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Властивості розв'язків лінійного рівняння KdV із φ-субгауссовими початковими умовами

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Properties of solutions to linear KdV equations with φ-sub-Gaussian initial conditions

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Важливий практичний аспект оцінювання статистичних властивостей фізичних систем спирається на ефективне представлення зв'язку між розв'язками рівнянь з частинними похідними та випадковими початковими умовами. У цій роботі досліджуються властивості траєкторій випадкових процесів, що задають розв'язки (в $L_2(\Omega)$) для рівняння Айрі з φ-субгауссовими стаціонарними випадковими початковими умовами. Властивості субгауссовості та φ-субгауссовості є важливими характеристиками випадкових процесів, оскільки вони дають можливість оцінити різні функціонали від цих процесів, і, зокрема, дослідити поведінку їх супремумів. Основні результати роботи – це оцінки для розподілів супремумів випадкових процесів, що задають розв'язки для рівняння Айрі, на обмежених множинах. Застосування отриманих результатів проілюстровано на прикладах у випадках гауссових початкових умов з різними допустимими функціями та φ-субгауссових початкових умов з певними функціями φ, зокрема $\varphi(x) = \exp\{|x|\} - |x| - 1, x \in \mathbb{R}$.

Ключові слова: φ-субгауссові процеси, рівняння Айрі, випадкові початкові умови, розподіл супремуму

In this paper, there are studied sample paths properties of stochastic processes representing solutions (in $L_2(\Omega)$ sense) to the linear Korteweg–de Vries equation (called also the Airy equation) with random initial conditions given by φ-sub-Gaussian stationary processes. The main results are the bounds for the distributions of the suprema for such stochastic processes considered over bounded domains. Also, there are presented some examples to illustrate the results of the study.

Key Words: φ-sub-Gaussian processes, Airy equation, random initial condition, distribution of supremum

1 Introduction

It is well known that effects of dispersion play the important role in the description of linear and nonlinear wave motion.

The Korteweg–de Vries equation is the one of the most popular and well studied nonlinear dispersive partial differential equation used in many areas of physics. In its classical form

$$\frac{\partial}{\partial t}u(t,x) + 6u(t,x)\frac{\partial}{\partial x}u(t,x) + \frac{\partial^3}{\partial^3 x}u(t,x) = 0,$$

$t > 0$, $x \in \mathbb{R}$, this equation was derived by Korteweg and de Vries in 1895 to model the unidirectional propagation of small amplitude long water waves in a shallow canal. Now many generalizations of this equation have been investigated and applied in various areas including fluid dynamics, acoustics, electrodynamics, plasma physics, in modeling shock waves formation, solitons, waves in elastic media, turbulence, traffic flows, mass transport. The above equation with nonlinear term droped is called the linear Korteweg–de Vries equation or the Airy equation.

The purpose of this paper is to investigate sample paths properties of stochastic processes representing solutions of the linear Korteweg–de Vries equation with random initial conditions given by φ-sub-Gaussian stationary processes.

The general theory of φ-sub-Gaussian processes is presented in the classical monograph [4] and its further development can be found in numerous recent studies (see, e.g., [3, 15, 19] and references therein). The properties of sub-Gaussianity and φ-sub-Gaussianity are important characteristics of random processes, as they make it possible to estimate different functionals from these processes, and, in particular, the behavior of their suprema. The theory of φ-sub-Gaussian random processes provides us with powerful techniques and tools suitable not only for obtaining asymptotic results, but also for deriving many useful bounds on the distributions of such processes.

Partial differential equations with random initial conditions have been intensively studied in the literature from different points of view, starting from the papers by J. Kampé de Feriet (1955) and M. Rosenblatt (1967) who introduced rigorous probabilistic tools in this area. In [13, 14] solutions to PDE subject to random initial conditions were investigated by means of Fourier methods, representations of solutions by uniformly convergent series and their approximations in different functional spaces were developed.

The present paper is most closely related to the papers [2, 5, 7, 8, 10, 11, 17]. We continue to investigate properties of solutions to different types of partial differential equations with random initial conditions, in particular, we derive estimates for the distribution of suprema of solutions.

The paper is organized as follows. Section 2 collects important definitions and facts on φ-sub-Gaussian processes needed for our study. In Section 3 we consider stochastic processes representing solutions (in $L_2(\Omega)$ sense) of the Airy equation with random φ-sub-Gaussian initial conditions and state the bounds for the distributions of the suprema for such stochastic processes. Section 4 presents some examples to illustrate the results.

2 φ-sub-Gaussian variables and processes

We present basic definitions and facts on φ-sub-Gaussian variables and processes which will be used in the paper.

Definition 2.1. [4, 16] Let $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ be a continuous even convex function. The function φ is an Orlicz N-function if $\varphi(0) = 0, \varphi(x) > 0$ as $x \neq 0$ and the following conditions hold: $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0,\ \lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

Condition Q. Let φ be an N-function which satisfies $\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0$, where the case $c = \infty$ is possible.

Definition 2.2. [4, 12] Let φ be an N-function satisfying condition Q and $\{\Omega, L, \mathbf{P}\}$ be a standard probability space. The random variable ζ belongs to the space $Sub_\varphi(\Omega)$, if $\mathsf{E}\zeta = 0$, $\mathsf{E}\exp\{\lambda\zeta\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$

$$E\exp\{\lambda\zeta\} \le \exp\{\varphi(\lambda a)\}.$$

The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\zeta) = \sup_{\lambda\neq 0} \frac{\varphi^{(-1)}\left(\ln \mathsf{E}\exp(\lambda\zeta)\right)}{|\lambda|},$$

which can be written equivalently as $\tau_\varphi(\zeta) = \inf\{a > 0 : E\exp\{\lambda\zeta\} \le \exp\{\varphi(a\lambda)\}$, and it is called the φ-sub-Gaussian standard of the random variable ζ.

Definition 2.3. [9] A family Δ of random variables $\zeta \in Sub_\varphi(\Omega)$ is called strictly φ-sub-Gaussian if there exists a constant C_Δ such that for all countable sets I of random variables $\zeta_i \in \Delta$, $i \in I$, the following inequality holds:

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\zeta_i\right) \le C_\Delta\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\zeta_i\right)^2\right)^{1/2}. \quad (2.1)$$

The constant C_Δ is called the *determining* constant of the family Δ.

The linear closure of a strictly φ-sub-Gaussian family Δ in the space $L_2(\Omega)$ is the strictly φ-sub-Gaussian with the same determining constant ([9]).

Definition 2.4. [9] Random process $\zeta = \{\zeta(t), t \in T\}$ is called (strictly) φ-sub-Gaussian if the family of random variables $\{\zeta(t), t \in T\}$ is (strictly) φ-sub-Gaussian with a determining constant C_ζ.

Let K be a deterministic kernel and suppose that the process $X = \{X(t), t \in T\}$ can be represented in the form $X(t) = \int_T K(t,s)\, d\xi(s)$, where $\xi(t)$, $t \in T$, is a strictly φ-sub-Gaussian random process and the integral above is defined in the mean-square sense. Then the process $X(t)$, $t \in T$, is strictly φ-sub-Gaussian random process with the same determining constant (see [9]).

Definition 2.5. [4, 16] Let $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ be an N-function. The function φ^* defined by

$$\varphi^*(x) = \sup_{y\in\mathbb{R}}(xy - \varphi(y))$$

is called the Young-Fenchel transform (or convex conjugate) of the function φ.

For a φ-sub-Gaussian random variable ζ the following estimate holds for its tail distribution:

$$P\{|\zeta| > u\} \le 2\exp\left\{-\varphi^*\left(\frac{u}{\tau_\varphi(\zeta)}\right)\right\}. \qquad (2.2)$$

For φ-sub-Gaussian random processes one can evaluate the distribution of their suprema in terms of the function φ^* using entropy methods (see [4]).

To derive the main results in Section 3 we will need additional notions and statements.

Lemma 2.1. *[6] Let $Z(u), u \ge 0$ be a continuous monotonically increasing function such that $Z(u) > 0$ and $\frac{u}{Z(u)}$ is nondecreasing for $u \ge u_0$, where $u_0 \ge 0$ is a constant. Then for $u > 0, v > 0$*

$$\min(\frac{u}{v}, 1) < \frac{Z(u+u_0)}{Z(v+u_0)}.$$

Definition 2.6. [10] The function $Z(u), u \ge 0$, is called admissible for the space $Sub_\varphi(\Omega)$, if for Z the conditions of Lemma 2.1 hold and for some $\varepsilon > 0$

$$\int_0^\varepsilon \Psi\Big(\ln\Big(Z^{(-1)}\Big(\frac{1}{s}\Big) - u_0\Big)\Big)ds < \infty,$$

where $\Psi(v) = \frac{v}{\varphi^{(-1)}(v)}, v > 0$.

Consider a separable φ-sub-Gaussian process defined on a separable metric space (T, d), where $T = \{a_i \le t_i \le b_i, i = 1, 2\}$ and $d(t,s) = \max_{i=1,2}|t_i - s_i|$, $t = (t_1, t_2)$, $s = (s_1, s_2)$.

Theorem 2.1. *[10] Assume that $X = \{X(t), t \in T\}$ is a separable φ-sub-Gaussian process such that*

$$\sup_{\substack{d(t,s)\le h,\\ t,s\in T}} \tau_\varphi(X(t) - X(s)) \le \sigma(h), \qquad (2.3)$$

where $\{\sigma(h),\ 0 < h \le \max_{i=1,2}|b_i - a_i|\}$ is a monotonically increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$ and for some $\varepsilon > 0$

$$\int_0^\varepsilon \Psi\Big(\ln\frac{1}{\sigma^{(-1)}(u)}\Big)\, du < \infty, \qquad (2.4)$$

where $\Psi(v) = \frac{v}{\varphi^{(-1)}(v)}$. Then

$$P\Big\{\sup_{t\in T}|X(t)| > u\Big\} \le 2A(u,\theta)$$

for all $0 < \theta < 1$ and

$$u > \frac{2I_\varphi(\min(\theta\varepsilon_0, \gamma_0))}{\theta(1-\theta)},$$

where
$A(u,\theta) =$
$$= \exp\Big\{-\varphi^*\Big(\frac{1}{\varepsilon_0}\Big(u(1-\theta) - \frac{2}{\theta}I_\varphi(\min(\theta\varepsilon_0, \gamma_0))\Big)\Big)\Big\},$$

and

$$\varepsilon_0 = \sup_{t\in T}\tau_\varphi(X(t)), \quad \gamma_0 = \sigma(\max_{i=1,2}|b_i - a_i|),$$

$\varphi^*(u)$ is the Young-Fenchel transform of the function φ,
$I_\varphi(\delta) =$

$$\int_0^\delta \Psi\left[\ln\left[\Big(\frac{b_1 - a_1}{2\sigma^{(-1)}(u)} + 1\Big)\Big(\frac{b_2 - a_2}{2\sigma^{(-1)}(u)} + 1\Big)\right]\right] du.$$

3 Results

Consider the Airy equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial^3 x},\ t > 0,\ x \in \mathbb{R}, \qquad (3.1)$$

subject to the random initial condition

$$u(0,x) = \eta(x),\ x \in \mathbb{R}, \qquad (3.2)$$

where η is a stochastic process satisfying the condition below.

A $\eta(x), x \in \mathbb{R}$ is a real, measurable, mean-square continuous stationary (in wide sense) stochastic process, which is strictly φ-sub-Gaussian with the determining constant c_η.

Let $B(x), x \in \mathbb{R}$, be a covariance function of η, therefore, we have the representation

$$B(x) = \int_{\mathbb{R}} \cos(\lambda x) dF(\lambda), \quad (3.3)$$

where $F(\lambda)$ is a spectral measure, and for η the spectral representation holds

$$\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} M(d\lambda). \quad (3.4)$$

The stochastic integral is considered as $L_2(\Omega)$ integral. The complex-valued orthogonal random measure M is such that $\mathsf{E}|M(d\lambda)|^2 = F(d\lambda)$.

Consider the process $u(t,x), t > 0, x \in \mathbb{R}$, defined by

$$u(t,x) = \int_{\mathbb{R}} g(t, x-y)\eta(y)dy, \quad (3.5)$$

where the function g is the fundamental solution to equation (3.1):

$$\begin{aligned} g(t,x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\alpha x - i\alpha^3 t}\, d\alpha \quad (3.6)\\ &= \frac{1}{\pi}\int_0^\infty \cos(\alpha x + \alpha^3 t)\, d\alpha \\ &= \frac{1}{\sqrt{\pi}\sqrt[3]{3t}} Ai\Big(\frac{x}{\sqrt[3]{3t}}\Big),\ t > 0,\ x \in \mathbb{R}, \end{aligned}$$

and

$$Ai(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\alpha x + \frac{\alpha^3}{3}\right) d\alpha,\ x \in \mathbb{R},$$

is the Airy function of the first kind.

Taking into account (3.4), we can write the following representation of the process given by (3.5):

$$u(t,x) = \int_{\mathbb{R}} \exp\Big\{i\lambda x + i\lambda^3 t\Big\} M(d\lambda). \quad (3.7)$$

The process (3.7) can be interpreted as the mean-square or $L_2(\Omega)$ solution to the Cauchy problem (3.1)–(3.2) (see [1]).

From the representation (3.7) the covariance of the field u can be calculated:

$$\begin{aligned} &Cov\big(u(t,x), u(s,y)\big) \\ &= \int_{\mathbb{R}} \exp(i\lambda(x-y) + i\lambda^3(t-s))dF(\lambda) \\ &= \int_{\mathbb{R}} \cos(\lambda(x-y) + i\lambda^3(t-s))dF(\lambda) \quad (3.8) \end{aligned}$$

From (3.8) we see that the random field u is stationary with respect to time and space variables.

Theorem 3.1. *Let $u(t,x), t > 0, x \in \mathbb{R}$, be the random field given by (3.7) and assumption A hold. Assume that $Z(u), u \ge 0$, is a function satisfying conditions of Lemma 2.1 and the following integral converges:*

$$C_Z^2 = \int_{\mathbb{R}} Z^2\Big(\frac{1}{2}|\lambda|(1+\lambda^2)+u_0\Big)F(d\lambda) < \infty. \quad (3.9)$$

Then

$$\sigma(h) := \sup_{\max\{|t-t_1|,|x-x_1|\}\le h} \tau_\varphi(u(t,x) - u(t_1,x_1)) \le$$

$$\le c_\eta C_Z \Big(Z\Big(\frac{1}{h} + u_0\Big)\Big)^{-1}. \quad (3.10)$$

Proof. Since the random field u is strictly φ-sub-Gaussian, we have:

$$\sup_{\max\{|t-t_1|,|x-x_1|\}\le h} \tau_\varphi(u(t,x) - u(t_1,x_1)) \le$$

$$\le c_\eta \sup_{\max\{|t-t_1|,|x-x_1|\}\le h} \Big(\mathsf{E}(u(t,x) - u(t_1,x_1))^2\Big)^{1/2}, \quad (3.11)$$

$$\mathsf{E}\Big(u(t,x) - u(t_1,x_1)\Big)^2 = \int_{\mathbb{R}} |b(\lambda)|^2 F(d\lambda), \quad (3.12)$$

where

$$b(\lambda) = \exp\{i(\lambda x + \lambda^3 t)\} - \exp\{i(\lambda x_1 + \lambda^3 t_1)\}.$$

$$|b(\lambda)|^2 \le 4\sin^2\Big(\frac{\lambda(x-x_1) + \lambda^3(t-t_1)}{2}\Big),$$

and for $|t - t_1| \le h, |x - x_1| \le h$:

$$|b(\lambda)|^2 \le 4\Big(\min\Big(\frac{h}{2}(\lambda+\lambda^3), 1\Big)\Big)^2.$$

Using Lemma 2.1 we can write the bound

$$|b(\lambda)|^2 \le 4\frac{Z^2(\frac{|\lambda+\lambda^3|}{2} + u_0)}{Z^2(\frac{1}{h} + u_0)}. \quad (3.13)$$

Substituting (3.13) in (3.12) and using inequality (3.11), we obtain (3.10).

Theorem 3.2. *Let $u(t,x), a \le t \le b, c \le x \le d$, be a separable modification of the stochastic process given by (3.7) and assumption A hold. Assume that $Z(u), u \ge 0$, is an admissible function for the space $Sub_\varphi(\Omega)$, and the integral (3.9) converges. Then for $0 < \theta < 1$ and*

$$u > \frac{2\hat{I}_\varphi(\min(\theta\Gamma, \gamma_0))}{\theta(1-\theta)}$$

the following inequality hold true:

$$\mathsf{P}\Big\{ \sup_{\substack{a\le t\le b;\\ c\le x\le d}} |u(t,x)| > u \Big\} \le 2A(u,\theta), \tag{3.14}$$

where

$$A(u,\theta) = \\ = \exp\Big\{ -\varphi^*\Big(\frac{1}{\Gamma}(u(1-\theta) - \frac{2}{\theta}\hat{I}_\varphi(\min(\theta\Gamma,\gamma_0)))\Big)\Big\};$$

$$\hat{I}_\varphi(\sigma) = \\ = \int_0^\sigma \Psi\Big(\ln\Big[\Big(\frac{b-a}{2}\Big(Z^{(-1)}\Big(\frac{c_\eta C_Z}{s}\Big) - u_0\Big) + 1\Big)\times \\ \times\Big(\frac{d-c}{2}\Big(Z^{(-1)}\Big(\frac{c_\eta C_Z}{s}\Big) - u_0\Big) + 1\Big)\Big]\Big)ds; \tag{3.15}$$

$$\Gamma = c_\eta\Big(\int_{\mathbb{R}} F(d\lambda)\Big)^{1/2}, \quad \Psi(u) = \frac{u}{\varphi^{(-1)}(u)},$$

$$\gamma_0 = \frac{c_\eta C_Z}{Z(\frac{1}{\varkappa} + u_0)}, \ \varkappa = \max(b-a, d-c);$$

C_Z is defined in (3.9).

Proof. The assertion of this theorem follows from Theorems 2.1 and 3.1.

Note that from Theorem 3.1 we have that condition (2.3) of Theorem 2.1 holds with $\sigma(h) = c_\eta C_Z\Big(Z(\frac{1}{h} + u_0)\Big)^{-1}$. We also have

$$\varepsilon_0 = \sup_{(t,x)\in[a,b]\times[c,d]} \tau_\phi(u(t,x)) \le \\ \le c_\eta \sup_{(t,x)\in[a,b]\times[c,d]} \Big(\mathsf{E}(u(t,x))^2\Big)^{1/2} \le \\ \le c_\eta\Big(\int_{\mathbb{R}} F(d\lambda)\Big)^{1/2} =: \Gamma < \infty.$$

Remark 3.1. On convergence of the integral (3.7). Following [1], in the present paper we treat the solution to the Airy equation with stationary random initial condition as a mean square solution, that is, integral (3.7) is in $L_2(\Omega)$. In the papers [2], [10] the integral functionals (with kernels of a particular form) of φ-sub-Gaussian random processes were studied as solutions of higher order partial differential equations with random initial conditions. It follows directly from the results in [2], that under the conditions of Theorem 3.2 the integral given by formula (3.7), that is,

$$u(t,x) = \int_{\mathbb{R}} \exp\Big\{ i\lambda x + i\lambda^3 t\Big\} M(d\lambda).$$

converges with probability 1 for $|x| \le A$, $0 \le t \le T$, where $A > 0$ and $T > 0$ are some constants.

4 Examples

Example 4.1. Let $\eta = \{\eta(u), u \in \mathbb{R}\}$ be a centered Gaussian random process satisfying Assumption A. Then $c_\eta = 1$, $\varphi(x) = \frac{x^2}{2}$, $\varphi^*(x) = \frac{x^2}{2}$, $\Psi(x) = \frac{1}{\sqrt{2}}x^{1/2}$. Consider the following admissible function

$$Z(u) = \ln^\alpha(u+1), \quad u \ge 0, \quad \alpha > 1/2.$$

In this case

$$u_0 = e^\alpha - 1, \quad Z^{(-1)}(v) = \exp\Big\{v^{\frac{1}{\alpha}}\Big\} - 1,$$

$$Z(v + u_0) = \ln^\alpha(v + e^\alpha),$$

$$C_Z^2 = \int_{\mathbb{R}} \ln^{2\alpha}\Big(\frac{|\lambda|(1+\lambda^2)}{2} + e^\alpha\Big)F(d\lambda).$$

The above integral converges if the following integral converges

$$\int_{\mathbb{R}} \ln^{2\alpha}\Big(|\lambda| + e^\alpha\Big)F(d\lambda) < \infty. \tag{4.16}$$

That is, if condition (4.16) holds true, then Theorem 3.2 holds. It follows from (3.15) that

$$\hat{I}_\varphi(\delta) = \int_0^\delta \frac{1}{\sqrt{2}}\Big(\ln\Big[\Big(\frac{b-a}{2}\Big(\exp\Big\{\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}\Big\} - \\ -e^\alpha\Big) + 1\Big) \times \Big(\frac{d-c}{2}\Big(\exp\Big\{\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}\Big\} - \\ -e^\alpha\Big) + 1\Big)\Big]\Big)^{\frac{1}{2}} ds.$$

Let $\frac{d-c}{2}e^\alpha > 1$ and $\frac{b-a}{2}e^\alpha > 1$, then

$$\hat{I}_\varphi(\delta) \le \frac{1}{\sqrt{2}} \\ \times \int_0^\delta \Big(\ln\Big(\frac{d-c}{2}\frac{b-a}{2}\exp\Big\{2\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}\Big\}\Big)\Big)^{\frac{1}{2}} ds \\ \le \frac{1}{\sqrt{2}}\int_0^\delta \ln\Big(\frac{(d-c)(b-a)}{4}\Big)^{\frac{1}{2}} ds \\ + \int_0^\delta \Big(\frac{C_Z}{s}\Big)^{\frac{1}{2\alpha}} ds \\ = \frac{\delta}{\sqrt{2}}\Big(\ln\Big(\frac{(d-c)(b-a)}{4}\Big)\Big)^{\frac{1}{2}} \\ + \delta\Big(\frac{C_Z}{\delta}\Big)^{\frac{1}{2\alpha}}\Big(1 - \frac{1}{2\alpha}\Big)^{-1}. \tag{4.17}$$

It follows from (3.14) that in this case for

$$u > \frac{2\hat{I}_\varphi(\min(\theta\Gamma,\gamma_0))}{\theta(1-\theta)}$$

we have

$$P\Big\{\sup_{\substack{a\le t\le b,\\ c\le x\le d}}|u(t,x)|>u\Big\}\le$$

$$\le \exp\Big\{-\frac{1}{2}\Big(\frac{1}{\Gamma}\Big(u(1-\theta)-\frac{2}{\theta}\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))\Big)\Big)^2\Big\},$$

$$\gamma_0=\frac{C_Z}{\ln^{\alpha}\left(\frac{1}{\varkappa}+e^{\alpha}\right)}.$$

If θ is such that $\theta\Gamma<\gamma_0$ $\left(\theta<\frac{\gamma_0}{\Gamma}\right)$, then for

$$u>\sup_{0<\theta<\frac{\gamma_0}{\Gamma}}\frac{2\hat{I}_{\varphi}(\theta\Gamma)}{\theta(1-\theta)}$$

we get the estimate

$$P\Big\{\sup_{\substack{a\le t\le b,\\ c\le x\le d}}|u(t,x)|>u\Big\}\le$$

$$\le \inf_{0<\theta<\frac{\gamma_0}{\Gamma}}\exp\Big\{-\frac{1}{2}\Big(\frac{1}{\Gamma}\Big(u(1-\theta)-\frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big)^2\Big\},$$

and if $\frac{\gamma_0}{\Gamma}>1$ then for

$$u>\sup_{0<\theta<1}\frac{2\hat{I}_{\varphi}(\theta\Gamma)}{\theta(1-\theta)}$$

we get

$$P\Big\{\sup_{\substack{a\le t\le b,\\ c\le x\le d}}|u(t,x)|>u\Big\}\le$$

$$\le \inf_{0<\theta\le 1}\exp\Big\{-\frac{1}{2}\Big(\frac{1}{\Gamma}\Big(u(1-\theta)-\frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big)^2\Big\}.$$

Example 4.2. Let $\eta=\{\eta(u),u\in\mathbb{R}\}$ be a centered Gaussian random process, as in example 4.1. Consider the admissible function $Z(u)=|u|^{\alpha}$, $0<\alpha\le 1$. In this case

$$u_0=0,\quad Z^{(-1)}(u)=u^{\frac{1}{\alpha}},\quad u>0,$$

$$C_Z^2=\int_{\mathbb{R}}\Big(\frac{|\lambda|(1+\lambda^2)}{2}\Big)^{2\alpha}F(d\lambda)=$$
$$=4^{-\alpha}\int_{\mathbb{R}}\lambda^{2\alpha}(1+\lambda^2)^{2\alpha}F(d\lambda).\qquad(4.18)$$

This integral converges if the next integral converges

$$\int_{\mathbb{R}}\lambda^{6\alpha}F(d\lambda)<\infty.\qquad(4.19)$$

That is, if (4.19) holds, then Theorem 3.2 holds. It follows from (3.15) that

$$\hat{I}_{\varphi}(\delta)=\frac{1}{\sqrt{2}}\int_0^{\delta}\Big(\ln\Big[\Big(\frac{b-a}{2}\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}+1\Big)\times\Big(\frac{d-c}{2}\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}+1\Big)\Big]\Big)^{\frac{1}{2}}ds.$$

For $0\le\beta<1$, $x>0$, $y>0$, we have

$$\ln((1+x)(1+y))\le\frac{1}{\beta}\big(x^{\beta}+y^{\beta}\big),$$

therefore, in the case of $\beta<\alpha$ we obtain

$$\hat{I}_{\varphi}(\delta)\le\frac{1}{\sqrt{2\beta}}\big(C_Z\big)^{\frac{\beta}{2\alpha}}\delta^{(1-\frac{\beta}{2\alpha})}\frac{1}{1-\frac{\beta}{2\alpha}}\Big[\Big(\frac{b-a}{2}\Big)^{\frac{\beta}{2}}+\Big(\frac{d-c}{2}\Big)^{\frac{\beta}{2}}\Big]=:\hat{J}_{\varphi}(\delta,\beta).\qquad(4.20)$$

It follows from (3.14) that in this case for

$$u>\frac{2\hat{J}_{\varphi}(\min(\theta\Gamma,\gamma_0),\beta)}{\theta(1-\theta)},\quad\gamma_0=C_Z\varkappa^{\alpha}$$

we get the estimate

$$P\Big\{\sup_{\substack{a\le t\le b\\ c\le x\le d}}|u(t,x)|>u\Big\}\le$$

$$\le\inf_{\theta,\beta}\exp\Big\{-\frac{1}{2}\Big(\frac{1}{\Gamma}\Big(u(1-\theta)-\frac{2}{\theta}\hat{J}_{\varphi}(\min(\theta\Gamma,\gamma_0),\beta)\Big)\Big)^2\Big\}.$$

Example 4.3. Let $y=\{y(u),\,u\in R\}$ be a φ-sub-Gaussian random process with

$$\varphi(x)=\begin{cases}\frac{x^2}{\alpha}, & |x|\le 1,\alpha>2,\\ \frac{|x|^{\alpha}}{\alpha}, & |x|\ge 1,\alpha>2.\end{cases}$$

In this case for p such that $\frac{1}{p}+\frac{1}{\alpha}=1$

$$\varphi^*(x)=\begin{cases}\alpha x^2/4, & 0\le|x|\le 2/\alpha,\\ |x|-1/\alpha, & 2/\alpha<|x|\le 1,\\ x^p/p, & |x|>1,\end{cases}$$

and

$$\Psi(u)=\begin{cases}\frac{1}{\alpha^{1/2}}u^{1/2}, & 0<u<\frac{1}{\alpha},\\ \frac{1}{\alpha^{1/\alpha}}u^{1-\frac{1}{\alpha}}, & u>\frac{1}{\alpha}.\end{cases}$$

Let $Z(u)$ be admissible function for this space. Then for

$$u>\max\Big(1,\frac{2\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))}{\theta(1-\theta)}\Big),$$

$$P\Big\{ \sup_{\substack{a\le t\le b\\ c\le x\le d}} |U(t,x)| > u \Big\}$$

$$\le 2\exp\Big\{ -\frac{1}{p}\Big(\frac{1}{\Gamma}\Big(u(1-\theta)-\frac{2}{\theta}\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))\Big)\Big)^{p}\Big\}.$$

Example 4.4. Consider
$\varphi(x) = \exp\{|x|\} - |x| - 1, x\in\mathbb{R}$.
Then
$\varphi^*(x) = (|x|+1)\ln(|x|+1) - |x|$.
Let us take the admissible function

$$Z(u) = \ln^{\alpha}(u+1), u \ge 0, \alpha > 1.$$

In this case

$$u_0 = e^{\alpha} - 1,$$
$$Z^{(-1)}(v) = \exp\Big\{ v^{\frac{1}{\alpha}} \Big\} - 1,$$
$$Z(v+u_0) = \ln^{\alpha}(v+e^{\alpha}),$$

$$C_Z^2 = \int_{\mathbb{R}} \ln^{2\alpha}\Big(\frac{|\lambda|(1+\lambda^2)}{2} + e^{\alpha}\Big) F(d\lambda).$$

The above integral converges if the following integral converges

$$\int_{\mathbb{R}} \ln^{2\alpha}\Big(|\lambda| + e^{\alpha}\Big) F(d\lambda) < \infty. \tag{4.21}$$

That is, if condition (4.21) holds true, then Theorem 3.2 holds. It follows from (3.15) that

$$\hat{I}_{\varphi}(\delta) = \int_0^{\delta} \Psi\Big(\ln\Big[\Big(\frac{b-a}{2}\Big(\exp\Big\{\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}\Big\} - e^{\alpha}\Big) + 1\Big) \times \Big(\frac{d-c}{2}\Big(\exp\Big\{\Big(\frac{C_Z}{s}\Big)^{\frac{1}{\alpha}}\Big\} - e^{\alpha}\Big) + 1\Big)\Big]\Big)\, ds.$$

It follows from (3.14) that in this case for

$$u > \frac{2\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))}{\theta(1-\theta)}$$

we have

$$P\Big\{ \sup_{\substack{a\le t\le b,\\ c\le x\le d}} |u(t,x)| > u \Big\} \le$$

$$\le \exp\Big\{ -\Big[\Big(\Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))\Big)\Big| + 1\Big)\ln\Big(\Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))\Big)\Big| + 1\Big) - \Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\min(\theta\Gamma,\gamma_0))\Big)\Big|\Big]\Big\},$$

where
$$\gamma_0 = \frac{C_Z}{\ln^{\alpha}\left(\frac{1}{\varkappa} + e^{\alpha}\right)}.$$

If θ is such that $\theta\Gamma < \gamma_0$ $\left(\theta < \frac{\gamma_0}{\Gamma}\right)$, then for

$$u > \sup_{0<\theta<\frac{\gamma_0}{\Gamma}} \frac{2\hat{I}_{\varphi}(\theta\Gamma)}{\theta(1-\theta)}$$

we get the estimate

$$P\Big\{ \sup_{\substack{a\le t\le b,\\ c\le x\le d}} |u(t,x)| > u \Big\} \le$$

$$\le \inf_{0<\theta<\frac{\gamma_0}{\Gamma}} \exp\Big\{ -\Big[\Big(\Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big| + 1\Big)\times\ln\Big(\Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big| + 1\Big) - \Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big|\Big]\Big\},$$

and if $\frac{\gamma_0}{\Gamma} > 1$ then for

$$u > \sup_{0<\theta<1} \frac{2\hat{I}_{\varphi}(\theta\Gamma)}{\theta(1-\theta)}$$

we get

$$P\Big\{ \sup_{\substack{a\le t\le b,\\ c\le x\le d}} |u(t,x)| > u \Big\} \le$$

$$\le \inf_{0<\theta\le 1} \exp\Big\{ -\Big[\Big(\Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big| + 1\Big)\times\ln\Big(\Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big| + 1\Big) - \Big|\frac{1}{\Gamma}\Big(u(1-\theta) - \frac{2}{\theta}\hat{I}_{\varphi}(\theta\Gamma)\Big)\Big|\Big]\Big\}.$$

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