Some properties of solutions of fully nonlinear partial differential inequalities

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1 Introduction

We consider scalar functions u satisfying second-order partial differential inequalities of the form

$$F(x, u, Du, D^2u) \geq 0 \ , \ x \in \Omega \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set, F is a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$ with values in $\mathbb{R}$ and $\mathcal{S}^N$ denotes the set of $N \times N$ symmetric matrices.
Due to the fully nonlinear character of F, the weak notion of viscosity solution is an appropriate one for the analysis of the inequality (1.1).
We recall for the convenience of the reader that a viscosity solution of (1.1) is any function $u \in LSC(\Omega)$, the set of lower semicontinuous functions $u : \Omega \to \mathbb{R}$, such that

$$F(x_0, u(x_0), D\zeta(x_0), D^2\zeta(x_0)) \geq 0$$

for all $\zeta \in C^2(\Omega)$ and all $x_0 \in \Omega$ such that $u - \zeta$ has a local minimum at x_0.
Viscosity solution of the inequality

$$F(x, u, Du, D^2u) \leq 0 \ , \ x \in \Omega \tag{1.2}$$

are similarly defined by replacing lower semicontinuity with upper semicontinuity and local minima with local maxima.

Finally, u is a viscosity solution of the equation

$$F(x, u, Du, D^2u) = 0 \ , \ x \in \Omega$$

if it is simultaneously a viscosity solution of (1.1) and (1.2).
We refer to [16] for the general theory of viscosity solutions and to [22], [12] and the references therein for surveys of subsequent developments of the theory.

In the special case $F(x, r, p, M) = -\mathrm{tr}\, M$, the inequality (1.1) becomes

$$-\Delta u \leq 0 \text{ in } \Omega \tag{1.3}$$

and it is known, see [12] for an explicit proof, that if u is a viscosity solution of (1.3) then for every ball $B \subset \Omega$ and every h such that $\Delta h = 0$ in B, the inequality $u \geq h$ on ∂B implies $u \leq h$ in B, and, conversely, that any u satisfying the above property is viscosity solution of (1.3).
This means that viscosity solutions of the inequality (1.3) are in fact superharmonic functions in the sense of potential theory, see [24] for example.
This remark suggests that several parts of the classical theory of superharmonic functions can be generalized to the framework of viscosity solutions. This has been, quite naturally indeed, a fruitful direction of investigation, leading to Perron type results about existence, see [25], comparison, uniqueness and stability results for possibly degenerate elliptic F, see [23], [16], [26], and strong maximum principle, Harnack inequalities, Hölder and Sobolev estimates for uniformly elliptic F, see [9], [10].

Our aim here is to report on a closely related aspect of this fundamental line of research, developed in recent years with A. Cutri and F. Leoni, whom I would like to thank for precious collaboration, aimed at establishing the validity of viscosity solutions versions of well - known classical results of the theory of superharmonic functions such as the Hadamard three - circles theorem, the Liouville's theorem and the Phragmen - Lindelöf principle.

2 Preliminaries

A continuous $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$ is uniformly elliptic if there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda\, \mathrm{tr}\,(Q) \leq F(x, t, p, M) - F(x, t, p, M + Q) \leq \Lambda\, \mathrm{tr}\,(Q) \tag{2.1}$$

for all $M, Q \in \mathcal{S}^N$ with Q nonnegative definite and for every fixed $t \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $x \in \Omega$. We denote here by $\mathrm{tr}\,(Q)$ the trace of a matrix Q.
We will consider functions F satisfying (2.1) for some fixed λ, Λ and

$$F(x,t,0,0) = 0 \tag{2.2}$$

$$F(x,t,p,0) \leq \sigma(|x|)|p| + h(x)t^\alpha \tag{2.3}$$

where $\alpha \geq 1$ and σ and h are continuous real valued functions such that

$$|x|\sigma(|x|) \geq -\Lambda(N-1)\ ,\ h(x) \leq 0 \tag{2.4}$$

for all $(x,t,p) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$.
The Pucci maximal operator, see [29],

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} (-\mathrm{tr}\,(AM))\ \ ,\ M \in \mathcal{S}^N$$

where

$$\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}^N\ :\ \lambda|\xi|^2 \leq A\,\xi \cdot \xi \leq \Lambda|\xi|^2,\ \forall\, \xi \in \mathbb{R}^N\}$$

is the fundamental example of operator satisfying (2.1). For $\lambda = \Lambda = 1$, the operator $\mathcal{M}^+_{\lambda,\Lambda}$ coincides with the Laplace operator $-\Delta$.
The following representation of $\mathcal{M}^+_{\lambda,\Lambda}(M)$, see [9], holds

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = -\lambda \sum e_i^+ - \Lambda \sum e_i^-\ , \tag{2.5}$$

where e_i^+, e_i^- are, respectively, the positive and negative eigenvalues of M.
Formula (2.5) implies that any uniformly elliptic F satisfies

$$F(x,t,p,M) \leq F(x,t,p,0) + \mathcal{M}^+_{\lambda,\Lambda}(M) \tag{2.6}$$

for all $x \in \Omega$, $t \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $M \in \mathcal{S}^N$.
As a consequence of the above inequality, if F satisfies conditions (2.1), (2.2), (2.3), (2.4), any viscosity solution of

$$u \geq 0\ ,\ F(x,u,Du,D^2u) \geq 0\ ,\ x \in \Omega$$

is also a viscosity solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| + h(x)u^\alpha \geq 0\ ,\ x \in \Omega$$

and, a fortiori, of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| \geq 0\ ,\ x \in \Omega. \tag{2.7}$$

A large class of fully nonlinear operators satisfying our set of assumptions is that of Bellman - Isaacs operators

$$\inf_{\gamma\in\mathcal{G}} \sup_{\beta\in\mathcal{B}} \left[-\mathrm{tr}\,(A^{\beta,\gamma}(x)D^2u) + b^{\beta,\gamma}(x)\cdot Du + c^{\beta,\gamma}(x)u\right] \tag{2.8}$$

arising in stochastic optimal control and differential games theory, see [19]. Indeed, it is easy to check this, provided that $A^{\beta,\gamma} \in \mathcal{A}_{\lambda,\Lambda}$ for some $0 < \lambda \leq \Lambda$, the vectorfields $b^{\beta,\gamma}$ and functions $c^{\beta,\gamma}$ satisfy

$$|b^{\beta,\gamma}(x)| \leq \sigma(|x|)\ ,\ c^{\beta,\gamma}(x) \leq h(x)$$

with h and σ as in (2.4) for any (β,γ) in the give parameters sets $\mathcal{B},\mathcal{G}$. Observe that the Pucci maximal operator can be trivially represented in the form (2.8) and also that our assumptions (2.1), (2.2), (2.3), (2.4) are satisfied, in particular, by linear operators in non divergence form

$$-\mathrm{tr}\,(A(x)D^2u) + b(x)\cdot Du + c(x)u$$

under suitable conditions on the data A, b, c.

3 A nonlinear three spheres theorem

The object of this section is the following version of the Hadamard three spheres theorem. Our result shows, roughly speaking, that the minimum on the ball of radius r of a lower semicontinuos viscosity solution of

$$u \geq 0\ , \qquad F(x,u,Du,D^2u) \geq 0$$

is a concave function of the "fundamental solution" of the Pucci maximal operator, thus extending to non smooth solutions of general differential inequalities with gradient dependence the classical linear result, see [28], as well as previous results for the Pucci operator, see [27].

Theorem 3.1 *Let $u \in LSC(\Omega)$ be a viscosity solution of*

$$u \geq 0\,, \qquad F(x,u,Du,D^2u) \geq 0\ ,\ \ x \in \Omega \tag{3.1}$$

in the annulus $\Omega = \{x \in \mathbb{R}^N : 0 < r_1 < |x| < r_2\}$.
If F satisfies conditions (2.1), (2.2), (2.3), (2.4), *then the function*

$$m(r) = \min_{r_1 \leq |x| \leq r} u(x)\ \ ,\ \ x \in [r_1, r_2] \tag{3.2}$$

satisfies

$$m(r) \geq \frac{\psi(r)}{\psi(r_2)} m(r_2) + \left(1 - \frac{\psi(r)}{\psi(r_2)}\right) m(r_1)\ ,\ \ \forall x \in [r_1, r_2] \tag{3.3}$$

where ψ is given by

$$\psi(r) = \int_{r_1}^{r} s^{\frac{-\Lambda}{\lambda}(N-1)} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s} \sigma(\tau)d\tau\right) ds\,. \tag{3.4}$$

The proof of this result, which has established for F independent of p in [17] and in [14] in the the present form, starts from two simple observations and relies on two fundamental principles in viscosity solution theory, namely the Comparison and the Strong Minimum Principle.
The first observation is, see (2.7), that in our assumptions any solution u of (3.1) is also a solution of

$$u \geq 0\ ,\ \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| \geq 0\ ,\ x \in \Omega\ .$$

The second one is that Pucci operator $\mathcal{M}^+_{\lambda,\Lambda}$ acts as a linear ordinary differential operators on smooth, radial, convex and non increasing functions Φ.
Indeed, it is easy to check, see [17], that the eigenvalues of the Hessian matrix

$$D^2\Phi(|x|) \equiv \frac{\Phi'(|x|)}{|x|} I_N + \left[\frac{\Phi''(|x|)}{|x|^2} - \frac{\Phi'(|x|)}{|x|^3}\right] x \otimes x$$

(here I_N denotes the $N \times N$ identity matrix) are $\Phi''(|x|)$ which is simple and $\frac{\Phi'(|x|)}{|x|}$ with multiplicity $N-1$. Hence, taking the representation formula (2.5) into account and setting $r = |x|$ we have

$$\mathcal{M}^+_{\lambda,\Lambda}(\Phi'') = -\lambda\Phi'' - \frac{\Lambda(N-1)}{r}\Phi'\ . \tag{3.5}$$

A simple computation shows that the function

$$\Phi(r)=\frac{\psi(r)}{\psi(r_2)}m(r_2)+\left(1-\frac{\psi(r)}{\psi(r_2)}\right)m(r_1)$$

and $\psi(r)$ and m are given respectively by (3.4), (3.2), is a smooth radial solution of the Dirichlet problem

$$\begin{array}{l}\mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi)+\sigma(|x|)|D\Phi|=0\ ,\ x\in\Omega\\ \Phi(x)=m(r_1)\ \text{ for } |x|=r_1,\quad \Phi(x)=m(r_2)\ \text{ for } |x|=r_2\ .\end{array}$$

Since, by construction, $\Phi(x)\leq u(x)$ on $\partial\Omega$, by the Comparison Principle, see [26], one deduces that

$$u(x)\geq\Phi(x)\qquad \text{in } \overline{\Omega}. \tag{3.6}$$

Observe now that the claim (3.3), which amounts in fact to

$$m(r)\geq\Phi(r)\qquad \text{for all}\quad r\in[r_1,r_2]\,, \tag{3.7}$$

is trivial if u (and consequently m) is a constant. If u is not a constant, by the Strong Minimum Principle, see [9], [4], u must attain its minimum value on the boundary of the compact set $\{x\in\mathbb{R}^N : r_1\leq|x|\leq r\ \}$ for each $r\in(r_1,r_2)$ and (3.7) easily follows using (3.6).

4 On the Liouville property

We discuss here the validity of Liouville type results for functions u satisfying

$$u\geq 0\,,\qquad F(x,u,Du,D^2u)\geq 0\ ,\ x\in\mathbb{R}^N \tag{4.1}$$

such as

(A) any solution of (4.1) is a constant

or

(B) the unique solution of (4.1) is $u\equiv 0$.

This questions have been the object of vaste attention for their own sake and for their applications to a priori estimates and existence theory for elliptic semilinear boundary value problems, see for example [8], [20], [21], [18], [5], [6] and the review paper [11], and also in connection with some asymptotic problem arising in ergodic stochastic control, see [1] and the references therein.

In a fully nonlinear setting, question (A) has been answered affirmatively in [9] for non-negative solutions of the equation $F(D^2u) = 0$.
Properties (A) and (B) have been proved to hold true in [17] for the a priori much wider set of viscosity solutions of the partial differential inequality (4.1) with $F = F(x, u, D^2u)$.

We announce in this section two Liouville type results taken from [14] which give affirmative answers to questions (A) and (B) when function F is allowed to depend also on first order derivatives, under the main structural condition

$$F(x, t, p, 0) \leq \sigma(|x|)|p| + h(x)t^\alpha \, .$$

Even under this restriction, the dependence of F on Du generates some interesting phaenomena depending on the behavior at infinity of the function

$$\psi(r) = \int_{r_1}^{r} s^{\frac{-\Lambda}{\lambda}(N-1)} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s}\sigma(\tau)d\tau\right) ds$$

occurring in the statement of the Hadamard Theorem in the previous section.

Theorems 4.1 and 4.2 below cover, respectively, the two possible cases of asymptotic behaviour, namely ψ divergent or bounded as $r \to +\infty$. The next one is about the case $\lim_{r\to+\infty} \psi(r) = +\infty$.

Theorem 4.1 *Let $u \in LSC(\Omega)$ be a viscosity solution of*

$$u \geq 0\,, \qquad F(x, u, Du, D^2u) \geq 0 \qquad \text{in } \mathbb{R}^N \tag{4.2}$$

with F satisfying (2.1), (2.2), (2.3), (2.4) *for all $x \in \mathbb{R}^N$.*
If $\lim_{r\to+\infty} \psi(r) = +\infty$, then u is a constant. If, in addition, the function h in (2.3) *is strictly negative at some point $x_0 \in \mathbb{R}^N$, then $u \equiv 0$.*

Before sketching the proof, let us observe that it is well - known, see [28] for a proof in the two dimensional case, that nonnegative solutions of the linear equation

$$-\mathrm{tr}\,(A(x)D^2u) + b(x)\cdot Du = 0 \qquad \text{in } \mathbb{R}^N\ ,\ N \geq 2$$

with $A(x)$ continuous and positive definite are necessarily constants, provided b satisfies the Fuchs type condition

$$(1 + |x|)|b(x)| \leq C$$

It is immediate to check that the assumptions of our Theorem 4.1 are trivially satisfied in that case.

Our proof of Theorem 4.1 seems to be simpler than the one based on the Krylov-Safonov-Harnack inequality for viscosity solutions of $F(D^2u) = 0$ suggested in [9]. Indeed, any viscosity solution of (4.2) is, a fortiori, a solution of the same inequality in any annulus Ω. Since $m(r_2) \geq 0$, by Theorem 3.1

$$m(r) \geq m(r_1)\left(1 - \frac{\psi(r)}{\psi(r_2)}\right) \ , \ r \in [r_1, r_2]$$

for arbitrary $0 < r_1 < r_2$. Keeping r fixed and letting r_2 go to $+\infty$ in the above we obtain, since ψ is divergent,

$$m(r) \geq m(r_1) \ , \ r \geq r_1 \, .$$

Since, by its very definition, $m(r)$ is nonincreasing, we conclude that

$$m(r) \equiv m(0) = u(0) \ ,$$

that is u attains its minimum on the closed ball $|x| \leq r$ at the interior point $x = 0$. By the Strong Minimum Principle, see [9], [4], u is a constant and the first claim is proved.

Finally, if C is a non-negative constant solution of (4.2), then from (2.3) it follows that

$$0 \leq F(x_0, C, 0, 0) \leq h(x_0)\, C^\alpha$$

which implies $u \equiv 0$, if $h(x_0) < 0$ at some x_0.

In the last part of this section we present a Liouville type result for the other possible case of behavior of function ψ, namely when $\psi(r)$ has a finite limit L as $r \to +\infty$. This situation arise, for example, when the function σ satisfies

$$|x|\sigma(|x|) \geq \lambda - \Lambda(N-1) + \delta \qquad \text{for some } \delta > 0,$$

a more stringent condition than the one in (2.4).

We also need to impose specific conditions on the behaviour at infinity of σ and of the zero order term in the operator, precisely

$$\sup_{\mathbb{R}^N} |x|\sigma(|x|) < +\infty \tag{4.3}$$

and

$$h(x) \leq -g(|x|) \quad \text{for } |x| \text{ large} \tag{4.4}$$

for some function g satisfying

$$\lim_{r\to+\infty} r^2 g(r)(L-\psi(r))^{\alpha-1} = +\infty \ . \tag{4.5}$$

Note that (4.4), (4.5) imply in particular $h < 0$ for large r, excluding therefore the case $h \equiv 0$.

We have then

Theorem 4.2 *Let $u \in LSC(\Omega)$ be a viscosity solution of*

$$u \geq 0\,, \qquad F(x,u,Du,D^2u) \geq 0 \qquad \textit{in } \mathbb{R}^N \tag{4.6}$$

with F satisfies conditions (2.1), (2.2), (2.3), (2.4), (4.3), (4.4), (4.5).
If $\lim_{r\to+\infty}\psi(r) = L < +\infty$, *then* $u \equiv 0$.

It is worth to observe and not very hard to check that the result applies in particular to C^2 solutions of the linear inequality

$$u \geq 0\,, \qquad -\Delta u - |x|^\gamma u^\alpha \geq 0 \qquad \text{in } \mathbb{R}^N$$

provided $\gamma > -2$ and $1 < \alpha < \frac{N+\gamma}{N-2}$. We recover then a result in [5], [6], established there by means of the integral estimate

$$\left(\int_{B_R\setminus B_{r_1}} |x|^{\gamma+1}\Phi\left(\frac{x}{|x|}\right)\zeta\left(\frac{|x|}{R}\right)dx\right)^{1-\frac{1}{\alpha}} \leq CR^{(N-\frac{\gamma}{\alpha-1})\frac{\alpha-1}{\alpha}-2}$$

where Φ is the first eigenfunction of the Laplace - Beltrami operator and ζ is a smooth cut - off function.

Our proof of Theorem 4.2, see [14], makes use instead of estimates of the type

$$\frac{m(R)}{L-\psi(R)} \le C\left(\frac{\Lambda(N+1)+R\sigma(R)}{R^2 g(R)(L-\psi(R))^{\alpha-1}}\right)^{\frac{1}{\alpha-1}} \tag{4.7}$$

where, as usual, $m(R)=\min_{r_1\le|x|\le R} u(x)$.

The estimate above is obtained by viscosity solutions techniques using the smooth radial test function

$$\zeta(|x|) = m(r)\left(1-\frac{[(|x|-r)^+]^3}{(R-r)^3}\right),$$

where r, R are parameters such that $R_0 \le r \le R$ and R_0 is chosen in such a way that $h(x)<0$ for $|x|\ge R_0$, see (4.4), (4.5).
Observe that the left-hand side in (4.7) is non-negative and that from Theorem 3.1 we have

$$m(r) \ge m(r_1)\left(1-\frac{\psi(r)}{\psi(r_2)}\right), \quad r\in[r_1,r_2].$$

Since $\psi(r)\to L<+\infty$, as $r\to\infty$, we obtain, after letting r_2 go to infinity and keeping r fixed in the above inequality, that

$$r \mapsto \frac{m(r)}{L-\psi(r)}$$

is non-decreasing on $[r_1,+\infty)$. Since the right-hand side of (4.7) tends to 0 as $R\to+\infty$ under the assumptions made, it follows that $m(R)\equiv 0$ and the conclusion follows as in Theorem 4.1 using the Strong Minimum Principle.

5 On the Phragmen - Lindelöf Principle

6 Some degenerate cases

The validity of Hadamard and Liouville type theorems for non-negative solutions of $F(x,u,Du,D^2u)\ge 0$ can be established also in some degenerate elliptic or completely degenerate cases. In this section we give some illustration of this issue with reference to two basic examples, see [15] for more complete results in this direction. The first example is

$$F(x, M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} (-\mathrm{tr}\,(AM)) \tag{6.1}$$

defined on matrices M of the form

$$M = \Sigma(x)\, B\, \Sigma^*(x)\ ,\, B \in \mathcal{S}^3$$

where the matrix $\Sigma(x)$ is given for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by

$$\begin{pmatrix} 1 & 0 & 2x_1 \\ 0 & 1 & -2x_2 \end{pmatrix}$$

and * denotes transposition. It easy to check that F satisfies the degenerate ellipticity condition

$$F(x, t, p, M + Q) \leq F(x, t, p, M)$$

for all $M, Q \in \mathcal{S}^3$ with Q nonnegative definite and for every fixed t, p, x.
For $\lambda = \Lambda = 1$, the operator defined by (6.1) reduces to

$$-\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 4(x_1^2 + x_2^2)\frac{\partial^2 u}{\partial x_3^2} + 4\left(x_2\frac{\partial^2 u}{\partial x_1 \partial x_3} - x_1\frac{\partial^2 u}{\partial x_1 \partial x_3}\right)\right)$$

The linear degenerate elliptic operator defined by the above formula is known as the Kohn Laplacian $-\Delta_{H^1}$ on the Heisenberg group H^1, see for example [30]. The nonlinear operator defined by (6.1) can therefore be seen as a degenerate analogue of the Pucci maximal operator considered in the previous sections.

The second example that we will exhamine here is the Hamilton-Jacobi-Bellman first order operator

$$F(x, t, p) = h(x)t + \inf_{\gamma \in \mathcal{G}} b^\gamma(x) \cdot p \tag{6.2}$$

a completely degenerate special case of the Bellman-Isaacs operator in Section 2.

Liouville type theorems for linear and semilinear partial differential inequalities involving the Kohn Laplacian have been established in [7], see also [13] for more general sublaplacians.

We report next on a three spheres theorem for non-negative supersolutions of (6.1).
The underlying Lie group structure suggest to consider annuli defined by the homogeneous norm

$$\rho(x) = \left((x_1^2 + x_2^2)^2 + x_3^3\right)^{\frac{1}{4}} .$$

Similar computations as those in Section 3 show that the operator F in (6.1) acts on functions $\Phi = \Phi(\rho)$ as

$$F(x, \Sigma(x)\, D^2\Phi(\rho)\, \Sigma^*(x)) = \left[\lambda\Phi'' + \frac{3\Lambda}{\rho}\Phi'\right] \frac{x_1^2 + x_2^2}{\rho^2} .$$

Observe that in the corresponding formula (3.5) for the uniformly elliptic case with $N = 3$ the coefficient of Λ is 2 instead of 3. Actually, it is well-known, [30], that in the analysis of the Kohn Laplacian the linear dimension N must replaced by the homogeneous dimension Q, $Q = 4$ in the case of Δ_{H^1}. Note also in this respect the definition of function ψ in the next result.

By the same method employed in the proof of Theorem 3.1 we obtain

Theorem 6.1 *Let $u \in LSC(\Omega)$ be a viscosity solution of*

$$u \geq 0 , \qquad \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} (-\mathrm{tr}\,(A\Sigma(x)\, D^2 u\, \Sigma^*(x))) \geq 0 , \quad x \in \Omega$$

in the annulus $\Omega = \{x \in \mathbb{R}^3 : 0 < r_1 < \rho(x) < r_2\}$. Then the function

$$m(r) = \min_{r_1 \leq \rho(x) \leq r} u(x) , \quad x \in [r_1, r_2]$$

satisfies

$$m(r) \geq \frac{\psi(r)}{\psi(r_2)} m(r_2) + \left(1 - \frac{\psi(r)}{\psi(r_2)}\right) m(r_1) , \quad \forall x \in [r_1, r_2]$$

where ψ is given by

$$\psi(r) = \int_{r_1}^{r} s^{\frac{-3\Lambda}{\lambda}} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s} \sigma(\tau) d\tau\right) ds .$$

We conclude the current section with a Liouville theorem in the spirit of Theorem 4.2 for the Hamilton-Jacobi-Bellman problem

$$u \geq 0 \,, \qquad h_0 u + \inf_{\gamma \in \mathcal{G}} b^\gamma(x) \cdot Du \geq 0 \;, \;\; x \in \mathrm{I\!R}^N \,. \tag{6.3}$$

Let us assume that $\mathcal{G}$ is a compact set in $\mathrm{I\!R}^M$ and that b^γ is continuous and bounded on $\mathrm{I\!R}^N \times \mathcal{G}$; we assume also that

$$|b^\gamma(x) - b^\gamma(y)| \leq L|x-y| \;, \quad (b^\gamma(x) - b^\gamma(y)) \cdot (x-y) \leq L|x-y|^2$$

for some constant L and all $x, y \in \mathrm{I\!R}^N$, $\gamma \in \mathcal{G}$. We assume also that $h_0 < 0$.
Let us associate to (6.3) the characteristic system

$$\dot{y}(s) = b^{\gamma(s)}(y(s)) \;, \;\; y(0) = x$$

where $\gamma(s)$ is any measurable function of $s \in [0, +\infty)$ valued in $\mathcal{G}$ and denote its solution by $y(s; x, \gamma)$.

It is known from optimal control theory, see [3], that if u is any viscosity solution of (6.3) then the function

$$\psi(s; x, \gamma) = e^{-h_0 s} \, u(y(s; x, \gamma))$$

is a non decreasing function of s for any choice of $x \in \mathrm{I\!R}^N$ and of the control $\gamma(\cdot)$.

Under the assumptions made we have

Theorem 6.2 *Let $u \in LSC(\mathrm{I\!R}^N)$ be a viscosity solution of (6.3).*
If for each $x \in \mathrm{I\!R}^N$ there exists a control function γ_x such that

$$\lim_{s \to +\infty} \psi(s; x, \gamma_x) = 0 \,,$$

then $u \equiv 0$.

The proof is very simple: since $\psi(s; x_0, \gamma_{x_0})$ is non decreasing, if $u(x_0) > 0$ at some x_0, then

$$0 = \lim_{s \to +\infty} \psi(s; x_0, \gamma_{x_0}) \geq \psi(0; x_0, \gamma_{x_0}) = u(x_0) > 0$$

showing that $u \equiv 0$.

Since $h_0 < 0$, the assumption made on the asymptotic behavior of ψ is clearly satisfied if for any initial point x there exist a control $\gamma = \gamma(s)$ and a constant M_x such that $|y(s; x, \gamma)| \leq M_x$ for all $s \geq 0$.
Several variants of the above result are of course possible by replacing the requirement on ψ by conditions on the growth of u at infinity, see [15].

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