect to the total metric $g_{\mu\nu}$, and a semicolon represents the covariant derivative with respect to the background metric $\gamma_{\mu\nu}$. Additionally the fluid velocity 4-vector is assumed to satisfy

$$g^{\mu\nu} u_{\mu} u_{\nu} = 1 , \qquad (3.317a)$$

and

$$\gamma^{\mu\nu} u^{(0)}_{\mu} u^{(0)}_{\nu} = 1 . \qquad (3.317b)$$

The equation (3.315) may be written explicitly as

$$T^{\mu\nu}{}_{,\mu} + \Gamma^{\mu}_{\sigma\mu}(g_{\alpha\beta})T^{\sigma\nu} + \Gamma^{\nu}_{\mu\sigma}(g_{\alpha\beta})T^{\mu\sigma} = 0.$$

Substitutions for $g_{\mu\nu}$ and $T_{\mu\nu}$ in this equation, from equations (2.401) and (3.212) respectively, give an expansion of the following form,

$$\begin{aligned} T^{(0)\mu\nu}{}_{;\mu} &+ \varepsilon \mathcal{T}^{(1)\nu} \\ &+ \varepsilon^2 \mathcal{T}^{(2)\nu} + \varepsilon^3 \mathcal{T}^{(3+)\nu} = 0, \end{aligned} \tag{3.318}$$

where

$$\begin{aligned} \mathcal{T}^{(1)}_{\nu} = \gamma^{\mu\theta} T^{(1)}_{\mu\nu;\theta} &- h^{\mu\theta} T^{(0)}_{\mu\nu;\theta} \\ &- \gamma^{\mu\theta}\left(\square^{\tau}_{\mu\theta} T^{(0)}_{\tau\nu} + \square^{\tau}_{\nu\theta} T^{(0)}_{\mu\tau}\right), \end{aligned} \tag{3.319}$$

$$\begin{aligned} \mathcal{T}^{(2)}_{\nu} = \gamma^{\mu\theta} T^{(2)}_{\mu\nu;\theta} &- h^{\mu\theta} T^{(1)}_{\mu\nu;\theta} \\ + \gamma^{\mu\theta}&\left(h^{\tau}{}_{\alpha} \square^{\alpha}_{\mu\theta} T^{(0)}_{\tau\nu} + h^{\tau}{}_{\alpha} \square^{\alpha}_{\nu\theta} T^{(0)}_{\mu\tau}\right) \\ - \gamma^{\mu\theta}&\left(\square^{\tau}_{\mu\theta} T^{(1)}_{\tau\nu} + \square^{\tau}_{\nu\theta} T^{(1)}_{\mu\tau}\right) \\ + h^{\mu\theta}&\left(\square^{\tau}_{\mu\theta} T^{(0)}_{\tau\nu} + \square^{\tau}_{\nu\theta} T^{(0)}_{\mu\tau}\right). \end{aligned} \tag{3.320}$$

Although the remainder term, $\varepsilon^3 \mathcal{T}^{(3+)}_{\nu}$, is not given explicitly, it is discussed in the subsequent analysis to indicate that it is too small to be of importance in the first order fluid flow equations. The $\square^{\alpha}_{\mu\nu}$ terms are as defined by equation (2.408).

Equation (3.316) substituted into equation (3.318) gives

$$\varepsilon \mathcal{T}^{(1)}_{\nu} + \varepsilon^2 \mathcal{T}^{(2)}_{\nu} + \varepsilon^3 \mathcal{T}^{(3+)}_{\nu} = 0. \tag{3.321}$$

Now consider the first term of this equation. If the following definition is made,

$$\Xi_{\mu\nu} \equiv T^{(1)}_{\mu\nu} + p^{(0)} h_{\mu\nu} ,$$

and the coordinate conditions given by equations (2.507) are imposed, then

$$\mathcal{T}^{(1)}_{\nu} = \Xi_{\mu\nu;}{}^{\mu} - h^{\mu\theta}\left(w^{(0)} u^{(0)}_{\mu} u^{(0)}_{\nu}\right)_{;\theta} \qquad (3.322)$$
$$- \frac{1}{2} w^{(0)} u^{(0)}_{\mu} u^{(0)}_{\tau} \left(h^{\tau}{}_{\nu;}{}^{\mu} + h^{\tau\mu}{}_{;\nu} - h_{\nu}{}^{\mu}{}_{;}{}^{\tau}\right).$$

It is now convenient to introduce a useful notation. If X denotes an arbitrary tensor of rank j, then let

$$\bar{X} \equiv \frac{\partial X}{\partial \phi} , \quad \nabla_{\mu} X \equiv \left(X_{;\mu}\right)_{\phi = \text{constant}}.$$

Thus, the covariant derivative of X may be written as a sum of two parts,

$$X_{;\mu} = \xi_{\mu} \bar{X} + \nabla_{\mu} X, \qquad (3.323)$$

where ξ_{μ} is defined in equation (2.701). In particular,

$$h_{\alpha\beta;\mu} = \xi_{\mu} \bar{h}_{\alpha\beta} + \nabla_{\mu} h_{\alpha\beta} .$$

Since $\bar{h}_{\alpha\beta} = O(1)$, $\nabla_{\mu} h_{\alpha\beta} = O(1)$ and $\xi_{\mu} = O(\varepsilon^{-1})$, it is possible to write

$$\varepsilon h_{\alpha\beta;\mu} = \varepsilon \xi_{\mu} \bar{h}_{\alpha\beta} + O(\varepsilon) .$$

Thus, to lowest order, the covariant derivative of $h_{\alpha\beta}$ is reduced to a partial derivative with respect to ϕ . It is also true that $X_{;\mu} = \nabla_{\mu} X$, for all tensors X satisfying $\bar{X} = 0$. For example, from equations (3.306)

$$u^{(0)}_{\nu;\mu} = \nabla_{\mu} u^{(0)}_{\nu}$$

These considerations show that equation (3.323) turns equation (3.322) into

$$\mathcal{T}_\nu^{(1)} = \{ w^{(0)} \xi^\alpha u_\alpha^{(0)} \bar{u}_\nu^{(1)} + (\bar{w}^{(1)} \xi^\alpha u_\alpha^{(0)} + w^{(0)} \xi^\alpha \bar{u}_\alpha^{(1)}) u_\nu^{(0)}$$
$$- (\bar{p}^{(1)} + \frac{w}{2}^{(0)} \bar{h}_{\alpha\beta} u^{(0)\alpha} u^{(0)\beta}) \xi_\nu \}$$
$$+ [\nabla^\mu \Xi_{\mu\nu} - h^{\mu\theta} \nabla_\theta (w^{(0)} u_\mu^{(0)} u_\nu^{(0)}) \qquad (3.324)$$
$$- \frac{1}{2} w^{(0)} u^{(0)\mu} u^{(0)\tau} (\nabla_\mu h_{\tau\nu} + \nabla_\nu h_{\tau\mu} - \nabla_\tau h_{\mu\nu})] .$$

If the expression in the first bracket is denoted by $a_{\nu 1}^{(1)}$, and that in the second bracket by $a_{\nu 2}^{(1)}$, then equation (3.324) may be written

$$\mathcal{T}_\nu^{(1)} = \{ a_{\nu 1}^{(1)} \} + [a_{\nu 2}^{(1)}] . \qquad (3.325)$$

Suppose it is now assumed that the fluid perturbations satisfy the following conditions,

$$u_\mu^{(1)},\ \bar{u}_\mu^{(1)} = O(1),\ p^{(1)}, \bar{p}^{(1)} = O(1), \qquad (3.326)$$
$$\rho^{(1)}, \bar{\rho}^{(1)} = O(1).$$

The equations of fluid motion are now investigated to discover whether this assumption provides a consistent solution.

By equations (2.702), (3.305) and (3.326), it follows from equation (3.324) that, at most,

$$a_{\nu 1}^{(1)} = O(\varepsilon^{-1}) ,\quad a_{\nu 2}^{(1)} = O(1). \qquad (3.327)$$

Also, if

$$\mathcal{T}_\nu^{(2)} = a_{\nu 1}^{(2)} + a_{\nu 2}^{(2)} ,$$
$$\mathcal{T}_\nu^{(3+)} = a_{\nu 1}^{(3+)} + a_{\nu 2}^{(3+)} ,$$

then, by a similar argument, it may be shown that

$$\left.\begin{aligned} a^{(2)}_{\nu 1} &= O(\varepsilon^{-1}), \quad a^{(2)}_{\nu 2} = O(1), \\ a^{(3+)}_{\nu 1} &= O(\varepsilon^{-1}), \quad a^{(3+)}_{\nu 2} = O(1), \end{aligned}\right\} \tag{3.328}$$

since equation (3.318) contains first derivatives only. Hence equation (3.321) may be written

$$\varepsilon(a^{(1)}_{\nu 1} + a^{(1)}_{\nu 2}) + \varepsilon^2(a^{(2)}_{\nu 1} + a^{(2)}_{\nu 2}) + \varepsilon^3(a^{(3+)}_{\nu 1} + a^{(3+)}_{\nu 2}) = 0 .$$

By inspection of the conditions (3.327) and (3.328), to first order this equation becomes

$$\varepsilon a^{(1)}_{\nu 1} + O(\varepsilon) = 0.$$

The approximate equation (3.321) of fluid flow may therefore be written explicitly as

$$\varepsilon\{w^{(0)}\xi^{\alpha}u^{(0)}_{\alpha}\bar{u}^{(1)}_{\nu} + (\bar{w}^{(1)}\xi^{\alpha}u^{(0)}_{\alpha} + w^{(0)}\xi^{\alpha}\bar{u}^{(1)}_{\alpha})u^{(0)}_{\nu} - (\bar{p}^{(1)} + \tfrac{1}{2}w^{(0)}\bar{h}_{\alpha\beta}u^{(0)\alpha}u^{(0)\beta})\xi_{\nu}\} + O(\varepsilon) = 0 . \tag{3.329}$$

We now consider the consequences of the restrictions (3.317) imposed upon the velocity 4-vectors. If equation (3.317a) is combined with equation (3.317b), it is found that

$$2u^{(0)}_{\mu}u^{(1)\mu} = h_{\alpha\beta}u^{(0)\alpha}u^{(0)\beta} + O(\varepsilon), \tag{3.330}$$

which also implies, since $u^{(0)}_{\mu}$ is assumed to be independent of ϕ, that

$$2u^{(0)}_{\mu}\bar{u}^{(1)\mu} = \bar{h}_{\alpha\beta}u^{(0)\alpha}u^{(0)\beta} + O(\varepsilon). \tag{3.331}$$

Thus, under the hypothesis (3.326), the motion of the irradiated perfect fluid is governed, in the first approximation, by equations (3.329) and (3.330). Moreover, it will be supposed throughout the remainder of this Section that the hypersurfaces of constant phase are null, to the following degree of approximation,

$$\varepsilon^2 \xi_\rho \xi^\rho + O(\varepsilon^2) = 0.$$

If equation (3.329) is divided by $\varepsilon u^{(0)}_\alpha \xi^\alpha$, then

$$\left\{ w^{(0)} \bar{u}^{(1)}_\nu + \bar{w}^{(1)} u^{(0)}_\nu + w^{(0)} \left(\frac{\xi^\alpha \bar{u}^{(1)}_\alpha}{\xi^\sigma u^{(0)}_\sigma} \right) u^{(0)}_\nu \right.$$
$$\left. - \frac{\bar{p}^{(1)}}{\xi^\alpha u^{(0)}_\alpha} \xi_\nu - \frac{w^{(0)}}{2} \frac{\bar{h}_{\alpha\beta} u^{(0)\alpha} u^{(0)\beta}}{\xi^\sigma u^{(0)}_\sigma} \xi_\nu \right\} + O(\varepsilon) = 0. \tag{3.332}$$

The divisor satisfies $u^{(0)}_\alpha \xi^\alpha \neq 0$, because $u^{(0)}_\mu$ must be a timelike vector, whereas ξ_μ is null.

By contracting equation (3.332) with $\varepsilon \xi^\nu$, and again using the null character of ξ_μ, we find that

$$w^{(0)} \left(\frac{\xi^\alpha \bar{u}^{(1)}_\alpha}{\xi^\sigma u^{(0)}_\sigma} \right) = -\frac{\bar{w}^{(1)}}{2} + O(\varepsilon). \tag{3.333}$$

If this result is substituted into equation (3.332), then a slight rearrangement gives

$$2w^{(0)} \bar{u}^{(1)}_\nu - \frac{w^{(0)} \bar{h}_{\alpha\beta} u^{(0)\alpha} u^{(0)\beta}}{\xi^\sigma u^{(0)}_\sigma} \xi_\nu$$
$$= \left\{ \frac{2\bar{p}^{(1)}}{\xi^\alpha u^{(0)}_\alpha} \xi_\nu - \bar{w}^{(1)} u^{(0)}_\nu \right\} + O(\varepsilon). \tag{3.334}$$

Equation (3.334) is now contracted with $u^{(0)\nu}$, and equation (3.317b) applied. This gives

$$2w^{(0)} \bar{u}^{(1)}_\nu u^{(0)\nu} - w^{(0)} \bar{h}_{\alpha\beta} u^{(0)\alpha} u^{(0)\beta}$$
$$= \left\{ 2\bar{p}^{(1)} - \bar{w}^{(1)} \right\} + O(\varepsilon). \tag{3.335}$$

However, by appeal to equation (3.331), it is seen that the LHS of this equation is zero in this approximation. Since $\bar{p}^{(1)}$, $\bar{w}^{(1)}$ are assumed initially to be of order unity, equation (3.335) therefore implies that

$$\bar{w}^{(1)} = 2\bar{p}^{(1)} + O(\mathcal{E}),$$

or

$$\bar{p}^{(1)} = \bar{\rho}^{(1)} + O(\mathcal{E}), \tag{3.336}$$

from the definition of $W^{(1)}$. Substitution of this result back into equation (3.334) produces

$$2w^{(0)}\bar{u}^{(1)}_{\nu} - w^{(0)}\frac{\bar{h}_{\alpha\beta}u^{(0)\alpha}u^{(0)\beta}}{\xi^{\sigma}u^{(0)}_{\sigma}}\xi_{\nu} \tag{3.337}$$
$$= \bar{w}^{(1)}\left\{\frac{\xi_{\nu}}{\xi^{\alpha}u^{(0)}_{\alpha}} - u^{(0)}_{\nu}\right\} + O(\mathcal{E}).$$

Consideration of the equation (3.330) implies that a possible solution for $u^{(1)}_{\mu}$ is given by

$$u^{(1)}_{\mu} = J\xi_{\mu} + F_{\mu}, \tag{3.338}$$

where

$$J = \frac{h_{\alpha\beta}u^{(0)\alpha}u^{(0)\beta}}{2\xi^{\sigma}u^{(0)}_{\sigma}}.$$

Substitution of this result back into equation (3.330) implies that F_{μ} must satisfy

$$u^{(0)}_{\alpha}F^{\alpha} = O(\mathcal{E}). \tag{3.339}$$

If the result (3.338) is now substituted into equation (3.337), then the following differential equation in F_{μ} arises,

$$\bar{F}_{\mu} = \frac{\bar{w}^{(1)}}{2w^{(0)}}\left\{\frac{\xi_{\mu}}{\xi^{\alpha}u^{(0)}_{\alpha}} - u^{(0)}_{\mu}\right\} + O(\mathcal{E}), \tag{3.340}$$

which is consistent with equation (3.339), as well as with equation (3.333).

The result (3.336) can now be shown to impose restrictions upon the functional relationship between $p^{(0)}$ and $\rho^{(0)}$, the unperturbed pressure and density of the fluid. Assume that the total pressure and density satisfy

$$p = g(w), \tag{3.341}$$

where g is arbitrary, and also that their unperturbed components are related by the same equation of state, thus

$$p^{(0)} = g(w^{(0)}). \tag{3.342}$$

If the expressions for p and w from equations (3.313) are used, then equation (3.341) may be expanded as a Taylor series,

$$p^{(0)} + \varepsilon p^{(1)} = g(w^{(0)}) + \varepsilon w^{(1)} \frac{d}{dw^{(0)}} g(w^{(0)}) + O(\varepsilon^2).$$

Now employ equation (3.342), and partially differentiate with respect to ϕ. Since $w^{(0)}$ is independent of ϕ,

$$\bar{p}^{(1)} = \left\{ \frac{d}{dw^{(0)}} g(w^{(0)}) \Big/ \left(1 - \frac{d}{dw^{(0)}} g(w^{(0)})\right) \right\} \bar{\rho}^{(1)} + O(\varepsilon). \tag{3.343}$$

Comparison of equations (3.336) and (3.343) provides a simple linear equation in g, given by

$$\frac{d}{dw^{(0)}} g(w^{(0)}) = \frac{1}{2},$$

so that

$$g(w^{(0)}) = \frac{1}{2}\{ w^{(0)} + \beta \},$$

where $\frac{1}{2}\beta$ is the constant of integration, Thus

$$p^{(0)} = \rho^{(0)} + \beta. \tag{3.344}$$

This result ensues from equation (3.341), in which the arbitrary functional dependence of p upon w is regarded as being of sufficient generality. The result (3.344) also arises in the event that equation (3.341) is replaced by

$p = g(\rho)$.

Thus, the imposition of the conditions (3.326) has finally led to equation (3.344), whereas the relationship between the pressure $p^{(0)}$, and the density $\rho^{(0)}$ of an isotropic perfect fluid medium is usually restricted by the inequalities.

$$0 \leqslant p^{(0)} \leqslant \frac{1}{3}\rho^{(0)}. \qquad (3.345)$$

The relationship

$$p^{(0)} = (i-1)\rho^{(0)}, \quad i = constant, \qquad (3.346)$$

has been studied by many authors, in particular, by Harrison (1965, 1967, 1968) and by Zel'dovich (1962). In order that equations (3.345) and (3.346) be consistent with one another, i must satisfy $1 \leqslant i \leqslant 4/3$. Moreover, given that the sound velocity, v_s, in the fluid is governed by $v_s^2 = \partial p^{(0)}/\partial\rho^{(0)}$, i must also satisfy $i \leqslant 2$ to ensure that v_s does not exceed the velocity of light.

In the limit of high material densities, Zel'dovich has claimed that indeed equation (3.346) becomes $p^{(0)} = \rho^{(0)}$; that is, $i \to 2$ as the densi ties become very large. Harrison (1965), however, refutes this conclusion and reinforces the claim that the maximum value of i is $4/3$. Without entering into the controversy one thing remains clear, and that is that an equation of state of the form (3.344) may be possible, but only in matter at 'supernuclear' densities. We do not wish to be restricted to such high densities and we accept Harrison's conclusion that (3.345) holds good.

Therefore the assumptions (3.326), which led to equation

(3.344) must be rejected. A possible alternative is the set

$$u^{(1)}_{\mu},\ \bar{u}^{(1)}_{\mu} = O(1),\quad p^{(1)},\ \bar{p}^{(1)} = O(\varepsilon),\quad \rho^{(1)},\ \bar{\rho}^{(1)} = O(\varepsilon). \tag{3.347}$$

It will now be shown that these lead to an internally consistent solution of the equations of fluid flow.

Substitution of the conditions (3.347) into equation (3.332) gives

$$\begin{aligned} w^{(0)}\bar{u}^{(1)}_{\nu} &+ w^{(0)}(\xi^{\alpha}\bar{u}^{(1)}_{\alpha}/\xi^{\sigma}u^{(0)}_{\sigma})\,u^{(0)}_{\nu} \\ &= \frac{w^{(0)}\bar{h}_{\alpha\beta}\,u^{(0)\alpha}\,u^{(0)\beta}}{2\,\xi^{\sigma}u^{(0)}_{\sigma}}\,\xi_{\nu} + O(\varepsilon). \end{aligned} \tag{3.348}$$

The contraction of this equation with ξ^{ν} will give

$$\xi^{\alpha}\bar{u}^{(1)}_{\alpha} = O(\varepsilon), \tag{3.349}$$

due to the null character of the ray vector, ξ_{ν}. If this result is substituted back into equation (3.348), then

$$w^{(0)}\bar{u}^{(1)}_{\nu} = \frac{w^{(0)}\bar{h}_{\alpha\beta}\,u^{(0)\alpha}\,u^{(0)\beta}}{2\,\xi^{\sigma}u^{(0)}_{\sigma}}\,\xi_{\nu} + O(\varepsilon).$$

This equation, and equation (3.330), imply that

$$u^{(1)}_{\mu} = \left(\frac{h_{\alpha\beta}\,u^{(0)\alpha}\,u^{(0)\beta}}{2\,\xi^{\sigma}u^{(0)}_{\sigma}}\right)\xi_{\mu} + O(\varepsilon),$$

which may be seen to be consistent with equation (3.349). This, of course, also implies that F_{μ} as defined in equation (3.338) satisfies

$$F_{\mu} = O(\varepsilon),$$

which is in accord with the equation (3.340) in $\bar{F}_{\mu}$, since $\bar{w}^{(1)} = O(\varepsilon)$ by definition.

Thus, the equations governing the motion of the fluid have a solution

$$\left.\begin{aligned} u_\mu &= u_\mu^{(0)} + \varepsilon\left(\frac{h_{\alpha\beta}\, u^{(0)\alpha} u^{(0)\beta}}{2\xi^\sigma u_\sigma^{(0)}}\right)\xi_\mu\,, \\ \rho &= \rho^{(0)} + O(\varepsilon^2), \\ p &= p^{(0)} + O(\varepsilon^2). \end{aligned}\right\} \qquad (3.350)$$

Further, it is possible to assert that the intuitive argument which gave rise to the statements (3.307) is acceptable. In particular

$$\hat{u}_\mu^{(1)} = \left(\frac{\mathcal{A} q_{\alpha\beta}\, u^{(0)\alpha} u^{(0)\beta}}{2\xi^\sigma u_\sigma^{(0)}}\right)\xi_\mu\,,$$

by comparison of equations (3.307) and (3.350).

In conclusion of this subsection, we return to the question posed at the opening of subsection 3.3(II). What is the form of equation (3.216) in matter, for the Isaacson approach?

For the case of the perfect fluid, the intuitive approach based on the equations (3.307), has been shown to be reasonable by a detailed analysis of the equations of fluid motion. Substitution of the results (3.350) into equations (3.314) implies that

$$T_{\mu\nu}^{(1)} = O(1)\,,\quad T_{\mu\nu}^{(2)} = O(1)\,,\quad T_{\mu\nu}^{(3+)} = O(\varepsilon).$$

Therefore, for the perfect fluid, equation (3.216) may be written

$$\varepsilon^2 \mathcal{G}_{\mu\nu}^{(1)} + O(\varepsilon^2) = 0, \qquad (3.351)$$

since $\varepsilon^2 \mathcal{G}_{\mu\nu}^{(1)} = O(1)$. To a good degree of approximation, equation (3.351) may be written

$$\mathcal{G}_{\mu\nu}^{(1)} = 0\,,$$

which becomes

$$\frac{1}{2}\left(h_{\mu\nu;}{}^{\rho}{}_{\rho} + h^{\sigma}{}_{\mu} R^{(0)}_{\nu\sigma} + h^{\sigma}{}_{\nu} R^{(0)}_{\mu\sigma} - 2R^{(0)}_{\rho\mu\nu\sigma} h^{\rho\sigma} - h_{\mu\nu} R^{(0)} + \gamma_{\mu\nu} h^{\alpha\beta} R^{(0)}_{\alpha\beta}\right) = 0\,, \quad (3.352)$$

if the coordinate conditions (2.507) are employed. Since the background Riemann-Christoffel tensor $R^{(0)}_{\alpha\beta\gamma\delta}$, and the background Ricci tensor $R^{(0)}_{\alpha\beta}$ have no functional dependence upon the $h_{\mu\nu}$, the wave equation (3.352) is seen to be linear in $h_{\mu\nu}$.

If an analysis similar to the one contained in Section 2.7 is now made of equation (3.352), then high frequency gravitational radiation in a perfect fluid is found to obey equations identical in form to those discovered in the vacuum case. That is, equation (3.352) gives rise to equations (2.716) and (2.717), provided that the Isaacson approach is adhered to. This should not be taken to imply, however, that the radiation traverses precisely similar geodesic paths through the background space-time regardless of whether or not material is present. Indeed, the background geometry of a manifold containing both matter and radiation would differ from the geometry of a space-time containing only radiation, as may be seen from equation (3.214).

The determination of the form of equation (3.216) for the case of a dissipative fluid is not so straightforward. When equation (3.216) is divided by $\mathcal{E}$, it is found that

$$G^{(1)}_{\mu\nu} + \mathcal{E}^2 G^{(3+)}_{\mu\nu} = -8\pi\left(\tilde{T}^{(1)}_{\mu\nu} + \mathcal{E}\tilde{T}^{(2)}_{\mu\nu} + \mathcal{E}^2\tilde{T}^{(3+)}_{\mu\nu}\right),$$

where $\widetilde{T}_{\mu\nu}$ is given by equation (3.309). Of the terms on the LHS of this equation $\mathcal{G}^{(1)}_{\mu\nu}$ is larger than $\varepsilon^2 \mathcal{G}^{(3+)}_{\mu\nu}$ by a factor ε^{-2}, which follows from equations (3.302) and (3.303). On the RHS, the equations (3.312) imply that the leading term $\widetilde{T}^{(1)}_{\mu\nu}$ dominates over the other terms by a factor ε^{-1}. Thus, maintaining the largest terms on either side of equation (3.216), we write

$$\mathcal{G}^{(1)}_{\mu\nu} \doteq -8\pi \widetilde{T}^{(1)}_{\mu\nu} . \tag{3.353}$$

The explicit form of the LHS of this equation is given in equation (3.352). If the WKB form $h_{\mu\nu} = A_{\mu\nu} e^{i\phi}$ is employed, we find that

$$\mathcal{G}^{(1)}_{\mu\nu} = \frac{1}{2}\left\{ \Phi^{[1]}_{\mu\nu} + \Phi^{[2]}_{\mu\nu} + \Phi^{[3]}_{\mu\nu} \right\},$$

where

$$\Phi^{[1]}_{\mu\nu} = -\xi^{\rho}\xi_{\rho} A_{\mu\nu} e^{i\phi},$$

$$\Phi^{[2]}_{\mu\nu} = \left\{ 2\xi_{\rho} A_{\mu\nu;}{}^{\rho} + \xi^{\rho}{}_{;\rho} A_{\mu\nu} \right\} i e^{i\phi},$$

$$\Phi^{[3]}_{\mu\nu} = \left\{ A_{\mu\nu;}{}^{\rho}{}_{\rho} + A^{\sigma}{}_{\mu} R^{(0)}_{\nu\sigma} + A^{\sigma}{}_{\nu} R^{(0)}_{\mu\sigma} - 2R^{(0)}_{\rho\mu\nu\sigma} A^{\rho\sigma} - A_{\mu\nu} R^{(0)} + \gamma_{\mu\nu} A^{\alpha\beta} R^{(0)}_{\alpha\beta} \right\} e^{i\phi}.$$

The equation (3.353) therefore becomes

$$\Phi^{[1]}_{\mu\nu} + \Phi^{[2]}_{\mu\nu} + \Phi^{[3]}_{\mu\nu} = -16\pi \widetilde{T}^{(1)}_{\mu\nu},$$

where

$$\Phi^{[1]}_{\mu\nu} = O(\varepsilon^{-2}), \quad \Phi^{[2]}_{\mu\nu} = O(\varepsilon^{-1}), \quad \Phi^{[3]}_{\mu\nu} = O(1),$$

by inspection of equations (2.702), and where at most $\widetilde{T}^{(1)}_{\mu\nu} = O(\varepsilon^{-1})$ from equations (3.312). Hence, equation (3.353) may be decomposed into the following pair of approximate equations,

$$\left.\begin{aligned} \varepsilon^2 \Phi^{[1]}_{\mu\nu} + O(\varepsilon) &= 0, \\ \varepsilon\, \Phi^{[2]}_{\mu\nu} &= -16\pi\varepsilon \tilde{T}^{(1)}_{\mu\nu} + O(\varepsilon). \end{aligned}\right\} \quad (3.354)$$

By an examination of the definition of $\Phi^{[1]}_{\mu\nu}$, the first of these equations implies that the ray vector ξ_μ is a null vector in the dissipative fluid, at least to the following degree of approximation,

$$\varepsilon^2 \xi^\rho \xi_\rho + O(\varepsilon) = 0.$$

The second equation in (3.354) indicates that the solution for the $A_{\mu\nu}$ is influenced by the presence of the fluid. It is sufficient for the time being to indicate that equation (3.216) becomes equation (3.353) to lowest order for radiation in a dissipative fluid. The consequences of equation (3.353) will be briefly discussed later, when the work of J. Madore is considered.

(III). <u>The effective energy in matter - the Isaacson method</u>.

In this final subsection on the Isaacson approach the effective energy of the radiation field in matter is discussed. By equation (3.215) we have

$$\begin{aligned} E_{\mu\nu}{}^{\mathrm{eff}} = (\varepsilon^2/8\pi)\Big\{ & R^{(2)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu} R^{(2)} \\ & - \tfrac{1}{2}\Delta^{(1)}_{\mu\nu} + \tfrac{1}{2} h_{\mu\nu} h^{\alpha\beta} R^{(0)}_{\alpha\beta} \Big\}, \end{aligned}$$

where the $R^{(i)}_{\mu\nu}$, $i = 0, 2$, are given by equations (2.405) and (2.407), and where $\Delta^{(1)}_{\mu\nu}$ is given by equation (3.205).

If Isaacson's definition of radiation is imposed, then the conditions (2.402) provide a means of assessing the orders of magnitude of the terms of equation (3.215). Since $R^{(2)}_{\mu\nu}$ contains terms of the form $h_{\mu\nu,\alpha} h_{\rho\sigma,\beta}$, and $R^{(1)}_{\mu\nu}$ contains terms of the form $h_{\mu\nu,\alpha\beta}$, then equations (2.402) imply that

$$R^{(2)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(2)} - \frac{1}{2}\Delta^{(1)}_{\mu\nu} = O(\varepsilon^{-2}).$$

Similar considerations in the case of the remaining term gives

$$h_{\mu\nu} h^{\alpha\beta} R^{(0)}_{\alpha\beta} = O(1).$$

Hence, equation (3.215) becomes

$$E^{\mathit{eff}}_{\mu\nu} = (\varepsilon^2/8\pi)\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(2)} - \frac{1}{2}\Delta^{(1)}_{\mu\nu}\right) + O(\varepsilon^2). \tag{3.355}$$

A comparison between this equationand its vacuum analogue, equation (2.802), indicates that $E^{\mathit{eff}}_{\mu\nu}$ in matter is modified, to lowest order, by the appearance of the term in $\Delta^{(1)}_{\mu\nu}$. We now wish to discoverthe form of the averaged energy tensor $E_{\mu\nu} \equiv \langle E^{\mathit{eff}}_{\mu\nu}\rangle$ in matter.

The effect of reapplying the averaging process to the terms containing $R^{(2)}_{\mu\nu}$ in equation (3.355) is known from Section 2.8. The effect of averaging the $\Delta^{(1)}_{\mu\nu}$ term is now investigated. By definition,

$$\varepsilon^2\Delta^{(1)}_{\mu\nu} = \varepsilon^2\left(h_{\mu\nu}R^{(1)} - \gamma_{\mu\nu}h^{\alpha\beta}R^{(1)}_{\alpha\beta}\right).$$

From equation (2.508), the leading term of $\Delta^{(1)}_{\mu\nu}$ is given by

$$h_{\mu\nu}R^{(1)} = \frac{1}{2}h_{\mu\nu}\left(h_{;}{}^{\rho}{}_{\rho} + 2h^{\sigma\alpha}R^{(0)}_{\alpha\sigma} - 2R^{(0)\alpha}{}_{\rho\sigma\alpha}h^{\rho\sigma}\right)$$
$$= \frac{1}{2}h_{\mu\nu}h_{;}{}^{\rho}{}_{\rho}.$$

However, from the TT coordinate condition (2.507b) we observe that this term vanishes. Moreover, by an inspection of equation(2.508), the second term comprising $\Delta^{(1)}_{\mu\nu}$ becomes

$$\gamma_{\mu\nu} h^{\alpha\beta} R^{(1)}_{\alpha\beta} = \frac{1}{2}\gamma_{\mu\nu}\{ h^{\alpha\beta} h_{\alpha\beta;}{}^{\rho}{}_{\rho}$$
$$+ h^{\alpha\beta}(h^{\sigma}{}_{\alpha} R^{(0)}_{\beta\sigma} + h^{\sigma}{}_{\beta} R^{(0)}_{\alpha\sigma} - 2R^{(0)}_{\rho\alpha\beta\sigma} h^{\rho\sigma})\}.$$

Then, using the equations (2.402), we find that

$$\varepsilon^2 \Delta^{(1)}_{\mu\nu} = -\frac{1}{2}\varepsilon^2 \gamma_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta;}{}^{\rho}{}_{\rho} + O(\varepsilon^2),$$

which, by a rearrangement of the covariant derivative, becomes

$$\varepsilon^2 \Delta^{(1)}_{\mu\nu} = \frac{1}{2}\varepsilon^2 \{ \gamma_{\mu\nu} h^{\alpha\beta}{}_{;\rho} h_{\alpha\beta;}{}^{\rho}$$
$$- (\gamma_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta;}{}^{\rho})_{;\rho}\} + O(\varepsilon^2).$$

Taking the average, and appealing to the Corollary to Rule (i), Section 2.8, we may conclude that

$$\varepsilon^2 \langle \Delta^{(1)}_{\mu\nu} \rangle = \frac{1}{2}\varepsilon^2 \langle \gamma_{\mu\nu} h^{\alpha\beta}{}_{;\rho} h_{\alpha\beta;}{}^{\rho} \rangle + O(\varepsilon).$$

Thus, the averaged high frequency energy tensor for gravitational radiation in matter may be written

$$E_{\mu\nu} = (\varepsilon^2/32\pi)\{ \langle h^{\alpha\beta}{}_{;\mu} h_{\alpha\beta;\nu} \rangle$$
$$- \langle \gamma_{\mu\nu} h^{\alpha\beta}{}_{;\rho} h_{\alpha\beta;}{}^{\rho} \rangle \} + O(\varepsilon).$$

Moreover, if the WKB assumption is adopted, then this equation becomes

$$E_{\mu\nu} = (\varepsilon^2 A^2/64\pi)\{ \xi_\mu \xi_\nu - \gamma_{\mu\nu} \xi_\rho \xi^\rho \} + O(\varepsilon).$$

However, due to the null character of the ray vector ξ_α in matter, we may write $E_{\mu\nu}$ finally as

$$E_{\mu\nu} = (\varepsilon^2 A^2/64\pi)\,\xi_\mu \xi_\nu + O(\varepsilon). \tag{3.356}$$

In conclusion, therefore, the definition (3.215) provides an expression for the wave energy in matter which is the same as that found in the vacuum case.

3.4 The general formalism and the 'Madore approach'.

The approximation scheme discussed in this, and the previous chapter, has been developed employing the assumption that the total metric $g_{\mu\nu}$ may be expressed as a sum of two functions, $\gamma_{\mu\nu}$ and $\varepsilon h_{\mu\nu}$. It is now apparent that the success of the technique is dependent upon this division of $g_{\mu\nu}$ into two parts being quite distinct, and this has been ensured by demanding that the radiation be of high frequency. To accomodate this hypothesis, the Isaacson approach introduces the array of magnitudes assessments (2.402). The Madore approach, however, demands only that the radiation satisfies the definition of high frequency given in Section 2.2, together with the equation (3.203).

Further, in contrast to Isaacson's use of one smallness parameter ε , Madore introduces and employs a second given by

$$\delta = \lambda/L , \tag{3.401}$$

where λ , L are defined in Section 2.2 and $\delta \ll 1$ by condition (2.204).

Equation (3.216) is now considered, to demonstrate that Madore's results arise naturally from the general formalism.

(I). Equation (3.216) - the Madore method.

First consider briefly the case when material energy is absent. The Madore approach applied to equation (3.216) gives, to lowest order, the now familiar equation

$$\mathcal{G}^{(1)}_{\mu\nu} = 0 .$$

Thus, the vacuum results of Madore (1972) may be shown to be contained within the general formalism of Section 3.2. The case of radiation travelling through a matter filled background requires more detailed comment.

Madore (1973) himself discussed the absorption of gravitational radiation by a dissipative fluid and found the rate of entropy production in the fluid resulting from the passage of the radiation. In this subsection attention is restricted to the absorption analysis. For this, Madore imposed the additional conditions

$$\delta \gg \varepsilon \ , \quad \varepsilon^{i} = 0 \text{ for } i \geq 2. \tag{3.402}$$

Application of the Madore approach to equation (3.216), together with the conditions (3.402), leads to the following expression,

$$\mathcal{G}^{(1)}_{\mu\nu} = -8\pi \tilde{T}^{(1)}_{\mu\nu} . \tag{3.403}$$

This equation is basic to Madore's absorption analysis (Madore, 1973. Equation (3.13)). He solves it for the dissipative fluid and finds that the radiation is absorbed by the fluid in a characteristic time η^{-1}, where η is the shear viscosity of the fluid, This provides verification of an earlier result of Hawking (1966). Thus, the general formalism of Section 3.2 is found to be sufficiently versatile to contain this result also.

If the Isaacson approach of Section 3.3 is reconsidered for a moment, a comparison of equation (3.403) with equation (3.353) indicates that radiation treated in a dissipative fluid by the Isaacson method is also absorbed. A rigourous proof of this statement was omitted from the previous Section, since to include it would have entailed duplicating much of what Madore has already stated in his 1973 paper. Moreover, when the general formalism of Section 3.2 is applied to cosmology in the subsequent chapters, the matter content of the cosmological models will be regarded as a perfect fluid, rather than as a dissipative fluid.

It is now apparent that the form of the lowest order approximation to equation (3.216) in matter is not dependent upon the mathematical approach employed. However, there are implicit physical assumptions associated with each approach, as we shall now demonstrate.

In Section 3.2, it was assumed that the effective radiation field energy was proportional to $\mathcal{E}^2 G^{(2)}_{\mu\nu}$. Thus, since $G^{(2)}_{\mu\nu}$ is a function of terms typically of the form $h_{\mu\nu,\alpha}\, h_{\rho\tau,\beta}$, from equation (2.201) it is possible to write

$$E_\omega \approx (8\pi)^{-1} \frac{\mathcal{E}^2}{\lambda^2} , \qquad (3.404)$$

where E_ω denotes the wave energy. Also, from equations (2.203) and (3.207), the background curvature may be estimated to be of order L^{-2}. Einstein's equations imply that the background curvature is proportional to the total energy source curving the background. Hence

$$L^{-2} \approx 8\pi \left\{ (8\pi)^{-1} \frac{\mathcal{E}^2}{\lambda^2} + E_f \right\},$$

where E_f represents the order of magnitude of the energy of the material fluid. From equation (3.401), this becomes

$$(\delta^2/\lambda^2) \approx 8\pi \left\{ (8\pi)^{-1} \frac{\mathcal{E}^2}{\lambda^2} + E_f \right\}. \qquad (3.405)$$

If an 'order of magnitude' argument is employed, it is possible to interpret physically the mathematical differences between the treatments of Isaacson and Madore. The original work of Isaacson (1968a,b) involved the vacuum case only, so that equation (3.405) reduces to

$$\delta^2 \approx \mathcal{E}^2. \qquad (3.406)$$

Comparison of equation (2.203) with the left hand column of Isaacson's equations (2.402) implies that $L \approx 1$, so that equation (3.406) gives $\mathcal{E} \approx \lambda$. Indeed, for simplicity Isaacson chose

$$L = 1, \quad \lambda = \mathcal{E}.$$

If the extention of Isaacson's treatment to cases in which matter is present is considered, then this new situation is not inconsistent with the use of a single smallness parameter, since equations (3.405) and (3.406) are compatible provided that

$$E_f \lesssim (8\pi)^{-1} \frac{\mathcal{E}^2}{\lambda^2}. \qquad (3.407)$$

Thus, from equation (3.404), the Isaacson approach applied to the general formalism of Section 3.2 implies a situation in which the order of magnitude of the energy of the radiation is at least comparable with the energy of the material medium.

The work of Madore in deriving the absorption results mentioned previously depended upon the additional conditions (3.402). Consideration of equation (3.405) shows that these conditions, interpreted physically, imply that the radiation energy is regarded as negligible by comparison to the energy of matter.

(II). <u>The effective energy in matter - the Madore method.</u>

Equation (3.215) states that the effective field energy of the radiation in matter is given by

$$E_{\mu\nu}^{\;eff} = (\varepsilon^2/8\pi)\Big\{R^{(2)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)} - \tfrac{1}{2}\Delta^{(1)}_{\mu\nu} + \tfrac{1}{2}h_{\mu\nu}h^{\alpha\beta}R^{(0)}_{\alpha\beta}\Big\}.$$

If the Madore approach is employed, then equations (2.201), (2.203) and (3.203) imply that

$$R^{(2)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)} - \tfrac{1}{2}\Delta^{(1)}_{\mu\nu} = O(\lambda^{-2}),$$
$$h_{\mu\nu}h^{\alpha\beta}R^{(0)}_{\alpha\beta} = O(L^{-2}).$$

Since the definition of high frequency demands that $L >> \lambda$, the last term on the RHS of equation (3.215) may be neglected and therefore

$$E_{\mu\nu}^{\;eff} = (\varepsilon^2/8\pi)\Big\{R^{(2)}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}R^{(2)} - \tfrac{1}{2}\Delta^{(1)}_{\mu\nu}\Big\},$$

for the wave energy in matter. The additional conditions (3.402) imposed by Madore in the treatment of equation (3.216) are abandoned here. The imposition of equations (2.602) and (2.507), and the definition (2.807), again gives the result that

$$E_{\mu\nu} \equiv \langle E_{\mu\nu}{}^{eff} \rangle = \mathcal{B}^2 \xi_\mu \xi_\nu , \tag{3.408}$$

where $\mathcal{B}$ is a constant. This result is not derived by Madore (1973). But in obtaining an expression for the rate of entropy production in the fluid, he assumes that the wave energy in matter is indeed given by equation (3.408), with the constant of proportionality taking the value $\mathcal{B} = (\varepsilon \mathcal{A})/(8\lambda\sqrt{\pi})$. Using this assumption, Madore provides a verification of the thermodynamic relation

$$T\,d\Sigma = dH ,$$

for a closed system, where $d\Sigma$ denotes the entropy per unit volume supplied to the fluid in unit time, dH is the heat content per unit volume of the fluid, and T is the temperature ($^{\circ}$K). The dependence of this result upon equation (3.408) may be regarded as a justification for the assumption (3.215) of Section 3.2.

In conclusion of this chapter it is possible to say that the general formalism of Section 3.2 for the description of high frequency gravitational radiation in matter, is of sufficient flexibility to contain the results of Isaacson and Madore.

Chapter 4.

Gravitational radiation in cosmology: a description of the radiation in a Friedmann universe.

4.1 Preliminaries.

Attention will now be turned to the problems encountered when gravitational radiation is introduced into relativistic models of the Universe,

To provide a basis for the following discussions, it is postulated that gravitational radiation exists as an active component of the energetic content of the physical Universe. This radiation, and the effects it may have upon the large-scale dynamics of the Universe, will be described by applying the high frequency approximation scheme which has been reviewed and developed in the preceeding chapters. To demonstrate that this scheme is well suited to the task, it is necessary to determine what is meant by high frequency radiation in the context of cosmology. Recalling the definition of Section 2.2, radiation is said to be of high frequency provided its wavelength is very much smaller than the characteristic length over which the background geometry changes significantly. Thus, radiation propagating upon a cosmological background may still be regarded as high frequency even if the wavelength is of the order of, say, a galactic diameter. The high frequency approximation is therefore applicable for a large range of possible values of the wavelength, λ .

To describe the dynamic properties of the physical Universe, the general relativistic formalism of cosmological models developed by Friedmann (1922, 1924) will be used. Thus the background metric of the Isaacson scheme, denoted by $\gamma_{\mu\nu}$, will be associated with the metric of a Friedmann universe. Since the material content of these cosmological models is represented by a perfect fluid distribution, clearly the theory extending Isaacson's vacuum treatment to matter, discussed in the last chapter, is applicable.

For simplicity, it will be assumed that

$$L = 1 \;, \quad \lambda = \varepsilon \;, \tag{4.101}$$

where L , λ are as defined in Chapter 2. The physical situation implied by this choice of 'smallness parameter' is discussed in Section 3.4. The consequences of this choice for a perfect fluid distribution are equations (3.352), (3.356) and (3.214). It is the intention in this Chapter to investigate the cosmological radiation by using the equations (3.352) and (3.356), employing a WKB analysis. Then in a subsequent chapter, it is hoped to show how the radiation affects the evolution of the cosmological model using equation (3.214), and to speculate upon possible generation mechanisms for the radiation.

Work on the cosmic effects of gravitational waves has been tackled recently by several authors. In this Chapter, however, our attention will be confined to the work of Isaacson and Winicour (1972, 1973). They have shown that, provided certain reasonable assumptions are made, the age of a world model containing gravitational radiation is not in conflict with the lower bounds set on the age of the physical Universe by other dating methods. To provide a starting point for their investigation, Isaacson and Winicour suppose that an isotropic gravitational radiation field in cosmology may be represented by a homogeneous perfect fluid of energy density ρ_r , and pressure $p_r = \frac{1}{3}\rho_r$.

Earlier Ehrenfest and Tolman (1930) had shown that the energy contributed by 'disordered electromagnetic radiation' in a cosmological model, could be represented by a perfect fluid distribution. Since Isaacson(1968a,b) has shown many similarities between gravitational radiation treated in the geometrical optics limit, and electromagnetic radiation, perhaps the assumption of Isaacson and Winicour is not unreasonable.

However, the theory of monodirectional gravitational radiation in a vacuum cannot be expected to give results applicable in an isotropic field situation with matter, without deserving some justification. Thus, the primary motivation in this chapter is to show that Isaacson's and Winicour's assumption is indeed reasonable, and this is done by the use of an example.

4.2 Discussion of the perturbation, and the background metric.

Let the total metric, $g_{\mu\nu}$, be of the form given by equation (2.102), and let

$$h_{\mu\nu} = \mathrm{Real}\left\{\mathcal{A} q_{\mu\nu} e^{i\phi}\right\} , \qquad (4.201)$$

where $\mathcal{A}$, $q_{\mu\nu}$ and ϕ are as defined in chapter 2.

To represent the physical Universe, the homogenous, isotropic cosmological models of general relativity are employed.

Physically, the assumptions of homogeneity and isotropy, which collectively constitute what is usually referred to as the 'cosmological principle', reflect the widely accepted view that macroscopically the Universe is the same to an observer wherever he may be within it. Mathematical formulations of these concepts provide the means of deriving the Friedmann model line element by symmetry arguments alone, which was originally shown by Robertson (1929) and Walker (1935). For a concise account, see also Peebles (1971).

So, without need of reference to Einstein's field equations, it can be shown that the line element of cosmological models whose large scale features are homogeneous and isotropic is given by

$$da^2 = dt^2 - \frac{R^2(t)}{c^2(1+Kr^2/4)^2}\left\{\sum_{i=1}^{3}(dx^i)^2\right\}, \qquad (4.202)$$

where $R(t)$ is a scale factor describing the dynamics of the models, c is the local velocity of light, K is the space curvature constant and $r^2 = \sum_{i=1}^{3}(x^i)^2$. Conveniently scaled, K may take either of the values -1, 0 or +1, depending upon whether the space is said to be hyperbolic, flat or spherical respectively. Here a, t have the physical dimensions of time, R the dimensions of length, and c the dimensions of velocity.

The particular background metric, $\gamma_{\mu\nu}$, to be considered is the flat space model, characterized by $K = 0$, with line element

$$da^2 = dt^2 - \frac{R^2(t)}{c^2}\left\{\sum_{i=1}^{3}(dx^i)^2\right\}. \qquad (4.203)$$

Inherent in this choice, however, is the assumption that a Friedmann line element can be used to describe a model containing both gravitational radiation and matter. That this is so, is apparent from equation (3.214), which displays both radiation and matter energy as sources of the background.

By inspection of equation (4.203), it can be seen that x^1, x^2, x^3 are dimensionless coordinates. A dimensionless temporal coordinate, τ, is introduced, as follows. Let $R_0 \equiv R(t_0)$ be an arbitrary constant length. Thus, if $t \equiv x^4$, $\tau \equiv \bar{x}^4$ and $Y \equiv R/R_0$, then the application of the transformation

$$x^4 = (R_0/c)\,\bar{x}^4, \quad x^i = \bar{x}^i, \quad i = 1, 2, 3 \tag{4.204}$$

to the background metric (4.203) will give

$$d\bar{a}^2 = d\tau^2 - Y^2(\tau)\left\{\sum_{i=1}^{3}(dx^i)^2\right\}, \tag{4.205}$$

where

$$\bar{a} = (c/R_0)\,a.$$

The non-vanishing components of this background metric are therefore given by

$$\gamma_{44} = 1, \quad \gamma_{ii} = -Y^2; \quad i = 1, 2, 3. \tag{4.206}$$

With this metric, the surviving Christoffel symbols are

$$\Gamma^i_{i4} = \Gamma^i_{4i} = \frac{1}{Y}\frac{dY}{d\tau}, \tag{4.207}$$

$$\Gamma^4_{ii} = Y\frac{dY}{d\tau}, \quad i = 1, 2, 3,$$

with respect to the coordinates x^1, x^2, x^3, τ.

Repeated indices in equations (4.206) and (4.207) do not imply summation. Moreover, the components of the background metric, and the Christoffel symbols, are of zero physical dimensions.

The choice $K = 0$ is made on the grounds of simplicity. Consideration of Einstein's field equations (see Section 5.1) implies that the results for the cases $K = -1, +1$ are essentially similar to the results for the $K = 0$ case provided that $(dR/dt) >> |K|$. That is, during the early history of the models.

4.3 Solution of the linear 'wave equation' in a Friedmann universe.

The equation describing the propagation of gravitational radiation in a perfect fluid is given, to lowest order, by equation (3.352). In the discussion following this equation in Chapter 3 we observed that it gave rise to the equations (2.716) and (2.717), which are expressions identical in form to those found to describe gravitational radiation in the vacuum case. Thus, the problem of solving equation (3.352) in this section reduces to the task of obtaining solutions to the set of vacuum equations (2.716), using the metric (4.206) and heeding the conditions (2.717). The first equation of (2.716) is expressible as

$$\gamma^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = 0 ,$$

which will be referred to as the 'eikonal equation'. It will be assumed that ϕ takes the form

$$\phi = \sum_{i=1}^{3} \tilde{\ell}_i x^i + \mathcal{P}(\tau), \tag{4.301}$$

where the $\tilde{\ell}_i$ are constants, and the coordinate identifications

$$x^1 = x\ ,\quad x^2 = y\ ,\quad x^3 = z\ ,$$

are made. This choice of ϕ, which is dictated by the character of the null geodesics of the particular Friedmann background, ensures that the wavefronts of the radiation are 'plane' in each of the family of spacelike surfaces, τ = constant. Substitution of the metric (4.206), and the equation (4.301), into the eikonal equation gives

$$\frac{d\mathcal{P}}{d\tau} = \frac{\theta \tilde{K}}{Y}\ , \tag{4.302}$$

where

$$\tilde{K}^2 = \sum_{i=1}^{3} \tilde{\ell}_i^2\ ,\quad \theta = \pm 1. \tag{4.303}$$

Since it is supposed that $\xi_\mu = O(\varepsilon^{-1})$, we define

$$\tilde{\ell}_i \equiv \frac{K}{\varepsilon}\ell_i\ ,\quad \tilde{K} \equiv \frac{K}{\varepsilon}\ ;\quad i = 1, 2, 3,$$

so that equation (4.303) becomes

$$\sum_{i=1}^{3} \ell_i^2 = 1. \tag{4.304}$$

Then, from equations (4.301) and (4.302), ϕ takes the form

$$\phi = \frac{K}{\varepsilon}\left\{\sum_{i=1}^{3} \ell_i x^i + \theta\left(\int \frac{d\tau}{Y} + C\right)\right\}, \tag{4.305}$$

where C is a constant of integration. If the definition of the ray vectors given by equation (2.701) is reconsidered, then equation (4.305) gives

$$\xi_i = \frac{K}{\varepsilon}\ell_i\ ,\quad \xi_4 = \frac{\theta K}{\varepsilon Y}\ ;\ i = 1, 2, 3. \tag{4.306}$$

Now consider the second equation of (2.716) given by

$$(A^2 \xi^\rho)_{;\rho} = 0.$$

From equations (4.207), this may be written explicitly as

$$2\left\{\frac{\partial\mathcal{A}}{\partial x}\xi_1 + \frac{\partial\mathcal{A}}{\partial y}\xi_2 + \frac{\partial\mathcal{A}}{\partial z}\xi_3 - Y^2\frac{\partial\mathcal{A}}{\partial\tau}\xi_4\right\}$$
$$+\mathcal{A}\left\{\frac{\partial\xi_1}{\partial x} + \frac{\partial\xi_2}{\partial y} + \frac{\partial\xi_3}{\partial z} - Y^2\frac{\partial\xi_4}{\partial\tau} - 3\Gamma^4_{11}\xi_4\right\} = 0. \qquad (4.307)$$

This is an equation in the amplitude $\mathcal{A}$ of the radiation, since the eikonal equation has provided suitable expressions for the ξ_μ. Is it now possible to restrict the functional dependence of $\mathcal{A}$ upon the coordinates in a manner which is physically reasonable, to simplify equation (4.307)?

In the traditional treatments of relativistic cosmology the important property of homogeneity, which is implicit in the metric (4.202), is reflected in the behaviour of the fluid density and pressure of the Friedmann models. Einstein's equations imply that those variables must be functions of time alone. In a situation of this type, the energy distribution is described as being 'uniform'. This concept can be usefully employed by assuming that $\mathcal{A}$ is a function of τ alone. Physically, this implies that at any instant $\tau = \tau_1$, the amplitude of the plane wave is the same on a surface of constant phase. ~~at each point along its path through the model universe~~. Additionally, this assumption provides the simplest way of obtaining a uniform energy distribution for the radiation, which can be seen by inspection of equation (3.356) and of the solutions (4.306) for ξ_μ.

It is henceforth assumed that $\mathcal{A} = \mathcal{A}(\tau)$. Further, substitution for the ξ_μ from equations (4.306) and for the Christoffel symbols from equations (4.207) into equation

(4.307) gives

$$\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\tau} + \frac{1}{Y}\frac{dY}{d\tau} = 0 ,$$

so that

$$\mathcal{A}(\tau) = \mathcal{A}^0 / Y(\tau) , \qquad (4.308)$$

where $\mathcal{A}^0$ is a constant of integration.

Now consider the polarization field tensor $q_{\mu\nu}$, which is governed by the third equation of (2.716),

$$q_{\mu\nu;\rho}\,\xi^{\rho} = 0 ,$$

subject to the additional conditions (2.717) given by

$$q_{\mu\nu}\,q^{\mu\nu} = 1 , \qquad \xi^{\nu} q_{\mu\nu} = 0 ,$$
$$\gamma^{\mu\nu} q_{\mu\nu} = 0 .$$

The general form of the third equation of (2.716) is

$$\xi_1\left\{\frac{\partial q_{\mu\nu}}{\partial x} - \Gamma^{\sigma}_{\mu 1} q_{\sigma\nu} - \Gamma^{\sigma}_{\nu 1} q_{\mu\sigma}\right\}$$
$$+ \xi_2\left\{\frac{\partial q_{\mu\nu}}{\partial y} - \Gamma^{\sigma}_{\mu 2} q_{\sigma\nu} - \Gamma^{\sigma}_{\nu 2} q_{\mu\sigma}\right\}$$
$$+ \xi_3\left\{\frac{\partial q_{\mu\nu}}{\partial z} - \Gamma^{\sigma}_{\mu 3} q_{\sigma\nu} - \Gamma^{\sigma}_{\nu 3} q_{\mu\sigma}\right\}$$
$$- Y^2 \xi_4\left\{\frac{\partial q_{\mu\nu}}{\partial \tau} - \Gamma^{\sigma}_{\mu 4} q_{\sigma\nu} - \Gamma^{\sigma}_{\nu 4} q_{\mu\sigma}\right\} = 0 ,$$

where μ, ν = 1, 2, 3, 4 and where $q_{\mu\nu} = q_{\nu\mu}$.
If it is again argued that $q_{\mu\nu} = q_{\mu\nu}(\tau)$, then equations (4.207) and (4.306) imply that

$$\theta Y \frac{dq_{44}}{d\tau} + \frac{2}{Y}\frac{dY}{d\tau}\sum_{k=1}^{3} \ell_k q_{k4} = 0 , \qquad (4.309)$$

$$\theta Y \frac{dq_{i4}}{d\tau} - \theta \frac{dY}{d\tau} q_{i4} + Y \frac{dY}{d\tau}\ell_i q_{44} \qquad (4.310)$$
$$+ \frac{1}{Y}\frac{dY}{d\tau}\sum_{k=1}^{3} \ell_k q_{ik} = 0 ,$$

$$\theta Y \frac{dq_{ii}}{d\tau} - 2\theta \frac{dY}{d\tau} q_{ii} + 2Y \frac{dY}{d\tau} l_i q_{i4} = 0, \qquad (4.311)$$

$$\theta Y \frac{dq_{ij}}{d\tau} - 2\theta \frac{dY}{d\tau} q_{ij} + Y \frac{dY}{d\tau}(l_i q_{j4} + l_j q_{i4}) = 0, \qquad (4.312)$$

where

$$i \neq j = 1, 2, 3.$$

Moreover, if the coordinate conditions (2.717) are considered it is found that $q_{\mu\nu} \xi^{\nu} = 0$ becomes

$$\sum_{k=1}^{3} l_k q_{k4} = \theta Y q_{44}, \qquad (4.313)$$

$$\sum_{k=1}^{3} l_k q_{ki} = \theta Y q_{i4}, \quad i = 1, 2, 3, \qquad (4.314)$$

and that $q_{\mu\nu} \gamma^{\mu\nu} = 0$ reduces to

$$\sum_{k=1}^{3} q_{kk} = Y^2 q_{44}. \qquad (4.315)$$

Solutions to the equations (4.309) to (4.312) will now be sought, employing the conditions (4.313) to (4.315). The substitution of condition (4.313) into equation (4.309) gives

$$\frac{dq_{44}}{d\tau} + \frac{2}{Y}\frac{dY}{d\tau} q_{44} = 0.$$

Thus q_{44} is given by

$$q_{44} = \alpha_{44} / Y^2, \qquad (4.316)$$

where α_{44} is a constant of integration.

The combination of the equations (4.310) with the conditions (4.314) gives

$$\theta Y \frac{dq_{i4}}{d\tau} + l_i Y \frac{dY}{d\tau} q_{44} = 0 , \quad i = 1, 2, 3.$$

Thus, substitution for q_{44} from the equation (4.316) leads to

$$\frac{dq_{i4}}{d\tau} = \frac{l_i \alpha_{44}}{\theta} \frac{d}{d\tau}\left(\frac{1}{Y}\right) ,$$

so that

$$q_{i4} = \theta\left\{ \frac{l_i \alpha_{44}}{Y} + \alpha_{i4} \right\} , \quad i = 1,2,3, \tag{4.317}$$

where the α_{i4} are constants of integration.

If this solution for the q_{i4} is now substituted into equation (4.311) then

$$\frac{d}{d\tau}\left(\frac{q_{ii}}{Y^2}\right) = l_i^2 \alpha_{44} \frac{d}{d\tau}\left(\frac{1}{Y^2}\right) + 2 l_i \alpha_{i4} \frac{d}{d\tau}\left(\frac{1}{Y}\right).$$

Therefore the q_{ii} are given by

$$\begin{aligned} q_{ii} = {} & \alpha_{ii} Y^2 + 2 l_i \alpha_{i4} Y \\ & + l_i^2 \alpha_{44} , \qquad i = 1, 2, 3, \end{aligned} \tag{4.318}$$

where the α_{ii} are constants of integration.

Finally, if the solution (4.317) is substituted into equation (4.312), then it is found that

$$q_{ij} = l_i l_j \alpha_{44} + (l_i \alpha_{j4} + l_j \alpha_{i4}) Y + \alpha_{ij} Y^2 , \tag{4.319}$$

where

$$i \neq j = 1, 2, 3,$$

and the α_{ij} are constants of integration.

The TT coordinate conditions (2.507) impose restrictions upon the constants $\alpha_{\mu\nu}$. The substitution of the polarization tensor $q_{\mu\nu}$, into the equations (4.313) to

(4.315) provides the following conditions,

$$\sum_{i=1}^{3} \alpha_{ii} = 0, \tag{4.320}$$

$$\sum_{i=1}^{3} l_i \alpha_{\mu i} = 0, \quad \mu = 1, 2, 3, 4. \tag{4.321}$$

Moreover, further restrictions are imposed upon the $\alpha_{\mu\nu}$, since the polarization tensor is also required to satisfy $q_{\mu\nu} q^{\mu\nu} = 1$. In the background under consideration, this condition becomes

$$\frac{1}{Y^4}\left\{\sum_{i=1}^{3} q_{ii}^2 + 2\left(q_{12}^2 + q_{13}^2 + q_{23}^2\right)\right\} - \frac{2}{Y^2}\sum_{i=1}^{3} q_{i4}^2 + q_{44}^2 = 1.$$

Consideration of the solutions (4.316) to (4.319), and the conditions (4.320), (4.321), implies that

$$\sum_{i=1}^{3} \alpha_{ii}^2 + 2\left(\alpha_{12}^2 + \alpha_{13}^2 + \alpha_{23}^2\right) = 1. \tag{4.322}$$

Thus, the third equation of (2.716), and equation (2.717), governing the polarization field have a general solution given by the equations (4.316) to (4.319), subject to the restrictions (4.320) to (4.322).

The polarization tensor $q_{\mu\nu}$ may therefore be regarded as a 4×4 symmetric matrix possessing, in general, ten independent components. It will now be shown, however, that only two of these represent physically significant degrees of freedom, and this arises by virtue of the non-uniqueness of the coordinate system, which was briefly discussed in Section 2.5. To distinguish between the physical manifestations of the radiation and spurious 'coordinate ripples' due to possible non-inertial motions of the reference frame, the components of the Riemann-

Christoffel tensor, $R_{\alpha\beta\gamma\delta}$, for the total field are examined. Only those components of $g_{\mu\nu}$ which provide a non-vanishing contribution to $R_{\alpha\beta\gamma\delta}$ may be regarded as physically 'real', in the sense that these components cannot be made to vanish by a suitable choice of coordinate transformation.

Consider again the monodirectional gravitational radiation, and choose an orientation of the coordinate axes such that radiation flows in the x direction only. Thus equation (4.305) becomes

$$\phi = \frac{K}{\varepsilon}\left\{ \ell_1 x + \theta\left(\int \frac{d\tau}{Y} + C \right) \right\}, \tag{4.323}$$

where the degree of freedom provided by $\theta = \pm 1$ corresponds to whether the radiation is positively or negatively directed along the x axis. Now since

$$\ell_1 = 1, \quad \ell_2 = \ell_3 = 0, \tag{4.324}$$

for radiation directed along the x axis, then

$$\xi_4 = \frac{\theta K}{\varepsilon Y}, \quad \xi_1 = \frac{K}{\varepsilon}, \quad \xi_2 = \xi_3 = 0. \tag{4.325}$$

Employing the equations (4.324), we find that the conditions (4.321) reduce to

$$\alpha_{\mu 1} = 0, \quad \mu = 1, 2, 3, 4, \tag{4.326}$$

whereas similar considerations with respect to the condition (4.320) imply that

$$\alpha_{22} = -\alpha_{33}. \tag{4.327}$$

Now consider the Riemann-Christoffel tensor. Substitution of the total metric (2.401) into $R_{\alpha\beta\gamma\delta}$ provides a power series in ε given by

$$R_{\alpha\beta\gamma\delta}(\gamma_{\mu\nu} + \varepsilon h_{\mu\nu}) \equiv R^{(0)}_{\alpha\beta\gamma\delta} + \varepsilon R^{(1)}_{\alpha\beta\gamma\delta} + \varepsilon^2 R^{(2)}_{\alpha\beta\gamma\delta} + \varepsilon^3 R^{(3+)}_{\alpha\beta\gamma\delta}, \tag{4.328}$$

where

$$\begin{aligned} R^{(0)}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta}(\gamma_{\mu\nu}), \\ R^{(1)}_{\alpha\beta\gamma\delta} &= \tfrac{1}{2}\big(h_{\beta\gamma;\alpha\delta} + h_{\alpha\delta;\beta\gamma} \\ &\quad - h_{\beta\delta;\alpha\gamma} - h_{\alpha\gamma;\beta\delta} \\ &\quad + R^{(0)}_{\alpha\sigma\gamma\delta} h^{\sigma}{}_{\beta} + R^{(0)}_{\beta\sigma\gamma\delta} h^{\sigma}{}_{\alpha}\big). \end{aligned} \tag{4.329}$$

For the sake of brevity the remaining coefficients are not expressed explicitly. By a similar argument to that applied to the Ricci tensor in Chapters 2, 3, the $\varepsilon R^{(1)}_{\alpha\beta\gamma\delta}$ term dominates the other terms on the RHS of equation (4.328) by a factor ε^{-1}. Thus the total Riemann-Christoffel tensor may be expressed as

$$R_{\alpha\beta\gamma\delta} = \varepsilon R^{(1)}_{\alpha\beta\gamma\delta}, \tag{4.330}$$

to a good degree of approximation.

Moreover, if the WKB form of $h_{\mu\nu}$ is substituted into equation (4.329), and the conditions (2.702) imposed, it is found that equation (4.330) becomes

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\varepsilon\{\xi_\alpha\xi_\delta h_{\beta\gamma} + \xi_\beta\xi_\gamma h_{\alpha\delta} - \xi_\alpha\xi_\gamma h_{\beta\delta} - \xi_\beta\xi_\delta h_{\alpha\gamma}\},$$

where quantities an order of ε smaller than the $\xi_\alpha\xi_\beta h_{\gamma\delta}$ terms are ignored.

Now if the ray vectors given by equations (4.325) are employed, then of the possible twenty independent components of $R_{\alpha\beta\gamma\delta}$, nine are non-zero. These are given by

$$R_{\mu ij\nu} = \frac{1}{2}\varepsilon(\xi_\mu \xi_\nu)h_{ij}, \tag{4.331}$$

where

$$\mu\nu = 11,\ 14,\ 44,$$
$$ij = 22,\ 23,\ 33.$$

Since h_{ij} is related to q_{ij} by the equation (4.201), the only components of $q_{\mu\nu}$ which contribute to the non-vanishing components of the approximate Riemann-Christoffel tensor are q_{22}, q_{23} and q_{33}. It follows that by the use of a suitable choice of coordinates all the components of $q_{\mu\nu}$, with the exception of those mentioned above, may be made to vanish. From the equations (4.318), (4.319) and the conditions (4.326), (4.327) the surviving components of $q_{\mu\nu}$ may be expressed as

$$q_{22} = -q_{33} = \alpha_{22} Y^2,$$
$$q_{23} = \alpha_{23} Y^2,$$

where, from equation (4.322) the $\alpha_{\mu\nu}$ must satisfy

$$\alpha_{22}^2 + \alpha_{23}^2 = \frac{1}{2}.$$

To demonstrate the 'transverse' nature of the radiation reconsider the total fundamental form of the space-time manifold given by

$$ds^2 = d\tau^2 + (\gamma_{ij} + \varepsilon h_{ij})dx^i dx^j$$
$$\equiv d\tau^2 - dR^2. \qquad i, j = 1, 2, 3.$$

The coefficient of $d\tau^2$ is such that $g_{44} = \gamma_{44} + \varepsilon h_{44} = 1$, since a reference frame may be chosen in which $q_{44} = 0$.

In each of the 3-spaces, τ = constant, the separation of neighbouring points is given by

$$d\mathcal{R}^2 = Y^2 dx^2 + (Y^2 - \varepsilon h_{22}) dy^2 + (Y^2 + \varepsilon h_{22}) dz^2 - 2\varepsilon h_{23}\, dy\, dz\,, \tag{4.332}$$

where

$$h_{22} = -h_{33} = \alpha_{22} \mathcal{H} Y e^{i\phi},$$
$$h_{23} = \alpha_{23} \mathcal{H} Y e^{i\phi},$$

for radiation travelling in the x direction. Here $\mathcal{H}$ is a constant of integration. From equation (4.332) it may be seen that the separation of two events along the direction of wave propagation ($dx \neq 0$, $dy = dz = 0$) is determined only by the instantaneous value of Y, the expansion scale factor. Points located in a plane perpendicular to the x direction ($dx = 0, dy \neq 0, dz \neq 0$) however, experience maximum relative accelerations due to the radiation, superimposed upon the effects of the cosmological expansion, thus demonstrating the transverse character of the waves.

4.4 The stress energy of monodirectional gravitational radiation in a Friedmann universe.

The conditions (4.101) adopted throughout this Chapter are characteristic of the Isaacson approach, which was discussed at length in Section 3.3. It was shown there that the Isaacson approach gives rise to equation (3.356),

$$E_{\mu\nu} \equiv \langle E_{\mu\nu}{}^{eff} \rangle = (\varepsilon^2 \mathcal{A}^2 / 64\pi)\, \xi_\mu \xi_\nu + O(\varepsilon)\,,$$

for the energy tensor of gravitational radiation in matter.

In this Section, the energy tensor $E_{\mu\nu}$ of monodirectional gravitational radiation in a Friedmann universe is briefly considered. It is important to point out before doing so that the result (3.356), and the associated conservation laws given by equation (2.822), arise only in the event that the equations (2.716) and (2.717) are satisfied. However, employing a Friedmann background, we have already found a solution to these equations in Section 4.3, which possesses all the necessary properties to be consistent with the derivation of equation (3.356). Thus, it is supposed that the energy tensor of the postulated cosmological radiation may be obtained from equation (3.356).

By the use of equation (4.308), equation (3.356) becomes

$$E_{\mu\nu} = \frac{\mathcal{F}\mathcal{E}^2}{y^2}\,\xi_\mu \xi_\nu\,, \qquad (4.401)$$

where

$$\mathcal{F} = \mathcal{H}^2/64\pi\,.$$

If the solutions for the ray vectors ξ_μ given by equations (4.306) are reconsidered, the energy tensor of the cosmological radiation may be presented as a 4×4 array in the following manner,

$$E_{\mu\nu} = \mathcal{F}K^2 \begin{pmatrix} \frac{l_1^2}{Y^2} & \frac{l_1 l_2}{Y^2} & \frac{l_1 l_3}{Y^2} & \frac{\theta l_1}{Y^3} \\ \frac{l_1 l_2}{Y^2} & \frac{l_2^2}{Y^2} & \frac{l_2 l_3}{Y^2} & \frac{\theta l_2}{Y^3} \\ \frac{l_1 l_3}{Y^2} & \frac{l_2 l_3}{Y^2} & \frac{l_3^2}{Y^2} & \frac{\theta l_3}{Y^3} \\ \frac{\theta l_1}{Y^3} & \frac{\theta l_2}{Y^3} & \frac{\theta l_3}{Y^3} & \frac{1}{Y^4} \end{pmatrix} \tag{4.402}$$

Some points to note about this tensor are, firstly that

$$E^{\alpha}{}_{\alpha} = 0. \tag{4.403}$$

Verification of this result is straightforward. Since

$$E^{\alpha}{}_{\nu} = \gamma^{\alpha\mu} E_{\mu\nu} ,$$

it follows, by inspection of equations (4.206) and (4.402), that

$$E^{i}{}_{i} = -\mathcal{F}K^2 l_i^2 / Y^4 , \quad i = 1, 2, 3 ,$$

where no summation is implied by repeated latin indices. By the same token

$$E^{4}{}_{4} = \mathcal{F}K^2/Y^4 .$$

Thus, using equation (4.304), we have

$$E^{\alpha}{}_{\alpha} = \frac{\mathcal{F}K^2}{Y^4}\left\{1 - \sum_{i=1}^{3} l_i^2\right\} = 0 .$$

Secondly, it is possible to attribute the quantities of radiation pressure P , and radiation energy density ρ_r , to the components of this tensor in the following way. Define

$$E^1{}_1 \equiv -p_{xx} \quad , \quad E^2{}_2 \equiv -p_{yy} \, , \qquad (4.404)$$
$$E^3{}_3 \equiv -p_{zz} \quad , \quad E^4{}_4 \equiv \rho_r \quad ,$$

with the off-diagonal components denoted by

$$E^1{}_2 = E^2{}_1 \equiv -p_{xy} \, , \; E^1{}_3 = E^3{}_1 \equiv -p_{xz} \, , \qquad (4.405)$$
$$E^2{}_3 = E^3{}_2 \equiv -p_{yz} \, .$$

The significance of the $E^i{}_4$, i = 1, 2, 3 components will become apparent in the course of discussion later. It must be emphasised that the identifications (4.404) and (4.405) refer to monodirectional radiation only. As may be expected in this situation, the diagonal pressure components are unequal,

$$p_{xx} \neq p_{yy} \neq p_{zz} \, ,$$

except in the particular case when the wave-front normal makes an equal angle with each of the coordinate axes. That is, when $\ell_1 = \ell_2 = \ell_3$.

The operation of identifying the physical quantities, p and ρ_r , with the components of $E_{\mu\nu}$ may be elucidated by consideration of the energy tensor of the null electromagnetic field. Although such a method of comparison is not, in general, adequate in bridging the gulf that exists between the theories of gravitation and electromagnetism, never-the-less it has played a useful role in clarifying the theory of gravitational radiation throughout its development.

For electromagnetic radiation propagating along the x

direction, in a flat space-time with metric $\eta_{\mu\nu} = +1, -1, -1, -1$, the surviving components of the electromagnetic energy tensor $T^{e.m.}_{\mu\nu}$ are

$$\left.\begin{aligned} T^{e.m.\,1}{}_{1} &= -\tfrac{1}{2}(E_y^2 + H_z^2) = -p^{e.m.}, \\ T^{e.m.\,4}{}_{4} &= \tfrac{1}{2}(E_y^2 + H_z^2) = \rho^{e.m.}, \\ T^{e.m.}_{14} &= T^{e.m.}_{41} = -E_y H_z, \\ \text{so that} & \\ p^{e.m.} &= \rho^{e.m.}. \end{aligned}\right\} \qquad (4.406)$$

Here, $\underset{\sim}{E} = (E_x, E_y, E_z)$ and $\underset{\sim}{H} = (H_x, H_y, H_z)$ are the electric and magnetic field strengths respectively. Since the x - axis is the direction of propagation, $\underset{\sim}{E}$ and $\underset{\sim}{H}$ have been chosen such that

$$E_x = E_z = H_x = H_y = 0, \quad E_y \neq 0, \quad H_z \neq 0.$$

Note that the parameter $p^{e.m.}$ is the pressure exerted by the electromagnetic radiation in the x direction, and that $\rho^{e.m.}$ represents the energy density of the electromagnetic radiation. For a description of the total tensor $T^{e.m.}_{\mu\nu}$ see, for example, Tolman (1934).

We return to the gravitational case. For gravitational radiation propagating in the x direction, equations (4.324) are satisfied. In this event, the surviving components of $E_{\mu\nu}$ are

$$\left.\begin{aligned} E^{1}{}_{1} &= -\mathcal{F}K^2/Y^4 \equiv -p_{xx}, \\ E^{4}{}_{4} &= \mathcal{F}K^2/Y^4 \equiv \rho_r, \\ E_{14} &= E_{41} = \theta\mathcal{F}K^2/Y^3, \end{aligned}\right\} \qquad (4.407)$$

so that

$$\left. p_{xx} = \rho_r . \right\} \quad (4.407)$$

The similarities between the gravitational and the electromagnetic situations for monodirectional radiation are immediately apparent, by comparison of equations (4.406) and (4.407). This strongly suggests that the interpretation of $E_{\mu\nu}$ given by equations (4.404) and (4.405) is acceptable.

A further point of interest, manifesting itself in the preceeding treatment of gravitational radiation, is the functional dependence of the energy density ρ_r, upon Y, the scale factor of the Friedmann model. From equations (4.407)

$$\rho_r \propto Y^{-4}$$

The factor Y^{-4} is what should be expected for the free expansion of any wave representing a massless particle.

4.5 The energy tensor of an isotropic gravitational radiation field.

In the preceeding Section, the WKB approximation has been employed to find an expression for the energy of monodirectional, plane waves in a Friedmann universe. It should be emphasised, however, that the monodirectional theory is not well suited to the task of solving problems related to the macroscopic structure of the cosmological background, since its application would require the radiation field to be globally aligned in a preferred direction. This unrealistic situation would be in conflict with the cosmological principle since the aligned field would introduce a degree of anisotropy into the energy distribution of the resulting world model. Thus, in what follows an isotropic radiation field will be examined.

In this Section, the question posed in Section 4.1 is reconsidered. It is wished to develop the theory required to provide a justification for the Isaacson and Winicour hypothesis that an isotropic gravitational radiation field may be represented by a perfect fluid energy distribution.

Consider a field of radiation containing more than a single plane wave perturbation. It is assumed that if the field contains j different wave components, the principle of the superposition of waves applies. That is, the total perturbation to the metric $\hat{h}_{\mu\nu}$ may be written

$$\hat{h}_{\mu\nu} = \sum_{k=1}^{j} h^{(k)}_{\mu\nu} , \qquad (4.501)$$

where

$$\left.\begin{aligned} h^{(k)}_{\mu\nu} &= \mathrm{Real}\{A^{(k)}_{\mu\nu}\, e^{i\phi^{(k)}}\} , \\ A^{(k)}_{\mu\nu} &= \mathcal{A}^{(k)} q^{(k)}_{\mu\nu} , \\ \xi^{(k)}_{\mu} &= \phi^{(k)}{}_{,\mu} . \quad k = 1, 2, \ldots, j . \end{aligned}\right\} \qquad (4.502)$$

Since each $h^{(k)}_{\mu\nu}$, $k = 1, 2, \ldots, j$, is a solution of the wave equation (3.352), then $\hat{h}_{\mu\nu}$ must also be a solution by virtue of the linearity of the wave equation.

It is further supposed that the energy tensor of a 'multiple field' of this nature may be written as

$$\hat{E}_{\mu\nu} = (\varepsilon^2/32\pi)\langle \hat{h}^{\rho\tau}{}_{;\mu}\,\hat{h}_{\rho\tau;\nu}\rangle + O(\varepsilon).$$

If the equations (2.507) and (2.702) are adopted, and the averaging procedure of Section 2.8 used, this becomes

$$\hat{E}_{\mu\nu} = \sum_{k=1}^{j} E^{(k)}_{\mu\nu} + (\varepsilon^2/32\pi)\sum_{\substack{k=1\\(k\neq i)}}^{j}\sum_{i=1}^{j}\{A^{(k)\rho\tau}A^{(i)}_{\rho\tau} \times \tilde{\xi}^{(k)}_{\mu}\tilde{\xi}^{(i)}_{\nu}\langle \sin\phi^{(k)}\sin\phi^{(i)}\rangle\} + O(\varepsilon), \quad (4.503)$$

where

$$E^{(k)}_{\mu\nu} = (\varepsilon^2/32\pi)\langle h^{(k)\rho\tau}{}_{;\mu}\,h^{(k)}_{\rho\tau;\nu}\rangle.$$

It is apparent that the double summation of 'cross-terms' in equation (4.503) prohibits, in general, the possibility of a superposition principle of the form

$$\hat{E}_{\mu\nu} = \sum_{k=1}^{j} E^{(k)}_{\mu\nu} + O(\varepsilon), \quad (4.504)$$

for the total energy tensor. This result may have been expected by virtue of the nonlinearity of equation (3.356).

However, there is a particular case when equation (4.504) is indeed valid. If it is supposed that the phase relations between the different wave components are random, then the offending double summation in equation (4.503) may be set equal to zero. Thus, the assumption that the wave components be generated by incoherent sources allows the total energy tensor $\hat{E}_{\mu\nu}$ to satisfy equation (4.504). Equation (4.504) may be written

$$\hat{E}_{\mu\nu} = (\varepsilon^2/64\pi)\sum_{k=1}^{j}(A^{(k)})^2\,\tilde{\xi}^{(k)}_{\mu}\tilde{\xi}^{(k)}_{\nu} + O(\varepsilon). \quad (4.505)$$

This is a result which has been obtained previously (Isaacson, 1968b). In the case of gravitational radiation in a Friedmann model, the summation in equation (4.501) will be identified with a sum of monodirectional, plane wave components, each in one of j different directions. It will be supposed that each component of the radiation is generated by one of j independent and therefore incoherent sources, so that equation (4.505) may be used to describe the total energy tensor.

From equation (4.308), each component of the total radiation field will have an amplitude given by

$$\mathcal{A}^{(k)} = \mathcal{H}^{(k)} / Y(\tau) , \quad k = 1, 2, \ldots, j ,$$

where the $\mathcal{H}^{(k)}$ are constants of integration. Hence, equation (4.505) becomes

$$\hat{E}_{\mu\nu} = \frac{\varepsilon^2}{Y^2} \sum_{k=1}^{j} \mathcal{F}^{(k)} \xi_\mu^{(k)} \xi_\nu^{(k)} , \tag{4.506}$$

where

$$\mathcal{F}^{(k)} = (\mathcal{H}^{(k)})^2 / 64\pi , \quad k = 1, 2, \ldots, j .$$

Moreover, from equations (4.306), the ray vectors may be expressed as

$$\xi_i^{(k)} = K^{(k)} \ell_i^{(k)} ,$$

$$\xi_4^{(k)} = \frac{\theta K^{(k)}}{\varepsilon Y} ,$$

where $i = 1, 2, 3$ and $k = 1, 2, \ldots, j$. Thus the total energy tensor describing a 'j - fold' gravitational radiation field in a Friedmann background is given by

$$\left.\begin{aligned}
\hat{E}_{ii} &= \frac{1}{Y^2}\sum_{k=1}^{j}\mathcal{F}^{(k)}\left(K^{(k)}\ell_i^{(k)}\right)^2 , \\
\hat{E}_{im} &= \frac{1}{Y^2}\sum_{k=1}^{j}\mathcal{F}^{(k)}\left(K^{(k)}\right)^2\ell_i^{(k)}\ell_m^{(k)} , \\
\hat{E}_{i4} &= \frac{\vartheta}{Y^3}\sum_{k=1}^{j}\mathcal{F}^{(k)}\left(K^{(k)}\right)^2\ell_i^{(k)} , \\
\hat{E}_{44} &= \frac{1}{Y^4}\sum_{k=1}^{j}\mathcal{F}^{(k)}\left(K^{(k)}\right)^2 ,
\end{aligned}\right\} \qquad (4.507)$$

where

$$i \neq m = 1, 2, 3.$$

If an isotropic field of radiation in a cosmological background is desired, the value of k in equations (4.507) may be allowed to run from one to infinity, each value of k corresponding to a different direction of flow. Thus, the energy tensor of an isotropic gravitational radiation field, $\overset{iso}{E}_{\mu\nu}$, may be denoted by the following expressions

$$\left.\begin{aligned}
\overset{iso}{E}_{ii} &= \frac{1}{Y^2}\sum_{k=1}^{\infty}\mathcal{F}^{(k)}\left(K^{(k)}\ell_i^{(k)}\right)^2 , \\
\overset{iso}{E}_{im} &= \frac{1}{Y^2}\sum_{k=1}^{\infty}\mathcal{F}^{(k)}\left(K^{(k)}\right)^2\ell_i^{(k)}\ell_m^{(k)} , \\
\overset{iso}{E}_{i4} &= \frac{\vartheta}{Y^3}\sum_{k=1}^{\infty}\mathcal{F}^{(k)}\left(K^{(k)}\right)^2\ell_i^{(k)} , \\
\overset{iso}{E}_{44} &= \frac{1}{Y^4}\sum_{k=1}^{\infty}\mathcal{F}^{(k)}\left(K^{(k)}\right)^2 ,
\end{aligned}\right\} \qquad (4.508)$$

where once again $i \neq m = 1, 2, 3$. It must be emphasised, however, that the infinite sum is only a notational device to express the notion of adding over all directions to obtain the desired isotropic field. The sum itself cannot contain the necessary information to uniquely evaluate the contributions made by the summands. In order to do this, it is necessary to express the infinite summations of equations (4.508) as integrals

From the eikonal equation, it is known that

$$\sum_{i=1}^{3} \left(\ell_i^{(k)} \right)^2 = 1 ,$$

for each value of k, where the $\ell_i^{(k)}$, $i = 1, 2, 3$, are given by

$$\ell_i^{(k)} = \cos \alpha_i^{(k)} . \tag{4.509}$$

Here $\alpha_i^{(k)}$ is the angle subtended by the direction of propagation of the wave $h_{\mu\nu}^{(k)}$ and the x^i axis, where the coordinate identifications employed in equation (4.301) are adopted.

Now consider one wave component of the total field which, locally and in the 3-space τ = constant, has its wave front normal passing through the coordinate origin. In order to establish an integral representation of equations (4.508), the polar angles ϖ, φ are introduced in Figure 1.

Each value of k is assigned to one of an infinity of different directions of radiative flow. So, as k varies, the continuum variables ϖ, φ change accordingly.

The propagation vector considered in Figure 1 corresponds to a wave component $h_{\mu\nu}^{(k)}$, for a particular value of k. Thus the superscript (k) is suppressed for convenience. From equation (4.509), the direction cosines of the propagation vector may be expressed in terms of the polar

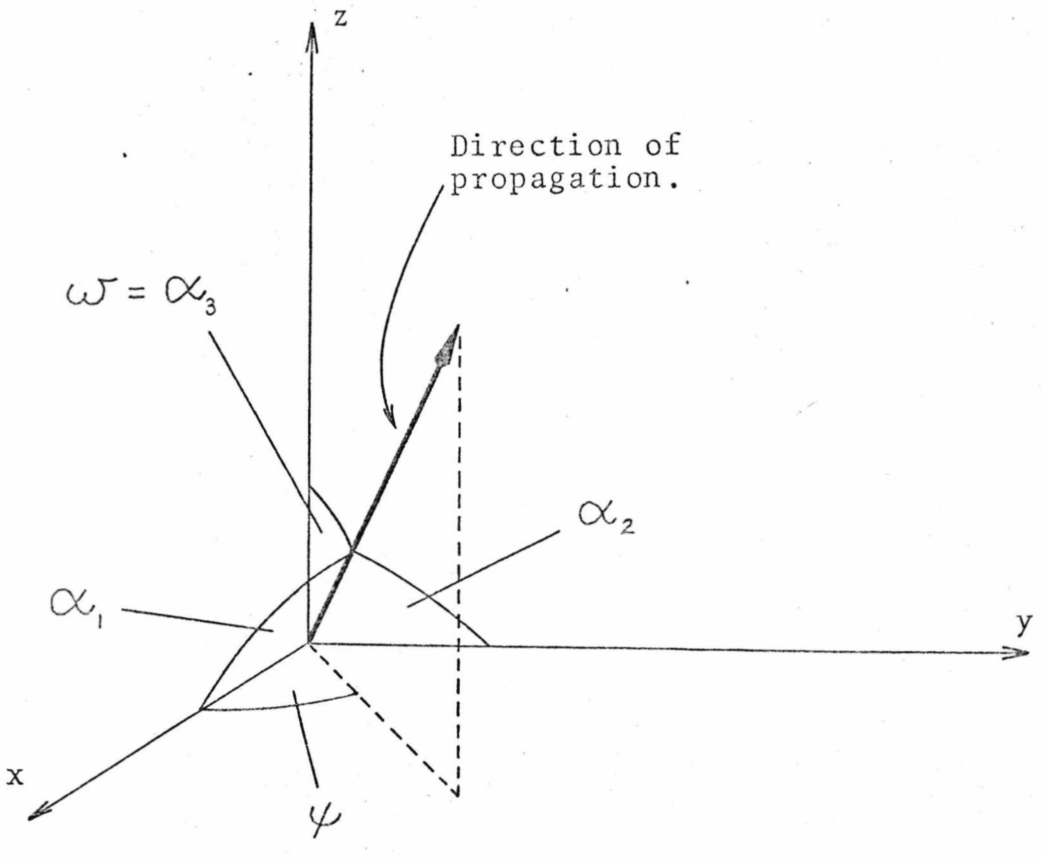
z
Direction of
propagation.
$\omega = \alpha_3$
α_2
α_1
y
x
ψ

Fig. 1.

angles ω, ψ as follows,

$$\begin{aligned} l_1 &= \cos\alpha_1 = \sin\omega\cos\psi, \\ l_2 &= \cos\alpha_2 = \sin\omega\sin\psi, \\ l_3 &= \cos\alpha_3 = \cos\omega. \end{aligned} \tag{4.510}$$

Finally, before evaluating the components of $E_{\mu\nu}^{iso}$, consider the quantities ~~quanties~~ $\mathcal{F}^{(k)}$, $K^{(k)}$ appearing in the equations (4.508). From Section 4.3, $\mathcal{F}^{(k)}$ and $K^{(k)}$ are, of course, constants for any one particular value of k. However, the assignment of superscripts to these constants implies that the values they may take are allowed to change, as k varies, or equivalently, as the direction of radiative flow changes. That is

$$\mathcal{F}^{(k)} = \mathcal{F}(\omega, \psi), \quad K^{(k)} = K(\omega, \psi).$$

In the following, however, it will be supposed that

$$\mathcal{F}^{(k)}(K^{(k)})^2 = \hat{\mathcal{L}}^2 = constant, \tag{4.511}$$

for all k.

The components of $E_{\mu\nu}^{iso}$ may now be determined by expressing the summations in equations (4.508) as integrals, and then integrating over all directions. Firstly, consider E_{11}^{iso}, given by

$$E_{11}^{iso} = \frac{1}{Y^2}\sum_{k=1}^{\infty} \mathcal{F}^{(k)}(K^{(k)} l_1^{(k)})^2.$$

If we employ the equations (4.510) and (4.511), this may be written

$$E_{11}^{iso} \equiv \frac{1}{Y^2}\int_0^{\pi}\int_0^{2\pi}\hat{\mathcal{L}}^2 \sin^2\omega \cos^2\psi\,(\sin\omega\, d\psi\, d\omega)$$

$$= \frac{\hat{\mathcal{L}}^2}{Y^2}\int_0^{\pi} \sin^3\omega \left(\int_0^{2\pi} \cos^2\psi\, d\psi\right) d\omega$$

$$= \frac{\pi\hat{\mathcal{L}}^2}{Y^2}\int_0^{\pi} \sin^3\omega\, d\omega = \frac{4\pi\hat{\mathcal{L}}^2}{3Y^2}.$$

Proceeding in this way, we may express E_{22}^{iso} as

$$E_{22}^{iso} \equiv \frac{1}{Y^2}\int_0^{\pi}\int_0^{2\pi}\hat{\mathcal{L}}^2 \sin^2\omega \sin^2\psi\,(\sin\omega\, d\psi\, d\omega)$$

$$= \frac{\hat{\mathcal{L}}^2}{Y^2}\int_0^{\pi} \sin^3\omega \left(\int_0^{2\pi} \sin^2\psi\, d\psi\right) d\omega$$

$$= \frac{\pi\hat{\mathcal{L}}^2}{Y^2}\int_0^{\pi} \sin^3\omega\, d\omega = \frac{4\pi\hat{\mathcal{L}}^2}{3Y^2}.$$

Similarly

$$E_{33}^{iso} \equiv \frac{1}{Y^2}\int_0^{\pi}\int_0^{2\pi}\hat{\mathcal{L}}^2 \cos^2\omega\,(\sin\omega\, d\psi\, d\omega)$$

$$= \frac{2\pi\hat{\mathcal{L}}^2}{Y^2}\int_0^{\pi} \cos^2\omega \sin\omega\, d\omega = \frac{4\pi\hat{\mathcal{L}}^2}{3Y^2}.$$

The remaining diagonal component E_{44}^{iso} may be evaluated in a similar fashion,

$$E_{44}^{iso} \equiv \frac{1}{Y^4}\int_0^{\pi}\int_0^{2\pi}\hat{\mathcal{L}}^2 (\sin\omega\, d\psi\, d\omega)$$

$$= \frac{2\pi\hat{\mathcal{L}}^2}{Y^4}\int_0^{\pi} \sin\omega\, d\omega = \frac{4\pi\hat{\mathcal{L}}^2}{Y^4}.$$

To complete the calculation, the off-diagonal components of $E_{\mu\nu}^{iso}$ are given by

$$\overset{iso}{E}_{12} \equiv \frac{1}{Y^2}\int_0^{\pi}\int_0^{2\pi} \hat{\mathcal{L}}^2 \sin^3\omega \sin\psi \cos\psi \, d\psi d\omega = 0 ,$$

$$\overset{iso}{E}_{13} \equiv \frac{1}{Y^2}\int_0^{\pi}\int_0^{2\pi} \hat{\mathcal{L}}^2 \sin^2\omega \cos\omega \cos\psi \, d\psi d\omega = 0 ,$$

$$\overset{iso}{E}_{23} \equiv \frac{1}{Y^2}\int_0^{\pi}\int_0^{2\pi} \hat{\mathcal{L}}^2 \sin^2\omega \cos\omega \sin\psi \, d\psi d\omega = 0 ,$$

$$\overset{iso}{E}_{14} \equiv \frac{\theta}{Y^3}\int_0^{\pi}\int_0^{2\pi} \hat{\mathcal{L}}^2 \sin^2\omega \cos\psi \, d\psi d\omega = 0 ,$$

$$\overset{iso}{E}_{24} \equiv \frac{\theta}{Y^3}\int_0^{\pi}\int_0^{2\pi} \hat{\mathcal{L}}^2 \sin^2\omega \sin\psi \, d\psi d\omega = 0 ,$$

$$\overset{iso}{E}_{34} \equiv \frac{\theta}{Y^3}\int_0^{\pi}\int_0^{2\pi} \hat{\mathcal{L}}^2 \sin\omega \cos\omega \, d\psi d\omega = 0 ,$$

where $\overset{iso}{E}_{\mu\nu} = \overset{iso}{E}_{\nu\mu}$. Thus the energy tensor of an isotropic gravitational radiation field in a Friedmann universe may be represented by the following 4 × 4 array,

$$\overset{iso}{E}_{\mu\nu} = \mathcal{L}^2 \begin{pmatrix} \frac{1}{3Y^2} & 0 & 0 & 0 \\ 0 & \frac{1}{3Y^2} & 0 & 0 \\ 0 & 0 & \frac{1}{3Y^2} & 0 \\ 0 & 0 & 0 & \frac{1}{Y^4} \end{pmatrix} , \qquad (4.512)$$

where

$$\mathcal{L}^2 = 4\pi \hat{\mathcal{L}}^2 = \text{constant}.$$

From equations (4.404), we may write

$$\overset{iso}{E}{}^1{}_1 \equiv -P_{xx} , \qquad \overset{iso}{E}{}^2{}_2 \equiv -P_{yy} ,$$

$$\overset{iso}{E}{}^3{}_3 \equiv -P_{zz} , \qquad \overset{iso}{E}{}^4{}_4 \equiv \rho_r .$$

By inspection of equation (4.512), the pressure components for the isotropic field tensor satisfy

$$p_{xx} = p_{yy} = p_{zz} \equiv p_r ,$$

so that the components $\overset{iso}{E}{}^{\alpha}{}_{\beta}$ may be written

$$\overset{iso}{E}{}^{i}{}_{i} = -\frac{\mathcal{L}^2}{3Y^4} \equiv -p_r ,$$
$$\overset{iso}{E}{}^{4}{}_{4} = \frac{\mathcal{L}^2}{Y^4} \equiv \rho_r , \quad i = 1,2,3, \tag{4.513}$$

where no summation is implied by repeated latin indices.
Moreover, from the equations (4.513) it is found that

$$\overset{iso}{E}{}^{\alpha}{}_{\alpha} = 0 ,$$

which implies that the radiation pressure p_r of the isotropic field is related to the radiation energy density ρ_r in the following manner

$$p_r = \frac{1}{3}\rho_r . \tag{4.514}$$

4.6 Properties of an isotropic gravitational radiation field in a Friedmann universe.

Now that we have the results (4.512) and (4.514) the equivalence of $\overset{iso}{E}_{\mu\nu}$ and the energy tensor of a perfect fluid distribution is easily demonstrated. Let the perfect fluid energy tensor be denoted by

$$T^{*}_{\mu\nu} = (p^{*}+\rho^{*})u^{*}_{\mu}u^{*}_{\nu} - \gamma_{\mu\nu}p^{*}, \tag{4.601}$$

where $\gamma_{\mu\nu}$ is given by equations (4.206), and where p^{*}, ρ^{*} and u^{*}_{μ} represent the pressure, density and energy flow vector respectively of the medium described by $T^{*}_{\mu\nu}$. This is not to be confused with equation (3.308). The components of $T^{*}_{\mu\nu}$ are given by

$$\left.\begin{aligned} T^*_{ii} &= (p^*+\rho^*)(u^*_i)^2 + Y^2 p^*, \\ T^*_{44} &= (p^*+\rho^*)(u^*_4)^2 - p^*, \\ T^*_{ij} &= (p^*+\rho^*)\, u^*_i\, u^*_j, \\ T^*_{i4} &= (p^*+\rho^*)\, u^*_i\, u^*_4, \end{aligned}\right\} \tag{4.602}$$

where $i, j = 1, 2, 3$ and $i \neq j$. If the components of $E^{iso}_{\mu\nu}$ given by equation (4.512) are equated to the corresponding components of $T^*_{\mu\nu}$, the following relationships result,

$$(u^*_i)^2 = \left\{ \frac{\mathcal{L}^2}{3Y^2} - Y^2 p^* \right\} (p^*+\rho^*)^{-1}, \tag{4.603}$$

$$(u^*_4)^2 = \left\{ \frac{\mathcal{L}^2}{Y^4} + p^* \right\} (p^*+\rho^*)^{-1}, \tag{4.604}$$

$$u^*_i\, u^*_j = 0, \tag{4.605}$$

$$u^*_i\, u^*_4 = 0, \tag{4.606}$$

where again $i, j = 1, 2, 3$ and $i \neq j$. Thus, if we let

$$p^* \equiv p_r = \frac{\mathcal{L}^2}{3Y^4}, \tag{4.607}$$

then equation (4.603) is satisfied provided that $u^*_i = 0$, $i = 1, 2, 3$. This result also ensures that equations (4.605) and (4.606) are satisfied. From equation (4.607), the remaining equation (4.604) becomes

$$(u^*_4)^2 = \frac{4\mathcal{L}^2}{3Y^4} (p_r + \rho^*)^{-1}. \tag{4.608}$$

Finally if we set

$$\rho^{*} \equiv \rho_r = \frac{\mathcal{L}^2}{Y^4} ,$$

then equation (4.608) gives

$$(u_4^*)^2 = 1 .$$

Thus, the energy representation given by equation (4.601) may be shown to be equivalent to $E_{\mu\nu}^{iso}$ provided a comoving coordinate system is chosen in which

$$u_4^* = 1 , \quad u_i^* = 0 , \quad i = 1, 2, 3. \qquad (4.609)$$

This implies that the coordinates are adjusted to ensure that the 'spherical' surface r = constant moves with the radiative energy distribution as it expands in the cosmological model. Thus the energy of an isotropic gravitational radiation field may be described by a perfect fluid energy tensor in a comoving system, where the equation of state is given by equation (4.514).

A further property of the radiation field may be demonstrated by consideration of the gravitational 'Poynting vector' proposed by Isaacson (1968b),

$$P^{\alpha} = (\delta^{\alpha}_{\mu} - u^{(ob)\alpha} u^{(ob)}_{\mu}) E^{\mu\nu} u^{(ob)}_{\nu} . \qquad (4.610)$$

It is introduced by analogy with the Poynting vector of electromagnetism. Here P^{α} describes the flow of gravitational energy measured by an observer with a time-like 4-vector $u^{(ob)}_{\mu}$. If $E^{\mu\nu}$ in equation (4.610) is given by equation (4.512) and the observer is regarded as participating in the universal expansion, that is $u^{(ob)}_{\mu} = u^{*}_{\mu}$, then

$$P^4 = (\overset{iso}{E}{}^{44} - \overset{iso}{E}{}^{44}) = 0 ,$$

$$P^i = \overset{iso}{E}{}^{i4} = 0 \ , \quad i = 1, 2, 3 .$$

Therefore at each comoving point there is no net energy transfer from one region of the model to another, a property to be expected of an isotropic cosmological radiation field.

The results obtained in this Chapter have been derived using a Friedmann background specified by $K = 0$, and apply only to free radiation since no interaction occurs between the radiation and the perfect fluid medium in this approximation, as shown in Section 3.3.

However, the conclusions drawn will be regarded as sufficient justification for the Isaacson and Winicour hypothesis.

Chapter 5.

Gravitational radiation in cosmology: a description of the effects of an isotropic gravitational radiation field upon a Friedmann universe.

5.1 The cosmological field equations.

In the preceeding Chapter a study of a cosmological gravitational radiation field has been attempted by consideration of the equations (3.352) and (3.356). Now we undertake an investigation of the effects that such a radiation field may have upon the cosmological model containing it. However with our present meagre knowledge of the mechanisms by which gravitational radiation is generated, we should point out that this analysis must necessarily be of an exploratory nature.

The manner in which the background curvature of space-time manifold is influenced by its material energy content $T^{(0)}_{\mu\nu}$, and by its radiative energy content $E^{\text{eff}}_{\mu\nu}$, is described by the equation (3.214), which is rewritten as

$$\mathcal{G}^{(0)}_{\mu\nu} = -8\pi G\left(T^{(0)}_{\mu\nu} + E^{\text{eff}}_{\mu\nu}\right).$$

Here $\mathcal{G}^{(0)}_{\mu\nu}$ is defined by the equations (3.207) and (2.405). It is convenient to abandon the dimensionless treatment adopted in Section 2.2. Henceforth natural c.g.s. units will be employed so that the Newtonian gravitational constant G, and the local velocity of light c are given by

$$G = 6.668 \times 10^{-8}\ \text{cm}^3\ \text{gm}^{-1}\ \text{sec}^{-2},$$
$$c = 2.998 \times 10^{10}\ \text{cm sec}^{-1}.$$

To discover the appropriate form of equation (3.214), it is necessary to discuss briefly what representations of the energy sources $T^{(0)}_{\mu\nu}$ and $E_{\mu\nu}^{eff}$, and of the background metric $\gamma_{\mu\nu}$, are suitable in a cosmological model.

The matter content will again be regarded as a perfect fluid given by

$$T^{(0)}_{\mu\nu} = \left(\rho_m + \frac{p_m}{c^2}\right) u^*_\mu u^*_\nu - \gamma_{\mu\nu} \frac{p_m}{c^2} , \qquad (5.101)$$

where u^*_μ , ρ_m and p_m are the 4-velocity, density and pressure respectively of the material medium.

Now we consider the radiative energy content of the model. From Section 2.8 we know that it is the averaged effective field energy $E_{\mu\nu} \equiv \langle E_{\mu\nu}^{eff} \rangle$ which is physically significant by virtue of its invariance properties in the high frequency limit. Therefore, the averaging technique defined by the equations (2.807) and (2.808) is applied to equation (3.214) to give

$$\langle \mathcal{G}^{(0)}_{\mu\nu} \rangle = -8\pi G \langle T^{(0)}_{\mu\nu} + E_{\mu\nu}^{eff} \rangle .$$

Since this average is taken effectively over a locally Euclidean region $\mathcal{D}$ of the manifold in which the background quantities $\gamma_{\mu\nu}$, u^*_μ , ρ_m and p_m remain approximately constant, the above equation may be written as

$$\mathcal{G}^{(0)}_{\mu\nu} = -8\pi G \left(T^{(0)}_{\mu\nu} + \langle E_{\mu\nu}^{eff} \rangle \right) ,$$

or

$$\mathcal{G}^{(0)}_{\mu\nu} = -8\pi G\left(T^{(0)}_{\mu\nu} + E^{iso}_{\mu\nu}\right), \qquad (5.102)$$

if the gravitational radiation field is isotropic. The form of $E^{iso}_{\mu\nu}$ for an isotropic field of free radiation in a Friedmann universe has been investigated for the case $K = 0$ in Chapter 4, where K is the space curvature constant. There it was possible to demonstrate that $E^{iso}_{\mu\nu}$ could be represented by a perfect fluid energy tensor in a comoving system, with the equations (4.513) and (4.514) describing the radiation energy density ρ_r, and the radiation pressure p_r. In what follows it is assumed that an isotropic gravitational radiation field which actively interacts with the matter content of a Friedmann model with K = -1, 0 or +1 may also be described by a perfect fluid in a comoving system. That is

$$E^{iso}_{\mu\nu} = \left(\rho_r + \frac{p_r}{c^2}\right)u^*_\mu u^*_\nu - \gamma_{\mu\nu}\frac{p_r}{c^2}, \qquad (5.103)$$

where the flow 4-vectors u^*_μ of the radiation energy distribution, given by the equations (4.609), correspond to the same comoving system as that which describes the material flow. For an interacting radiation field the $\rho_r \propto R^{-4}$ relationship is abandoned for reasons which will become evident later, whereas it is assumed that the equation of state given by

$$p_r = \frac{1}{3}\rho_r c^2, \qquad (5.104)$$

is preserved.

For a general value of the space curvature constant

K, the background metric $\gamma_{\mu\nu}$ representing the cosmological model is given by equation (4.202). In terms of spherical polar coordinates r, ω, ψ and cosmic time t, this becomes

$$da^2 = dt^2 - \frac{R^2(t)}{c^2}\left\{\frac{dr^2 + r^2 d\omega^2 + r^2 \sin^2\omega \, d\psi^2}{(1 + Kr^2/4)^2}\right\}, \qquad (5.105)$$

where the mean motion of energy in the model is given by the space-time geodesics r = constant, ω = constant, ψ = constant. Substitution of the equations (5.101), (5.103) and (5.105) into the equation (5.102) gives the cosmological field equations,

$$\frac{8\pi G}{3}(\rho_m + \rho_r) = \left(\frac{\dot{R}}{R}\right)^2 + \frac{Kc^2}{R^2}, \qquad (5.106)$$

$$\frac{8\pi G}{c^2} p_r = -\frac{2\ddot{R}}{R} - \frac{Kc^2}{R^2} - \left(\frac{\dot{R}}{R}\right)^2, \qquad (5.107)$$

relating the scale factor $R(t)$, to the energy content of the model. Here $\dot{R} = dR/dt$ and the conditions $p_m \approx 0$ and $\Lambda = 0$ are assumed, where Λ is the cosmical constant. If equation (5.106) is multiplied by R^3, differentiated with respect to time and then combined with equation (5.107), it is found that

$$\frac{d}{dt}\left\{(\rho_m + \rho_r)R^3\right\} + \frac{p_r}{c^2}\frac{dR^3}{dt} = 0. \qquad (5.108)$$

This will be referred to as the 'conservation of energy' equation.

We now introduce four parameters which will be useful in the subsequent analysis of the equations (5.106) and (5.107). Let $t = t_i$ be some preassigned instant in cosmic time. The Hubble parameter at time $t = t_i$ is defined as follows,

$$H_i = \frac{\dot{R}(t_i)}{R(t_i)} \equiv \frac{\dot{R}_i}{R_i} , \tag{5.109}$$

where H_i has the dimensions of sec^{-1}. If $\dot{R}_i > 0$, the 'Hubble time' H_i^{-1} may be regarded as an approximate value of the age of the cosmological model at the instant $t = t_i$.

The acceleration parameter at time $t = t_i$ is given by

$$q_i = -\frac{\ddot{R}_i}{R_i}\left(\frac{R_i}{\dot{R}_i}\right)^2 , \tag{5.110}$$

where q_i is a pure number. Thus if

$$q_i > 0 , \tag{5.111}$$

then the expansion of the model is decelerating at time $t = t_i$, whereas

$$q_i < 0 ,$$

implies an acceleration at time $t = t_i$.

Finally, it is convenient to introduce the following density parameters,

$$\sigma_{ri} = \frac{4\pi G}{3H_i^2} \rho_{ri} , \tag{5.112}$$

$$\sigma_{mi} = \frac{4\pi G}{3H_i^2} \rho_{mi} , \tag{5.113}$$

where

$$\rho_{ri} = \rho_r(t_i) , \quad \rho_{mi} = \rho_m(t_i).$$

5.2 On the origin of gravitational radiation in a Friedmann universe.

If the total mean density of the cosmological model is denoted by

$$\rho = \rho_m + \rho_r ,$$

then at the present epoch $t = t_0$, the equation (5.106) may be written

$$\frac{8\pi G}{3}\rho_0 = \left(\frac{\dot{R}_0}{R_0}\right)^2 + \frac{Kc^2}{R_0^2} , \tag{5.201}$$

where a subscript zero implies that the variable to which it is assigned is evaluated at $t = t_0$. Let the total density parameter at the present time be given by

$$\sigma_0 = \frac{4\pi G}{3H_0^2}\rho_0 = \sigma_{m0} + \sigma_{r0} , \tag{5.202}$$

where H_0 is the Hubble parameter defined by equation (5.109), evaluated at time $t = t_0$. Rearranging equation (5.201), we find that the sign of K in the model is determined by the present value of the total density ρ_0 since

$$K = \left(\frac{R_0 H_0}{c}\right)^2 (2\sigma_0 - 1) . \tag{5.203}$$

Thus, if the density parameter satisfies $\sigma_0 \leq \frac{1}{2}$, the model will be an open one in which the expansion phase continues indefinitely. On the other hand, if the density parameter is such that $\sigma_0 > \frac{1}{2}$ the resulting model will be closed, and will recontract after reaching some maximum 'radius'. The critical value that corresponds to the $K = 0$ case is given by

$$\sigma_0 = \frac{1}{2} .$$

If a Hubble time of $H_0^{-1} = 2 \times 10^{10}$ years (Sandage and Tammann, 1974) is employed, this implies that the critical density is

$$\rho(\text{critical}) = \frac{3H_0^2}{8\pi G} \approx 4 \times 10^{-30}\ gm\ cm^{-3}. \qquad (5.204)$$

In an effort to determine whether the observed mean density parameter of the Universe is larger or smaller than the critical value, Oort (1958) and Shapiro (1971) among others, have estimated its value for luminous matter with the result that

$$\bar{\sigma}_0(\text{galaxies}) \approx 0.01 .$$

In a more recent study however, Field (1972) concludes that the density parameter of inter-galactic matter could be as high as

$$\bar{\sigma}_0(\text{I.M.}) \approx 0.5 ,$$

where

$$\sigma_{mo} = \bar{\sigma}_0(\text{galaxies}) + \bar{\sigma}_0(\text{I.M.}).$$

Further, the density parameter of the electromagnetic radiation content of the Universe has been estimated to be of the order 10^{-4}. Throughout the remainder of this discussion however electromagnetic radiation will not be considered.

Thus, the evidence from observations of the galactic material content of the Universe alone suggests that

$$\sigma_0 < \frac{1}{2} ,$$

which implies, from equation (5.203), that $K = -1$.

At this point in the discussion however the so-called 'missing mass' problem arises. This is adequately summarised by Partridge (1969) when he comments that "some cosmologies feel that the Universe is, of philosophically ought to be, closed". Those who subscribe to this idea argue that there must therefore be 'missing matter' in amounts sufficient to bring σ_0 up to a value greater than the critical value, to ensure that $K = +1$. The notion that a gravitational radiation field may pervade the cosmos has from time to time been regarded as a way to resolve this 'problem', since the radiation field may be attributed with an energy density sufficient to ensure that $\sigma_0 > \frac{1}{2}$.

Kafka (1970), Bertotti and Cavaliere (1971) and Isaacson and Winicour (1972, 1973) have presented studies of the cosmological effects of gravitational radiation. In each of these, a large radiation energy density has been adopted although not for any reason related to the missing mass enigma. During the early 1970's theoretical models of this kind were influenced by the only observational evidence then available, that obtained by Weber (1967, 1969, 1970) from his system of two radiation detectors, one sited at the University of Maryland and the other 1000 kilometers away in Chicago. He had reported strong bursts of gravitational radiation which appeared to be eminating from the Galactic Centre, and which suggested that the Galaxy was being converted into gravitational waves at a rate of approximately $10^2 M_\odot \, yr^{-1}$. Thus, as well as accounting for the common use of large radiation density parameters in models, Weber's results also provided stimulus to the hypothesis that the postulated Universal gravitational radiation field had been generated by processes occuring within galaxies. We will henceforth refer to this as the 'galactic hypothesis'. If the radiation field was created in this way, contributions of the radiative energy to the evolution of the Universe can be significant only during the galactic era. We will have reason to consider the galactic hypothesis in more detail later.

An alternative to the galactic hypothesis is that of the 'cosmological hypothesis', which supposes that the gravitational radiation field was created during the 'fireball' phase of the Universe and has thus had an influence upon the cosmological evolution ever since. This alternative hypothesis has, however, been forcibly argued against by Kafka in light of the large radiation density parameter suggested by Weber's experiment. Adopting the cosmological hypothesis, and assuming that $\sigma_{r0} = 5$ and $\Lambda = 0$, he assessed the present age of the resulting cosmological model to be $0{\cdot}24\, H_0^{-1}$. This corresponds to an age of approximately 4.8×10^9 years which Kafka concluded was an uncomfortably short estimate. In further defence of the cosmological hypothesis, he then invoked the possibility that $\Lambda \neq 0$ in order to find models with ages compatible with other dating methods. However, an investigation of these models uncovered further difficulties in that they exhibited antipoles at redshifts $z < 0{\cdot}5$. Assuming the large radiation density suggested by Weber's observations, Kafka found no way to reconcile the cosmological hypothesis with models possessing suitable age estimates, and therefore concluded "that a cosmological origin of gravitational radiation should be abandoned". This gave a further, theoretical, impetus to the galactic hypothesis which was later persued by Bertotti and Cavaliere, and Isaacson and Winicour.

The present observational climate suggests that Kafka's estimate of σ_{ro} was undu~~l~~ly large. Many experiments in gravitational radiation detection have been attempted since Weber's pioneering efforts, with instruments of greater sensitivity, and as yet Weber's results have not been confirmed. In fact, null results have been reported, see for example Tyson (1973). Hence if we abandon Kafka's premise that $\sigma_{ro} = 5$, his objections to the cosmological origin of gravitational radiation are overcome, and the question as to whether we adopt the cosmological hypothesis or the galactic hypothesis is again an open one.

In what follows both possibilities will be considered, although the galactic hypothesis will receive the greater attention.

5.3 Some remarks on the Isaacson and Winicour model of the Universe.

The numerical analysis presented by Isaacson and Winicour (IW) in their 1973 paper was a thorough investigation of the consequences of the galactic hypothesis in a cosmological model for the range of cosmic time up to the present epoch. Although the results of such a treatment will depend to some extent upon the way in which the galactic hypothesis is expressed mathematically, the initial assumptions of IW provide a sufficiently general formulation of the problem in light of the present ignorance of the rate at which radiative energy might accumulate in the Universe. To specify the 'profile' of the radiation energy during the expansion, IW chose the following functional dependence of ρ_r upon time,

$$\rho_r = \rho_{ro}\left(\frac{R}{R_o}\right)^{\nu-4}, \qquad (5.301)$$

where the index ν is a constant, and where ρ_{ro} is the present gravitational radiation energy density. This in itself is not, however, a statement of the galactic hypothesis. If we define

$$E_r(t) = \rho_r R^3, \quad E_m(t) = \rho_m R^3, \qquad (5.302)$$

to be the radiation energy content and the matter energy content of the model respectively, then the galactic hypothesis may be given expression by writing

$$E_r \to 0 \text{ as } R \to 0. \qquad (5.303)$$

That is, the amount of energy present due to the radiation at the initial instant is zero.

The equations (5.301) and (5.303) provide the basis of the IW model. The second of these imposes a restriction upon the value of ν in equation (5.301) given by

$$\nu > 1, \qquad (5.304)$$

which may be regarded as a statement of the galactic hypothesis in these models.

What is the physical significance of assigning a value to the parameter ν outside this range? To answer this we consider a mass M of the material content of the model, which is contained within a coordinate volume V. Here V is a time independent quantity, once the values of r, ϖ, ψ defining its boundary are specified. The physical volume v corresponding to V is given by

$$v(t) = R^3(t)\,V .$$

If a situation is envisaged in which matter energy does not undergo conversion into any other form of energy, then V will always contain a mass M of material. Thus, consideration of the volume v gives the density of matter as a function of time,

$$\rho_m(t) = \frac{M}{v(t)} = \frac{M}{R^3(t)V} .$$

Hence,

$$\rho_m R^3 = \frac{M}{V} = \text{constant}. \tag{5.305}$$

The substitution of equation (5.104) into equation (5.108) modifies the conservation of energy equation to give

$$\frac{d}{dt}\left(\rho_m R^3\right) + \frac{1}{R}\frac{d}{dt}\left(\rho_r R^4\right) = 0. \tag{5.306}$$

If equation (5.305) is now substituted into this, then

$$\rho_r R^4 = \text{constant}, \tag{5.307}$$

which, from equation (4.513), is the time dependence of the energy density of a 'free' isotropic gravitational radiation field. Comparing equation (5.301) with (5.307) we may conclude that the value $\nu = 0$ corresponds to a model in which there is no conversion of matter into radiation, and in which radiation contributes to the energy content throughout its entire history. Moreover, restricting ν to the range

$$0 \leqslant \nu \leqslant 1 , \tag{5.308}$$

in equation (5.301) provides a condition representing the imposition of the cosmological hypothesis in an IW universe, since the statement (5.303) is not satisfied.

Following IW, in the remainder of this chapter we will restrict our attention to the galactic hypothesis. That is, ν is given by (5.304).

Some information regarding the ultimate fate of an IW model universe may be obtained by consideration of equation (5.306). If equation (5.301) is introduced into equation (5.306) it is found that

$$\frac{d}{dt}(\rho_m R^3) + \frac{1}{R}\frac{d}{dt}\left\{\rho_{ro}\left(\frac{R}{R_o}\right)^{\nu-4} R^4\right\} = 0 ,$$

which becomes

$$\frac{d}{dt}(\rho_m R^3) + \frac{\rho_{ro}}{R_o^{\nu-4}}\left(\frac{\nu}{\nu-1}\right)\frac{dR^{\nu-1}}{dt} = 0 .$$

This may be integrated to give

$$\rho_m R^3 + \frac{\rho_{ro}}{R_o^{\nu-4}}\left(\frac{\nu}{\nu-1}\right) R^{\nu-1} = R_o^3 \beta_o ,$$

where β_o is a constant of integration given by

$$\beta_o = \rho_{mo} + \left(\frac{\nu}{\nu-1}\right)\rho_{ro} .$$

Thus

$$\rho_m = \left(\frac{R_o}{R}\right)^3\left\{\rho_{mo} + \rho_{ro}\left(\frac{\nu}{\nu-1}\right)\left[1-\left(\frac{R}{R_o}\right)^{\nu-1}\right]\right\} . \tag{5.309}$$

It follows from this equation that the continued generation of the radiation at a rate given by equation (5.301) could lead to a negative matter density in the model. To see this, we observe that there exists an instant $t = t_c$ in cosmic time at which the matter density ρ_m falls to zero, provided that the scale factor satisfies

$$\dot{R}(t) > 0 \; , \quad t_0 \leqslant t \leqslant t_c \; ,$$

and given that both ρ_{mo}, ρ_{ro} are positive. This type of scale factor behaviour will certainly be exhibited by models in which $K = 0, -1$, as well as in some $K = +1$ cases. From equation (5.309), the value of $R(t_c) \equiv R_c$ is given by

$$R_c = R_0 \left\{ \frac{(\nu - 1)\rho_{mo} + \nu \rho_{ro}}{\nu \rho_{ro}} \right\}^{\frac{1}{\nu - 1}} . \tag{5.310}$$

By the definition of the radiation energy density (5.301), 'generation' will continue in the IW model for values of $R > R_c$ and, provided that R continues to increase, equation (5.309) implies that

$$\rho_m(t) < 0 \; , \quad t > t_c \; . \tag{5.311}$$

So, although the model has proved to be an adequate description of the situation examined by IW, its final fate under certain circumstances is one of physical unacceptability.

If such a generation of gravitational radiation was occuring in the physical Universe, a situation characterized by equation (5.311) would be unlikely to arise in any case, since it would be difficult to conceive of any mechanism capable of 'driving' the generation as the value of the density of matter approached zero. In an attempt to account for this, and to overcome the problems outlined above, the following scheme is proposed to produce a model containing gravitational radiation which is physically acceptable for all values of the cosmic time.

We will envisage a point source cosmological model in which the condition (5.303) is satified. As the model expands, the radiation energy density will be described by a 'profile' equation of the form (5.301) until some moment $t = t_1 < t_c$ is reached. At this instant it is supposed that the generation of radiation ceases, and this is followed by a period of time $t > t_1$ during which the remaining matter and the radiation coexist in a noninteracting state. In IW models which formally suffered from problems of the type expressed in equation (5.311), in particular the $K = 0, -1$ cases, the instant $t = t_1$ would presumeably correspond to an epoch in the model's history by which time much of its material content had finally settled into 'dead' condensations - dark dwarfs, collapsed stars, perhaps even collapsed galaxies. A schematic representation of the proposed model is given in Figure 2.

Since the energy distribution of the proposed model is different on either side of the time boundary $t = t_1$, correspondingly the model's behaviour on either side of this boundary is described by different solutions of the cosmological equations (5.106) and (5.306). That is, the Universe is represented by two different Friedmann models, one describing the 'generation era' up until $t = t_1$ and

$R(0) = 0.$ $E_r(0) = 0.$	FORMATION OF THE GALAXIES. THE MATERIAL CONTENT OF THE MODEL IS CONVERTED TO GRAVITATIONAL RADIATION, AS IN IW UNIVERSES.	TIME AT WHICH GENERATION CEASES. $\downarrow$	MATTER AND RADIATION COEXIST IN NONINTERACTING STATE.
$t = 0.$	$t = t_o.$	$t = t_1 < t_c.$	COSMIC TIME $\longrightarrow$

Fig. 2. The proposed model.

the other describing the 'noninteraction era' thereafter. Thus, before we may begin to construct the model universe outlined in Figure 2, it is necessary to discuss the continuity conditions required to 'mate' the two Friedmann models at $t = t_1$.

5.4 Continuity of two Friedmann models at any instant $t = t_1$.

Let the region of space-time bounded by $0 \leqslant t < t_1$ be denoted by S, and let the region for which $t > t_1$ be denoted by $\tilde{S}$. We will suppose that the energy distribution that specifies the geometry of S differs from that of $\tilde{S}$, and that both S and $\tilde{S}$ possess the properties of homogeneity and isotropy. Thus, in S, the space-time interval may be expressed as

$$da^2 = dt^2 - \frac{R^2(t)}{c^2}\left\{\frac{dr^2 + r^2 d\omega^2 + r^2 \sin^2\omega \, d\psi^2}{(1 + Kr^2/4)^2}\right\}, \tag{5.401}$$

where it will be supposed that $\rho(t)$, $p(t)$ represent the total density and pressure respectively for $0 \leqslant t < t_1$. Similarly, the metric in $\tilde{S}$ will be given by

$$d\tilde{a}^2 = dt^2 - \frac{\tilde{R}^2(t)}{\tilde{c}^2}\left\{\frac{d\tilde{r}^2 + \tilde{r}^2 d\tilde{\omega}^2 + \tilde{r}^2 \sin^2\tilde{\omega} \, d\tilde{\psi}^2}{(1 + \tilde{K}\tilde{r}^2/4)^2}\right\}, \tag{5.402}$$

where the total density and pressure in $\tilde{S}$ will be denoted by $\tilde{\rho}(t)$, $\tilde{p}(t)$.

It is assumed that, in equations (5.401) and (5.402), the cosmic time t is continuous. In the 'interconnecting' three dimensional space $t = t_1$, which we shall call S_1, it will be supposed that each point may be uniquely specified by three spatial coordinates r_1, ϖ_1, ψ_1. In addition, the space curvature constant and the fundamental velocity will be required to satisfy

$$c = \tilde{c}, \quad K = \tilde{K}. \tag{5.403}$$

We now consider a comoving observer O in the space S at the point $P^{(0)}(r^{(0)}, \varpi^{(0)}, \psi^{(0)}, t^{(0)})$. As t increases, the coordinates of O will be given by $(r^{(0)}, \varpi^{(0)}, \psi^{(0)}, t)$ so that as

$$t \to t_1 \text{ from } S, \quad P^{(0)} \to P_1^{(0)},$$

where $P_1^{(0)}$ is a point belonging to S_1 with the coordinates

$$(r_1, \varpi_1, \psi_1) = (r^{(0)}, \varpi^{(0)}, \psi^{(0)}).$$

Now at some time $t = \tilde{t}^{(0)} > t_1$, the observer O is located at a point $\tilde{P}^{(0)}(\tilde{r}^{(0)}, \tilde{\varpi}^{(0)}, \tilde{\psi}^{(0)}, \tilde{t}^{(0)})$ belonging to $\tilde{S}$. To define a suitable continuation of the coordinate system in $\tilde{S}$, the point $\tilde{P}^{(0)}$ will be chosen such that

$$\tilde{P}^{(0)} \to P_1^{(0)} \text{ as } t \to t_1 \text{ from } \tilde{S}.$$

This implies that the position of O at all times in $\tilde{S}$ is given by

$$(\tilde{r}, \tilde{\varpi}, \tilde{\psi}, t) = (r^{(0)}, \varpi^{(0)}, \psi^{(0)}, t),$$

since $\tilde{r}$, $\tilde{\varpi}$, $\tilde{\psi}$ are comoving coordinates. Hence, a single coordinate system (r, ϖ, ψ, t) may be employed to specify space-time positions for all values of the cosmic time.

It is now supposed that the space-time interval between two comoving observers O and O' is given by $a^{(0)}$ in S and by $\tilde{a}^{(0)}$ in $\tilde{S}$. If

$$a^{(0)} \to a_1^{(0)} \text{ as } t \to t_1 \text{ from } S,$$

then the value of $\tilde{a}^{(0)}$ in $\tilde{S}$ is chosen so that

$$\tilde{a}^{(0)} \to a_1^{(0)} \text{ as } t \to t_1 \text{ from } \tilde{S},$$

thus ensuring the continuity of a at the boundary $t = t_1$. Adopting these results and suppositions, we may rewrite equation (5.402) as

$$da^2 = dt^2 - \frac{\tilde{R}^2(t)}{c^2}\left\{\frac{dr^2 + r^2 d\omega^2 + r^2 \sin^2\omega \, d\varphi^2}{(1 + Kr^2/4)^2}\right\}, \tag{5.404}$$

for the metric of space-time in $\tilde{S}$. From a comparison of equations (5.401) and (5.404), it is possible to infer that the scale factor is continuous at $t = t_1$ as a consequence of the continuity of a at $t = t_1$. That is

$$\left.\begin{array}{l} R(t) \to R(t_1) \text{ as } t \to t_1 \text{ from } S, \\ \tilde{R}(t) \to R(t_1) \text{ as } t \to t_1 \text{ from } \tilde{S}. \end{array}\right\} \tag{5.405}$$

For the sake of brevity, we will henceforth adopt the following notation. If a function $Z(t)$ in S, and its continuation $\tilde{Z}(t)$ in $\tilde{S}$, is continuous at $t = t_1$ such that

$$Z(t) \to Z(t_1) \text{ as } t \to t_1 \text{ from } S,$$
$$\tilde{Z}(t) \to Z(t_1) \text{ as } t \to t_1 \text{ from } \tilde{S},$$

then we write

$$Z_1 = \tilde{Z}_1 .$$

Thus, the continuity of the scale factor as expressed in equation (5.405) is written as

$$R_1 = \tilde{R}_1 . \tag{5.406}$$

The transition from the space S to the space $\tilde{S}$ will be regarded as physically acceptable provided that the density and the pressure of the energy distribution are continuous at $t = t_1$. The continuity conditions at $t = t_1$ required to ensure such a transition will now be discussed. The cosmological field equations for the density are

$$\begin{aligned} \frac{8\pi G}{3}\rho &= \left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2}{R^2}, \quad 0 \leq t < t_1 , \\ \frac{8\pi G}{3}\tilde{\rho} &= \left(\frac{\dot{\tilde{R}}}{\tilde{R}}\right)^2 + \frac{kc^2}{\tilde{R}^2}, \quad t > t_1 , \end{aligned} \tag{5.407}$$

whereas those that describe the pressure are given by

$$\begin{aligned} \frac{8\pi G}{c^2} p &= -\frac{2\ddot{R}}{R} - \frac{kc^2}{R^2} - \left(\frac{\dot{R}}{R}\right)^2, \quad 0 \leq t < t_1 , \\ \frac{8\pi G}{c^2}\tilde{p} &= -\frac{2\ddot{\tilde{R}}}{\tilde{R}} - \frac{kc^2}{\tilde{R}^2} - \left(\frac{\dot{\tilde{R}}}{\tilde{R}}\right)^2, \quad t > t_1 . \end{aligned} \tag{5.408}$$

Hence, given equation (5.406), a sufficient condition to ensure that the density is continuous at $t = t_1$ is, from the equations (5.407),

$$\dot{R}_1 = \dot{\tilde{R}}_1 . \tag{5.409}$$

Similarly, by an inspection of equations (5.408), the equations (5.406) and (5.409) imply that the pressure is continuous at $t = t_1$ provided that

$$\ddot{R}_1 = \ddot{\tilde{R}}_1 . \tag{5.410}$$

Alternatively, by consideration of the conservation of energy equation,

$$\frac{d}{dt}(\rho R^3) = -\frac{p}{c^2}\frac{dR^3}{dt}, \quad 0 \leq t < t_1,$$

$$\frac{d}{dt}(\tilde{\rho}\tilde{R}^3) = -\frac{\tilde{p}}{c^2}\frac{d\tilde{R}^3}{dt}, \quad t > t_1,$$

and given equations (5.406) and (5.409), another condition that provides a continuous pressure at $t = t_1$ is given by

$$\dot{\rho}_1 = \dot{\tilde{\rho}}_1, \tag{5.411}$$

which is a statement equivalent to the condition (5.410). Thus the continuity conditions (5.403), (5.406), (5.409) and (5.410) or (5.411) are those required to ensure a physically acceptable passage from the space S to the space $\tilde{S}$.

The problem of joining two solutions of Einstein's equations across a boundary

$$x^\alpha - constant = 0, \quad \alpha = 1, 2, 3, 4,$$

has been examined previously. For instance, Lichnerowicz (1955) has claimed that suitable junction conditions are that the metric tensor $\gamma_{\mu\nu}$, and all of its first partial derivatives $\gamma_{\mu\nu,\alpha}$, should be continuous at the boundary. A similar investigation led O'Brien and Synge (1952) to the conclusion that the $\gamma_{\mu\nu}$ and the $\gamma_{\mu\nu,\alpha}$, with the possible exception of $\gamma_{\mu 4,4}$, should be continuous at the boundary. Robson (1972) has, however, demonstrated that these sets of junction conditions are equivalent to one another provided that the 'boundary space' is a nonnull three dimensional space defined by an equation of the form

$$t - constant = 0.$$

Since the transition from S to $\tilde{S}$ is of this nature, we need only discuss one set of conditions, say the Lichnerowicz set, to describe the junction of two Friedmann models at $t = t_1$. By an inspection of the metrics (5.401) and (5.404), the junction conditions of Lichnerowicz imply that

$$R_1 = \tilde{R}_1 \,, \quad \dot{R}_1 = \dot{\tilde{R}}_1 \,. \tag{5.412}$$

With reference to our discussion of 'physically acceptable' continuity conditions, the equations (5.412) show that the Lichnerowicz conditions require only the density of the Friedmann models to be continuous at $t = t_1$. That is, in general they do not imply that the pressure need be continuous at the boundary.

However, in the particular case of the model universe proposed in Figure 2, the imposition of the Lichnerowicz conditions (5.412) are sufficient to ensure continuity of the pressure. To see this, we note that the radiation density in this model is defined by the following equations

$$\rho_r \propto R^{\nu-4}, \qquad 0 \leq t < t_1 \,,$$

$$\tilde{\rho}_r \propto \tilde{R}^{-4} \,, \qquad t > t_1 .$$

Hence, from equations (5.412) we have that

$$\rho_{r1} = \tilde{\rho}_{r1} \,. \tag{5.413}$$

Moreover the pressure in this model is defined by equation (5.104),

$$p = p_r = \frac{1}{3}\rho_r c^2, \quad 0 \leq t < t_1,$$
$$\tilde{p} = \tilde{p}_r = \frac{1}{3}\tilde{\rho}_r c^2, \quad t > t_1.$$

Thus, from the condition (5.413) it is possible to conclude that

$$p_1 = \tilde{p}_1 .$$

Previously we have demonstrated that this result leads to the condition (5.410), or the equivalent statement (5.411). Hence the imposition of the Lichnerowicz conditions (5.412) at $t = t_1$ in the particular case of the model proposed in Figure 2 automatically ensures a physically acceptable transition from S to $\tilde{S}$.

5.5 Uniform models of the Universe containing matter and gravitational radiation.

(I) Introduction.

In this Section we investigate further the model of the Universe outlined briefly in Section 5.3. With reference to Figure 2, the cosmological parameters defined previously are now expressed in terms of time $t = t_1$, the instant at which the generation of gravitational radiation ceases. We have

$$H_1 = \frac{\dot{R}(t_1)}{R(t_1)} \equiv \frac{\dot{R}_1}{R_1}, \tag{5.501}$$

$$q_1 = -\frac{\ddot{R}_1}{R_1}\left(\frac{R_1}{\dot{R}_1}\right)^2, \tag{5.502}$$

$$\sigma_{m1} = \frac{4\pi G}{3H_1^2}\rho_{m1}, \quad \sigma_{r1} = \frac{4\pi G}{3H_1^2}\rho_{r1}. \tag{5.503}$$

At the instant $t = t_1$, the cosmological field equations (5.106) and (5.107) become

$$6(\sigma_{m1} + \sigma_{r1}) = 3 + \frac{3\kappa c^2}{R_1^2 H_1^2},$$

$$2\sigma_{r1} = 2q_1 - 1 - \frac{\kappa c^2}{R_1^2 H_1^2},$$

which imply that

$$\kappa\left(\frac{c}{R_1 H_1}\right)^2 = -(q_1 + 1 - 3\sigma_{m1} - 4\sigma_{r1}), \tag{5.504}$$

$$q_1 = \sigma_{m1} + 2\sigma_{r1}. \tag{5.505}$$

Substituting equation (5.505) into equation (5.504), we find that

$$\kappa\left(\frac{c}{R_1 H_1}\right)^2 = -1 + 2(\sigma_{m1} + \sigma_{r1}),$$

so that the sign of κ in the model is now determined by the value of the total energy density at time $t = t_1$. **Thus**

$$\left.\begin{aligned} \kappa &= -1 \quad \text{if} \quad \sigma_{m1} + \sigma_{r1} < \tfrac{1}{2}, \\ \kappa &= 0 \quad \text{if} \quad \sigma_{m1} + \sigma_{r1} = \tfrac{1}{2}, \\ \kappa &= +1 \quad \text{if} \quad \sigma_{m1} + \sigma_{r1} > \tfrac{1}{2}. \end{aligned}\right\} \tag{5.506}$$

To describe the generation of radiative energy during the period $0 \leq t < t_1$, the IW equation (5.301) is expressed in terms of the instant $t = t_1$ as

$$\rho_r = \rho_{r1}\left(\frac{R}{R_1}\right)^{\nu-4}, \quad 0 \leq t < t_1, \tag{5.507}$$

and it is further assumed that the condition (5.304) is satisfied. That is, the galactic hypothesis is adopted. This immediately gives rise to questions as to which sources of gravitational waves within galaxies are capable of generating a significant Universal energy density. A number of astrophysical processes which are promising candidates in this respect have been investigated by many authors. Boccaletti, De Sabbata, Gualdi and Fortini (BSGF) (1968) have provided a review of the situation, and have made an estimate of the energy density of gravitational radiation resulting from such processes. Although many possible sources were discussed, including QSOs, supernovae, orbital encounters between stars in densely populated stellar regions and 'gravitational bremsstrahlung' due to thermal motions within stellar interiors, the authors conclude that the main contributors to the postulated gravitational radiation field are close binary systems. Working with the assumption that each galaxy contributes similar amounts of radiation in this way, BSGF estimate the mean energy density of gravitational radiation to be approximately 10^{-40} gm cm^{-3} for the present epoch, a value insignificant by comparison with the present mean density of matter.

It is reasonable to suppose, however, that this value represents only a lower bound. For instance, how reliable an estimate may be obtained of the rate at which gravitational waves are generated within QSOs when there is no real appreciation of their physical nature? Further, other likely candidates such as neutron stars and blackholes have been omitted from BSGF's discussion. The former of these has been examined in great detail by Thorne (1969), who has concluded that neutron stars which pulsate in a nonspherical manner can give rise to considerably stronger gravitational emission than binary stars. It should be admitted however that, since there is no reliable way of assessing the spatial distribution of such objects within the Galaxy, their contribution to the Universal radiation field is difficult to estimate.

The latter candidate, the black hole, is predicted by general relativity theory, but as yet it has not been detected observationally. Despite this however, the presence of massive black holes in the nuclei of galaxies has been suggested many times. For example Hills (1975) discusses the possibility that the observed energy output of QSOs and Seyfert galaxies is supplied by violent astrophysical events associated with the presence of a black hole at their centres. Although Hills neglects effects due to the emission of gravitational radiation, a crude estimate of the radiative power of such objects is forthcoming from his calculations. If the observed velocity dispersion and density of the stellar population at the centre of our own galaxy are adopted, then Hills' argument suggests that a 'seed blackhole' of mass $10\, m_\odot$, formed in the early history of the Galaxy, has had sufficient time to accumulate a mass of at least $3.\, 10^2\, m_\odot$. Any massive object in orbit around this blackhole would be expected to spiral slowly inward, because of loss of energy through the release of gravitational radiation, before being captured.
Ruffini and Wheeler (1971) have shown that the rest mass of the material accreted by the blackhole would be converted

into gravitational emission with a minimum efficiency of 5%. Hence, the contribution to the mean energy density of gravitational radiation of this very modest object can be estimated to be at least of the same order of magnitude as the combined emission of all the afore mentioned binary star systems.

Another example cited by Hills is the Seyfert galaxy NGC 4151. This object, being considerably denser than our own galaxy, provides an enviroment in which a blackhole with an initial mass of 10 $m_\odot$ has had time to grow into a blackhole of mass $10^8\ m_\odot$. Again, assuming a modest 5% efficiency for the radiative process, we may conclude that NGC 4151 has a gravitation wave luminosity some six orders of magnitude larger than the binary system luminosity of our own galaxy.

Hence it is possible to infer that the galactic hypothesis may have provided an amount of gravitational radiation which is cosmologically significant at the present epoch. Moreover, during the period of time $t_0 < t < t_1$, gravitational radiation energy will continue to accumulate since, as BSGF point out, "the destruction rate of the gravitons is exceedingly smaller than the production rate".

(II) <u>On the conversion of mass energy into gravitational radiation $(0 \leq t < t_1)$.</u>

As we have seen from Section 5.3, the use of equation (5.507) to describe the gravitational radiation energy density in the model implies, from equation (5.309), that the matter density is given by

$$\rho_m = \left(\frac{R_1}{R}\right)^3 \left\{ \rho_{m1} + \rho_{r1}\left(\frac{\nu}{\nu-1}\right)\left[1 - \left(\frac{R}{R_1}\right)^{\nu-1}\right] \right\}. \qquad (5.508)$$

Hence the matter energy content of the model during the 'generation era' is governed by the equation

$$\rho_m R^3 = R_1^3 \left\{ \rho_{m1} + \rho_{r1}\left(\frac{\nu}{\nu-1}\right)\left[1 - \left(\frac{R}{R_1}\right)^{\nu-1}\right] \right\}.$$

From this equation, the amount of material energy present initially may be expressed as

$$(\rho_m R^3)_i = R_1^3 \left\{ \rho_{m1} + \rho_{r1}\left(\frac{\nu}{\nu-1}\right) \right\}, \quad \nu > 1,$$

since we have supposed that $R(0) = 0$. Moreover, the amount present at time $t = t_1$, when the generation ceases, is given by

$$(\rho_m R^3)_1 = \rho_{m1} R_1^3.$$

Hence if we express the mass energy lost to radiation during the generation era, as a fraction of the initial mass energy

$$\mu = \frac{(\rho_m R^3)_i - (\rho_m R^3)_1}{(\rho_m R^3)_i}, \qquad (5.509)$$

then

$$\mu = \frac{\nu \rho_{r1}}{\{(\nu-1)\rho_{m1} + \nu\rho_{r1}\}}. \qquad (5.510)$$

We note that this equation is not valid if ρ_{m1}, ρ_{r1} are such that

$$(\nu-1)\rho_{m1} + \nu\rho_{r1} = 0.$$

Since $\nu > 1$, this situation may only arise in one of the following ways. Either (a) one density is negative and the other positive, of (b) both density values are zero. The former situation we may dismiss on physical grounds. If the latter statement is true, then equations (5.507) and (5.508) imply that $\rho_m(t)$ and $\rho_r(t)$ will be zero for all values of time $0 \leq t < t_1$. This can occur in $\nu > 1$ models only in the event that $(\rho_m R^3)_i = 0$.

Hence equation (5.510) is sound provided that we restrict our attention to models in which

$$(\rho_m R^3)_i > 0 .$$

This initial matter content acts as an energy source for the subsequent generation of gravitational radiation, so that some of it will be converted to radiation during the period $0 \leq t < t_1$. Hence

$$(\rho_m R^3)_i > (\rho_m R^3)_1 \geq 0 ,$$

which in turn implies, from equation (5.509), that

$$0 < \mu \leq 1 . \tag{5.511}$$

The case $(\rho_m R^3)_1 = 0$ arises in the event that $t_1 = t_c$, where the instant $t = t_c$ corresponds to the epoch in the model's history at which the matter density falls to zero by virtue of the generation process.

Finally, multiplying equation (5.510) top and bottom by $4\pi G/(3H_1^2)$, we may express μ in its most convenient form,

$$\mu = \frac{\nu\sigma_{r1}}{\{(\nu-1)\sigma_{m1} + \nu\sigma_{r1}\}} \tag{5.512}$$

This equation, and equation (5.505), provide a set of simultaneous equations in σ_{m1}, σ_{r1} given by

$$\mu(\nu-1)\sigma_{m1} + \nu(\mu-1)\sigma_{r1} = 0 ,$$

$$\sigma_{m1} + 2\sigma_{r1} = q_1 .$$

Their solutions are

$$\sigma_{m1} = \frac{q_1 \nu (1-\mu)}{[2\mu(\nu-1) - \nu(\mu-1)]} , \tag{5.513}$$

$$\sigma_{r1} = \frac{q_1 \mu (\nu-1)}{[2\mu(\nu-1) - \nu(\mu-1)]} , \tag{5.514}$$

where, given the conditions (5.304) and (5.511),

$$\mathcal{N} \equiv 2\mu(\nu-1) - \nu(\mu-1) > 0 .$$

If the matter and radiation densities are to be non-negative at $t = t_1$, equation (5.505) or equations (5.513), (5.514) imply that

$$q_1 \geq 0 , \tag{5.515}$$

in these models. That is, the expansion is decelerating at the instant $t = t_1$, when the generation of radiation ceases. Models in which

$$q_1 < 0 ,$$

are possible if the cosmical constant Λ is nonzero.

(III) <u>The scale factor $R(t)$ of the models for $0 \leq t < t_1$.</u>

The cosmological field equations are given by (5.106) and (5.306). We have seen that if ρ_r is given by (5.507), then equation (5.306) implies that ρ_m is given by equation (5.508). Thus, the total density is given by

$$\rho_m + \rho_r = \left(\frac{R_1}{R}\right)^3 \left\{ \rho_{m1} + \frac{\rho_{r1}}{(\nu-1)} \left[\nu - \left(\frac{R}{R_1}\right)^{\nu-1} \right] \right\} .$$

We now use this result to solve equation (5.106) for $0 \leq t < t_1$.

If the dimensionless variables

$$y = R/R_1 , \quad X = tH_1 ,$$

are introduced, then the total density becomes

$$\rho_m + \rho_r = \frac{1}{y^3}\left\{\rho_{m1} + \frac{\rho_{r1}}{(\nu-1)}\left[\nu - y^{\nu-1}\right]\right\}. \tag{5.516}$$

Similarly, a change of variables in the field equation (5.106) gives

$$\frac{8\pi G}{3H_1^{\,2}}(\rho_m + \rho_r) = \left(\frac{dy}{dX}\right)^2 \frac{1}{y^2} + K\left(\frac{c}{R_1 H_1}\right)^2 \frac{1}{y^2}.$$

Introducing (5.516) into this equation, and using equations (5.503) and (5.504), we find that

$$\frac{1}{y}\left\{2\sigma_{m1} + \frac{2\sigma_{r1}}{(\nu-1)}\left[\nu - y^{\nu-1}\right]\right\}$$
$$= \left(\frac{dy}{dX}\right)^2 - q_1 - 1 + 3\sigma_{m1} + 4\sigma_{r1}.$$

A further substitution from equation (5.513), (5.514) reduces this equation to its most convenient form,

$$\left(\frac{dy}{dX}\right)^2 = \frac{2q_1}{\mathcal{N}}\left\{\frac{\nu}{y} - \mu y^{\nu-2}\right\} - \frac{2q_1}{\mathcal{N}}(\nu - \mu) + 1, \tag{5.517}$$

which describes the behaviour of the scale factor as a function of time, during the period $0 \leqslant t < t_1$. Particular models are specified by assigning a value to each of the parameters (q_1, μ, ν), where the permitted ranges of values they may take are given by (5.515), (5.511) and (5.304) respectively. The value of the space curvature constant K in each model may be found by the use of equations (5.506), (5.513) and (5.514).

(IV) <u>The scale factor $R(t)$ of the models for $t > t_1$.</u>

During the period following the termination of radiation generation, we have supposed that the matter and radiation do not interact. As discussed in Section 5.3, this implies that

$$\tilde{\rho}_m \tilde{R}^3 = \rho_{m1} R_1^3 \ , \quad \tilde{\rho}_r \tilde{R}^4 = \rho_{r1} R_1^4 .$$

These relationships may be seen to satisfy the field equation (5.306). Introducing the dimensionless variables

$$\tilde{y} = \tilde{R}/R_1 \ , \quad X = tH_1 \ ,$$

we may write the total density as

$$\tilde{\rho}_m + \tilde{\rho}_r = \frac{1}{\tilde{y}^3}\left\{\rho_{m1} + \frac{\rho_{r1}}{\tilde{y}}\right\} .$$

Hence the remaining field equation (5.106) for $t > t_1$ becomes

$$\frac{2\sigma_{m1}}{\tilde{y}} + \frac{2\sigma_{r1}}{\tilde{y}^2} = \left(\frac{d\tilde{y}}{dX}\right)^2 + K\left(\frac{c}{R_1 H_1}\right)^2 .$$

Finally, the substitution of equations (5.504), (5.513) and (5.514) into this equation gives

$$\left(\frac{d\tilde{y}}{dX}\right)^2 = \frac{2q_1}{N}\left\{\frac{\nu(1-\mu)}{\tilde{y}} + \frac{\mu(\nu-1)}{\tilde{y}^2}\right\} - \frac{2q_1}{N}(\nu-\mu) + 1 \ , \qquad (5.518)$$

which describes the behaviour of the scale factor during the period $t > t_1$.

(V) <u>Continuity conditions at $t = t_1$</u>.

It was shown in Section 5.4 that a physically acceptable transition from $S(0 \leq t < t_1)$ to $\tilde{S}(t > t_1)$ is ensured provided that the scale factor, and its first time derivative, are continuous across the boundary $S_1\ (t = t_1)$.

The behaviours of $R(t)$ in S, and $\tilde{R}(t)$ in $\tilde{S}$, are described by the equations (5.517) and (5.518) respectively.

Since it is supposed that the models have a point source origin, the constant of integration associated with the first order equation (5.517) is chosen to ensure that $R(0) = 0$. Then, as time increases, there will exist a number R_1 in S_1 such that

$$R(t) \to R_1 \quad \text{as } t \to t_1 \text{ from } S.$$

The continuity of the scale factor across $t = t_1$ is ensured by choosing the constant of integration that arises in the solution of equation (5.518) for $\tilde{R}(t)$ in $\tilde{S}$, in such a way that

$$\tilde{R}(t) \to R_1 \quad \text{as } t \to t_1 \text{ from } \tilde{S}.$$

Moreover, the equation (5.517) is a direct statement of the behaviour of dR/dt in S, whereas equation (5.518) is the corresponding statement for $d\tilde{R}/dt$ in $\tilde{S}$. From equation (5.517), since $y \to 1$ as $t \to t_1$ from S, it follows that

$$dR/dt \to \pm H_1 R_1 \quad \text{as } t \to t_1 \text{ from } S,$$

where the positive sign is taken if the scale factor is increasing at $t = t_1$. Consideration of the equation (5.518) for $\tilde{R}(t)$ in $\tilde{S}$ shows that

$$d\tilde{R}/dt \to \pm H_1 R_1 \quad \text{as } t \to t_1 \text{ from } \tilde{S}.$$

Hence, the first derivative of the scale factor is also continuous across S_1, so that the conditions (5.412) are satisfied by the models. As a consequence of this (see Section 5.4), both the density and the pressure in the models are continuous across the boundary space $t = t_1$.

5.6 Solutions.

(I) Comments on the method of solution for $y(X)$ in $S(0 \leq t < t_1)$.

An inspection of equation (5.517), which describes the behaviour of y in S, shows that for general values of the index ν, analytic solutions of this equation are not available. Hence, the solutions will be sought numerically. The remainder of this subsection is devoted to a brief description of the iterative technique used to obtain these.

For the sake of brevity, let equation (5.517) be denoted by

$$\left(\frac{dy}{dX}\right)^2 = \mathcal{J}(y).$$

To generate the numerical solution, the y axis of the X, y plane is divided into intervals of equal width Δ, so that the boundary values of the i^{th} interval satisfy

$$y^{i+1} - y^{i} = \Delta,$$

where $y^i = y(X^i)$. Now, if the values of (X^i, y^i) on the solution curve of equation (5.517) are known for some value of i, then the value of X^{i+1} may be found by the use of the following Taylor expansion,

$$X^{i+1} = X^i + \Delta \frac{dX^i}{dy}(y^i) + O(\Delta^2), \quad (5.601)$$

where the value of y at $X = X^{i+1}$ is given by

$$y^{i+1} = y^i + \Delta.$$

Hence, another point (X^{i+1}, y^{i+1}) on the solution curve of equation (5.517) has been obtained. By repeating these operations a sufficient number of times, we are able to obtain the numerical solution of y in S.

Since a value of $\Delta = 10^{-3}$ was chosen to perform the calculation, the error per evaluation of X^{i+1} by the use of equation (5.601) may be expected to be of the order of $\Delta^2 = 10^{-6}$. Moreover the range of y in S is $0 \leq y < 1$ so that the number of such evaluations made is of the order of 10^3. Hence, we may estimate that the maximum deviation of the computed solution, $y(X)_{comp}$, from the true solution $y(X)$ of equation (5.517) will be given by

$$y(X)_{comp} = y(X) + O(10^{-3}).$$

Combining equation(5.517) and (5.601), we find that the computed solution is generated by

$$\left.\begin{aligned} X^{i+1} &= X^i + 10^{-3}\{\mathcal{S}^{-\frac{1}{2}}(y^i)\}, \\ y^{i+1} &= y^i + 10^{-3}, \end{aligned}\right\} \quad (5.602)$$

where the positive square root of equation (5.517) has been taken. To provide a starting point for the calculation, suitable initial values (X^0, y^0) are required. Although we have asserted that the models have a point source origin, the initial values $(X^0, y^0) = (0, 0)$ are of no use in general since the explicit form of $\mathcal{S}(y)$ is not well behaved at $y = 0$. To overcome this problem it is observed from equation (5.517) that, if $\nu \geq 2$, the value of y as $X \rightarrow 0$ is governed by the differential equation

$$\left(\frac{dy}{dX}\right)^2 = \frac{2q_1^{\nu}}{N}\,\frac{1}{y},$$

which has a solution

$$y = \left(\frac{9\nu q_1}{2N} X^2\right)^{\frac{1}{3}}.$$

Hence, if we choose an initial y value of $y^{\circ} = 10^{-3}$ and substitute it into the above equation, we obtain a suitable initial X value of

$$X^{\circ} = \left(\frac{2N}{9\nu q_1} \times 10^{-9}\right)^{\frac{1}{2}}. \qquad (5.603)$$

If values are assigned to the parameters (ν, μ, q_1), a value for X^{o} is found from equation (5.603). The substitution of (X°, y°) into equations (5.602) then gives another, subsequent point (X', y') on the solution curve. By the repeated use of equations (5.602) the numerical solution y in S may be generated. The iterative process is terminated when $t = t_1$ which, from the definition of y, corresponds to the instant at which

$$y = 1 \;, \; X = X_1(\nu, \mu, q_1) = t_1 H_1 . \qquad (5.604)$$

As indicated by the notation, the value of X at $y = 1$ is dependent upon the parameters (ν, μ, q_1). The value of the dimensionless time variable X_1, should not be confused with the superscripted X values associated with the equations (5.601).

(II) <u>Analytic solutions of $\tilde{y}(X)$ in $\tilde{S}(t > t_1)$</u>.

The analytic solutions of equation (5.518) are readily available. Rearranging this equation slightly we find that

$$\tilde{y}^2 \left(\frac{d\tilde{y}}{dX}\right)^2 = a_1 \tilde{y}^2 + a_2 \tilde{y} + a_3 \;,$$

where

$$\left.\begin{aligned} a_1 &= 1 - \frac{2q_1}{\mathcal{N}}(\nu-\mu), \\ a_2 &= \frac{2q_1}{\mathcal{N}}\nu(1-\mu), \\ a_3 &= \frac{2q_1}{\mathcal{N}}\mu(\nu-1). \end{aligned}\right\} \qquad (5.605)$$

From the equations (5.513) and (5.514), it is noted that the a_i may be re-expanded in terms of the density parameters σ_{m1}, σ_{r1} as follows

$$\left.\begin{aligned} a_1 &= 1 - 2(\sigma_{m1} + \sigma_{r1}), \\ a_2 &= 2\sigma_{m1}, \\ a_3 &= 2\sigma_{r1}. \end{aligned}\right\} \qquad (5.606)$$

The equation (5.518) is conveniently expressed as

$$\int dX = \int \frac{\tilde{y}\, d\tilde{y}}{(a_1\tilde{y}^2 + a_2\tilde{y} + a_3)^{\frac{1}{2}}}, \qquad (5.607)$$

where the positive square root has been taken (i.e. the solution curves of equation (5.607) will describe the expansion phase of the models). To solve equation (5.607) we consider two cases.

Case A. Models in which $\rho_{m1} = 0$.

This condition applies to models in which the initial matter content has been completely converted to gravitational radiation during the period $0 \leq t < t_1$. From equations (5.503) and (5.512), $\rho_{m1} = 0$ implies that

$$\sigma_{m1} = 0, \quad \mu = 1.$$

Hence, from equations (5.605), the coefficients a_i become

$$a_1 = 1 - q_1 \ , \quad a_2 = 0 \ , \quad a_3 = q_1 \ ,$$

so that equation (5.607) may be written

$$\int dX = \int \frac{\tilde{y}\, d\tilde{y}}{([1-q_1]\,\tilde{y}^2 + q_1)^{\frac{1}{2}}} \ . \tag{5.608}$$

This integral is most conveniently performed if the cases $q_1 > 1$, $q_1 = 1$, $q_1 < 1$ are considered separately. The value of the space curvature constant K for each solution is easily calculated by the use of equations (5.504) to (5.506). Solving equation (5.608), we find that for

$$\left.\begin{aligned}
&K = +1 \ (q_1 > 1), \\
&\quad \tilde{y} = \frac{1}{(q_1 - 1)^{\frac{1}{2}}}\left\{q_1 - (X + X_{(+1)})^2 (q_1 - 1)^2\right\}^{\frac{1}{2}}, \\
&K = 0 \ (q_1 = 1), \\
&\quad \tilde{y} = \left\{2(X + X_{(0)})\right\}^{\frac{1}{2}}, \\
&K = -1 \ (q_1 < 1), \\
&\quad \tilde{y} = \frac{1}{(1 - q_1)^{\frac{1}{2}}}\left\{(X + X_{(-1)})^2 (1 - q_1)^2 - q_1\right\}^{\frac{1}{2}},
\end{aligned}\right\} \tag{5.609}$$

where the constants of integration $X_{(K)}$ are arranged so as to produce continuity of the scale factor at $t = t_1$,

$$\left.\begin{aligned}
X_{(+1)} &= -\left\{\frac{1}{(q_1 - 1)} + X_1(\nu, 1, q_1)\right\}, \\
X_{(0)} &= \frac{1}{2} - X_1(\nu, 1, 1) \ , \\
X_{(-1)} &= \frac{1}{(1 - q_1)} - X_1(\nu, 1, q_1) \ .
\end{aligned}\right\} \tag{5.610}$$

Here the numbers $X_1(\nu, 1, q_1)$ are as defined in equation (5.604).

Case B. Models in which $\rho_{m1} > 0$.

This condition applies to models in which the initial mass content has only partially been converted to gravitational radiation before the generation ceases at $t = t_1$. From the equations (5.503) and (5.509), $\rho_{m1} > 0$ will imply that

$$\sigma_{m1} > 0 \;, \quad \mu < 1 .$$

Hence the conditions (5.506), combined with the equations (5.606), provide the following statements,

$$a_1 < 0 \iff K = +1 ,$$

$$a_1 = 0 \iff K = 0 ,$$

$$a_1 > 0 \iff K = -1 ,$$

$$a_2 > 0 ,$$

$$a_3 > 0 .$$

Using these conditions to solve (5.607), we find that for

$$\left.\begin{aligned} &K = +1 ,\\ &\quad X + X^{(+1)} = \frac{1}{a_1}\left(a_1 \tilde{y}^2 + a_2 \tilde{y} + a_3\right)^{\frac{1}{2}} \\ &\qquad - \frac{a_2}{2(-a_1)^{\frac{3}{2}}} \sin^{-1}\left\{\frac{2a_1\tilde{y} + a_2}{(a_2^2 - 4a_1a_3)^{\frac{1}{2}}}\right\} ,\\ &K = 0 ,\\ &\quad X + X^{(0)} = \frac{2}{3a_2^2}\left\{\left(a_2\tilde{y} + a_3\right)^{\frac{3}{2}} \right.\\ &\qquad \left. - 3a_3\left(a_2\tilde{y} + a_3\right)^{\frac{1}{2}}\right\} , \end{aligned}\right\} \quad (5.611)$$

$$\left.\begin{aligned} & K = -1, \\ & X + X^{(-1)} = \frac{1}{a_1}(a_1\tilde{y}^2 + a_2\tilde{y} + a_3)^{\frac{1}{2}} \\ & -\frac{a_2}{2(a_1)^{\frac{3}{2}}} \ln\left|2(a_1[a_1\tilde{y}^2 + a_2\tilde{y} + a_3])^{\frac{1}{2}} + 2a_1\tilde{y} + a_2\right| \end{aligned}\right\} \quad (5.611)$$

where the constants of integration $X^{(K)}$ are arranged so as to ensure continuity of the scale factor at $t = t_1$,

$$\left.\begin{aligned} X^{(+1)} &= \frac{1}{a_1} - \frac{a_2}{2(-a_1)^{\frac{3}{2}}} \sin^{-1}\left\{\frac{2a_1 + a_2}{(a_2^2 - 4a_1a_3)^{\frac{1}{2}}}\right\} \\ &\quad - X_1(\nu, \mu, q_1), \\ X^{(0)} &= \frac{2}{3a_2^2}(a_2 - 2a_3) - X_1(\nu, \mu, q_1), \\ X^{(-1)} &= \frac{1}{a_1} - \frac{a_2}{2(a_1)^{\frac{3}{2}}} \ln\left|2(a_1)^{\frac{1}{2}} + 2a_1 + a_2\right| \\ &\quad - X_1(\nu, \mu, q_1). \end{aligned}\right\} \quad (5.612)$$

For the case $K = +1$, the principle values of $\sin^{-1}$, between $-\pi/2$ and $\pi/2$, are to be taken.

The numbers $X_1(\nu, \mu, q_1)$ are as defined in equation (5.604).

(III) Solution Curves

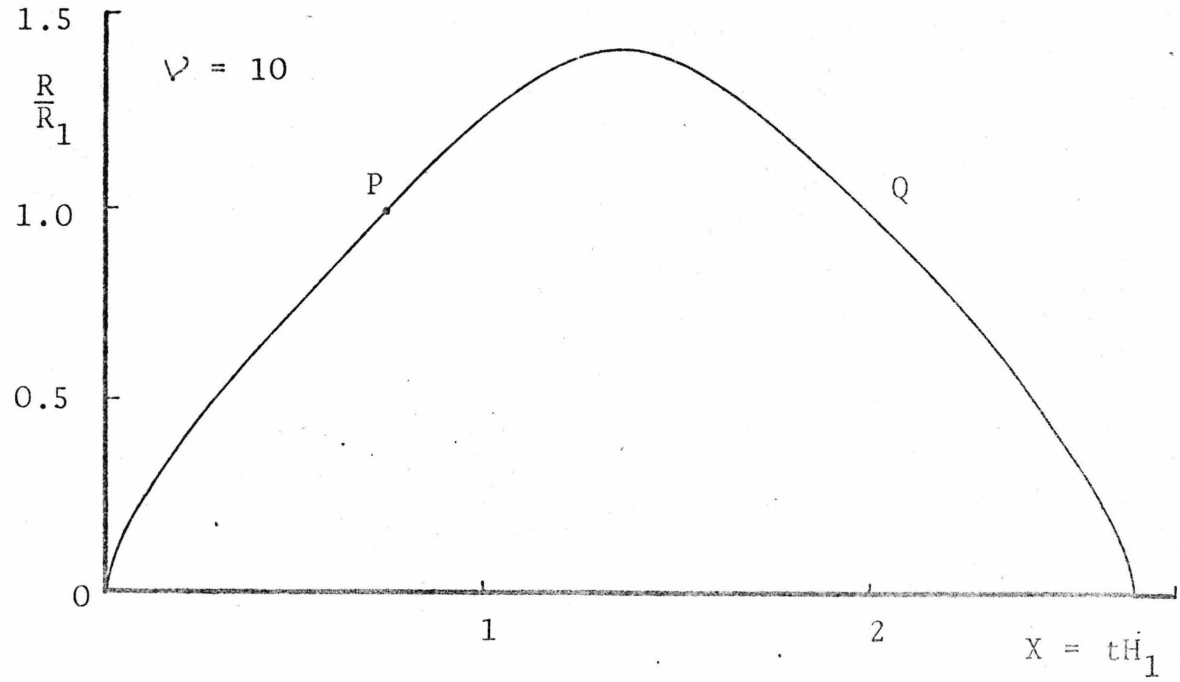

Fig. 3. The time dependence of the scale factor for an IW model in which $q_c = \frac{1}{2}$ ($\kappa = -1$). The mass density satisfies $\rho_m > 0$ along OP, and $\rho_m < 0$ along PQ.

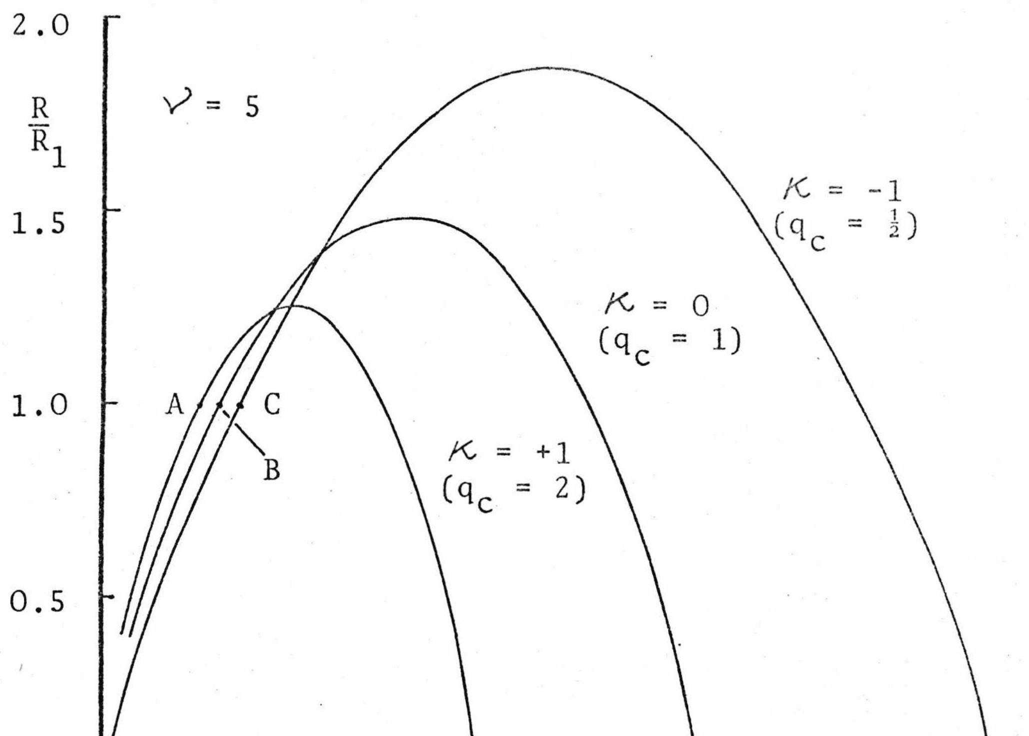

Fig. 4. The time dependence of the scale factor for three IW models. The points A, B, C mark the instants at which the density falls to zero, for each curve.

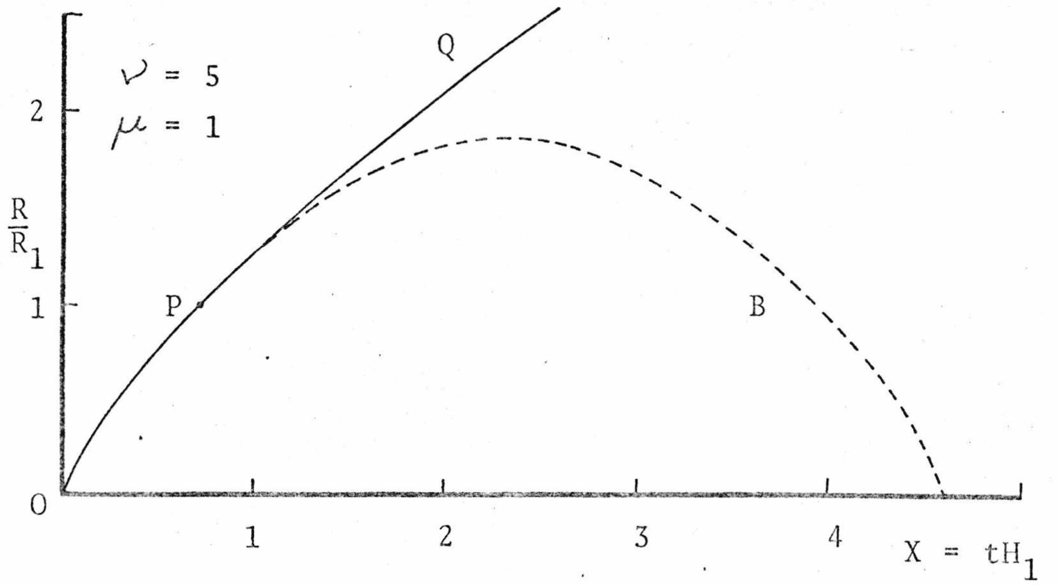

Fig. 5. The time dependence of the scale factor for a model in which $\kappa = -1$ ($q_1 = \frac{1}{2}$). The mass density satisfies $\rho_m > 0$ along OP, $\rho_m = 0$ along PQ, and $\rho_m < 0$ along PB. The curve PQ tends asymptotically to $\tilde{y} = X/\sqrt{2}$.

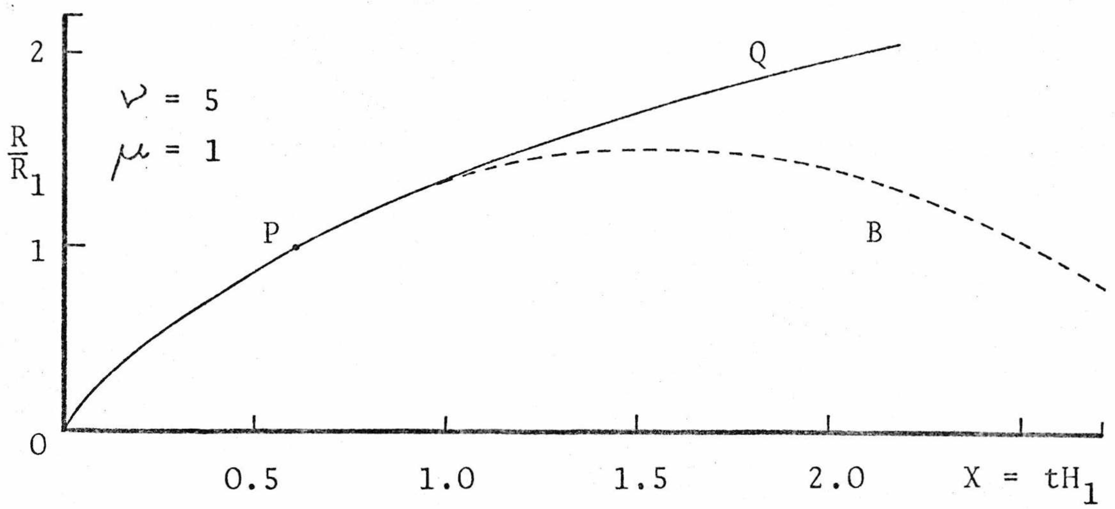

Fig. 6. The time dependence of the scale factor for a model in which $\kappa = 0$ ($q_1 = 1$). The mass density satisfies $\rho_m > 0$ along OP, $\rho_m = 0$ along PQ, and $\rho_m < 0$ along PB. The curve PQ tends asymptotically to $\tilde{y} = \sqrt{2X}$.

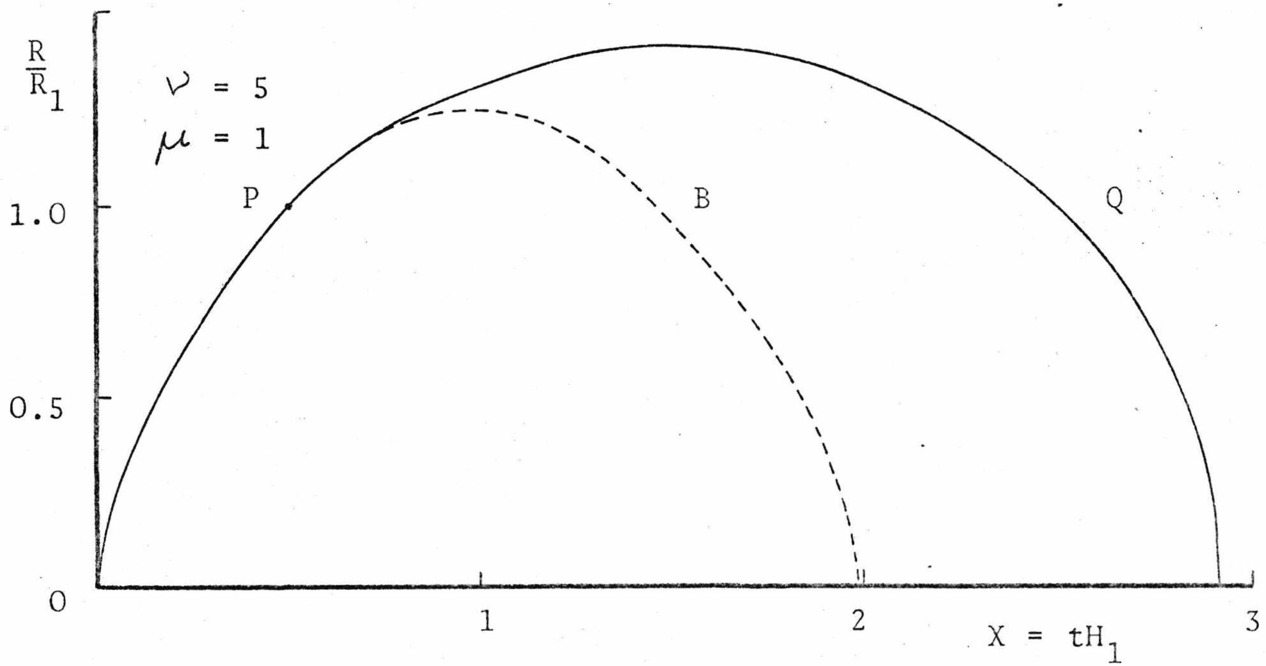

Fig. 7. The time dependence of the scale factor for a model in which $\kappa = +1$ ($q_1 = 2$). The mass density satisfies $\rho_m > 0$ along OP, $\rho_m = 0$ along PQ, and $\rho_m < 0$ along PB.

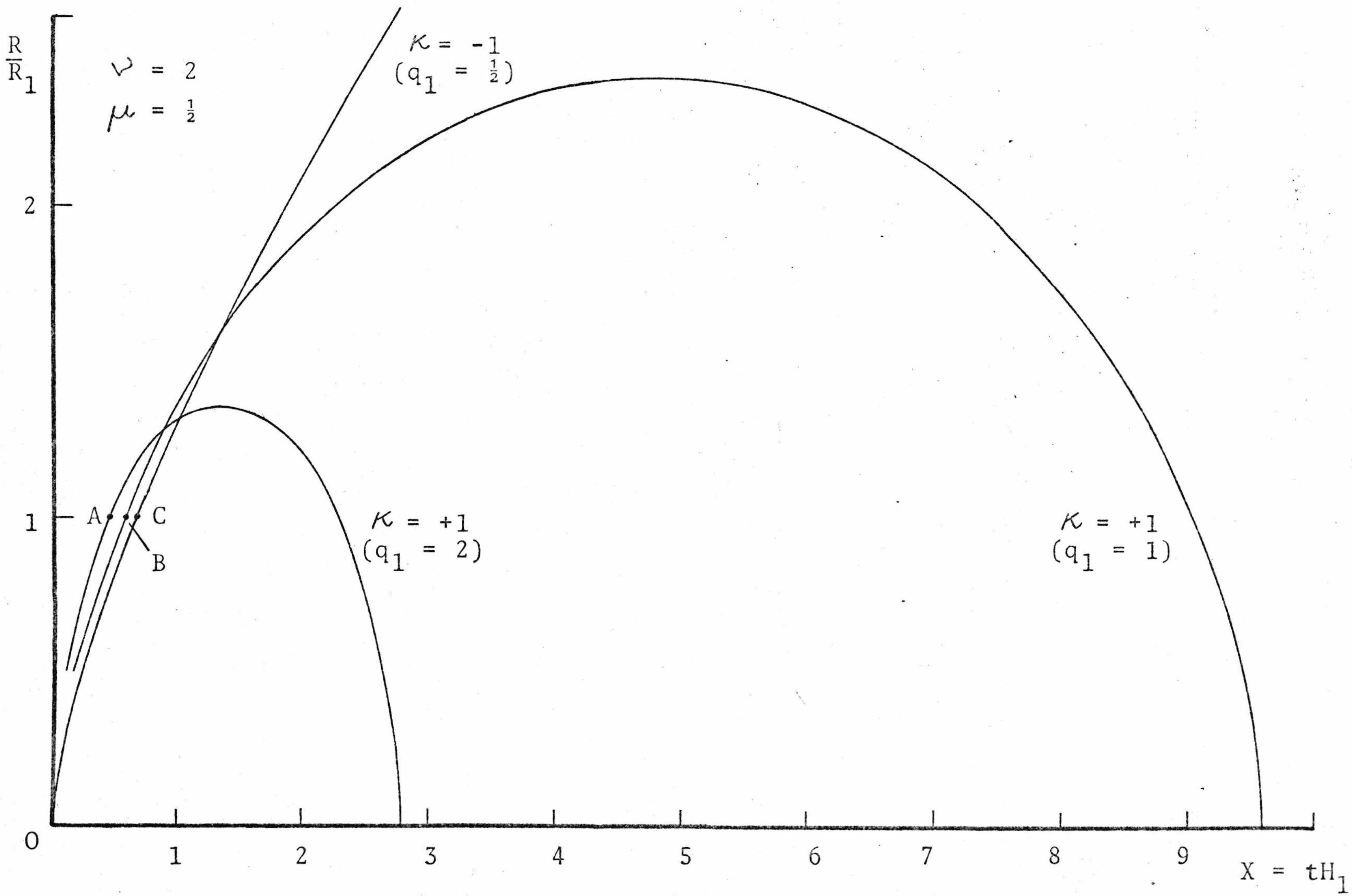

Fig. 8. The time dependence of the scale factor. The points A, B, C mark the instants at which the radiative generation ceases for each curve.

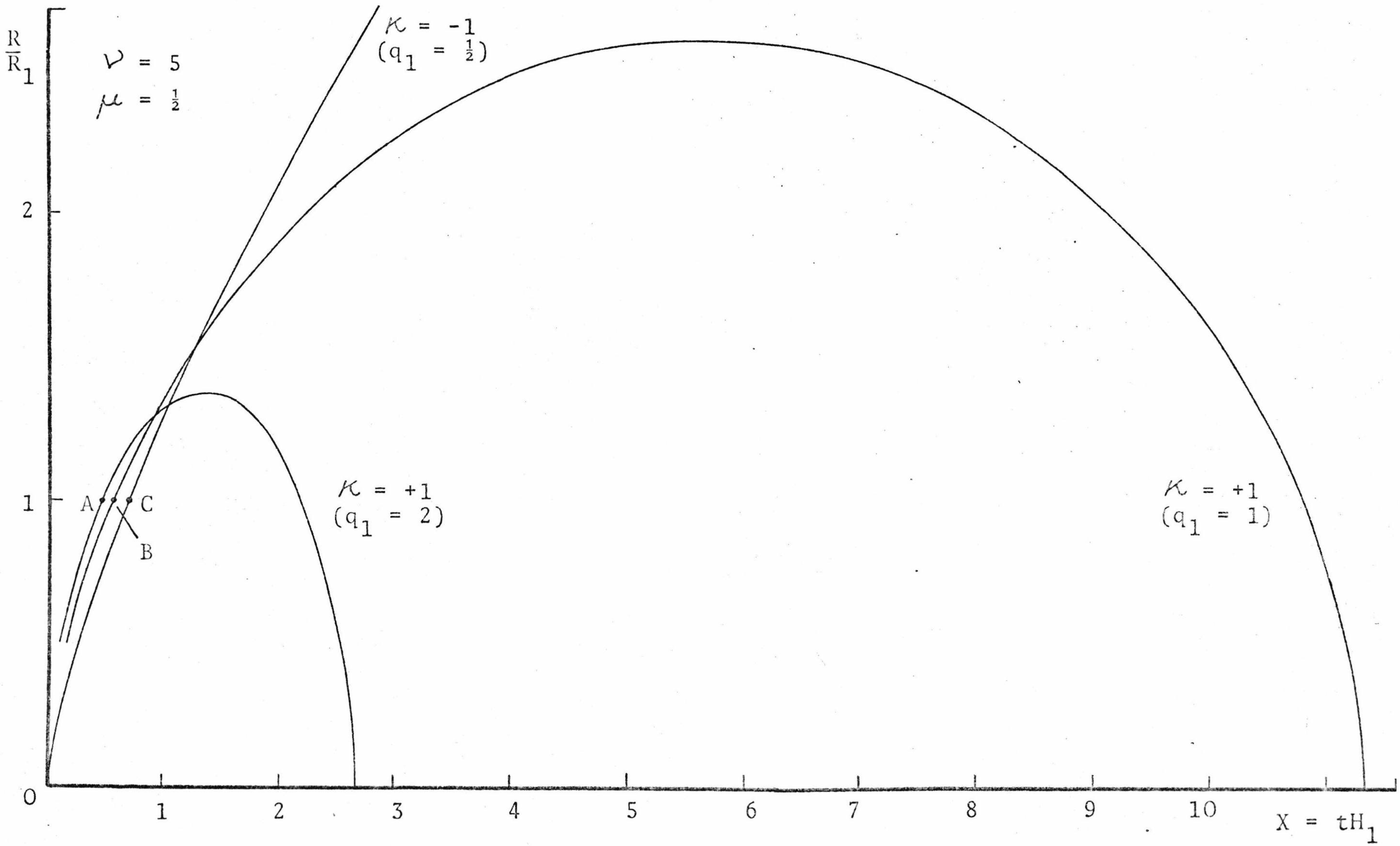

Fig. 9. The time dependence of the scale factor. The points A, B, C mark the instants at which the radiative generation ceases for each curve.

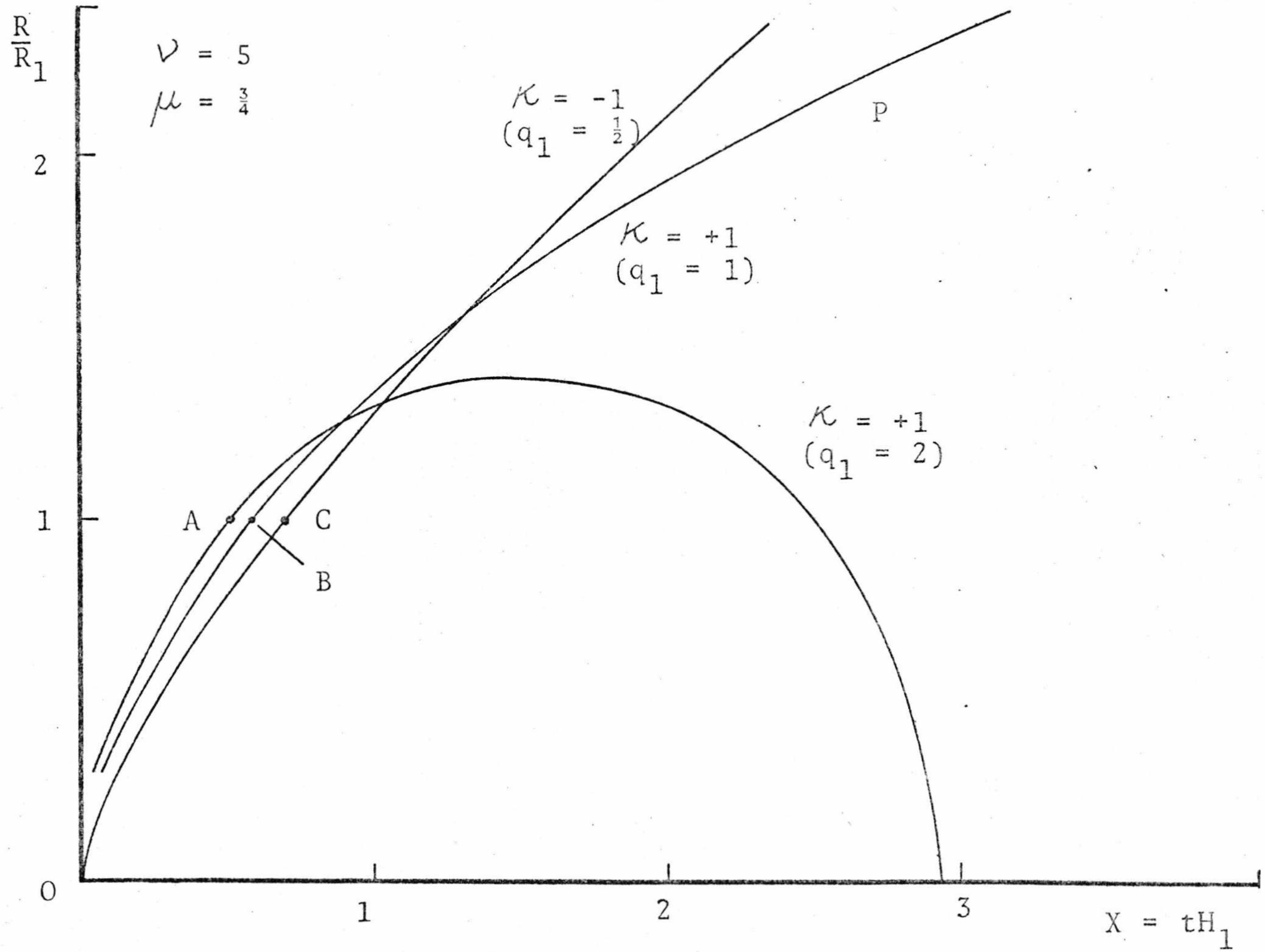

Fig. 10. The time dependence of the scale factor. The points A, B, C mark the instants at which the radiative generation ceases for each curve. The curve OBP has a maximum at $\tilde{y} = 3.408$, $X = 10.188$.

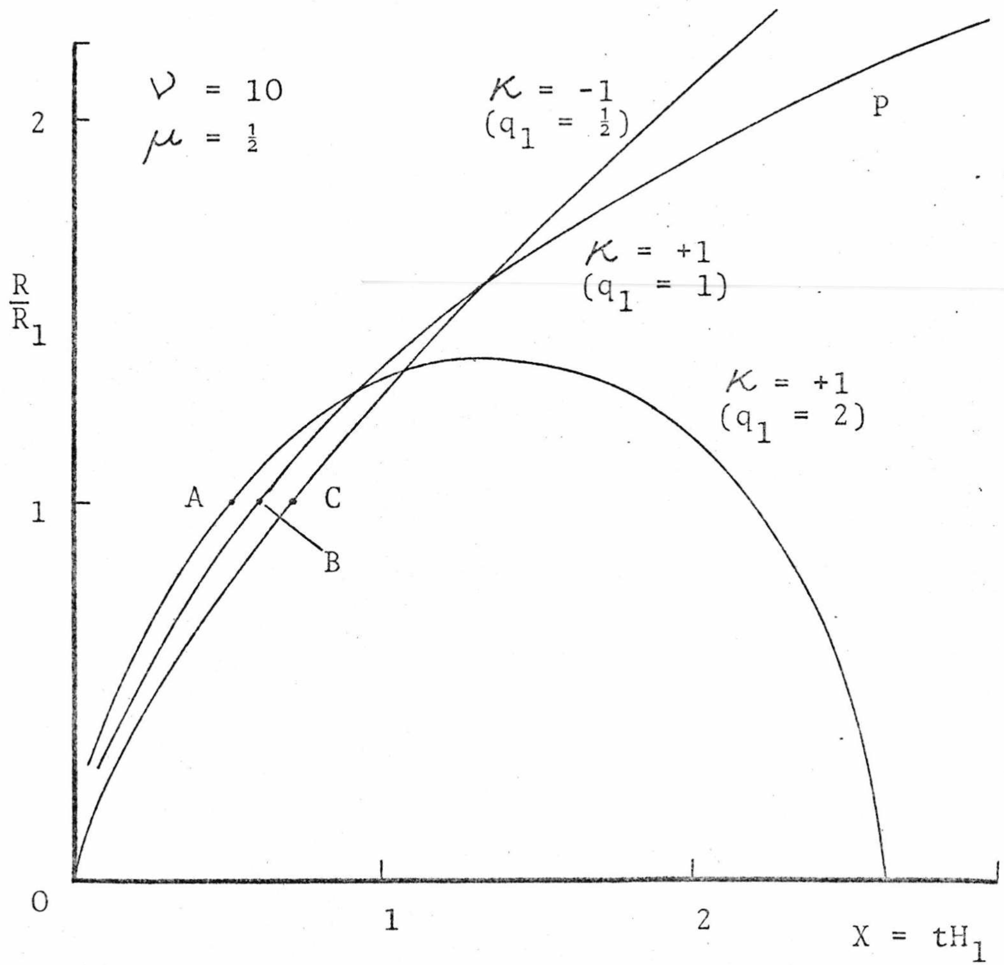

Fig. 11. The time dependence of the scale factor. The points A, B, C mark the instants at which the radiative generation ceases for each curve. The curve OBP has a maximum at $\tilde{y} = 2.647$, $X = 6.032$.

(IV) Discussion.

In the preceeding subsection, I have set out a number of solution curves of the equations (5.517) and (5.518) to exhibit the difficulties that can arise if the IW models of the Universe are temporally extrapolated, and to demonstrate how the new formulation of the models can overcome these.

Figures 3 and 4, each of which is labelled by the appropriate value of the radiation index ν , show the behaviour of the scale factor as a function of cosmic time for IW models. Figure 3 displays the nature of the 'negative mass' difficulty associated with these models, in that the curve represents an acceptable solution, in which $\rho_m \geqslant 0$ only for values of $X \leqslant X_1$. A similar situation occurs in Figure 4. The solution curves exhibited there indicate that the IW model universes may be 'closed' independently of whether the space curvature constant K takes the value +1, 0 or -1.

In contrast to these results, Figures 5 to 7 provide examples of IW models in which the generation of radiation has been terminated at the instant $t = t_c$ when the matter density has fallen to zero. The continuous lines describe the behaviour of the scale factor for the physically acceptable models in which $\rho_m > 0$ for $0 \leqslant t < t_c$ and $\rho_m = 0$ for $t \geqslant t_c$. The dotted lines, on the other hand, exhibit the behaviour for the models if the radiative generation is sustained beyond the instant $t = t_c$. The acceptable solutions now possess the open or closed behaviour appropriate to the value assigned to K in each case.

The remaining graphs, Figures 8 to 11, are each labelled by values of ν and μ. These figures represent the behaviour of the scale factor for models in which the radiative generation is terminated at some arbitrary time $t = t_1$ where $0 < t_1 < t_c$. The energy distribution in these models is such that radiation is created during the period $0 < t < t_1$. Thereafter the generation ceases, and the remaining material energy content of the models, $\rho_m R^3$, is conserved.

We would now like to demonstrate how a particular model may be specified by inserting numerical values of the parameters μ, ν into the formulae. Moreover, by assigning values to σ_{mo}, σ_{ro} and H_o, we will be able to show how the present epoch $t = t_o$, is related to the future time $t = t_1$ at which the postulated conversion of matter into gravitational waves ceases.

From equations (5.508) and (5.510), the matter density in the models may be expressed as

$$\rho_m(t) = \frac{\nu \rho_{r1} y^{-3}}{\mu(\nu-1)} (1 - \mu y^{\nu-1}).$$

If the value of y at $t = t_o$ is denoted by

$$y_o = R_o / R_1 ,$$

then the present mass density is given by

$$\rho_{mo} = \frac{\nu \rho_{r1} y_o^{-3}}{\mu(\nu-1)} (1 - \mu y_o^{\nu-1}).$$

Multiplying both sides of this equation by $4\pi G/(3H_o^2)$, we find that

$$\sigma_{mo} = \frac{H_1^2}{H_o^2} \frac{\nu \sigma_{r1} y_o^{-3}}{\mu(\nu-1)} (1 - \mu y_o^{\nu-1}),$$

which becomes, from equation (5.514),

$$\sigma_{mo} = \left(\frac{q_1 y_o^{-3} H_1^2}{N H_o^2}\right) \nu \left[1 - \mu y_o^{\nu-1}\right]. \qquad (5.613)$$

A similar procedure applied to equation (5.507) leads to the following expression for the radiation density parameter at $t = t_o$,

$$\sigma_{ro} = \left(\frac{q_1 y_o^{-3} H_1^2}{N H_o^2}\right) \mu \left[\nu - 1\right] y_o^{\nu-1}. \qquad (5.614)$$

If the equations (5.613) and (5.614) are now combined, by the elimination of the expression in round brackets from each, then

$$y_o = \left\{\frac{1}{\mu}\left(\frac{\nu\sigma_{ro}}{(\nu-1)\sigma_{mo} + \nu\sigma_{ro}}\right)\right\}^{\frac{1}{\nu-1}}. \qquad (5.615)$$

A rearrangement of this equation expresses $R(t_1)$ in terms of $R(t_o)$

$$R_1 = R_o\left\{\frac{\mu\left[(\nu-1)\sigma_{mo} + \nu\sigma_{ro}\right]}{\nu\sigma_{ro}}\right\}^{\frac{1}{\nu-1}},$$

which is a generalisation of the equation (5.310). Furthermore, as a consequence of the equations (5.504) and (5.505), we find that

$$\frac{H_o^2}{H_1^2} = \frac{1}{y_o^2}\left\{\frac{2(\sigma_{m1} + \sigma_{r1}) - 1}{2(\sigma_{mo} + \sigma_{ro}) - 1}\right\}, \qquad (5.616)$$

and if equation (5.614) is rearranged, then

$$q_1 = \frac{H_o^2}{H_1^2}\frac{\sigma_{ro} N}{\mu(\nu-1) y_o^{\nu-4}}. \qquad (5.617)$$

Now, by choosing values for the parameters μ, ν, σ_{mo} and σ_{ro}, the equation (5.615) allows us to calculate a value for y_o. The system of equations (5.513), (5.514), (5.616) and (5.617) then provides a means of evaluating the unknowns H_o^2/H_1^2, q_1, σ_{m1} and σ_{r1}. After some straightforward manipulation we find that

$$q_1 = \mathcal{N}\sigma_{ro}/\mathcal{U}, \tag{5.618}$$

where

$$\mathcal{U} = 2(\nu-\mu)\sigma_{ro} + \mu(\nu-1)y_o^{\nu-2}\{1-2(\sigma_{mo}+\sigma_{ro})\}. \tag{5.619}$$

From this result, equations (5.513) and (5.514) then give

$$\sigma_{m1} = \sigma_{ro}\,\nu(1-\mu)/\mathcal{U}, \tag{5.620}$$

$$\sigma_{r1} = \sigma_{ro}\,\mu(\nu-1)/\mathcal{U}, \tag{5.621}$$

so that equation (5.616) becomes

$$\frac{H_o^2}{H_1^2} = \mu(\nu-1)y_o^{\nu-4}/\mathcal{U}. \tag{5.622}$$

These results, however, are mathematically and physically acceptable only in the event that

$$\mathcal{U} > 0. \tag{5.623}$$

An inspection of equation (5.619) reveals that this condition is automatically satisfied for models in which $K = 0, -1$, and also for models in which $K = +1$ provided that the inequality

$$\sigma_{mo} + \sigma_{ro} < \frac{\sigma_{ro}(\nu-\mu)}{\mu(\nu-1)\,y_o^{\nu-2}} + \frac{1}{2},$$

holds. When considering particular examples we will suppose that the cessation of the radiative generation occurs at some future time in the model's history, so that

$$\mathcal{R}_1 > \mathcal{R}_o.$$

Hence, from equation (5.615), we will restrict our attention to $K = 0, -1$ models the parameters of which satisfy

$$\sigma_{mo} + \sigma_{ro} > \frac{\sigma_{ro}(\nu-\mu)}{\mu(\nu-1)} ,$$

and to $K = +1$ models in which

$$\frac{\sigma_{ro}(\nu-\mu)}{\mu(\nu-1)} < \sigma_{mo} + \sigma_{ro} < \frac{\sigma_{ro}(\nu-\mu)}{\mu(\nu-1)y_o^{\nu-2}} + \frac{1}{2} .$$

We will now consider some examples. If a Hubble time of $H_o^{-1} = 2\times 10^{10}$ years is used, the value of H_1^{-1} is easily computed from equation (5.622). This provides a means of evaluating t_o and t_1 for a particular model, since

$$t_o = X_o H_1^{-1} , \quad t_1 = X_1 H_1^{-1},$$

where X_o is the value of X corresponding to $y = y_o$, and X_1 is again the value of X corresponding to $y = 1$ on the solution curve of equation (5.517). Tables 1 and 2 set out examples of models in which the present epoch $(t = t_o)$ is specified by the values assigned to the parameters σ_{mo}, σ_{ro}. In Table 1 a value for σ_{mo} that corresponds to Oort's and Shapiro's observational estimates of luminous material in the Universe is used, whereas in Table 2 a value of ten times this is employed. In both cases the density parameter for gravitational radiation at $t = t_o$ is 10^{-4}. This value, which incidently is of a similar order of magnitude to the contributions due to electromagnetic radiation, has been regarded as reasonable in light of the discussion contained in Section 5.5(I). It is however very much smaller than the values considered by IW.

Table 1.

$H_0^{-1} = 2\times 10^{10} yr,\ \sigma_{mo} = 0{\cdot}01,\ \sigma_{ro} = 0{\cdot}0001.$					
μ	ν	$\sigma_{m1}\times 10^{-4}$	$\sigma_{r1}\times 10^{-4}$	$t_0\times 10^{10} yr$	$t_1\times 10^{10} yr$
0.1	3	35.56	2.63	1.93	5.15
	4	46.86	3.91	1.93	3.87
	5	54.47	4.84	1.93	3.31
	10	71.52	7.15	1.93	2.49
0.2	3	22.41	3.73	1.93	7.31
	4	33.14	6.21	1.93	4.89
	5	40.80	8.16	1.93	3.95
	10	58.94	13.26	1.93	2.69
0.3	3	16.03	4.58	1.93	8.98
	4	25.36	8.15	1.93	5.62
	5	32.29	11.07	1.93	4.38
	10	49.34	19.03	1.93	2.82

Table 2.

$H_0^{-1} = 2\times 10^{10} yr,\ \sigma_{mo} = 0{\cdot}1,\ \sigma_{ro} = 0{\cdot}0001.$					
μ	ν	$\sigma_{m1}\times 10^{-4}$	$\sigma_{r1}\times 10^{-4}$	$t_0\times 10^{10} yr$	$t_1\times 10^{10} yr$
0.1	3	133.95	9.92	1.69	17.18
	4	252.46	21.04	1.69	8.52
	5	348.07	30.94	1.69	5.87
	10	593.94	59.39	1.69	3.02
0.2	3	84.97	14.16	1.73	24.66
	4	180.35	33.82	1.69	10.91
	5	263.69	52.73	1.69	7.09
	10	494.19	111.19	1.69	3.29
0.3	3	60.96	17.42	1.78	30.41
	4	138.75	44.59	1.69	12.59
	5	210.02	72.01	1.69	7.91
	10	416.09	160.49	1.69	3.47

If indeed the matter content of the Universe is undergoing a process of conversion into gravitational radiation by the acretion of material into black holes at galactic centres, then the efficiency of the process would lie in the range 5% to 40% (Ruffini and Wheeler, 1971). From equation (5.509) the value of μ in the model would be numerically equal to the efficiency of the emission process. Hence, we have considered models in which μ takes the values 0.1, 0.2, 0.3.

In conclusion of this chapter, I would like to point out that the technique of joining two Friedmann models may also be of use in situations other than that examined here. Indeed, if the equations of state describing the energy distribution in a model universe change at some instant t in its history, for any reason, then the cosmological equation may be solved on either side of the time boundary. The models that result from this procedure may then be 'joined' together by the use of the junction conditions set out in Section 5.4.

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APPENDIX

Comments on the primeval origin of gravitational radiation in a Friedmann universe.

A.1. Introduction.

In this Appendix the cosmological hypothesis is briefly considered. That is, a situation is envisaged in which the postulated universal gravitational radiation field is generated during the early, pre-galactic era of the Universe, as opposed to the view, investigated throughout Chapter 5, that the radiation was galactic in origin.

Although the subsequent analysis is broadly based upon that contained within Chapter 5, a number of differences in detail are imposed upon the model. The primary differences between this and previous work may be summarised as follows. Firstly, it will be supposed that the radiative generation takes place during the period in the model's history usually referred to as the prestellar of 'fireball' era. Secondly, to describe the energy field in the model a particulate description of matter will be adopted, and finally it is assumed that the pressure of matter may no longer be regarded as negligible during this phase. An examination of the resulting model leads us to draw conclusions about the rate of entropy production, and the particle number density.

Although the analysis of the model is not fully developed here, a check is made to ensure that it satisfies the criteria of physical acceptability, which were introduced in Chapter 5.

A.2. The description of energy in the model.

A model of the Universe is considered in which gravitational and electromagnetic radiation and matter are present. For simplicity, the material content will be envisaged as a distribution of a single species of particle of mass m and of its corresponding antiparticle. The number densities of the particle and antiparticle populations are denoted by $n^-(t)$ and $n^+(t)$ respectively, where the total number density of the matter field is given by

$$n(t) = n^- + n^+ . \tag{A.201}$$

Although this arrangement does not rule out the possibility that the particle gas may possess a net charge per unit volume, we will henceforth neglect the small gravitational effect due to the electromagnetic energy of such a charge distribution (see Eddington, 1923).

The electromagnetic and gravitational radiation components of the energy field will be taken to be isotropic. The energy of the gravitational radiation will again be represented by a perfect fluid of density ρ_r and pressure p_r , where equation (5.104) is assumed valid. In addition, we define

$$\rho_r = \mathcal{B}(t)/c^2 , \tag{A.202}$$

so that

$$p_r = \frac{1}{3}\mathcal{B}(t) ,$$

where $\mathcal{B}(t)$ is a general function of the cosmic time t. Similarly, the electromagnetic radiation will be described by a perfect fluid of density ρ_e , pressure p_e and blackbody temperature T_e , where

$$p_e = \frac{1}{3}\rho_e c^2 = \frac{1}{3} a^* T_e^4 ,$$
$$a^* = 7.564 \times 10^{-15} \text{erg cm}^{-3} \text{deg}^{-4}. \quad \text{(A.203)}$$

The specific energy of the material particles, that is their energy per unit mass, is given by

$$e_m = c^2 + e_{int} , \quad \text{(A.204)}$$

where e_{int} denotes their specific internal energy in ergs per gram. We will assume that each particle possesses an internal energy of

$$kT_m/(\gamma-1),$$
$$k = 1.381 \times 10^{-16} \text{erg deg}^{-1},$$

where k is Boltzmann's constant, T_m is the temperature of the matter in ^oK , and γ is the ratio of the specific heats of the particle gas. Then, since there are numerically m^{-1} particles per unit mass, the specific internal energy may be written

$$e_{int} = \frac{kT_m}{m(\gamma-1)} . \quad \text{(A.205)}$$

The matter energy density is related to the specific energy of the particles by the following equation,

$$e_m = \rho_m \mathcal{V} c^2 , \quad \text{(A.206)}$$

where $\mathcal{V}$ is the specific volume, given by

$$\mathcal{V} = (mn)^{-1}. \quad \text{(A.207)}$$

Thus, introducing equations (A.204), (A.205) and (A.207) into equation (A.206), we find that

$$\rho_m = mn + \frac{nkT_m}{c^2(\gamma-1)} . \quad \text{(A.208)}$$

Moreover, the pressure of matter will be given by

$$p_m = nkT_m . \tag{A.209}$$

Since the early history of the Universe is to be examined, it will further be assumed that the electromagnetic radiation and matter contents of the model are in thermal equilibrium with one another. As a consequence of this, we are able to write

$$T_e = T_m \equiv T . \tag{A.210}$$

Hence, by the substitution of this condition into the equations (A.203), (A.208) and (A.209), the total pressure and density in the model are given by

$$p = nkT + \frac{1}{3} a^* T^4 + \frac{1}{3}\mathcal{B} , \tag{A.211}$$

$$\rho = mn + \frac{nkT}{c^2(\gamma - 1)} + \frac{a^* T^4}{c^2} + \frac{\mathcal{B}}{c^2} . \tag{A.212}$$

Finally, if the metric (5.105) is assumed valid, the equations governing the model are given by equations (5.106) and (5.108), where the pressure and density contributions are now given by (A.211) and (A.212).

A.3. <u>The conservation of energy equation, and the particle number density.</u>

By an arguement similar to that following equation (5.304), it may be shown that if particles are neither created, nor destroyed, during the history of the model the particle number density satisfies

$$nR^3 = n_2 R_2^3 , \tag{A.301}$$

where

$$R_2 = R(t_2)\ , \quad n_2 = n(t_2)\ ,$$

are the values of the scale factor and the number density respectively, at some arbitrarily chosen instant $t = t_2$ in cosmic time.

In what follows we wish to consider a model in which the gravitational radiation content interacts with the particle content. In this situation the equation (A.301) is not valid, since we require the material particles to act as a source to enhance the energy of the gravitational radiation. We shall now investigate how the time dependence of $n(t)$ is modified in this situation.

The introduction of the new pressure and density equations, (A.211) and (A.212), into the conservation of energy equation (5.508) gives

$$\frac{d}{dt}\left\{(\rho_m + \rho_e + \rho_r) R^3\right\} + \frac{(p_m + p_e + p_r)}{c^2}\frac{dR^3}{dt} = 0\ . \tag{A.302}$$

Since it is wished to examine the interaction between the material and gravitational radiation components of the energy, it is assumed that the electromagnetic radiation field is noninteracting. This assumption may be expressed as

$$\frac{d}{dt}(\rho_e R^3) + \frac{p_e}{c^2}\frac{dR^3}{dt} = 0\ . \tag{A.303}$$

When (A.203) is substituted into this equation, it is found that the temperature has a time dependence given by

$$TR = T_2 R_2\ . \tag{A.304}$$

Now, if the following definition is made,

$$\mathcal{Y} = R(t)/R_2 ,$$

then the equation (A.303) implies that (A.302) reduces to

$$\frac{d}{d\mathcal{Y}}(\rho_m + \rho_r) + \frac{3}{\mathcal{Y}}\left(\rho_m + \rho_r + \frac{p_m}{c^2} + \frac{p_r}{c^2}\right) = 0 .$$

From the equations (A.211) and (A.212), this equation becomes

$$\frac{d}{d\mathcal{Y}}\left\{mn + \frac{nkT}{c^2(\gamma-1)} + \frac{\mathcal{B}}{c^2}\right\}$$

$$+ \frac{3}{\mathcal{Y}}\left\{mn + \frac{nkT}{c^2(\gamma-1)} + \frac{\mathcal{B}}{c^2} + \frac{nkT}{c^2} + \frac{1}{3}\frac{\mathcal{B}}{c^2}\right\} = 0.$$

Substituting for T from equation (A.304), we find, after some calculation, that the number density has a time dependence of the form

$$n\mathcal{Y}^3 = \left(\frac{\mathcal{Y}+b_1}{\mathcal{Y}}\right)^{3\gamma-4}\left\{-b_2\int\frac{\mathcal{Y}^{3\gamma-4}}{(\mathcal{Y}+b_1)^{3\gamma-3}} \times \frac{d}{d\mathcal{Y}}(\mathcal{Y}^4\mathcal{B})\,d\mathcal{Y} + b_3\right\}, \tag{A.305}$$

where

$$b_1 = \frac{kT_2}{(\gamma-1)mc^2} ,$$

$$b_2 = (mc^2)^{-1} ,$$

$$b_3 = \text{constant of integration.}$$

Moreover, since it is reasonable to suppose that the 'primeval gas' was extremely relativistic, we may assign the value

$$\gamma = 4/3 , \tag{A.306}$$

to the ratio of the specific heats (see Weinberg, 1972a)

Thus equation (A.305) reduces to

$$n\mathcal{Y}^3 = -b_2 \int \frac{1}{(\mathcal{Y}+b_1)} \frac{d}{d\mathcal{Y}}(\mathcal{Y}^4 \mathcal{B})\, d\mathcal{Y} + b_3 \,, \tag{A.307}$$

where b_1 now becomes

$$b_1 = 3kT_2/(mc^2).$$

The equation (A.307) therefore provides a relationship between the density of the gravitational radiation and the particle number density in the early history of the model. Although it is difficult to make a physically significant choice of the function $\mathcal{B}(t)$, which describes the radiation density, we may impose restrictions upon it by introducing thermodynamical considerations.

A.4. <u>Entropy production through the mechanism of gravitational radiation generation.</u>

For a change from one thermodynamical state to an immediately neighbouring one, we have

$$T d\Sigma^* = de + p\, d\mathcal{T} \,, \tag{A.401}$$

where Σ^* and e are the specific entropy and the total specific energy in the model respectively. Since the total density and the total specific energy are related by

$$e = \rho \mathcal{T} c^2,$$

the equation (A.401) implies that

$$\frac{T}{c^2} \frac{d\Sigma^*}{d\mathcal{Y}} = \mathcal{T} \frac{d\rho}{d\mathcal{Y}} + \left(\rho + \frac{p}{c^2}\right) \frac{d\mathcal{T}}{d\mathcal{Y}} \,. \tag{A.402}$$

However, the conservation of energy equation (A.302) implies that

$$\frac{d\rho}{d\mathcal{R}} = -\frac{3}{\mathcal{R}}\left(\rho + \frac{p}{c^2}\right).$$

If this equation is now combined with equation (A.402), it is found that

$$\frac{T}{c^2}\frac{d\Sigma^*}{d\mathcal{R}} = -\left(\rho + \frac{p}{c^2}\right)\left(\frac{\pi}{\mathcal{R}^3}\right)^2 \frac{d}{d\mathcal{R}}\left(\frac{\mathcal{R}^3}{\pi}\right). \qquad (A.403)$$

Since the specific entropy Σ^* must satisfy

$$\frac{d\Sigma^*}{d\mathcal{R}} \geqslant 0, \qquad (A.404)$$

the equation (A.403) implies that

$$\frac{d}{d\mathcal{R}}\left(\frac{\mathcal{R}^3}{\pi}\right) \leqslant 0,$$

provided that $[\rho + (p/c^2)] \geqslant 0$ is satisfied. Hence, from equation (A.207) the condition (A.404) is satisfied provided that

$$\frac{d}{d\mathcal{R}}(n\mathcal{R}^3) \leqslant 0. \qquad (A.405)$$

Also, by inspection of equation (A.307), the function $\mathcal{B}(t)$ governing the gravitational radiation density must satisfy

$$\frac{d}{d\mathcal{R}}(\mathcal{R}^4\mathcal{B}) \geqslant 0, \qquad (A.406)$$

in order that the entropy be constant or increasing.

If the equality holds in the condition (A.404), then correspondingly the equalities in the conditions (A.405) and (A.406) also hold. Thus, a model in which the entropy remains constant has, by an inspection of equation (A.301), particle conservation, and a radiation energy density of the form $\mathcal{B} \propto \mathcal{R}^{-4}$. On the other hand, if the entropy is strictly increasing, then there will be an annihilation of particles accompanied by a corresponding enhancement in the density of the radiation.

The reactions that are envisaged as taking place in the model are the annihilation of particle - antiparticle pairs, which result in the production of gravitons (gravitational radiation). This category of event has been considered by Vladimirov (see Vladimirov, 1964) on the basis of a quantization of the weak (linearized) gravitational field. Although this framework is not well suited to examine the situation under discussion, it does provide some indication of the probability of such reactions occuring. Vladimirov concludes that the reaction cross section of, say, the annihilation of an electron - positron pair, to produce two gravitons, is very small indeed. However, the reaction rate may still be significant since the probability of such an event occurring is considerably amplified by a very high number density, which is to be expected during the early 'fireball phase' of the model. Other reactions, such as particle annihilation to produce a photon - graviton pair, have a considerably higher probability of occuring, but the assumption, expressed in equation (A.303), of the non-interaction of the electromagnetic radiation with the other energy components forbids any consideration of these in such a greatly simplified model.

However, despite its shortcomings in this respect, the model dictates that particle annihilation must take place provided that entropy increases. This result is valid independently of the extent to which the three energy fields, matter, electromagnetic radiation and gravitational radiation, are allowed to interact with one another.

In conclusion of this Section it is interesting to examine what the conditions (A.404) and (A.406) imply about the form of $\mathcal{B}(t)$ used throughout Chapter 5. From equation (5.507), we write

$$\mathcal{B} = \mathcal{B}_2 \, \mathcal{Y}^{\nu-4}, \qquad (A.407)$$

where

$$\mathcal{B}_2 = \rho_{n_1} c^2 \left(\frac{R_2}{R_1}\right)^{\nu-4} \geqslant 0,$$

so that

$$\frac{d}{d\mathcal{Y}}\left(\mathcal{Y}^4 \mathcal{B}\right) = \frac{d}{d\mathcal{Y}}\left(\mathcal{B}_2 \mathcal{Y}^{\nu}\right) = \nu \mathcal{B}_2 \mathcal{Y}^{\nu-1}.$$

Thus, with a radiation 'profile' of the form (A.407), it can be seen from equation (A.406) that the model possesses a satisfactory thermodynamical behaviour provided that

$$\nu \geqslant 0 .$$

A.5. The termination of radiation generation.

In a situation, envisaged in the preceeding Section, when gravitational radiation is generated by the mechanism of particle annihilation, it can be foreseen that the particle number density will ultimately fall to zero if the process is allowed to continue indefinitely. Thereafter, $n(t)$ will take a continuum of physically unacceptable negative values, which may be demonstrated by the following simple example. Suppose the radiation density is described by equation (A.407), where the index ν takes the value unity. Then

$$\frac{d}{d\mathcal{Y}}\left(\mathcal{Y}^4 \mathcal{B}\right) = \mathcal{B}_2 ,$$

so that equation (A.307) becomes

$$nY\!\!/^3 = -b_2\mathcal{B}_2 \int \frac{dY\!\!/}{(Y\!\!/ + b_1)} + b_3 .$$

Integrating this equation, we find that

$$nY\!\!/^3 = b_2\mathcal{B}_2 \ln\left(\frac{1+b_1}{Y\!\!/ + b_1}\right) + n_2 .$$

Thus, as $Y\!\!/$ becomes greater than unity the logarithmic term on the RHS becomes negative. Indeed, as $Y\!\!/$ continues to increase there exists a range of $Y\!\!/$ given by

$$Y\!\!/ > \exp\left\{\frac{1}{b_2\mathcal{B}_2}\left[n_2 + b_2\mathcal{B}_2 \ln(1+b_1)\right]\right\} - b_1 ,$$

for which

$$n < 0.$$

The difficulties encountered here are similar to those found in Chapter 5, and we use the techniques developed there to help overcome them. It is supposed that at the instant characterised by $Y\!\!/ = 1$, the generation of gravitons in the model ceases. Hence for $Y\!\!/ < 1$, when there is particle destruction, we impose a number density of the form (A.307), whereas for $Y\!\!/ > 1$, when there is particle conservation, we impose equation (A.301).

By an appeal to equations (A.211) and (A.212) it can be appreciated that the cosmological equations will give rise to different models of the Universe on either side of the boundary 3-space $t = t_2$. We will conclude this Appendix by briefly demonstrating that the description of a universe in which the 'cosmological hypothesis' is adopted satisfies the necessary continuity conditions at the boundary $t = t_2$.

The pressure p , and the density ρ , are now shown to be continuous at $t = t_2$. If the quantities in the spaces

$t<t_2$ and $t>t_2$ are distinguished from each other by the use of a tilde, for example

$$\mathcal{Y} = R/R_2 \;, \quad t < t_2 \,,$$

$$\tilde{\mathcal{Y}} = \tilde{R}/R_2 \;, \quad t > t_2 \,,$$

and if the notation of Section 5.4 is used, then the density for $t<t_2$ is given by

$$\rho = n(\mathcal{Y})\left\{m + \frac{kT_2}{(\gamma-1)c^2}\frac{1}{\mathcal{Y}}\right\} + \frac{a^* T_2{}^4}{c^2}\frac{1}{\mathcal{Y}^4} + \frac{\mathcal{B}}{c^2} \,.$$

Here $n(\mathcal{Y})$ is chosen arbitr arily provided that equation (A.405) is satisfied, and $\mathcal{B}$ is the corresponding radiation profile satisfying equation (A.307). At $t=t_2$, the constant of integration b_3 in equation (A.307) is chosen such that

$$n(t_2) = n_2 \;, \quad \mathcal{B}(t_2) = \mathcal{B}_2 \,.$$

Thus, as $t \to t_2$ from the space $t<t_2$,

$$\rho_2 = n_2\left\{m + \frac{kT_2}{(\gamma-1)c^2}\right\} + \frac{a^* T_2{}^4}{c^2} + \frac{\mathcal{B}_2}{c^2} \,. \tag{A.501}$$

Now, in the space $t>t_2$, the number density and radiation profile are given by

$$\tilde{n} = n_2 \tilde{\mathcal{Y}}^{-3} \,, \quad \tilde{\mathcal{B}} = \mathcal{B}_2 \tilde{\mathcal{Y}}^{-4} \,,$$

so that equation(A.212) becomes

$$\tilde{\rho} = \frac{n_2}{\tilde{\mathcal{Y}}^3}\left\{m + \frac{kT_2}{(\gamma-1)c^2}\frac{1}{\tilde{\mathcal{Y}}}\right\} + \frac{a^* T_2{}^4}{c^2}\frac{1}{\tilde{\mathcal{Y}}^4} + \frac{\mathcal{B}_2}{c^2}\frac{1}{\tilde{\mathcal{Y}}^4}$$

Hence, as $t \to t_2$ from the space $t>t_2$,

$$\tilde{\rho}_2 = \rho_2 \,. \tag{A.502}$$

Moreover, a similar argument, based upon the equation (A.211) leads to the result that

$$\tilde{p}_2 = p_2 \,. \tag{A.503}$$

Upon substituting the new pressure and density distributions (A.211) and (A.212) into the cosmological equations (5.106) and (5.107), we find that the conditions (A.502) and (A.503) imply that

$$\dot{\tilde{R}}_2 = \dot{R}_2 \,, \quad \ddot{\tilde{R}}_2 = \ddot{R}_2 \,,$$

given that the scale factor satisfies

$$\tilde{R}_2 = R_2 \,. \tag{A.504}$$

The condition (A.504) may be arranged by a suitable choice of the constant of integration associated with the solution of equation (5.106).

Hence the transition from $t < t_2$ to $t > t_2$ is physically acceptable in the sense discussed in Section 5.4.