Partition functions

Zeros, unstable dynamics and complexity

Buys, P.

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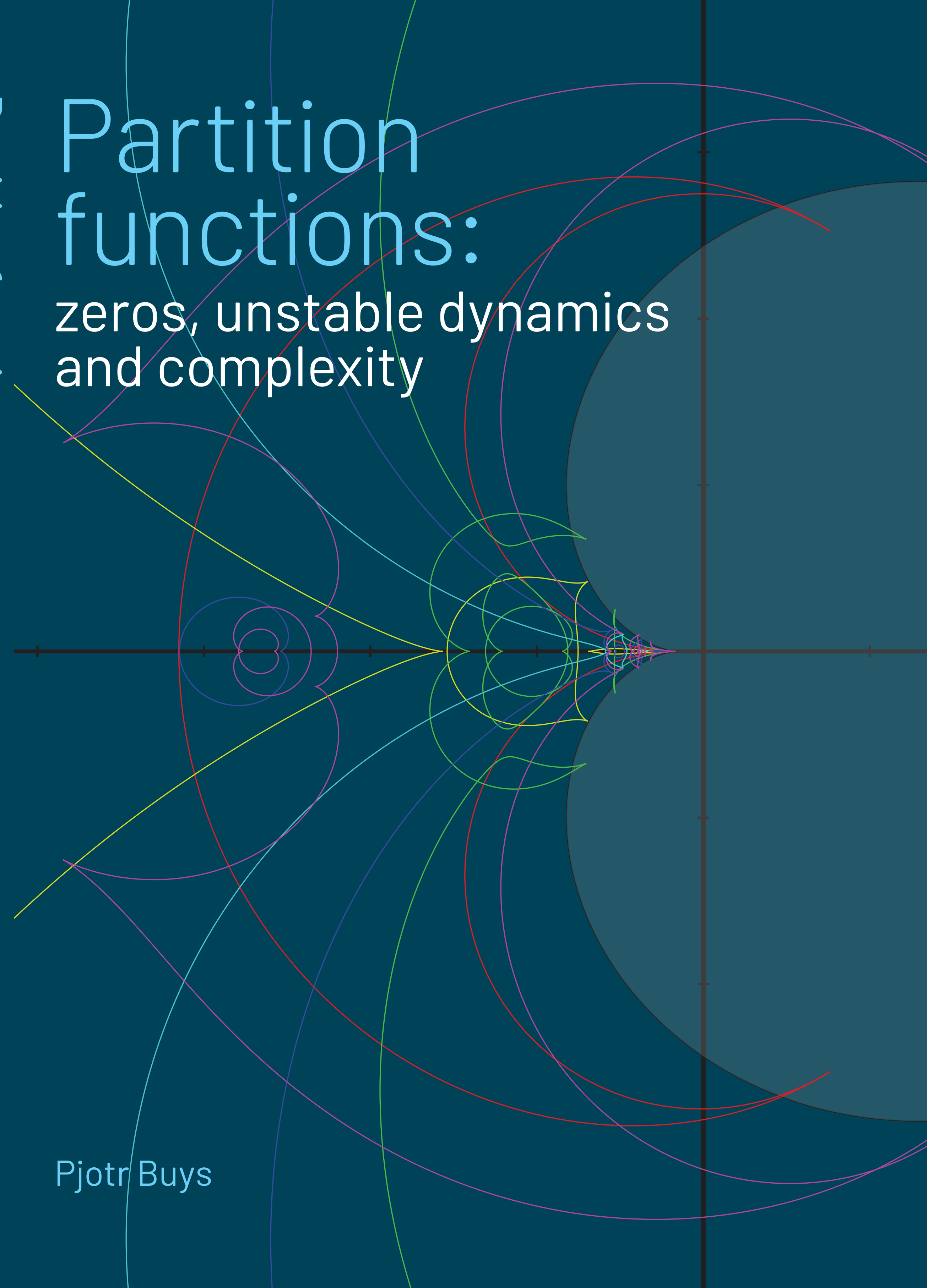

Pjotr Buys

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Partition functions: zeros, unstable dynamics and complexity

ACADEMISCH PROEFSCHRIFT

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prof. dr. ir. P.P.C.C. Verbeek
ten overstaan van een door het College voor Promoties ingestelde commissie,
in het openbaar te verdedigen in de Agnietenkapel
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door Peter Jacob Willem Buys
geboren te Amsterdam

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[1] Waarvan één keer officieus.

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Chapter 1

Introduction

The words in the title of this thesis refer to concepts that at first glance belong to vastly different areas of mathematics and physics. *Partition functions* originate in statistical physics, *zeros* and *unstable dynamics* are notions from complex analysis/dynamics and *complexity* refers to computational complexity, which is studied by computer scientists. What follows is a brief introduction to these concepts and how they relate to each other in the context of this thesis.

1.1. Partition functions

Physicists use models to describe the physical world. In this thesis we deal with discrete models, i.e. models where the system can only be in one of finitely many states. Fundamental assumptions about the universe allow statistical physicists to assign to each state σ a positive weight $w(\sigma)$ such that the probability of observing σ is proportional to $w(\sigma)$. Therefore, the exact probability of observing σ is $w(\sigma)/Z$, where $Z = \sum_\sigma w(\sigma)$ summing over all possible states. This normalizing constant, Z, is called the *partition function* of the model, it being a function of the underlying parameters (e.g. temperature or external magnetic field). The corresponding probability distribution is called the *Boltzmann distribution*.

The goal is to understand the macroscopic properties, for example energy or magnetization, the system is likely to posses. This means that we want to know what kind of states are likely to occur. It turns out that these types of questions can be answered by having a good understanding of the partition function and in particular how its properties change as the parameters change.

We will now introduce the two models that are studied in this thesis.

1.1.1. The Ising model

The Ising model, introduced by Lenz and Ising [Len20, Isi25], is used to study magnetism. The underlying structure is a finite simple graph $G = (V, E)$, where the vertices represent particles and the edges represent those pairs of particles that can interact. Each particle can be in one of two states, either spin + or spin -. The possible states of the system are therefore functions $\sigma : V \to \{+, -\}$.

In the version of the model that is discussed in this thesis there are two parameters: a magnetic field parameter λ and an interaction parameter b. The weight of a state σ is $w(\sigma) = \lambda^{|n_+(\sigma)|} b^{\delta(\sigma)}$, where $n_+(\sigma)$ is the set of vertices that get assigned the spin + under σ, and $\delta(\sigma)$ is the number of edges with differing spins. The partition function is therefore

$$Z_G^{\mathrm{Is}}(\lambda, b) := \sum_{\sigma: V \to \{+,-\}} \lambda^{|n_+(\sigma)|} b^{\delta(\sigma)}.$$

The parameter λ represents a magnetic field. If $\lambda \in (0, 1)$ the magnetic field points down such that states with more particles that have spin - are generally more likely. If $\lambda \in (1, \infty)$ the magnetic field points up and states with more particles with spin + are more likely. Note that as λ goes to either 0 or ∞, keeping b fixed, the Boltzmann distribution converges to a degenerate distribution supported on the all - or all + state respectively.

The parameter b represents an interaction between neighboring particles. If $b \in (0, 1)$, neighboring particles will generally have a higher probability to have the same spin. In this case the model is called *ferromagnetic*. If $b \in (1, \infty)$, neighboring particles have a tendency to have opposite spins. In this case the model is called *antiferromagnetic*. As b goes to 0 the Boltzmann distribution converges to a distribution that depends on λ and is supported on the all - and the all + state. As b goes to ∞ the Boltzmann distribution converges to a distribution on those states that are closest to representing a proper two-coloring. In this thesis the ferromagnetic Ising model is studied and thus the parameter b lies in the interval $(0, 1)$. We will henceforth assume that this parameter is fixed and think of Z_G^{Is} as a univariate polynomial in λ.

1.1.2. The independence polynomial

In the hard-core model the underlying structure is again a finite simple graph $G = (V, E)$. The vertices represent lattice sites that may or may not be occupied by particles. These particles have a certain size, which prevents two neighboring sites being occupied. Therefore the possible states are subsets $I \subseteq V$ such that no pair $u, v \in I$ forms an edge. Such a subset is called an *independent set*. The weight of a state I is equal to $w(I) = \lambda^{|I|}$ for some positive parameter λ, referred to as the fugacity parameter. We can thus write the partition function as

$$Z_G^{\mathrm{ind}}(\lambda) = \sum_{\substack{I \subseteq V: \\ I \text{ is independent}}} \lambda^{|I|}.$$

This function is also known as the *independence polynomial.* We refer to [SS05] and the references therein for more background on the hard-core model.

Note that as λ goes to 0 the Boltzmann distribution converges to a degenerate distribution supported on the empty set, while, as λ goes to ∞, it converges to a distribution supported on the maximal independent sets of G.

Remark. In the remainder of this introduction we will write Z_G when a particular statement applies to both models and write either Z_G^{Is} or Z_G^{ind} when we want to distinguish between the two.

1.1.3. Complex zeros and analyticity of the pressure

Recall that we claimed that properties predicted by the model can be derived from the partition function. An important quantity is called the *pressure* and it is defined by

$$p_G(\lambda) = \frac{1}{|V|} \log(Z_G(\lambda)).$$

The *density* is then defined as $\rho_G(\lambda) = \lambda \cdot p_G'(\lambda)$. It follows from a short calculation that for the independence polynomial $\rho_G(\lambda)$ is equal to the expected fraction of sites occupied by an independent set that is drawn from the Boltzmann distribution and for the Ising model $\rho_G(\lambda)$ is equal to the expected fraction of vertices that get assigned a +.

The graphs that are relevant from a physical perspective usually are quite structured; the vertices might form part of a lattice for example. Moreover, these graphs typically contain a very large number of vertices (a small piece of iron contains more than 10^{23} particles). To analyse such large quantities mathematically we would like to speak about the pressure and the density on infinite graphs (usually a lattice $\mathcal{L}$). To make this formal a sequence of finite graphs $G_1, G_2, G_3, \ldots$ converging[1] to $\mathcal{L}$ is considered. Lee and Yang [LY52a] showed that the functions p_{G_n} converge to a continuous function $p_{\mathcal{L}}$.

While $p_{\mathcal{L}}$ is continuous on the whole positive real axis, its derivatives may not be. A model is said to undergo a *phase transition* at a parameter λ_0 if $p_{\mathcal{L}}$ is not analytic at λ_0. Lee and Yang give a condition that is sufficient to guarantee the absence of a phase transition. This condition requires us to consider what

[1] There are some conditions on the manner of convergence that we will ignore for the sake brevity.

happens to Z_G when we allow λ to be a complex parameter. The polynomials Z_G have positive coefficients for any graph G and therefore they do not vanish for any $\lambda > 0$. However, for a sequence of graphs $G_1, G_2, \ldots$, the complex zeros of Z_{G_n} might accumulate on the real axis. The Lee–Yang Theorem states that if these zeros do not accumulate on λ_0 there is no phase transition at λ_0.

Theorem ([LY52a])**.** *If $G_1, G_2, G_3, \ldots$ is a sequence of graphs converging to a lattice $\mathcal{L}$ such that there is a complex domain U that is zero-free for Z_{G_n} for all n then the limit $p_{\mathcal{L}}$ is analytic on U.*

We will not use this theorem in the remainder of this thesis and in fact we will seldom think about Z_G as having anything to do with a probability distribution. Nevertheless, we consider this theorem as motivation that it is natural to examine what happens to Z_G for general complex parameters λ and especially to study its complex zeros. In the subsequent sections λ will denote an arbitrary complex number.

1.2. Computational complexity

Central to this thesis are variants of the following question:

$$\textit{How difficult is it to calculate } Z_G(\lambda)\textit{?} \tag{1.1}$$

This question is motivated by the importance of partition functions in physics and started to receive attention from a computer science perspective in the late 1980s. It is still a very active area of research with many recent developments. We first spend some time on making question (1.1) more precise.

1.2.1. Complexity classes

Before we can start answering question (1.1) we need to agree on some way to measure the difficulty of a computational problem. Computational complexity theory is about classifying computational problems into complexity classes and relating these classes to each other. What follows is a very brief introduction to the complexity classes that are relevant to this thesis. It is largely based on [Jer03, Chapter 2].

The most basic type of computational problem is a *decision problem*. These types of problems are often posed as yes/no questions. For example, the independent set decision problem asks, on input of a graph G and an integer k, whether G has an independent set of size k.

Formally, such a problem is modeled by a *predicate* $\phi : \Sigma^* \to \{0, 1\}$, where Σ^* is the set of all finite strings of a finite alphabet Σ. The alphabet Σ is chosen such that all instances of the problem can be encoded into Σ^*. For example, a graph might be encoded by its adjacency matrix and an integer by its decimal expansion. The predicate ϕ satisfies the property that on input of encoded data $x \in \Sigma^*$ one has $\phi(x) = 1$ if and only if the answer to the question with the corresponding input is *yes*.

The class P. A predicate ϕ is said to belong to the complexity class P if an algorithm exists that solves the decision problem and whose running time is polynomial in the size of the input. Formally, this means that there exists a polynomial p and a deterministic Turing machine T that, on input of an $x \in \Sigma^*$, terminates in at most $p(|x|)$ steps and accepts if and only if $\phi(x) = 1$. In the rest of the thesis we will always speak of the running time of an algorithm rather than the number of steps taken by a Turing machine. In general one should think of polynomial time algorithms as fast and thus of problems in P as easy.

The class NP. Roughly speaking, the class NP consists of those decision problems for which any solution (henceforth called witness), if it exists, can be easily verified. For example, a witness to the independent set decision problem with input G and k is an explicit independent set of size k in G. Since it is easy to verify whether or not a given subset is independent (only a quadratic number of comparisons are needed), this problem is in NP.

Again, more formally, a predicate is in NP if and only if there exists a *witness-checking* predicate $\chi : \Sigma^* \times \Sigma^* \to \{0, 1\}$ and a polynomial p such that the following hold:

1. for each $x \in \Sigma^*$ it is the case that $\phi(x) = 1$ if and only if there exists a $w \in \Sigma^*$ with $|w| \leq p(|x|)$ for which $\chi(x, w) = 1$;
2. the decision problem which, on input of $x, w \in \Sigma^*$, yields the output $\chi(x, w)$ is in P.

One should think of the first item as saying that for each encoded input x the answer to the decision problem ϕ is *yes* if and only if there exists a witness w of comparable size attesting to that fact. Furthermore, the second item states that verifying whether a witness w is indeed correct is an easy problem.

The class $\#\mathsf{P}$. More general than decision problems are *counting problems*. Usually these problems correspond to questions of the type *"How many … do there exist?"* Formally, such a problem is a function $f : \Sigma^* \to \mathbb{Z}_{\geq 0}$. The complexity class $\#\mathsf{P}$, introduced by Valiant [Val79], consists of the counting versions of problems in NP. By definition f is in $\#\mathsf{P}$ if and only if

$$f(x) = |\{w \in \Sigma^* : \chi(x, w) = 1 \text{ and } |w| \leq p(|x|)\}|,$$

where χ and p are as described above. The counting variant of the independent set decision problem, which asks how many independent sets of size k the graph G has, is thus a problem in $\#\mathsf{P}$.

A related problem in #P is to determine the total number of independent sets of an input graph G. Recall that the number of independent sets is $Z_G^{\text{ind}}(1)$. This problem therefore falls under the scope of the question laid out in (1.1).

1.2.2. Hardness

We say that a problem A *reduces* to a problem B if there is a polynomial time algorithm that solves A while being allowed to use a polynomial number of instances of problem B with input whose size is polynomial in the size of the original input. We think of this as saying that B is at least as hard to solve as A. Any problem in P is also in NP, furthermore, any NP problem trivially reduces to its corresponding #P problem. We can thus say that each subsequent complexity class in general contains problems which are at least as hard as those in the previous one. Intuitively one would expect to find problems of strictly increasing difficulty as one moves up in complexity classes. Certainly there are problems inside NP for which there is no known polynomial time algorithm. Perhaps surprisingly however, there are currently no methods known to rigorously prove that a given problem in either NP or #P is not solvable in polynomial time. In fact, it is one of the most famous open problems in mathematics to either prove or disprove that P equals NP (see also [Coo06]).

A computational problem A is called NP-hard, respectively #P-hard, if any problem in NP, respectively #P, reduces to A. These problems are at least as hard as any problem in the corresponding complexity class. A polynomial time algorithm for such a problem would therefore lead to a proof of either P = NP or P = #P. The general consensus among complexity theorists is that the class NP should be much larger than P and that #P should be larger still. Exhibiting a polynomial time algorithm for a #P-hard problem would therefore not only solve one of the major open problems in mathematics, but also shock the mathematical community. This is the reason why we think of #P-hard problems as being difficult.

1.3. The hardness of approximation

The problem of counting the number of independent sets of an input graph G, i.e. calculating $Z_G^{\text{ind}}(1)$, is a #P-hard problem. Because it is also in #P, it is called a #P-complete problem. It remains #P-complete if we restrict the input graph to have maximum degree at most 3 [Gre00]. In fact, exactly calculating $Z_G(\lambda)$ for any non-zero parameter λ within the class of graphs with maximum degree at most 3 is #P-hard for both the Ising model and the independence polynomial [KC16]. We can thus rightly say that the answer to question (1.1), when pertaining to exact calculation, is: very difficult.

Whenever exact calculation is infeasible it is common to study a relaxation, namely that of approximate calculation. In this setting the output of the algorithm is required to be correct within a certain given margin of error. We can distinguish between two categories: deterministic approximation and randomised approximation. In the latter category the algorithm is allowed to make random choices (proverbially it is allowed to *flip a coin*). The output is required to be a good approximation with probability exceeding some fixed constant (say $\frac{3}{4}$). A very successful method for randomised approximation is the Markov chain Monte Carlo (MCMC) approach. For example, this approach was used to obtain an efficient algorithm for approximating $Z_G^{\mathrm{Is}}(\lambda)$ for $\lambda > 0$ [JS93]. For graphs with maximum degree at most Δ there is a threshold $\lambda_c(\Delta)$ such that there exist efficient deterministic algorithms for approximating $Z_G^{\mathrm{ind}}(\lambda)$ if $0 < \lambda < \lambda_c(\Delta)$ [Wei06] and randomised approximation is difficult if $\lambda > \lambda_c(\Delta)$ [Sly10, SS12, GGv$^+$14].

In this thesis we consider the problem of deterministic approximation. The input is a graph with a vertex set of size n and an allowed error ϵ and the output should be a number that is at most a multiplicative error of $1 + \epsilon$ away from $Z_G(\lambda)$ (see sections 2.1.1 and 3.1.3 for precise problem statements for Z_G^{Is} and Z_G^{ind} respectively). We say that there is a *fully polynomial time approximation scheme* (FPTAS) for approximating $Z_G(\lambda)$ if there is an algorithm that solves this problem whose running time is polynomial in n/ϵ.

Determining the set of parameters λ for which there exists an FPTAS for approximating $Z_G(\lambda)$ is a question that has received a lot of attention in recent years for both real and complex parameters; see e.g. [PR19, GGHP20] for the independence polynomial and [GG17, LSS19c] for the Ising model. Many recent results rely on a sufficient condition, formulated by Barvinok [Bar16] and refined by [PR17], for the existence of an efficient approximation algorithm. We elaborate in the next section.

1.3.1. Approximation and zeros

We have already seen that, rather than considering any graph as input, the input is often restricted to graphs belonging to a certain family $\mathcal{G}$. An important family for this thesis is the one consisting of bounded degree graphs. For an integer $\Delta \in \mathbb{Z}_{\geq 2}$ we let $\mathcal{G}_\Delta$ denote the class of graphs with maximum degree at most Δ. Patel and Regts, building upon work by Barvinok, proved that the following condition is sufficient to imply that approximation is easy.

Theorem ([Bar16, PR17]). *Let $\Delta \in \mathbb{Z}_{\geq 2}$, $\mathcal{G} \subseteq \mathcal{G}_\Delta$ and suppose the parameter λ lies in a complex domain U containing* 0 *that is zero-free for $\mathcal{G}$. Then there is an* FPTAS *for approximating $Z_G(\lambda)$ for $G \in \mathcal{G}$.*

The domain U being zero-free for $\mathcal{G}$ means that $Z_G(\mu) \neq 0$ for any $\mu \in U$ and

$G \in \mathcal{G}$. Note the similarities between this theorem and the Lee–Yang Theorem discussed in Section 1.1.3.

Momentarily ignoring the condition that the zero-free neighborhood of the given parameter needs to contain 0, this result can be thought of as saying that zero-freeness implies that approximation is easy. In this thesis it is shown that, for both the ferromagnetic Ising model and the independence polynomial, within the class of bounded degree graphs an inverse statement also holds. Define

$$\mathcal{Z}_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

We show the following.

Theorem (2.1.1 and 3.1.1)**.** *The closure of the zeros $\overline{\mathcal{Z}_\Delta}$ is contained in the closure of parameters λ for which approximating $Z_G(\lambda)$ for $G \in \mathcal{G}_\Delta$ is #P-hard.*

1.3.2. The location of the zeros

In light of this result we would like to have a good understanding of the closure of the complex zeros of bounded degree graphs. For the ferromagnetic Ising model this problem is fully solved. The Lee-Yang circle theorem [LY52b] says that the complex zeros of Z_G^{Is} all lie on the unit circle in the complex plane for any graph G; these zeros are therefore often referred to as *Lee-Yang zeros.* Within the class of bounded degree graphs this result was refined by Peters and Regts [PR20] who showed that, depending on the degree bound Δ and the interaction parameter b, the closure of the zeros is either equal to the unit circle or equal to a circular arc strictly contained in the unit circle. The algorithmic approach in [Bar16, PR17] gives a polynomial time algorithm for approximating Z_G^{Is} in the complement of this closure. The results of this thesis thus show that, in the case of the ferromagnetic Ising model, unless #P = P, the closure of of the complex zeros is exactly equal to the closure of parameters for which approximating Z_G^{Is} is #P-hard.

For the independence polynomial the closure of the complex zeros of bounded degree graphs is much more complicated and an explicit description has not yet been found. The zeros of all graphs lie dense in the complex plane [BHN04], however within the class of bounded degree graphs many results on zero free regions exist (see e.g. [She85, SS05, BC18, PR19]). In [BGGv20] it is shown that parameters for which approximating Z_G^{ind} for $G \in \mathcal{G}_\Delta$ is #P-hard lie dense outside a certain explicit bounded region Λ_Δ, often referred to as *the cardioid.* In Chapter 3 of this thesis their results are used to show that zeros lie dense outside Λ_Δ. If it were true that Λ_Δ is actually zero-free, one could again apply the approximability results of [Bar16, PR17] to find equality between the closure of zeros and the closure of parameters for which approximating is #P-hard (unless

$\#\mathsf{P} = \mathsf{P}$). In Chapter 4 of this thesis it is shown that Λ_Δ is in general not zero-free and thus it remains an open problem whether or not the closure of the zero parameters is equal to or strictly contained in the closure of hardness parameters.

1.4. Ratios

Let $G = (V, E)$ be a finite simple graph with a marked vertex $v \in V$. The Ising model partition function is a sum over all $\{\texttt{+}, \texttt{-}\}$ assignments to V and the independence polynomial is a sum over independent sets. In both cases the summands can be split up into two categories according to whether v is assigned a + or - for the Ising model, or whether v lies in or out of the independent set for the independence polynomial. We denote the partition functions that restrict the sum to one of these categories by $Z_{G,+v}$ or $Z_{G,-v}$ and $Z^{\text{in}}_{G,v}$ or $Z^{\text{out}}_{G,v}$ respectively. Therefore, we have

$$Z^{\text{Is}}_G(\lambda) = Z_{G,+v}(\lambda) + Z_{G,-v}(\lambda) \quad \text{and} \quad Z^{\text{ind}}_G(\lambda) = Z^{\text{in}}_{G,v}(\lambda) + Z^{\text{out}}_{G,v}(\lambda). \tag{1.2}$$

A useful quantity to define is the ratio between these contributions. To that effect we let $R^{\text{Is}}_{G,v}(\lambda) = Z_{G,+v}(\lambda)/Z_{G,-v}(\lambda)$ and $R^{\text{ind}}_{G,v}(\lambda) = Z^{\text{in}}_{G,v}(\lambda)/Z^{\text{out}}_{G,v}(\lambda)$. We will write $R_{G,v}(\lambda)$ if a given statement is applicable to both models. If λ is a positive real number the Ising model and the independence polynomial describe a probability distribution on $\{\texttt{+}, \texttt{-}\}$ assignments and independent sets respectively. In this case $R^{\text{Is}}_{G,v}(\lambda)$ denotes the odds that v gets assigned a + and $R^{\text{ind}}_{G,v}(\lambda)$ denotes the odds that v is in the observed independent set, where the odds of an event mean the probability of an event happening divided by the probability of that event not happening. In Chapter 2 these ratios are called *fields*, while in the subsequent chapters they are called *(occupation) ratios*.

In general $R_{G,v}$ is a rational function in λ. We can thus interpret $R_{G,v}$ as a function $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. It follows from equation (1.2) that complex zeros λ_0 of Z_G correspond[2] one to one with parameters λ_0 for which $R_{G,v}(\lambda_0) = -1$. Understanding the zeros of Z_G is thus essentially equivalent to understanding the -1-parameters of $R_{G,v}$.

1.4.1. Recursion

An advantage of using ratios is that they behave very nicely under certain graph constructions. A construction that is used very frequently in this thesis is the following. Suppose $(G_1, v_1), \ldots, (G_k, v_k)$ are graphs with marked vertices. We construct a new rooted graph (G, v) by taking a disjoint union of $G_1, \ldots, G_k$ and

[2]The technicality that common zeros in the quotient might cancel, leading to values different from -1, is swept under the rug.

subsequently adding a new vertex v whose neighbors are the roots $v_1, \ldots, v_k$. Then the following equalities of rational functions hold

$$R^{\text{ind}}_{G,v}(\lambda) = \lambda \cdot \prod_{i=1}^{k} \frac{1}{1 + R^{\text{ind}}_{G_i,v_i}(\lambda)} \quad \text{and} \quad R^{\text{Is}}_{G,v}(\lambda) = \lambda \cdot \prod_{i=1}^{k} \frac{R^{\text{Is}}_{G_i,v_i}(\lambda) + b}{bR^{\text{Is}}_{G_i,v_i}(\lambda) + 1}.$$

This is especially useful when combined with the following fact.

A construction, due to Weitz [Wei06] and refined by Bencs [Ben18], shows that, in the case of the independence polynomial, for any rooted graph (G, v) there is a rooted tree (T, u) that has the same maximum degree as G for which $R^{\text{ind}}_{G,v} = R^{\text{ind}}_{T,u}$. For the Ising model this is not exactly true, however it is true that for any (G, v) there exists a rooted tree (T, u) with the same maximum degree as G for which $R^{\text{Is}}_{G,v} = R^{\text{Is}}_{T,u,\tau}$ (see [PR20]). Here the τ in the subscript indicates a boundary condition on a subset of the leaves of T, i.e. a particular assignment of + and - to these leaves of T. This means that, when defining $R^{\text{Is}}_{T,u,\tau}(\lambda)$, one considers the probability distribution that is obtained when conditioning on those states that agree with τ. Any rooted tree can be generated by recursively applying the construction described in the previous paragraph, starting with single vertices. It therefore follows[3] that the ratio $R_{G,v}(\lambda)$ of any rooted graph (G, v) of degree at most $d + 1$ can be written as a composition of maps $F_{1,\lambda}, \ldots, F_{d,\lambda}$, where

$$F^{\text{ind}}_{k,\lambda}(z_1, \ldots, z_k) = \lambda \cdot \prod_{i=1}^{k} \frac{1}{1 + z_i} \quad \text{and} \quad F^{\text{Is}}_{k,\lambda}(z_1, \ldots, z_k) = \lambda \cdot \prod_{i=1}^{k} \frac{z_i + b}{bz_i + 1},$$

applied to initial values. These initial values are the occupation ratios of single vertices.

The iteration of functions with complex parameters is a setting that is common in the field of complex dynamics. The recursive nature of these ratios make it possible to use insights from this branch of mathematics to study both the zeros and computational complexity within the class of bounded degree graphs.

1.5. Complex dynamics

It turns out that both the presence of zeros and the computational complexity of approximating Z_G is related to the local dynamical behaviour of the family of ratios. We use this section to make this more precise and to provide a very brief introduction to the most important notions from the field of holomorphic dynamics that are used in this thesis.

[3] This is almost true. One application of $F_{d+1,\lambda}$ to obtain the ratio of a tree with root degree $d + 1$ should be allowed.

1.5.1. Iteration of univariate functions

Let $\mathcal{F}$ be a family of holomorphic functions from an open set $U \subseteq \widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ (so for example a set of rational functions). We say that $\mathcal{F}$ is *normal* on U if any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U. If $\mathcal{F}$ is not normal on U, one can think of $\mathcal{F}$ as behaving chaotically on U. A common type of family to study is the iteration of a given rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The set of values in the complex plane around which there does not exist an open neighborhood on which $\{f^{\circ n}\}_{n\geq 1}$ is normal is called the *Julia set* of f. This set consists of those points for which an arbitrarily small perturbation to the initial value can lead to drastically different orbits under repeated application of the map f.

Instead of describing the dynamical behaviour of a single function one can study how these dynamics change as the function changes. In an example of this setting one has an open connected subset $X \subseteq \widehat{\mathbb{C}}$ and a holomorphic map $f : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that for every fixed $\lambda \in X$ the map $z \mapsto f(\lambda, z)$ is a rational function. The value λ is often referred to as the *parameter* and this map is usually denoted by f_λ. Every f_λ has a Julia set J_λ associated to it and thus there is a map from X to the compact subsets of $\widehat{\mathbb{C}}$ which sends λ to J_λ. We say that the Julia set moves continuously at λ_0 if this map is continuous at λ_0. Parameters at which the Julia set moves continuously are called *stable.*

1.5.2. Critical orbits

A point around which a holomorphic function f is not locally injective is called a *critical point* of f. Its orbit under iteration of f is called a *critical orbit* of f. In the case of a holomorphic family of rational functions $f : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the function f_λ has a fixed amount of critical points for each λ. If there are holomorphic maps $c_i : X \to \widehat{\mathbb{C}}$ parameterizing these critical points, then λ_0 being a stable parameter is equivalent to all critical orbits $\{\lambda \mapsto f_\lambda^{\circ n}(c_i(\lambda))\}_{n\geq 1}$ forming a normal family around λ_0 (see e.g. [McM94, Theorem 4.2]). As an example of these concepts and their relation to the zeros of partition functions of graph classes we briefly describe the class of Cayley trees.

1.5.3. Zeros of Cayley trees

A *perfect d-ary tree* is a rooted tree such that every node that is not a leaf has exactly d children, and all the leaves have the same distance to the root. Such trees are sometimes also referred to as *Cayley trees.* In the case of the independence polynomial, the occupation ratios of such trees at a parameter λ are given by the orbit $\{f_{d,\lambda}^{\circ n}(0)\}_{n\geq 1}$, where $f_{d,\lambda}(z) = \lambda/(1+z)^d$. The critical points of $f_{d,\lambda}$ are -1 and ∞. Under iteration of $f_{d,\lambda}$ the point -1 goes to ∞ which in turn goes to 0. Therefore there is only one critical orbit, which consists of the ratios of Cayley trees. It thus follows that λ_0 is not stable for this parameterized family of holomorphic functions if and only if there is no neighborhood of λ_0 on which the ratios of Cayley trees form a normal family.

Montel's Theorem states that if a family of holomorphic functions is not normal, it can avoid at most two values in $\widehat{\mathbb{C}}$. This allows us to conclude that on every open neighborhood of a non-stable parameter λ_0 there must be a λ_1 and a positive integer n such that $f_{d,\lambda_1}^{\circ n}(0) = -1$. It thus follows that the zeros of the independence polynomials of Cayley trees accumulate on the set of non-stable parameters of $f_{d,\lambda}$. The connected component containing 0 of the stable parameters[4] of $f_{\lambda,d}$ can be described explicitly and has already been mentioned: the cardioid Λ_{d+1}.

The cardioid Λ_{d+1} is zero-free for Cayley trees. In Section 3.7 a more precise description of the set of zeros of Cayley trees is given. In the case of the ferromagnetic Ising model the closure of the zeros of graphs of perfect d-ary trees is equal to the non-stable parameters of the parameterized family $z \mapsto \lambda \cdot ((z+b)/(bz+1))^d$. This turns out to be equal to the closure of the zeros of all graphs with degree at most $d+1$. Therefore Cayley trees determine the location of the zeros of all bounded degree graphs. In Chapter 4 it is shown that the property that the maximal zero-free domain for Cayley trees is also zero-free for general bounded degree graphs does not hold for the independence polynomial.

1.5.4. From iteration to families of rational functions

Moving from the family of Cayley trees to the family of all graphs with degree at most Δ, the property that the set of ratios can be described as the iterates of a single univariate function is lost. However, because of the tree construction described in the previous section, the set of ratios is still equal to the orbit of a semi-group generated by a finite set of multivariate maps (see Definition 5.2.8). Motivated by the connection between zeros and non-stable parameters for Cayley trees we generalize this notion to all bounded degree graphs. Namely, we say that λ_0 is an *active* parameter if the set of ratios of graphs with degree at most Δ is not normal on any open neighborhood of λ_0. We call the set of all active parameters the *activity locus*, denoted by $\mathcal{A}_\Delta$. In Chapter 3 we will show that for the independence polynomial the activity locus $\mathcal{A}_\Delta$ is equal to the closure of the zeros $\overline{\mathcal{Z}}_\Delta$. For the Ising model this also holds (see [PR20]).

[4]To be very precise the parameter $\lambda = 0$ should actually be excluded because there the degree of $f_{d,\lambda}$ is lower than the generic degree.

1.6. Activity and computational complexity

In this section we give a brief outline of the steps involved in relating the activity locus to parameters for which approximating is #P-hard. These steps are followed for both the Ising model in Chapter 2 and the independence polynomial in Chapter 3.

1.6.1. Density

The first step is relating parameters inside the activity locus to a different type of chaotic behaviour, namely *density*. Montel's Theorem says that if λ_0 lies in the activity locus, then any neighborhood of λ_0 gets mapped essentially everywhere when applying all ratios. We show that, instead of a whole neighborhood, there are even single parameters arbitrarily close to λ_0 for which this happens. To that effect we define the *density locus* $\mathcal{D}_\Delta$ to consist of those parameters λ such that the set of ratios

$$\{R_{G,v}(\lambda) : (G,v) \text{ rooted graph with degree at most } \Delta \text{ and } \deg(v) < \Delta\}$$

is dense. For the Ising model we consider only those λ for which $|\lambda| = 1$, from which it follows that any ratio is contained inside the unit circle $\mathbb{S}$; therefore the word dense here means dense in $\mathbb{S}$ for the Ising model and dense in $\widehat{\mathbb{C}}$ for the independence polynomial.

For both models we show that the density locus $\mathcal{D}_\Delta$ is a dense open subset of $\mathcal{A}_\Delta$. For the Ising model and certain values of the parameter b a considerable amount of time is spent on proving the statement that any parameter $x+iy \in \mathcal{A}_\Delta$ not equal to ± 1 with $x, y \in \mathbb{Q}$ lies in $\mathcal{D}_\Delta$ (in Chapter 2 the notation $\mathbb{S}_{\mathbb{Q}}(d,b)$ is used instead of $\mathcal{D}_\Delta$).

1.6.2. Exponentially fast implementation

In the next step we show that parameters inside the density locus actually satisfy a stronger type of density, namely the following. For any $\lambda \in \mathcal{D}_\Delta$ we show that there is an algorithm that, on input of a parameter P and an $\epsilon > 0$, yields a rooted tree (T,u) that has maximum degree at most Δ, satisfies $|R_{T,u}(\lambda) - P| < \epsilon$ and is small in terms of the sizes of the parameters ϵ and P (see Lemma 2.7.1 and Corollary 3.6.5 for precise statements). Let us briefly describe how this tree is constructed. Suppose we are given a set of rooted trees $(T_1, v_1), \ldots, (T_n, v_n)$ and we consider the tree constructed by attaching these trees by the root to each vertex of a path of length n respectively. Then the ratio at the endpoint is given by the composition

$$(f_{R_{T_n,v_n}(\lambda)} \circ \cdots \circ f_{R_{T_1,v_1}(\lambda)})(0),$$

where $f_\zeta(z)$ is the Möbius transformation $\zeta/(1+z)$ for the independence polynomial and $\zeta(z+b)/(bz+1)$ for the Ising model. Thus any ratio corresponds to a Möbius transformation. We show that, if $\lambda \in \mathcal{D}_\Delta$, we can find a finite set of rooted trees such that for any given parameter P we can construct a sequence of the corresponding Möbius transformations for which the orbit of the initial value converges to P exponentially fast. The corresponding graph is a *short* path whose vertices have any tree from a fixed set of trees attached to them and thus this graph is *small.* We call this algorithm *exponentially fast implementation.*

1.6.3. Reduction

For the final we step we show that, in the presence of an algorithm for exponentially fast implementation, known #P-hard problems reduce to the problem of approximating $Z_G(\lambda)$. For the independence polynomial this was already done in [BGGv20], where they show that, with an algorithm for exponentially fast implementation, exactly calculating $Z_G^{\text{ind}}(1)$ reduces to approximating $Z_G^{\text{ind}}(\lambda)$. In Section 2.8.2 of this thesis it is shown that an algorithm for exponentially fast implementation and polynomially many applications of an approximation algorithm for $Z_G^{\text{Is}}(\lambda, b)$ lead to a polynomial algorithm for exactly calculating $Z_G^{\text{Is}}(\lambda, \hat{b})$, where $\hat{b} \in (0,1)$ is an altered edge activity depending on λ and b. This problem is known to be #P-hard [KC16].

The rough idea behind the reduction is that, on input of a graph G, the graph G is altered to obtain graphs $\hat{G}$ by attaching certain gadgets to G. These gadgets are obtained by implementing well chosen parameters P. The graphs $\hat{G}$ are constructed in such a way that approximating $Z_{\hat{G}}$ with the original parameters yields approximations of expressions involving the value Z_G with the altered parameters. It turns that only a small number of these approximations are needed to exactly pin down the value of Z_G with the altered parameters. Because of exponentially fast implementation, the gadgets are are also small, i.e. polynomial in the size of G, and thus the whole algorithm runs polynomially in the size of G.

This rough sketch of the argument leaves out many details. One such detail is that, to use the approximation algorithm on the graphs $\hat{G}$, these graphs need to remain inside class of graphs with degree at most Δ. Furthermore, the idea that a small number of approximations can lead to an exact calculation might seem counter-intuitive. For details concerning the independence polynomial we refer to [BGGv20]. For the Ising model this argument is given in Section 2.8.

1.7. Organization of the thesis

- **Chapter 2:** The strategy outlined in Section 1.6 is used to study the problem of approximating the ferromagnetic Ising partition function within the class of graphs with maximum degree at most $\Delta \geq 3$ for a complex external field parameter λ with $|\lambda| = 1$ and an interaction parameter $b \in (0, 1)$. When $b \in (0, \frac{\Delta-2}{\Delta}]$ it is known that the complex zeros in terms of the parameter λ form a dense subset of the unit circle. We show that in this regime, for any $\lambda = x + iy$ unequal to ± 1 with $x, y \in \mathbb{Q}$, approximating the partition function is a #P-hard problem. When $b \in (\frac{\Delta-2}{\Delta}, 1)$ the closure of the zeros is a circular arc strictly contained in the unit circle. We show that the parameters λ for which approximating the partition function is #P-hard are dense in this arc. The main result of this chapter is contained in Theorem 2.1.1. This chapter is based on [BGPR22].

- **Chapter 3:** The strategy outlined in Section 1.6 is used to study the problem of approximating the independence polynomial within the class of graphs with maximum degree at most $\Delta \geq 3$ for complex fugacity parameters λ. We show that the parameters λ for which approximating the partition function is #P-hard are dense inside the closure of the complex zeros. In the final section a precise description is given of the complex zeros of Cayley trees. The main result of this chapter is contained in Theorem 3.1.1. This chapter is based on [dBG$^+$21].

- **Chapter 4:** The maximum zero-free region containing 0 for Cayley trees with maximum degree Δ is denoted by Λ_Δ. In this chapter it is shown that Λ_Δ is not zero-free within the class of all graphs with maximum degree at most Δ. The main result of this chapter is contained in Theorem 4.1.3. This chapter is based on [Buy21].

- **Chapter 5:** For any finite Δ the closure of the zeros of the independence polynomial of graphs with degree at most Δ is an elusive object. There is for example no known algorithm to make accurate computer images of it. By letting the degree bound go to infinity while appropriately rescaling the parameter λ, we give an explicit description of the limit set of the zeros in terms of a dynamical system. By analysing this dynamical system we prove topological properties of this limit set and make accurate computer images of it. The main result of this chapter is contained in Theorems 5.1.1 and 5.1.2. This chapter is based on [BBP21].

The chapters, while occasionally referring to each other, are self-contained and can therefore be read separately.

1.8. Publications

This dissertation is primarily based on the work in the following four papers.

Pjotr Buys, Andreas Galanis, Viresh Patel, and Guus Regts, *Lee–Yang zeros and the complexity of the ferromagnetic Ising model on bounded-degree graphs*, Forum of Mathematics, Sigma **10** (2022), e7

David de Boer, Pjotr Buys, Lorenzo Guerini, Han Peters, and Guus Regts, *Zeros, chaotic ratios and the computational complexity of approximating the independence polynomial*, arXiv e-prints (2021), arXiv:2104.11615

Pjotr Buys, *Cayley Trees do Not Determine the Maximal Zero-Free Locus of the Independence Polynomial*, Michigan Mathematical Journal **70** (2021), no. 3, 635 – 648

Ferenc Bencs, Pjotr Buys, and Han Peters, *The limit of the zero locus of the independence polynomial for bounded degree graphs*, arXiv e-prints (2021), arXiv:2111.06451

The following papers have also appeared during the author's time as a PhD student.

Ferenc Bencs, Pjotr Buys, Lorenzo Guerini, and Han Peters, *Lee-Yang zeros of the antiferromagnetic Ising model*, Ergodic Theory and Dynamical Systems (2021), 1–35

David de Boer, Pjotr Buys, and Guus Regts, *Uniqueness of the Gibbs measure for the* 4*-state anti-ferromagnetic Potts model on the regular tree*, arXiv e-prints (2020), arXiv:2011.05638, to appear in Combinatorics, Probability and Computing

Ferenc Bencs, David de Boer, Pjotr Buys, and Guus Regts, *Uniqueness of the Gibbs measure for the anti-ferromagnetic Potts model on the infinite* Δ*-regular tree for large* Δ, arXiv e-prints (2022), arXiv:2203.15457

To each of these papers all authors contributed an equal amount.

Chapter 2

Lee-Yang zeros and the complexity of the ferromagnetic Ising model on bounded-degree graphs

2.1. Introduction

The Ising model is a classical model from statistical physics that arises in multiple sampling and inference tasks across computer science. The model has an edge-interaction parameter b and a vertex parameter λ, known as the external field. For a graph $G = (V, E)$, configurations of the model are all possible assignments of two spins +, - to the vertices of G. Each configuration $\sigma : V \to \{+, -\}$ has weight $\lambda^{|n_+(\sigma)|} b^{\delta(\sigma)}$, where $n_+(\sigma)$ is the set of vertices that get the spin + under σ and $\delta(\sigma)$ is the number of edges that get different spins. The *partition function* is the aggregate weight of all configurations, i.e.,

$$Z_G(\lambda, b) = \sum_{\sigma: V \to \{+,-\}} \lambda^{|n_+(\sigma)|} b^{\delta(\sigma)}.$$

In this chapter, we consider the problem of approximating the partition function when $b \in (0, 1]$, known as the ferromagnetic case, and when the parameter λ is in the complex plane. Complex parameters for the Ising model have been studied in the computation of probability amplitudes of quantum circuits, see, e.g., [lCDdNMD11, MB19, BMS16]. Recall that complex parameters are also fundamental in understanding the complexity of approximation even for real-valued parameters (Section 1.3). We partially repeat the background given in the introduction of this thesis, focusing on the ferromagnetic Ising model.

Many of the recent advances on the development of approximation algorithms for counting problems have been based on viewing the partition function as a polynomial of the underlying parameters in the complex plane, and using refined interpolation techniques from [Bar16, PR17] to obtain fully polynomial time approximation schemes (FPTAS; see below for the technical definition), even for real values [GLLZ19, GLL20, LSS19a, BDPR21, BR19, LSS19b, SS21, PR19, PR20]. The bottleneck of this approach is establishing zero-free regions in the complex plane of the polynomials, which in turn requires an in-depth understanding of the models with complex-valued parameters. This framework of designing approximation algorithms aligns with the classical statistical physics perspective on phase transitions, where zeros in the complex plane have long been studied in the context of phase transitions (see, e.g., [LY52a, HL72]), and several of these classical results have recently been used to obtain efficient approximation algorithms ([LSS19c, PR17]).

The celebrated Lee-Yang circle theorem [LY52b] says that, when regarding the partition function of the ferromagnetic Ising model as a polynomial in the external field parameter λ, all its zeros, referred to as *Lee-Yang zeros*, lie on the unit circle in the complex plane. (These Lee-Yang zeros have actually been observed in quantum experiments [PZW$^+$15].) The Lee-Yang theorem was recently used by Liu, Sinclair, and Srivastava [LSS19c] to obtain an FPTAS for approximating the partition function for values $\lambda \in \mathbb{C}$ that do not lie on the unit circle. This result can be viewed as a derandomisation of the Markov chain based randomised algorithm by Jerrum and Sinclair [JS93] for $\lambda > 0$ (see also [GJ18, CGHT16]), solving a longstanding problem.[1]

As noted in [LSS19c, Remark p.290], the "no-field" case $|\lambda| = 1$ is unclear, since on the one hand we have the algorithm by [JS93] for $\lambda = 1$, on the other hand it is known that Lee-Yang zeros are dense on the unit circle. The density picture was further explored in [PR20] for graphs of bounded maximum degree Δ, by establishing for each $b \in (0, 1)$ a symmetric arc around $\lambda = 1$ on the unit circle where the partition function does not vanish for all graphs of maximum degree at most Δ and showing density of the Lee-Yang zeros on the complementary arc. See also [CHJR19] for the density result.

In this chapter we resolve the complexity picture of the ferromagnetic Ising model. We show that for graphs of maximum degree Δ approximately computing the partition function is #P-hard[2], on the arc of the unit circle where the Lee-Yang zeros are dense. See Theorem 2.1.1 below for a precise statement of the main results. Since on the complementary arc there exists an FPTAS, by the results of [PR20] (in combination with [Bar16, PR17]), this gives a direct connection between hardness of approximation and the presence of Lee-Yang zeros. Combined with the results of [JS93, LSS19c], our work therefore classifies the complexity of approximating the partition function of the ferromagnetic Ising

[1]Notably, the correlation decay approach, which also yields deterministic approximation algorithms and was key in the full classification of antiferromagnetic 2-spin systems [LLY13, SST14, SS14, GvV15], somewhat surprisingly does not perform as well for ferromagnetic systems; see [GL18] for the state-of-the-art on this front.

[2]The complexity class #P may be seen as the counting version of the complexity class NP. See Section 1.2 or [Val79, Jer03, AB09] for further background.

model on the complex plane.

It should be noted that the existence of zeros does not imply hardness in a straightforward manner.[3] We obtain the connection between the Lee-Yang zeros and computational complexity via tools from complex dynamical systems. The partition function on trees naturally gives rise to a dynamical system; cf. Section 2.2.1. Both the hardness of approximating the partition function as well as the density of the Lee-Yang zeros originate from chaotic behavior of the dynamics, while normal behavior is linked to absence of zeros and hence the existence of efficient approximation algorithms [PR20].

This chapter falls into the broader context of showing how zeros in the complex plane actually relate to the existence and design of approximation algorithms. This connection has been well studied for general graphs; see e.g. [GJ14, GG17, GGHP20]. For bounded-degree graphs, the picture is less clear, but the key seems to lie in understanding the underlying complex dynamical systems [BGGv20, Buy21, CHJR19, PR20, BBGP21, BGGv19]. A general theory is so far elusive, but it seems that the chaotic behavior of the underlying complex dynamical system is linked to the presence of zeros of the partition function and to the #P-hardness of approximation. In the next chapter we will show that this connection is also there for the independence polynomial.

2.1.1. The main results

To state the inapproximability results, we first formally define the computational problems that we consider. For $z \in \mathbb{C}$, we let $|z|$ be the norm of z, $\mathrm{Arg}(z)$ be its argument in the interval $[0, 2\pi)$, and $\arg(z) = \{\mathrm{Arg}(z) + 2k\pi \mid k \in \mathbb{Z}\}$ be the set of all of its arguments. We will consider the problems of approximating the norm of the partition function $Z_G(\lambda, b)$ within a rational factor $K > 1$ and its argument within an additive rational constant $\rho > 0$. For the computational problems, we moreover assume that $b \in (0, 1)$ is rational and λ has rational real and imaginary parts. The rationality assumption is mainly for convenience (representation issues), and it simplifies some of the proofs.

Name $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$.

Instance A graph $G = (V, E)$ with maximum degree $\leq \Delta$.

Output If $Z_G(\lambda, b) = 0$, the algorithm may output any rational. Otherwise, it must return a rational $\widehat{N}$ such that $\widehat{N}/K \leq |Z_G(\lambda, b)| \leq K\widehat{N}$.

[3] For example, the graphs in [PR20] whose partition function is shown to be zero are trees, and these can be clearly detected in polynomial time. More generally, it is hard to imagine a construction of graphs with vanishing partition function which can directly yield hardness. In any case, our results, following the framework of [GJ14, BGGv20, BGGv19], show hardness for a relaxed version of the problems where zeros do not need to be detected, making all these considerations irrelevant.

We remark here that the explicit constant $K > 1$ in the problem definition above is only for convenience, having $K = 2^{n^{1-\epsilon}}$ for any constant $\epsilon > 0$ does not change the complexity of the problem using standard powering arguments (here, n is the size of the input graph).

Name $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$.

Instance A graph $G = (V, E)$ with maximum degree $\leq \Delta$.

Output If $Z_G(\lambda, b) = 0$, the algorithm may output any rational. Otherwise, it must return a rational $\widehat{A}$ such that $|\widehat{A} - a| \leq \rho$ for some $a \in \arg(Z_G(\lambda, b))$.

A *fully polynomial time approximation scheme* (FPTAS) for approximating $Z_G(\lambda, b)$ for given λ and b and positive integer Δ is an algorithm that for any n-vertex graph G of maximum degree at most Δ and any rational $\varepsilon > 0$ solves both probems $\#\mathsf{IsingNorm}(\lambda, b, \Delta, 1 + \varepsilon)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \varepsilon)$ in time polynomial in n/ε.

We use $\mathbb{Q}$ to denote the set of rational numbers and $\mathbb{C}_{\mathbb{Q}}$ to denote the set of complex numbers with rational real and imaginary part. We denote by $\mathbb{S}$ the unit circle in the complex plane, and $\mathbb{S}_{\mathbb{Q}} = \mathbb{S} \cap \mathbb{C}_{\mathbb{Q}}$. It is well-known that numbers in $\mathbb{S}_{\mathbb{Q}}$ are dense on the unit circle.[4] For $\theta \in (0, \pi)$ we denote

$$I(\theta) := \{e^{i\vartheta} \mid -\theta < \vartheta < \theta\}.$$

For $\Delta \geq 3$ and $b \in (\frac{\Delta-2}{\Delta}, 1)$ we denote by $\theta_b \in (0, \pi)$ the angle from [PR20, Theorem A] for which the following holds:

(i) for any graph G of maximum degree at most Δ and any $\lambda \in I(\theta_b)$ it holds that $Z_G(\lambda, b) \neq 0$;

(ii) for each $\lambda \in \mathbb{S} \setminus I(\theta_b)$ there exists $\lambda' \in \mathbb{S}$ arbitarily close to λ and a tree T of maximum degree Δ for which $Z_T(\lambda', b) = 0$.

The main result of this chapter is as follows.

Theorem 2.1.1. *Let $\Delta \geq 3$ be an integer and let $K = 1.001$ and $\rho = \pi/40$.*

(a) Let $b \in \left(0, \frac{\Delta-2}{\Delta}\right]$ be a rational, and $\lambda \in \mathbb{S}_{\mathbb{Q}}$ such that $\lambda \neq \pm 1$. Then the problems $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard.

(b) Let $b \in \left(\frac{\Delta-2}{\Delta}, 1\right)$ be a rational. Then the collection of complex numbers $\lambda \in \mathbb{S}_{\mathbb{Q}}$ for which $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard is dense in the arc $\mathbb{S} \setminus I(\theta_b)$.

[4] See for example the upcoming Lemma 2.8.3.

Combined with [LSS19c], part (a) completely classifies the hardness of approximating the partition function $Z_G(\lambda, b)$ (as per the two computational problems stated above) for $b \in (0, \frac{\Delta-2}{\Delta}]$. Combined with [PR20, Corollary 1], part (b) 'essentially' classifies the hardness of approximating the partition function for $b \in (\frac{\Delta-2}{\Delta}, 1)$ and answers a question from [PR20]. Technically, we do not rule out that there may be an efficient algorithm for these problems for some $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus I(\theta_b)$, but such an algorithm must be specifically tailored to such a particular λ (unless of course $\mathsf{P} = \#\mathsf{P}$). We in fact conjecture that, when $b \in (\frac{\Delta-2}{\Delta}, 1)$, approximating the partition function (as in Theorem 2.1.1) is #P-hard for all non-real $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus I(\theta_b)$. See Remark 2.1.2 below for a discussion of the antipodal case $\lambda = -1$.

We should further remark that the open interval $b \in \big(0, \frac{\Delta-2}{\Delta}\big)$ for positive λ corresponds to the so-called non-uniqueness region of the infinite Δ-regular tree. For the antiferromagnetic Ising model and positive λ, non-uniqueness leads to computational intractability [SS14, GvV16], in contrast to the ferromagnetic case. As we explain in Section 2.2, the phenomenon which underpins our proofs for $|\lambda| = 1$ with $\lambda \neq \pm 1$ is the chaotic behaviour of the underlying complex dynamical system, which resembles in rough terms a complex-plane analogue of non-uniqueness. Interestingly, at criticality, i.e., when $b = \frac{\Delta-2}{\Delta}$, while the model is in uniqueness for $\lambda = 1$, the chaotic behaviour is nevertheless present for non-real λ, and we show #P-hardness for this case too.

Remark 2.1.2. We further discuss the real cases $\lambda = \pm 1$ which are not explicitly covered by Theorem 2.1.1. The case $\lambda = 1$ admits an FPRAS [JS93, GJ18, CGHT16], but the existence of a deterministic approximation scheme is open. We study the case $\lambda = -1$ in more detail in Section 2.9 where we show that the problem is not #P-hard (assuming $\#\mathsf{P} \neq \mathsf{NP}$): using the "high-temperature" expansion of the model, we show an odd-subgraphs formulation of the partition function (Lemma 2.9.1), which is then used to conclude (Theorem 2.9.2) that the sign of the partition function can be determined trivially, while the problem of approximating the norm of the partition function for all $\Delta \geq 3$ is equivalent to the problem of approximately counting the number of perfect matchings (even on unbounded-degree graphs). The complexity of the latter is an open problem in general, but it can be approximated with an NP-oracle [JVV86], therefore precluding #P-hardness.

In the next section, we give an outline of the key pieces to obtain our inapproximability results; the details of these pieces will be filled in in the forthcoming sections (see also the upcoming Section 2.2.5 for the organisation of the chapter).

2.2. Proof outline

Let $\Delta \geq 3$ be an integer, $b \in (0, \frac{\Delta-2}{\Delta}]$, and $\lambda \in \mathbb{S}_{\mathbb{Q}}$ with $\lambda \neq \pm 1$. It will be convenient to work sometimes with $d = \Delta - 1$. For $z_1, z_2 \in \mathbb{S}$ let $\mathrm{Arc}\,[z_1, z_2]$ and $\mathrm{Arc}\,(z_1, z_2)$ denote the counterclockwise arc in $\mathbb{S}$ from z_1 to z_2 including and excluding the endpoints respectively. For an arc A on the unit circle $\mathbb{S}$, we let $\ell(A)$ denote the length of A. We use $\overline{z}$ to denote the conjugate of z.

2.2.1. Rooted-tree gadgets and complex dynamical systems

Our reductions are based on gadgets that are rooted trees, whose analysis will be based on understanding the dynamical behaviour of certain complex maps on the unit circle, given by[5]

$$f_{\lambda,k} : z \mapsto \lambda \cdot \left(\frac{z+b}{bz+1}\right)^k, \text{ for integers } k = 1, \ldots, d. \tag{2.1}$$

We will sometimes drop λ when it is clear from the context. To connect these maps with rooted-tree gadgets, for a graph $G = (V, E)$ and a vertex u of G, we let $Z_{G,+u}$ be the contribution to the partition function from configurations with $\sigma(u) = +$, i.e.,

$$Z_{G,+u}(\lambda, b) := \sum_{\sigma: V \to \{+,-\}; \sigma(u)=+} \lambda^{|n_+(\sigma)|} b^{\delta(\sigma)},$$

and we define analogously $Z_{G,-u}$.

Definition 2.2.1. Let λ, b be arbitrary numbers and T be a tree rooted at r. We say that T implements the field λ' if $Z_{T,-r}(\lambda, b) \neq 0$ and $\lambda' = \frac{Z_{T,+r}(\lambda,b)}{Z_{T,-r}(\lambda,b)}$. We call λ' the field of T.

The next lemma explains the relevance of the maps $f_{\lambda,1}, \ldots, f_{\lambda,d}$ for implementing fields.

Lemma 2.2.2. *Let $b \in (0, 1)$ and $\lambda \in \mathbb{S}$. Let T_1, T_2 be rooted trees with roots r_1, r_2 and fields $\xi_1, \xi_2 \in \mathbb{S}$, respectively. Then, the tree T rooted at r_2 consisting of T_2 and k distinct copies of T_1 which are attached to r_2 via an edge between r_2 and r_1, implements the field $\xi = f_{\xi_2,k}(\xi_1) \in \mathbb{S}$.*

Proof. Omitting for convenience the arguments λ, b from the partition functions, we have

$$Z_{T,+r_2} = Z_{T_2,+r_2}(Z_{T_1,+r_1} + bZ_{T_1,-r_1})^k, \quad Z_{T,-r_2} = Z_{T_2,-r_2}(bZ_{T_1,+r_1} + Z_{T_1,-r_1})^k.$$

Dividing these yields the result (note, $Z_{T_2,-r_2} \neq 0$ and $\xi_1 = \frac{Z_{T_1,+r_1}}{Z_{T_1,-r_1}} \in \mathbb{S}$, so $Z_{T,-r_2} \neq 0$); the fact that $\xi \in \mathbb{S}$ follows from footnote 5. $\square$

[5] Note that, for real b and $\lambda \in \mathbb{S}$, if $z \in \mathbb{S}$ then $f_{\lambda,k}(z) \in \mathbb{S}$ as well.

Note in particular that all fields implemented by trees lie on the unit circle $\mathbb{S}$; see footnote (5). The following theorem, which lies at the heart of the construction of the gadgets, asserts that throughout the relevant range of the parameters we can in fact implement a field arbitrarily close to any number in $\mathbb{S}$. We use $\mathcal{T}_{d+1}$ to denote the set of all rooted trees with maximum degree $\leq d+1$ whose roots have degree $\leq d$.

Definition 2.2.3. Given $b \in (0,1)$ and $d \geq 2$ we denote by $\mathbb{S}_{\mathbb{Q}}(d,b)$ the collection of $\lambda \in \mathbb{S}_{\mathbb{Q}}$ for which the set of fields implemented by trees in $\mathcal{T}_{d+1}$, whose roots have degree 1, is dense in $\mathbb{S}$.

Theorem 2.2.4. *Let $d \geq 2$ be an integer.*

(a) Let $b \in \big(0, \frac{d-1}{d+1}\big]$ be a rational. Then $\mathbb{S}_{\mathbb{Q}}(d,b) = \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$.

(b) Let $b \in \big(\frac{d-1}{d+1}, 1\big)$ be a rational. Then $\mathbb{S}_{\mathbb{Q}}(d,b)$ is dense in $\mathbb{S} \setminus I(\theta_b)$.

Theorem 2.2.4 (b) is in stark contrast to what happens for $\lambda \in I(\theta_b)$, where it is known that fields are confined in an arc around 1 [PR20]. We conjecture that in item (b) it is true that $\mathbb{S}_{\mathbb{Q}}(d,b) = \mathbb{S}_{\mathbb{Q}} \setminus (I(\theta_b) \cup \{-1\})$. Moreover, while in Theorem 2.2.4 we focus on rational b, which is most relevant for our computational problems, we note that for any real $b \in (0,1)$ $\mathbb{S}_{\mathbb{Q}}(d,b)$ is dense in $\mathbb{S}$ in case (a) and dense in $\mathbb{S} \setminus I(\theta_b)$ in case (b). We suspect that case (a) is true when $\mathbb{S}_{\mathbb{Q}}$ is replaced by the collection of algebraic numbers on the unit circle and $b \in (0, \frac{d-1}{d+1}]$ is algebraic, but this seems to be challenging to prove.

Later, in Section 2.7, we bootstrap Theorem 2.2.4 to obtain fast algorithms to implement fields with arbitrarily small error, see Lemma 2.7.1 for the exact statement. Roughly, these fields are then used as "probes" in our reductions to compute exactly the ratio $\frac{Z_{G,+v}(\lambda,b)}{Z_{G,-v}(\lambda,b)}$ for any graph G and vertex v; we say more about this in Section 2.2.4. For now, we focus on the key Theorem 2.2.4 and the ideas behind its proof.

2.2.2. Hardness via Julia-set density

To prove Theorem 2.2.4, we will be interested in the set of values obtained by successive composition of the maps $f_{\lambda,k}$ in (2.1) starting from the point $z = 1$; the main challenge is to prove that, for λ, b as in Theorem 2.2.4, these values are dense on the unit circle $\mathbb{S}$. Part (b) is relatively easy to prove, but the real challenge lies in proving part (a).

To understand the reason that this is challenging let us consider the properties of the map $f_{\lambda,k}$ for some root degree $k \geq 1$ viewed as a dynamical system; cf. the upcoming Lemmas 2.3.2 and 2.3.4 for details. Then, for all $b \in (0,1)$ the following hold.

1. **The "well-behaved" regime:** When $b \in (\frac{k-1}{k+1}, 1)$, there exists $\lambda_k = \lambda_k(b) \in \mathbb{S}$ with $\operatorname{Im} \lambda_k > 0$ such that for all λ in an arc around 1 given by $\operatorname{Arc}[\overline{\lambda_k}, \lambda_k]$, the iterates of the point $z = 1$ under the map $f_{\lambda,k}$ converge to a value $R_k(\lambda) \in \mathbb{S}$. In fact, the map $f_{\lambda,k}$ has nice convergence/contracting properties in an arc around $z = 1$: the iterates of any point in $\operatorname{Arc}[1, R_k(\lambda)]$ converge to $R_k(\lambda)$.

2. **The "chaotic" regime:** On the other hand, when $b \in (\frac{k-1}{k+1}, 1)$ and $\lambda \in \operatorname{Arc}(\lambda_k, \overline{\lambda_k})$ or $b \in (0, \frac{k-1}{k+1}]$, all points in $\mathbb{S}$ belong to the so-called Julia set of the map; roughly, this means that the iterates under $f_{\lambda,k}$ of two distinct but arbitrarily close points in $\mathbb{S}$ will be separated by some absolute constant infinitely many times. In other words, the map $f_{\lambda,k}$ has a chaotic behaviour on $\mathbb{S}$.

For $b \in (0, 1)$, we use $\Lambda_k(b)$ to denote the set of $\lambda \in \mathbb{S}$ where the degree-k map $f_{\lambda,k}$ exhibits the behaviour in (2), see the relevant Definition 2.3.1 and Lemma 2.3.2. Based on item (1), it was shown in [PR20] that the iterates of the point $z = 1$ under the successive composition of the maps in (2.1) stay "trapped" in a small arc around 1 when $b \in (\frac{d-1}{d+1}, 1)$ and $\lambda \in \operatorname{Arc}(\overline{\lambda_d}, \lambda_d)$.

Our goal is to tame the chaotic behaviour in item (2) to get density on $\mathbb{S}$ for fixed $b \in (0, \frac{d-1}{d+1}]$ and $\lambda \in \mathbb{S}\backslash\{\pm 1\}$. We should emphasise here that, in the range of b, λ we consider, the map $f_{\lambda,d}$ has the chaotic behaviour described in item (2) throughout $\mathbb{S}$, so by default it is hopeless to aim for any fine analytical understanding, and this is the major technical obstacle we need to address.

An analogous setting has been previously considered in [BGGv20] in the context of approximating the independence polynomial. The bottleneck of determining the desired density is to first argue density around a point x^* in the Julia set of the degree-d map. Once this is done, the chaotic behaviour of the degree-d map around the Julia-set point x^* can be utilised to bootstrap the density to the whole complex plane. The key challenge here is arguing the initial density around the Julia-set point of the degree-d map, since the degree-d map itself is useless for creating density in the Julia set. In [BGGv20], an auxiliary Fibonacci-style recursion was used to converge to such a point x^*; the density around x^* is then achieved by utilising the convergence to further obtain a set of contracting maps around a neighbourhood N of x^* such that the images of N under the maps form a covering of N.

While the contracting/covering maps framework can be adapted to our setting (see Lemma 2.4.1), the bottleneck step of obtaining the initial density around the Julia-set point requires a radically different argument: the convergence of the recursion in [BGGv20] relies on a certain linearisation property, which is not present in the case of the ferromagnetic Ising model; even worse, the recursion does not converge for all the relevant range of b, λ.

2.2.3. Our approach to obtain density around a Julia-set point

We devise a new technique to tackle the problem of showing density around a point in the Julia set of $f_{\lambda,d}$. The main idea is to exploit the chaotic behaviour of the iterates of $f_{\lambda,k}$ when $\lambda \in \Lambda_k(b)$ to obtain an iterate ξ of 1 with an expanding derivative, i.e., $|f'_{\lambda,k}(\xi)| > 1$. The existence of ξ follows by general arguments from the theory of complex dynamical systems; see the relevant Lemma 2.3.9. The lower bound on the derivative is then used in careful inductive constructions to obtain families of contracting maps that cover an appropriate arc of the circle.

To illustrate the main idea of this inductive construction, let us assume that the degree $d + 1$ is odd. Then, using Lemma 2.3.9 and the fact that $\lambda \in \Lambda_d(b)$, we obtain an iterate of the point $z = 1$ under the map $f_{\lambda,d}$, say ξ, so that $|f'_{\lambda,d}(\xi)| > 1$. The key point is to consider the map $f_{\xi,k}$ for $k = d/2$. On one hand, if it happens that $\xi \notin \Lambda_k(b)$ lies in the "well-behaved" regime of the degree-k map, it can be shown that the maps $f_{\xi,k}$, $f_{\xi,k+1}$ are contracting/covering maps in an appropriate arc of $\mathbb{S}$, yielding the required density as needed (details of this argument can be found in Lemma 2.4.3). On the other hand, if $\xi \in \Lambda_k(b)$ lies in the "chaotic" regime of the degree-k map, then we can proceed inductively by finding an iterate ν of 1 under the map $f_{\xi,k}$ with expanding derivative $|f'_{\xi,k}(\nu)| > 1$ and recurse.

Technically, to carry out this inductive scheme we have to address the various integrality issues, while at the same time being careful to maintain the degrees of the trees bounded by Δ. More importantly, we need to consider pairs/triples/quadruples of maps to ensure the contraction/covering property in the inductive step; to achieve this, we need to understand the dependence of the derivative at the fixpoint of the k-ary map with k. Here, things turn out to be surprisingly pleasant, since $|f'_{\lambda,k}(z)|$ depends linearly on the degree k and is independent of λ, see item (i) of Lemma 2.3.4; this fact is exploited in the arguments of Section 2.4.2. These considerations cover almost all cases, but a few small degrees d remain that we cover by a Cantor-style construction, see Section 2.5.2 for details.

2.2.4. The reduction

The arguments discussed so far can be used to show that rooted trees in $\mathcal{T}_{d+1}$ implement any field ξ on the unit circle $\mathbb{S}$ within arbitrarily small error $\epsilon > 0$, see Lemma 2.7.1 for the form that we actually need. We now discuss in slightly more detail the high-level idea behind the final reduction argument in Section 2.8.

The key observation to utilise the gadgets is that for any graph G and vertex v with $Z_{G,\text{-}v}(\lambda, b) \neq 0$, the "field" at a vertex v satisfies $\frac{Z_{G,+v}(\lambda,b)}{Z_{G,\text{-}v}(\lambda,b)} \in \mathbb{S}$, cf. Lemma 2.8.2, and hence we can use our rooted-tree gadgets as probes to compute exactly the ratio $Q_{G,v} := \frac{Z_{G,+v}(\lambda,b)}{Z_{G,\text{-}v}(\lambda,b)}$. The straightforward way to do this would be to attach a tree on v which implements a field $x \in \mathbb{S}$ and use oracle calls to either $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ and look for $x = x^*$ that makes the partition function of the resulting graph equal to zero. From the key observation earlier, we know that such an x^* exists, namely $x^* = -1/Q_{G,v}$, and, to determine it, we can use binary search.

This is the main idea behind the reduction, though there are a couple of caveats. First of all, there is no way to know whether the ratio $Q_{G,v}$ is well-defined, i.e., whether $Z_{G,\text{-}v}(\lambda, b) \neq 0$, even using oracle calls to the approximation problems we study: $Z_{G,\text{-}v}(\lambda, b)$ is not a partition function of a graph (since v's spin is fixed), and even if we managed to cast this as a partition function, the oracles cannot detect zeros (cf. Section 2.1.1). The second caveat is that attaching the tree increases the degree of v which is problematic when, e.g., G is Δ-regular, and the peeling-vertices argument does not quite work since there is no simple way to utilise the oracles after the first step.

The first point is addressed by replacing the edges of G with paths of appropriate length which has the effect of "changing" the value of the parameter b to some value $\hat{b}$ close but not equal to 1 where the partition function is zero-free. We actually need to attach to internal vertices of the paths rooted trees with fields close to $1/\lambda$ so that the complex external field λ is almost cancelled. Then, using oracle calls to $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ or $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$, our algorithm aims to determine the value of $Z_G(\lambda, \hat{b})$ which is a $\#\mathsf{P}$-hard problem ([KC16], Theorem 1.1). The key-point is that now we have zero-freeness of the partition function which allows us to assert that the quantities we compute during the course of the algorithm are actually well-defined.

The second point is addressed by doing the peeling-argument at the level of edges by trying to figure out, for an edge e of G, the value of the ratio $\hat{Q}_{G,e} = \frac{Z_G(\lambda,\hat{b})}{Z_{G,\setminus e}(\lambda,\hat{b})}$. We do this by subdividing the edge and use a field gadget on the middle vertex; this has the benefit that it does not increase the maximum degree of the graph but certain complications arise since instead of $\hat{Q}_{G,e}$ we retrieve a slightly different ratio; see Lemma 2.8.5 in Section 2.8.2. Some extra work is required to finish off the proof of Theorem 2.1.1; see Section 2.8.3 for details.

2.2.5. Outline

The next section details the dynamical properties of the maps $f_{\lambda,k}$ and elaborates on the inductive proof of Theorem 2.2.4, which is based on the upcoming Lemma 2.3.11. Section 2.4 explains in more detail the contracting/covering maps framework and how we utilise the degree/derivative interplay to cover the bulk of the cases in Lemma 2.3.11. Section 2.5 contains the remaining pieces needed to complete the proof of Lemma 2.3.11, which is given in Section 2.6. In Section 2.7, we bootstrap Theorem 2.2.4 to obtain fast algorithms to implement fields on the unit circle with arbitrarily small precision error, which is used in the reduction arguments of Section 2.8, where the proof of Theorem 2.1.1 is completed. Finally, in Section 2.9, we study the case $\lambda = -1$ (cf. Remark 2.1.2), and show the equivalence with the problem of approximately counting perfect matchings.

2.3. Complex dynamics preliminaries and the inductive step in Theorem 2.2.4

In this section, we set up some preliminaries about the maps $f_{\lambda,k}$ in (2.1) that will be used to prove Theorem 2.2.4. We first consider the general case $k \geq 1$ in Section 2.3.1 and then further study the $k = 1$ case separately in Section 2.3.2. In Section 2.3.3, we use these properties to obtain points with expanding derivatives using tools from complex dynamics. Then, in Section 2.3.4, we give the main lemma that lies at the heart of the inductive proof of Theorem 2.2.4 and conclude the proof of the latter.

2.3.1. Results on $f_{\lambda,k}$ for general k

This section contains relevant properties of the maps $f_{\lambda,k} : z \mapsto \lambda\cdot\left(\frac{z+b}{bz+1}\right)^k$ that we will need; these were discussed informally in Section 2.2.2, and here we formalise them. Almost all results of this section follow from arguments in [PR20].

We begin by defining formally the set $\Lambda_k(b)$.

Definition 2.3.1. Let $k \geq 1$ be an integer and $b \in (0, 1)$. We let $\Lambda_k(b)$ be the set of $\lambda \in \mathbb{S}$ such that all fixed points z of the map $f_{\lambda,k}$ with $z \in \mathbb{S}$ are repelling, i.e., $|f'_{\lambda,k}(z)| > 1$.

The following lemma gives a description of the set $\Lambda_k(b)$ and characterises the Julia set of $f_{\lambda,k}$. We have described informally the dynamical properties of the map $f_{\lambda,k}$ that the Julia set captures, see item (2) in Section 2.2.2. The reader is referred to [Mil06, Chapter 4] for more details on the general theory.

Lemma 2.3.2. *Let $k \geq 1$ be an integer. Then,*

- *if $b \in (0, \frac{k-1}{k+1})$, $\Lambda_k(b) = \mathbb{S}$. For $b = \frac{k-1}{k+1}$, $\Lambda_k(b) = \mathbb{S}\backslash\{+1\}$.*
- *if $b \in (\frac{k-1}{k+1}, 1)$, there is $\lambda_k = \lambda_k(b) \in \mathbb{S}$ with $\mathrm{Im}\,(\lambda_k) > 0$ such that $\Lambda_k(b) = \mathrm{Arc}(\lambda_k, \overline{\lambda_k})$.*

Moreover, if $k > 1$, then for all $\lambda \in \Lambda_k(b)$, the Julia set of $f_{\lambda,k}$ is equal to the unit circle $\mathbb{S}$.

Proof. For $b \in (\frac{k-1}{k+1}, 1)$, the range of $\Lambda_k(b)$ follows from [PR20, Theorem 14]. For $b \in (0, \frac{k-1}{k+1}]$, the range of $\Lambda_k(b)$ follows from item (i) in Lemma 2.3.4 below. The characterisation of the Julia set for $\lambda \in \Lambda_k(b)$ is shown in [PR20, Proof of Proposition 17]. □

Remark 2.3.3. The $\lambda_k(b)$ of the lemma is equal to $e^{i\theta_b}$ from the statement of Theorem 2.1.1 (where we replace Δ by $k+1$).

Let A be an arc of $\mathbb{S}$. A map $f : A \to \mathbb{S}$ is orientation-preserving if for any $z, z_1, z_2 \in A$ with $z \in \text{Arc}\,[z_1, z_2]$ it holds that $f(z) \in \text{Arc}\,[f(z_1), f(z_2)]$. The orbit of a point $z_0 \in \mathbb{S}$ under the map $f_{\lambda,k}$ is the sequence of the iterates $\{f^n_{\lambda,k}(z_0)\}_{n\geq 0}$. A fixed point z of the map $f_{\lambda,k}$ is called attracting if $|f'_{\lambda,k}(z)| < 1$ and parabolic if $|f'_{\lambda,k}(z)| = 1$.

The following lemma captures properties of the maps $f_{\lambda,k}$ when $k \in \{1, \dots, d-1\}$ in the regime $b \in \left(\frac{d-2}{d}, \frac{d-1}{d+1}\right]$, which turns out to be the hard part of the proof of Theorem 2.2.4 (the lemma is stated more generally for $b \in \left(\frac{d-2}{d}, 1\right)$).

Lemma 2.3.4. *Let $d \in \mathbb{Z}_{\geq 2}$ and let $b \in \left(\frac{d-2}{d}, 1\right)$. Then the following holds:*

(i) For all $\lambda \in \mathbb{S}$ and $k \in \mathbb{Z}_{\geq 1}$ the map $f_{\lambda,k} : \mathbb{S} \to \mathbb{S}$ is orientation-preserving. Also, the magnitude of the derivative at a point $z \in \mathbb{S}$ does not depend on λ and equals $|f'_k(z)|$ where

$$|f'_k(z)| = k \cdot |f'_1(z)| = \frac{k(1-b^2)}{b^2 + 2b \cdot \text{Re}\,(z) + 1}. \tag{2.2}$$

(ii) For $k \in \{1, \dots, d-1\}$, let $\lambda_k = \lambda_k(b) \in \mathbb{S}$ be as in Lemma 2.3.2. Then,

- *if $\lambda \in \text{Arc}\,(\overline{\lambda_k}, \lambda_k)$, then $f_{\lambda,k}$ has a unique attracting fixed point $R_k(\lambda) \in \mathbb{S}$.*
- *if $\lambda = \overline{\lambda_k}$ or λ_k, then $f_{\lambda,k}$ has a unique parabolic fixed point $R_k(\lambda) \in \mathbb{S}$.*

(iii) The fixed point maps $R_k : \text{Arc}\,[\overline{\lambda_k}, \lambda_k] \to \mathbb{S}$ are continuously differentiable on $\text{Arc}\,(\overline{\lambda_k}, \lambda_k)$ and orientation-preserving with the property that $R_k(1) = 1$ and $R_k(\overline{\lambda}) = \overline{R_k(\lambda)}$.

(iv) For $\lambda \in \text{Arc}\,(1, \lambda_k]$, the fixed point $R_k(\lambda)$ is in the upper half-plane. For $z_0 \in \text{Arc}\,[1, R_k(\lambda)]$ the orbit of z_0 under iteration of $f_{\lambda,k}$ converges to $R_k(\lambda)$ and is contained in $\text{Arc}\,[z_0, R_k(\lambda)]$.

(v) The following inequalities hold:

$$Arg(\lambda_{d-1}) < Arg(\lambda_{d-2}) < \cdots < Arg(\lambda_1),$$

while for $\lambda \in \mathrm{Arc}\,(1, \lambda_m]$*, with* $m \leq d-1$*, we have*

$$Arg(R_1(\lambda)) < Arg(R_2(\lambda)) < \cdots < Arg(R_m(\lambda)),$$

with the additional property that, for $i \in \{1, \ldots, m-2\}$*,*

$$\ell\left(\mathrm{Arc}\left[R_i(\lambda), R_{i+1}(\lambda)\right]\right) \leq \ell\left(\mathrm{Arc}\left[R_{i+1}(\lambda), R_{i+2}(\lambda)\right]\right). \tag{2.3}$$

Proof. We refer to [PR20] for proofs of items (i)–(iv). Specifically, item (i) follows from [PR20, Lemma 8 & Equation (3.1)], item (ii) from [PR20, Lemma 13, Theorem 14, Proof of Proposition 17], item (iii) from [PR20, Proof of Theorem 14], and item (iv) from [PR20, Theorem 14, Proof of Theorem 5(i)].

We will prove item (v). By taking the derivative of both sides of the equality $f_{\lambda,k}(R_k(\lambda)) = R_k(\lambda)$ with respect to λ and rewriting we obtain

$$R_k'(\lambda) = \frac{R_k(\lambda)}{\lambda\Big(1 - f_{\lambda,k}'\big(R_k(\lambda)\big)\Big)}.$$

Using equation (2.2) for $z = R_m(1) = 1$ we obtain that $R_{i+1}'(1) > R_i'(1)$ for $i \in \{1, \ldots, d-2\}$ and thus for $\lambda \in \mathbb{S}$ in the upper half-plane near 1 we find that $\mathrm{Arg}(R_{i+1}(\lambda)) > \mathrm{Arg}(R_i(\lambda))$.

The derivative at a fixed point of a map of the unit circle to itself is real (see also [PR20, Lemma 11]). Furthermore, if such a map is orientation-preserving the derivative at a fixed point is positive. Because the map $f_{\lambda,i}$ is orientation-preserving with attracting fixed point $R_i(\lambda)$ we find that $f_{\lambda,i}'(R_i(\lambda)) \in (0,1)$ for $\lambda \in \mathrm{Arc}\,(\overline{\lambda_i}, \lambda_i)$. From this we deduce that we can write

$$|R_k'(\lambda)| = \frac{1}{1 - f_{\lambda,k}'\left(R_k(\lambda)\right)}. \tag{2.4}$$

From equation (2.2) it can be seen that $|f_i'(z)|$ is increasing both with respect to $\mathrm{Arg}(z)$ when $\mathrm{Im}(z) > 0$ and with respect to the index i and thus, as long as $R_i(\lambda)$ and $R_{i+1}(\lambda)$ are both defined and $\mathrm{Arg}(R_{i+1}(\lambda)) > \mathrm{Arg}(R_i(\lambda))$, we deduce that $|R_{i+1}'(\lambda)| > |R_i'(\lambda)|$. Since $\mathrm{Arg}(R_{i+1}(\lambda)) > \mathrm{Arg}(R_i(\lambda))$ for λ in the upper half-plane close to 1, we conclude that there cannot be any λ in the upper half-plane such that $\mathrm{Arg}(R_{i+1}(\lambda)) \leq \mathrm{Arg}(R_i(\lambda))$.

Now suppose that there is some index i such that $\mathrm{Arg}(\lambda_i) \leq \mathrm{Arg}(\lambda_{i+1})$. Note that $R_i(\lambda_i)$ is a parabolic fixed point of $f_{\lambda_i,i}$, which means that $f_{\lambda_i,i}'(R_i(\lambda_i)) = 1$. Because we assumed that $\mathrm{Arg}(\lambda_i) \leq \mathrm{Arg}(\lambda_{i+1})$ we see from item (ii) that $R_{i+1}(\lambda_i)$ must be well defined. We already deduced that $\mathrm{Arg}(R_{i+1}(\lambda_i)) > \mathrm{Arg}(R_i(\lambda_i))$ and thus $f_{\lambda_i,i+1}'(R_{i+1}(\lambda_i)) > f_{\lambda_i,i}'(R_i(\lambda_i)) = 1$, which contradicts the fact that $R_{i+1}(\lambda_i)$ is an attracting fixed point of $f_{\lambda_i,i+1}$. This concludes the proof of the first two claims of item (v).

Finally, we show the final claim of item (v). For indices $0 \leq i \leq j \leq m$, it will be convenient to denote by $A_{i,j,\lambda}$ the arc $\text{Arc}\,[R_i(\lambda), R_j(\lambda)]$, under the convention that $R_0(\lambda) = 1$. For $\lambda \in \text{Arc}\,(1, \lambda_m]$, our goal is hence to show that $\ell\,(A_{i,i+1,\lambda}) \leq \ell\,(A_{i+1,i+2,\lambda})$ for all $i \in \{1, \ldots, m-2\}$.

For any $\lambda \in \text{Arc}\,(1, \lambda_k]$ we observe for $i = 1, \ldots, k$ that

$$\ell(A_{0,i,\lambda}) = \int_{\text{Arc}\,[1,\lambda]} |R_i'(z)|\,|dz|$$

and thus for $i \in \{1, \ldots, k-1\}$ we have $\ell(A_{i,i+1,\lambda}) = \int_{\text{Arc}\,[1,\lambda]} |R_{i+1}'(z)| - |R_i'(z)|\,|dz|$.

We first show item (v) for λ near 1. As we let λ approach 1 along the circle, we obtain that

$$\begin{aligned}\lim_{\lambda\to 1} \frac{\ell(A_{i,i+1,\lambda})}{\ell(\text{Arc}\,[1,\lambda])} &= |R_{i+1}'(1)| - |R_i'(1)| = \frac{1}{1 - f_{\lambda,i+1}'(1)} - \frac{1}{1 - f_{\lambda,i}'(1)} \\ &= \frac{(1+b)/(1-b)}{\big(i - (1+b)/(1-b)\big)\big(i - 2b/(1-b)\big)}.\end{aligned}$$

The second equality can be obtained by using (2.4) and the third by using (2.2) and simplifying. If we denote this expression by $g(i)$ then it is not hard to see that $g(i+1) > g(i)$ as long as $i + 1 < 2b/(1-b)$. Because $b \in (\frac{d-2}{d}, 1)$, we have $2b/(1-b) > d-2$. So indeed $g(i+1) > g(i)$ for $i \in \{1, \ldots, d-3\}$, which contains $\{1, \ldots, k-2\}$. This shows that inequality in (2.3) is true for λ near 1.

Now suppose that there is $\lambda \in \mathbb{S}$ and index i for which the inequality in (2.3) does not hold. Then, by continuity, because the inequality does hold near 1, there is $\lambda \in \mathbb{S}$ for which the inequality is an equality, i.e.,

$$\ell\,(A_{i,i+1,\lambda}) = \ell\,(A_{i+1,i+2,\lambda})\,. \tag{2.5}$$

For convenience, we will henceforth drop the subscript λ from the notation for the arcs $A_{i,j,\lambda}$ and simply write $A_{i,j}$. The maps $f_{\lambda,j}$ are orientation-preserving for any j and thus

$$\ell(\text{Arc}[\lambda, R_j(\lambda)]) = \ell(f_{\lambda,j}(A_{0,j})) = \int_{A_{0,j}} |f_j'(z)|\,|dz|.$$

Using this equality we can write

$$\ell(A_{j,j+1}) = \int_{A_{0,j+1}} |f_{j+1}'(z)|\,|dz| - \int_{A_{0,j}} |f_j'(z)|\,|dz|.$$

We use this equality for $j = i$ and $j = i+1$ and rearrange (2.5) to obtain

$$2\int_{A_{0,i+1}} |f_{i+1}'(z)|\,|dz| = \int_{A_{0,i+2}} |f_{i+2}'(z)|\,|dz| + \int_{A_{0,i}} |f_i'(z)|\,|dz|.$$

We use (2.2) to rewrite the left-hand side of this equation as

$$2(i+1)\int_{A_{0,i}} |f_1'(z)|\,|dz| + 2\int_{A_{i,i+1}} |f_{i+1}'(z)|\,|dz|$$

and we rewrite the right-hand side as

$$(i+2)\int_{A_{0,i}} |f_1'(z)|\,|dz| + \int_{A_{i,i+2}} |f_{i+2}'(z)|\,|dz| + i\cdot\int_{A_{0,i}} |f_1'(z)|\,|dz|.$$

These two being equal implies that

$$2\int_{A_{i,i+1}} |f_{i+1}'(z)|\,|dz| = \int_{A_{i,i+2}} |f_{i+2}'(z)|\,|dz|.$$

We will show that this yields a contradiction. We rewrite the right-hand side as

$$\int_{A_{i,i+2}} |f_{i+2}'(z)|\,|dz| = \int_{A_{i,i+1}} |f_{i+2}'(z)|\,|dz| + \int_{A_{i+1,i+2}} |f_{i+2}'(z)|\,|dz|.$$

and we will show that both summands are greater than $\int_{A_{i,i+1}} |f_{i+1}'(z)|\,|dz|$, which will yield the contradiction. The inequality for the first summand follows easily from the fact that $|f_{i+2}'(z)| > |f_{i+1}'(z)|$ for all $z \in \mathbb{S}$; cf. (2.2). The second inequality uses the fact that $|f_{i+2}'(z)|$ increases as $\operatorname{Arg}(z)$ increases (when $\operatorname{Im} z > 0$) and thus

$$\begin{aligned}\int_{A_{i+1,i+2}} |f_{i+2}'(z)|\,|dz| &> |f_{i+2}'(R_{i+1}(\lambda))|\cdot\ell(A_{i+1,i+2}) > |f_{i+1}'(R_{i+1}(\lambda))|\cdot\ell(A_{i,i+1})\\ &> \int_{A_{i,i+1}} |f_{i+1}'(z)|\,|dz|.\end{aligned}$$

The second inequality of this derivation uses the assumed equality in (2.5). This yields the desired contradiction. □

Remark 2.3.5. For any $\lambda \in \mathbb{S}$ and $k \in \mathbb{Z}_{\geq 1}$ for which the fixed point $R_k(\lambda)$ is defined we have $|f_k'(R_k(\lambda))| > |f_k'(\lambda)|$. To see this when $\operatorname{Im}\lambda > 0$, note from item (iv) and the fact that the maps $f_{\lambda,k}$ are orientation-preserving that $\operatorname{Arg}(R_k(\lambda)) \in (\operatorname{Arg}(\lambda), \pi)$ and hence by (2.2) that $|f_k'(R_k(\lambda))| > |f_k'(\lambda)|$. When $\operatorname{Im}\lambda < 0$, the inequality follows from the above since $\overline{R_k(\lambda)} = R_k(\overline{\lambda})$ from item (iii) and the expression in (2.2) depends only the real part of z.

2.3.2. Results on $f_{\lambda,1}$

Note that $f_{\lambda,1}$ is a Möbius transformation; we will extract some relevant information about it using the theory of Möbius transformations, following [Bea95, Section 4.3].

There is a natural way to relate each Möbius transformation g with a 2×2 matrix A. Formally, let $\mathrm{GL}_2(\mathbb{C})$ be the group of 2×2 invertible matrices with complex entries (with the multiplication operation) and $\mathcal{M}$ be the group of Möbius transformations (with the composition operation $\circ$). The following map gives a surjective homomorphism between the groups $\mathrm{GL}_2(\mathbb{C})$ and $\mathcal{M}$:

$$\Phi : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (z \mapsto \frac{az+b}{cz+d}).$$

For $g \in \mathcal{M}$, take any $A \in \Phi^{-1}(g)$ and define $\mathrm{tr}^2(g) = \mathrm{tr}(A)^2/\det(A)$. This value does not depend on the choice of preimage and thus tr^2 is a well defined operator on $\mathcal{M}$. In the following theorem it is stated how this operator is used to classify Möbius transformations. We say that $f, g \in \mathcal{M}$ are conjugate if there is an $h \in \mathcal{M}$ such that $f = h \circ g \circ h^{-1}$.

Theorem 2.3.6 ([Bea95, Theorem 4.3.4]). *Let $g \in \mathcal{M}$ not equal to the identity, then g is conjugate to*

1. *a rotation $z \mapsto e^{i\theta} z$ for some $\theta \in (0, \pi]$ if and only if* $\mathrm{tr}^2(g) \in [0, 4)$, *in which case* $\mathrm{tr}^2(g) = 2 \cdot (\cos(\theta) + 1)$;
2. *a multiplication $z \mapsto e^{\theta} z$ for some $\theta \in \mathbb{R}_{>0}$ if and only if* $\mathrm{tr}^2(g) \in (4, \infty)$, *in which case* $\mathrm{tr}^2(g) = 2 \cdot (\cosh(\theta) + 1)$.

In case (1), g is said to be *elliptic*, while in case (2) g is called *hyperbolic*. If $\mathrm{tr}^2(g) = 4$ the map is called *parabolic*.

Corollary 2.3.7. *Let $b \in (0, 1)$ and let $\lambda_1 = \lambda_1(b) \in \mathbb{S}$ be as in Lemma 2.3.2. The map $f_{\lambda,1}$ is hyperbolic when $\lambda \in \mathrm{Arc}\,(\overline{\lambda_1}, \lambda_1)$ and $f_{\lambda,1}$ is elliptic when $\lambda \in \mathrm{Arc}\,(\lambda_1, \overline{\lambda_1})$.*

Proof. Write $\lambda = x + iy$ with $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$. A short calculation gives that

$$\mathrm{tr}^2(f_{\lambda,1}) = \frac{2\,(x+1)}{1-b^2}.$$

The value of $tr^2(f_{\lambda,1})$ strictly increases from 0 to $4/(1-b^2)$ as x increases from -1 to 1. It follows that there is a unique value $x \in (-1, 1)$ such that $\mathrm{tr}^2(f_{\lambda,1}) = 4$. This value must coincide with $\mathrm{Re}(\lambda_1)$, where $\lambda_1 = \lambda_1(b) \in \mathbb{S}$ as in Lemma 2.3.2, completing the proof. □

Lemma 2.3.8. *Let $b \in (0,1)$ be a rational. Suppose that $\xi \in \mathbb{S}_{\mathbb{Q}}$ with $\xi \neq \pm 1$ is such that $f_{\xi,1}$ is elliptic. Then $f_{\xi,1}$ is conjugate to an irrational rotation.*

Proof. Let $\xi = x + iy$ with $x, y \in \mathbb{Q}$ such that $x^2 + y^2 = 1$. Because $f_{\xi,1}$ is elliptic it is conjugate to a rotation $z \mapsto e^{i\theta} z$ with

$$2 \cdot (\cos(\theta) + 1) = \frac{2\,(x+1)}{1 - b^2}. \tag{2.6}$$

Let $t = 2 \cdot (\cos(\theta) + 1)$. Suppose θ is an angle corresponding to a rational rotation, i.e., if we let $z = e^{i\theta}$ then there is a natural number n such that $z^n = 1$. It follows that then also $\overline{z}^n = 1$ and thus both z and $\overline{z}$ are algebraic integers. Therefore $z + \overline{z} = 2\cos(\theta)$ is an algebraic integer. It follows that t is an algebraic integer, while the right-hand side of (2.6) shows that t must also be rational. Because the only rational algebraic integers are integers we can conclude that t must be an integer and thus $t \in \{0, 1, 2, 3\}$. If $t = 0$ we see that $\xi = x = -1$, which we excluded, so only three possible values of t remain. Let $X = t(1+b)/(1-b)$ and $Y = 2ty/(1-b)^2$ then (X, Y) is a rational point on the elliptic curve E_t given by the following equation:

$$E_t : Y^2 = X^3 - (t-2)t \cdot X^2 + t^2 \cdot X.$$

The set of rational points of an elliptic curve together with an additional point has a group structure that is isomorphic to $\mathbb{Z}^r \times \mathbb{Z}/N\mathbb{Z}$. The number $r \geq 0$ is called the rank of the curve and the subgroup isomorphic to $\mathbb{Z}/N\mathbb{Z}$ is called the torsion subgroup. The rank and the torsion subgroup of a particular curve can be found using a computer algebra system. Using *Sage*, if the variable `t` is declared to be either $1, 2$ or 3, the curve E_t can be defined with the code `Et = EllipticCurve([0, -(t-2)*t, 0, t**2, 0])`. The rank and the torsion subgroup can subsequently be found with the commands `Et.rank()` and `Et.torsion_subgroup()`. We find that E_t has rank 0 for all $t \in \{1, 2, 3\}$. For $t \in \{1, 3\}$ the torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and for $t = 2$ it is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This means that there is one rational point on E_1 and E_3, which we can see is the point $(0, 0)$, and there are three rational points on E_2, namely $\{(0, 0), (2, \pm 4)\}$. These points do not correspond to values of b within the interval $(0, 1)$, which means that θ cannot correspond to a rational rotation. $\square$

2.3.3. Obtaining points with expanding derivatives

In this section we use the dynamical study of the maps $f_{\lambda,k}$ from previous sections to conclude the existence of points with expanding derivatives. More precisely, we show the following.

Lemma 2.3.9. *Let $b \in (0,1)$, $k \geq 1$ be an integer, and $\xi \in \Lambda_k(b)$ with $\xi \neq -1$. Let $z_0 \in \mathbb{S}$ and let $z_n = f^n_{\xi,k}(z_0)$ for $n > 0$. Then there is some index m such that $|f'_{\xi,k}(z_m)| > 1$.*

Proof. For $k = 1$ it follows from Corollary 2.3.7 that $f_{\xi,1}$ is conjugate to a rotation. If $f_{\xi,1}$ is conjugate to an irrational rotation then the orbit of any initial point z_0 will get arbitrarily close to -1 for which $|f'_1(-1)| = \frac{1+b}{1-b} > 1$. Otherwise, if $f_{\xi,1}$ is conjugate to a rational rotation, there is an integer $N > 1$ such that $f^N_{\xi,1}(z) = z$ for all z; consider the smallest such integer N. Let $\theta \in (0, \pi]$ be the angle such that $f_{\xi,1}$ is conjugate to the rotation $z \mapsto e^{i\theta} \cdot z$. Equation (2.6) then states that

$$2 \cdot (\cos(\theta) + 1) = \frac{2\,(\mathrm{Re}(\xi) + 1)}{1 - b^2}.$$

If $N = 2$, then $\theta = \pi$ and thus $\mathrm{Re}(\xi) = -1$ contradicting $\xi \neq -1$. Hence, $N > 2$. From $f^N_{\xi,1}(z) = z$, we obtain

$$\prod_{n=0}^{N-1} f'_{\xi,1}(z_n) = (f^N_{\xi,1})'(z_0) = 1. \tag{2.7}$$

From (2.2) there are precisely two values of $w \in \mathbb{S}$ such that $|f'_{\xi,1}(w)| = 1$. Because $N > 2$ and N is the smallest integer such that $f^N_{\xi,1}(z_0) = z_0$, we conclude there is at least one term, say with index m, of the product in (2.7) for which $|f'_{\xi,1}(z_m)| > 1$.

Consider now the case $k \geq 2$ and denote $f = f_{\xi,k}$. By Lemma 2.3.2, for $\xi \in \Lambda_k(b)$ the Julia set of f is the circle $\mathbb{S}$. In [PR20, Proof of Proposition 17], it is shown that the two Fatou components of f, denoted by $\mathbb{D}$ and $\overline{\mathbb{D}}^c$, are attracting basins and contain the critical points $-b$ and $-1/b$. From [Mil06, Theorem 19.1], we therefore conclude that the map f is hyperbolic, i.e., there exists a conformal metric μ on a neighborhood U of $\mathbb{S}$ such that $||Df_z||_\mu \geq \kappa > 1$ for a constant κ and all $z \in \mathbb{S}$. We will briefly expound the meaning of this notation.

For any $z \in U$ the metric μ induces a norm $||\cdot||_\mu$ on the tangent space of U at z denoted by TU_z. For $z \in \mathbb{S}$ the map f induces a linear map $Df_z : TU_z \to TU_{f(z)}$. The norm $||Df_z(v)||_\mu / ||v||_\mu$ is independent of the choice of nonzero vector $v \in Df_z$ and is denoted by $||Df_z||_\mu$.

Because $\mathbb{S}$ is compact and the metric μ is conformal there is a constant $c > 0$ such that $|g'(z)| > c \cdot ||Dg_z||_\mu$ for all $z \in \mathbb{S}$ and maps $g : \mathbb{S} \to \mathbb{S}$. If follows that for all $N > 0$

$$\prod_{n=0}^{N-1} |f'(z_n)| = |(f^N)'(z_0)| > c \cdot ||Df^N_{z_0}||_\mu \geq c \cdot \kappa^N.$$

There is an $N > 0$ such that the right-hand side of this equation is greater than 1. The product on the left-hand side of the equation shows that for such an N there must be at least one index $m \in \{0, \dots, N-1\}$ such that $|f'(z_m)| > 1$. $\square$

Lemma 2.3.10. *Let $b \in (0,1)$, $\lambda \in \mathbb{S} \setminus \{\pm 1\}$ and $d, k \in \mathbb{Z}_{\geq 1}$. Supppose there exists a rooted tree in $\mathcal{T}_{d+1}$ whose root degree m is at most $d-k$ and which implements a field $\xi \in \Lambda_k(b) \backslash \{-1\}$.*

Then there is $\sigma \in \mathbb{S}$ with $|f_k'(\sigma)| > 1$ and a sequence of rooted trees $\{T_n\}_{n \geq 1}$ in $\mathcal{T}_{d+1}$ with root degrees at most $m+k$ which implement a sequence of fields $\{\zeta_n\}_{n\geq 1}$ such that ζ_n approaches σ without being equal to σ.

Proof. Consider the orbit

$$\mathcal{S} = \left\{ f_{\lambda,1}^n(1) : n \geq 1 \right\}.$$

Note that the elements of $\mathcal{S}$ are fields of paths. We have seen in Section 2.3.2 that either $f_{\lambda,1}$ is conjugate to an irrational rotation or the orbit of 1 tends towards an attracting or a parabolic fixed point. In either case there is $\sigma_0 \in \mathbb{S}$ such that $\sigma_0 \notin \mathcal{S}$ and the elements of $\mathcal{S}$ accumulate on σ_0. It follows from Lemma 2.3.9 that there is a positive integer N such that $\sigma := f_{\xi,k}^N(\sigma_0)$ has the property $|f_k'(\sigma)| > 1$. Now define

$$\mathcal{R} = \left\{ f_{\xi,k}^N(s) : s \in \mathcal{S} \right\}.$$

By assumption, ξ can be implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree $m \leq d-k$, so by applying inductively Lemma 2.2.2, the elements of $\mathcal{R}$ are fields of trees in $\mathcal{T}_{d+1}$ whose root degrees is $m+k \leq d$. There is a sequence $\{\zeta_n\}_{n\geq 1} \subseteq \mathcal{R}$ accumulating on σ without being equal to σ, which is what we wanted to show. $\square$

2.3.4. The main lemma to carry out the induction: proof of Theorem 2.2.4

We are now ready to state the following lemma that will imply Theorem 2.2.4. In this section we will show how Theorem 2.2.4 follows from this lemma, and the next couple of sections are dedicated to proving Lemma 2.3.11.

Lemma 2.3.11. *Let $k, d \in \mathbb{Z}_{\geq 2}$ with $k \leq d$, $b \in \left(\frac{d-2}{d}, \frac{d-1}{d+1}\right] \cap \mathbb{Q}$ and $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$. Suppose there exists a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-k$ that implements a field $\xi \neq 1$ with the property that $|f_k'(\xi)| \geq 1$ and $\xi \in \mathrm{Arc}\,[\overline{\lambda_{\lfloor k/2 \rfloor}}, \lambda_{\lfloor k/2 \rfloor}]$. Then the set of fields implemented by trees in $\mathcal{T}_{d+1}$ is dense in $\mathbb{S}$.*

Using this lemma we can prove Theorem 2.2.4, which we restate here for convenience.

Theorem 2.2.4. *Let $d \geq 2$ be an integer.*

(a) Let $b \in \left(0, \frac{d-1}{d+1}\right]$ be a rational. Then $\mathbb{S}_{\mathbb{Q}}(d,b) = \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$.

(b) Let $b \in \left(\frac{d-1}{d+1}, 1\right)$ be a rational. Then $\mathbb{S}_{\mathbb{Q}}(d,b)$ is dense in $\mathbb{S} \setminus I(\theta_b)$.

Proof. We start with the proof of part (b). Let $\lambda' \in \mathbb{S} \setminus I(\theta_b)$. By [PR20, Corollary 4 and Theorem 5] it follows that there exists $\lambda \in \mathbb{S}$ arbitrarily close to λ' for which there exists a tree $T \in \mathcal{T}_{d+1}$ such that $Z_T(\lambda, b) = 0$. Choose such a tree T with the minimum number of vertices and let v be a leaf of T, from now on referred to as the root of T. Denote $T' = T - v$ and let u be the unique neighbour of v in T. Then $Z_{T,\text{-}v}(\lambda, b) \neq 0$. Indeed, if $Z_{T,\text{-}v}(\lambda, b) = 0$, then $Z_{T,+v}(\lambda, b) = 0$. Since

$$\begin{pmatrix} \lambda & \lambda b \\ b & 1 \end{pmatrix} \begin{pmatrix} Z_{T',+u}(\lambda, b) \\ Z_{T',\text{-}u}(\lambda, b) \end{pmatrix} = \begin{pmatrix} Z_{T,+v}(\lambda, b) \\ Z_{T,\text{-}v}(\lambda, b) \end{pmatrix}$$

and since the matrix is invertible (as $|b| \neq 1$) this would imply $Z_{T',+u}(\lambda, b) = Z_{T',-u}(\lambda, b) = 0$ and hence $Z_{T'}(\lambda, b) = 0$ contradicting the minimality of T. Therefore $R_{T,v} = -1$.

Since the map $z \mapsto \xi(z) := \frac{Z_{T,+v}(z,b)}{Z_{T,\text{-}v}(z,b)}$ is holomorphic near $z = \lambda$, it follows that there exists $\lambda'' \in \mathbb{S}_{\mathbb{Q}}$ arbitrarily close to λ' such that $\xi = \xi(\lambda'') \in \mathrm{Arc}(\lambda_1(b), \overline{\lambda_1(b)}) \setminus \{-1\}$. Therefore, by Lemma 2.3.8 and Theorem 2.3.6 the orbit $\{f^n_{\xi,1}(1)\}$ is dense in $\mathbb{S}$. So from Lemma 2.2.2, by using paths with the tree T attached to all but one of its vertices at the root v of T, we obtain a collection of trees contained in $\mathcal{T}_{d+1}$ whose fields are dense in $\mathbb{S}$.

We next prove part (a) for all $d \geq 2$ and $b \in (\frac{d-2}{d}, \frac{d-1}{d+1}]$. The case $b \in (0, \frac{d-2}{d}]$ follows from this by invoking smaller values for d. Let $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$.

Let $k_0 = d$ and $m_0 = 0$ and define the sequences k_n and m_n by $k_{n+1} = \left\lfloor \frac{k_n}{2} \right\rfloor$ and $m_{n+1} = m_n + k_{n+1}$. Inductively we show that $m_n \leq d - k_n$: we have $m_0 = d - k_0$ and then

$$m_{n+1} = m_n + k_{n+1} \leq d - k_n + k_{n+1} = d - \left(k_n - \left\lfloor \tfrac{k_n}{2} \right\rfloor\right) \leq d - \left\lfloor \tfrac{k_n}{2} \right\rfloor = d - k_{n+1}.$$

Clearly, there is an integer N such that $k_{N+1} = 1$. We claim that for every $n \in \{0, \ldots, N\}$ there is a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most m_n that implements a field ξ_n so that $|f'(\xi_n)| > 1$ and at least one of the following holds.

1. There is a tree in $\mathcal{T}_{d+1}$ with root degree at most m_n that implements a field inside $\mathrm{Arc}\left(\lambda_{k_{n+1}}, \overline{\lambda_{k_{n+1}}}\right) \setminus \{-1\}$, or else

2. The set of fields implemented by trees in $\mathcal{T}_{d+1}$ is dense in $\mathbb{S}$.

To show this for $n = 0$ we consider the tree consisting of a single vertex. This tree implements the field λ and its root degree is 0. By equation (2.2) of

Lemma 2.3.4 and since $b \leq \frac{d-1}{d+1}$, we have that $|f_d'(z)| > 1$ for all $z \in \mathbb{S}\setminus\{1\}$, and in particular we have $|f_d'(\lambda)| > 1$. If $\lambda \in \mathrm{Arc}[\overline{\lambda_{k_1}}, \lambda_{k_1}]$ then we apply Lemma 2.3.11 to obtain Item (2). If $\lambda \in \mathrm{Arc}\,(\lambda_{k_1}, \overline{\lambda_{k_1}})$ then the tree consisting of a single vertex satisfies the conditions of Item (1).

Now suppose that we have shown the claim for $n-1$ for some $n \geq 1$, and assume that we are in the case of Item (1) (otherwise we are done), i.e., there is a tree in $\mathcal{T}_{d+1}$ with root degree m_{n-1} that implements a field ξ inside $\mathrm{Arc}\,(\lambda_{k_n}, \overline{\lambda_{k_n}}) \setminus \{-1\}$. We can apply Lemma 2.3.10 to obtain a tree T in $\mathcal{T}_{d+1}$ with root degree at most $k_n + m_{n-1} = m_n$ that implements a field $\zeta \neq -1$ such that $|f_{k_n}'(\zeta)| > 1$. If $\zeta \in \mathrm{Arc}[\overline{\lambda_{k_{n+1}}}, \lambda_{k_{n+1}}]$ we can apply Lemma 2.3.11 to obtain Item (2). We can apply this lemma because $m_n \leq d - k_n$. If $\zeta \in \mathrm{Arc}\,(\lambda_{k_{n+1}}, \overline{\lambda_{k_{n+1}}})$ then T itself satisfies the conditions of Item (1), which proves the claim.

To finish the proof, it remains to consider the case of Item (1), where we can find a tree in $\mathcal{T}_{d+1}$ with root degree at most $m_N < d-1$ which implements a field ξ inside $\mathrm{Arc}\,(\lambda_1, \overline{\lambda_1}) \setminus \{-1\}$. We have shown in Lemma 2.3.8 that $f_{\xi,1}$ is conjugate to an irrational rotation and thus the orbit $\{f_{\xi,1}^n(1)\}_{n\geq 1}$ is dense in $\mathbb{S}$. The elements of this orbit correspond to rooted trees in $\mathcal{T}_{d+1}$, and hence we can conclude Item (2) in this case as well.

Let $\mathcal{R}$ denote the set of all fields implemented by rooted trees in $\mathcal{T}_{d+1}$. Let $\zeta \in \mathcal{R}$ and T be a tree in $\mathcal{T}_{d+1}$ that implements ζ. We construct the tree $\tilde{T}$ with root r obtained by attaching r to the root of T with an edge. Then, the root of $\tilde{T}$ has degree 1 and the field implemented by $\tilde{T}$ is $f_{\lambda,1}(\zeta)$. So the set of fields implemented by rooted trees in $\mathcal{T}_{d+1}$ whose root degrees are 1 contains $f_{\lambda,1}(\mathcal{R})$. Since $f_{\lambda,1}(\mathbb{S}) = \mathbb{S}$ and $\mathcal{R}$ is dense in $\mathbb{S}$, we conclude that $f_{\lambda,1}(\mathcal{R})$ is dense in $\mathbb{S}$ as well. □

Remark 2.3.12. We note that our proof of part (b) rests on the existence of zeros for trees proved in [PR20], which in turn depends on the chaotic behaviour of the map $f_{d,\lambda}$. Alternatively one could also prove part (b) directly from Lemma 2.3.9. The same proof also yields a dense set of $\lambda \in \mathbb{S}_{\mathbb{Q}}$ for which the collection of fields of trees in $\mathcal{T}_{d+1}$ with root degree 1 is dense in $\mathbb{S}$ when $b \in (0, \frac{d-1}{d+1}]$.

2.4. Contracting maps that cover

In this section we adapt the contracting/covering maps framework of [BGGv20] in our setting and show how to apply it using the degree-derivative interplay alluded to in Section 2.2.3. Section 2.4.1 gives the details of the framework, and Section 2.4.2 gives the main lemmas that exploit this interplay.

2.4.1. Density on circular arcs via contracting maps that cover

The contracting maps that cover framework is captured by the following lemma on the interval $[0,1]$, which yields Corollary 2.4.2 on circular arcs of the unit circle $\mathbb{S}$.

Lemma 2.4.1. *Let $f_1, \ldots, f_k$ be continuously differentiable maps from the interval $[0,1]$ to itself such that $0 < f_m'(x) < 1$ for each index m and $x \in (0,1)$ and such that $\bigcup_{m=1}^{k} f_m([0,1]) = [0,1]$.*

Then for any open interval $J \subseteq [0,1]$ there is a sequence of indices $m_1, \ldots, m_N$ such that

$$(f_{m_1} \circ \cdots \circ f_{m_N})([0,1]) \subset J.$$

Proof. For every $m \in \{1, \ldots, k\}$ define the closed interval $I_m = f_m([0,1]) = [f_m(0), f_m(1)]$ and note that $f_m : [0,1] \to I_m$ is bijective with a differentiable inverse. We define a sequence of intervals in the following way. Let $J_0 = J$ and as long as there exists an index m such that $J_n \subseteq I_m$ we define $J_{n+1} = f_m^{-1}(J)$. We will show that this can not be done indefinitely, i.e., there will be some interval J_n such that $J_n \not\subseteq I_m$ for all m.

For an interval $I \subseteq [0,1]$ let $\ell(I)$ denote the length of the interval and denote $\ell(J)$ by ϵ. For each index m choose a partition $I_m = I_{m,L} \cup I_{m,M} \cup I_{m,R}$, where $I_{m,L}, I_{m,M}, I_{m,R}$ are of the form $[f_m(0), a), [a,b], (b, f_m(1)]$ respectively for a choice of $a, b \in \operatorname{int}(I_m)$ such that $a < b$ and both $\ell(I_{m,L})$ and $\ell(I_{m,R})$ are less than $\epsilon/4$. We can choose $C > 1$ such that ${f_m^{-1}}'(x) > C$ for all indices m and $x \in I_{m,M}$. We will show inductively that for all $n \geq 0$ for which J_n is defined it is the case that $\ell(J_n) \geq \epsilon \cdot (1 + C^n)/2$. For $n = 0$ the statement is true. Suppose that the statement is true for $n \geq 0$ for which J_{n+1} is defined. By definition there is an index m such that $J_n \subseteq I_m$ and $J_{n+1} = f_m^{-1}(J_n)$. We find

$$\begin{aligned}
\ell(J_{n+1}) &= \ell(f_m^{-1}(J_n \cap I_{m,M})) + \ell(f_m^{-1}(J_n \cap I_{m,L})) + \ell(f_m^{-1}(J_n \cap I_{m,R})) \\
&\geq C \cdot \ell(J_n \cap I_{m,M}) + \ell(J_n \cap I_{m,L}) + \ell(J_n \cap I_{m,R}) \\
&= C\,(\ell(J_n) - \ell(J_n \cap I_{m,L}) - \ell(J_n \cap I_{m,R})) + \ell(J_n \cap I_{m,L}) + \ell(J_n \cap I_{m,R}),
\end{aligned}$$

where we have used that ${f_m^{-1}}'(x) \geq 1$ for $x \in I_m$. Because $\ell(J_n \cap I_{m,L}) + \ell(J_n \cap I_{m,R}) \leq \epsilon/2$ this is again at least equal to

$$C(\ell(J_n) - \epsilon/2) + \epsilon/2 \geq C(\epsilon \cdot (1 + C^n)/2 - \epsilon/2) + \epsilon/2 = \epsilon \cdot (1 + C^{n+1})/2.$$

It follows that there is an index n such that J_n is not totally contained inside I_m for any index m. This means that there is an m such that J_n contains at least one of the endpoints of I_m; without loss of generality we can assume that J_n contains the left endpoint of I_m. It follows that there is an $a > 0$ such that $f_m([0,a]) \subset J_n$ and thus there is a sequence $m_1, \ldots, m_n$ such that $(f_{m_1} \circ \cdots \circ f_{m_n} \circ f_m)([0,a]) \subset J$.

We complete the proof by showing that for at least one of the maps f_i there is an index N_a for any $a > 0$ such that $f_i^{N_a}([0,1]) \subset [0,a]$.

Observe that there must be at least one map f_i such that $f_i(0) = 0$. We obtain an inclusion of intervals $[0,1] \supset f_i([0,1]) \supset f_i^2([0,1]) \supset \cdots$, where $f_i^N([0,1]) = [0, f_i^N(1)]$. This shows that the sequence $\{f_i^N(1)\}_{N\geq 0}$ is decreasing and thus has a limit L. If $L \neq 0$ we would have $f_i([0,L]) = [0,L]$, which contradicts the fact that $f_i'(x) < 1$ for all $x \in (0,L)$, so $L = 0$. This concludes the proof. □

Corollary 2.4.2. *Let $A \subset \mathbb{S}$ be a closed circular arc and let $f_1, \ldots, f_k$ be orientation preserving continuously differentiable maps from A, such that $\bigcup_{m=1}^{k} f_m(A) = A$ and $0 < |f_m'(x)| < 1$ for each index m and $x \in A$ not equal to either of the endpoints of A.*

Then for any open circular arc $J \subseteq A$ there is a sequence of indices $m_1, \ldots, m_N$ such that

$$(f_{m_1} \circ \cdots \circ f_{m_N})(A) \subset J.$$

2.4.2. Exploiting the dependence of derivatives on the degrees

In this section, we prove a few key lemmas that demonstrate how we employ the contracting maps that cover idea, by exploiting the dependence of derivatives on the degrees.

Lemma 2.4.3. *Let $k \in \mathbb{Z}_{\geq 1}$ and $b \in \left[\frac{k}{k+2}, 1\right)$. Let $\xi \in \operatorname{Arc}[\overline{\lambda_{k+1}}, \lambda_{k+1}]$ with $\xi \neq 1$ be such that $|f_{2k+1}'(R_k(\xi))| \geq 1$. Then there is an arc A of $\mathbb{S}$ such that the orbit of 1 under the action of the semigroup generated by $f_{\xi,k+1}$ and $f_{\xi,k}$ is dense in A.*

Proof. We can assume that ξ lies in the upper half-plane. Since all maps in this argument use the parameter ξ we will write f_m instead of $f_{\xi,m}$ for all m. Define the arc $A = \operatorname{Arc}[R_k(\xi), R_{k+1}(\xi)]$. Using equation (2.2) we find that for every m

$$|f_m'(R_k(\xi))| = \frac{m}{2k+1} \cdot |f_{2k+1}'(R_k(\xi))| \geq \frac{m}{2k+1}.$$

By using the fact that $R_{k+1}(\xi)$ is either a parabolic or an attracting fixed point of f_{k+1} we deduce that for all $z \in A$

$$|f_k'(z)| < |f_{k+1}'(z)| \leq |f_{k+1}'(R_{k+1}(\xi))| \leq 1,$$

where the second inequality is strict when $z \neq R_{k+1}(\xi)$. It follows that for all $z \in A$ not equal to $R_{k+1}(\xi)$ we have $k/(2k+1) \leq |f_k'(z)| < 1$ and $(k+1)/(2k+1) \leq |f_{k+1}'(z)| < 1$. Therefore:

$$\ell(A) > \ell(f_k(A)) > \frac{k}{2k+1}\ell(A) \quad \text{and} \quad \ell(A) > \ell(f_{k+1}(A)) > \frac{k+1}{2k+1}\ell(A).$$

From this we deduce that $\ell(f_k(A)) + \ell(f_{k+1}(A)) > \ell(A)$. Thus, because $f_k(A)$ is of the form $\mathrm{Arc}\,[R_k(\xi), a]$ and $f_{k+1}(A)$ is of the form $\mathrm{Arc}\,[b, R_{k+1}(\xi)]$ for some $a, b \in A$, we conclude that $f_k(A) \cup f_{k+1}(A) = A$.

It follows from item (iv) of Lemma 2.3.4 that there is some M such that $f_{k+1}^M(1) \in A$. Let $J \subseteq A$ be any open arc. According to Corollary 2.4.2 there is a sequence of indices $m_1, \ldots, m_N \in \{k, k+1\}$ such that $\left(f_{m_1} \circ \cdots \circ f_{m_N} \circ f_{k+1}^M\right)(1)$ lies in J. The fact that J was chosen as an arbitrary open arc in A concludes the proof. □

Lemma 2.4.4. *Let $k \in \mathbb{Z}_{\geq 1}$ and $b \in \left[\frac{k-1}{k+1}, 1\right)$. Let $\xi_1, \xi_2 \in \mathrm{Arc}\,[\overline{\lambda_k}, \lambda_k]$ such that ξ_1, ξ_2 are distinct and lie in the same half-plane, i.e., both in the upper or lower half-plane, and such that $|f'_{2k}(R_k(\xi_i))| \geq 1$ for $i \in \{1, 2\}$. Then there is an arc A of $\mathbb{S}$ such that the orbit of 1 under the action of the semigroup generated by $f_{\xi_1,k}$ and $f_{\xi_2,k}$ is dense in A.*

Proof. We can assume that ξ_1 and ξ_2 lie in the upper half-plane with $\mathrm{Arg}(\xi_1) < \mathrm{Arg}(\xi_2)$. Let $A = \mathrm{Arc}\,[R_k(\xi_1), R_k(\xi_2)]$, then for all $z \in A$ we have

$$\frac{1}{2} \leq \frac{1}{2} \cdot |f'_{2k}(R_k(\xi_1))| = |f'_k(R_k(\xi_1))| \leq |f'_k(z)| \leq |f'_k(R_k(\xi_2))| \leq 1.$$

where the second to last inequality is strict when $z \neq R_k(\xi_2)$. Therefore for $i \in \{1, 2\}$ we have

$$\ell(A) > \ell(f_{\xi_i,k}(A)) > \frac{1}{2} \cdot \ell(A)$$

and from this we deduce that $\ell(f_{\xi_1,k}(A)) + \ell(f_{\xi_2,k}(A)) > \ell(A)$. The rest of the proof proceeds exactly as the proof of Lemma 2.4.3. □

Lemma 2.4.5. *Let $k \geq 5$, $b \in \left(\frac{k-1}{k+1}, 1\right)$ and $\xi \in \mathrm{Arc}\,[\overline{\lambda_k}, \lambda_k]$ with $\xi \neq 1$ such that there is an integer $2k \leq p \leq 3k - 5$ for which $|f'_p(\xi)| \geq 1$. Then at least one of the following two statements holds:*

(i) The orbit of 1 under the action of the semigroup generated by $f_{\xi,k-2}, f_{\xi,k-1}$ and $f_{\xi,k}$ is dense in an arc of $\mathbb{S}$.

(ii) We have $|f'_k(R_k(\xi))| > 1 - \frac{p-k+2}{p} \cdot \frac{p-2k+1}{k}$.

Proof. We may assume that ξ lies in the upper half plane. We write $f_m = f_{\xi,m}$ for all indices m. Define the arcs $A_1 = \mathrm{Arc}\,[R_{k-2}(\xi), R_{k-1}(\xi)]$ and $A_2 = \mathrm{Arc}\,[R_{k-1}(\xi), R_k(\xi)]$. Analogously to the proofs of Lemmas 2.4.3 and 2.4.4 we can use Corollary 2.4.2 to show that the orbit of 1 under the action of the semigroup generated by f_{k-2}, f_{k-1} and f_k is dense in $A_1 \cup A_2$ if

$$f_{k-2}(A_1 \cup A_2) \cup f_{k-1}(A_1 \cup A_2) \cup f_k(A_1 \cup A_2) = A_1 \cup A_2. \tag{2.8}$$

We will assume that this is not the case and show that this leads to statement (ii). First we will show that the left-hand side of Equation (2.8) does cover A_1. For any arc A in the upper half-plane such that $\operatorname{Arg}(x) \geq \operatorname{Arg}(\xi)$ for all $x \in A$ and index m we have

$$\ell(f_m(A)) > |f_m'(\xi)| \cdot \ell(A) = \frac{m \cdot |f_p'(\xi)|}{p} \cdot \ell(A) \geq \frac{m}{p} \cdot \ell(A). \tag{2.9}$$

We use this and the fact that $\ell(A_2) \geq \ell(A_1)$, which follows from item (v) of Lemma 2.3.4, to conclude the following

$$\begin{aligned} \ell(f_{k-2}(A_1 \cup A_2)) + \ell(f_{k-1}(A_1)) &\geq \frac{k-2}{p}\ell(A_1 \cup A_2) + \frac{k-1}{p}\ell(A_1) \\ &\geq \frac{2(k-2)}{p}\ell(A_1) + \frac{k-1}{p}\ell(A_1) \\ &= \frac{3k-5}{p}\ell(A_1) \geq \ell(A_1). \end{aligned}$$

Because $f_{k-2}(A_1 \cup A_2)$ is of the form $\operatorname{Arc}[R_{k-2}(\xi), a]$ and $f_{k-1}(A_1)$ is of the form $\operatorname{Arc}[b, R_{k-1}(\xi)]$ we have that A_1 is covered by $f_{k-2}(A_1 \cup A_2) \cup f_{k-1}(A_1)$. Our assumption can be formulated as

$$\ell(A_2) \geq \ell(f_{k-1}(A_2)) + \ell(f_k(A_1 \cup A_2)).$$

Note that

$$\ell(f_{k-1}(A_2)) + \ell(f_k(A_1 \cup A_2)) \geq \frac{k-1}{p}\ell(A_2) + \frac{k}{p}\left(\ell(A_1) + \ell(A_2)\right).$$

Combining the previous two inequalities we get

$$\ell(A_1) \leq \frac{p-2k+1}{k} \cdot \ell(A_2). \tag{2.10}$$

Let $A_0 = \operatorname{Arc}[1, R_{k-2}(\xi)]$. By using the fact that $R_m(\xi)$ is a fixed point of f_m and $f_m(1) = \xi$ for every m we see that $f_{k-2}(A_0) = \operatorname{Arc}[\xi, R_{k-2}(\xi)]$, $f_{k-1}(A_0 \cup A_1) = \operatorname{Arc}[\xi, R_{k-1}(\xi)]$ and $f_k(A_0 \cup A_1 \cup A_2) = \operatorname{Arc}[\xi, R_k(\xi)]$. It follows from the relation between the derivative of different maps given in item (i) of Lemma 2.3.4 that for any arc A on which f_{m_1} and f_{m_2} are injective we have

$$\ell(f_{m_1}(A)) = m_1 \cdot \ell(f_1(A)) = \frac{m_1}{m_2} \cdot \ell(f_{m_2}(A)).$$

These observations can be used to write $\ell(A_1)$ and $\ell(A_2)$ as follows:

$$\begin{aligned}\ell(A_1) &= \ell(f_{k-1}(A_0 \cup A_1)) - \ell(f_{k-2}(A_0)) \\ &= \frac{k-2}{k-1}\ell(f_{k-1}(A_0 \cup A_1)) + \frac{1}{k-1}\ell(f_{k-1}(A_0 \cup A_1)) - \frac{k-2}{k-1}\ell(f_{k-1}(A_0)) \\ &= \frac{k-2}{k-1}\ell(f_{k-1}(A_1)) + \frac{1}{k-1}\ell(f_{k-1}(A_0 \cup A_1))\end{aligned}$$

and

$$\begin{aligned}\ell(A_2) &= \ell(f_k(A_0 \cup A_1 \cup A_2)) - \ell(f_{k-1}(A_0 \cup A_1)) \\ &= \ell(f_k(A_2)) + \frac{1}{k-1} \cdot \ell(f_{k-1}(A_0 \cup A_1)).\end{aligned}$$

By combining our way of writing $\ell(A_1)$ and the inequalities given in (2.9) and (2.10) we obtain the following inequalities

$$\begin{aligned}\frac{1}{k-1}\ell(f_{k-1}(A_0 \cup A_1)) = \ell(A_1) - \frac{k-2}{k-1}\ell(f_{k-1}(A_1)) &< \ell(A_1) - \frac{k-2}{k-1}\,\frac{k-1}{p}\ell(A_1) \\ = \frac{p-k+2}{p} \cdot \ell(A_1) &< \frac{p-k+2}{p} \cdot \frac{p-2k+1}{k} \cdot \ell(A_2).\end{aligned}$$

It follows from the fact that $\text{Arg}(R_k(\xi)) \geq \text{Arg}(x)$ for all $x \in A_2$ that $\ell(f_k(A_2)) < f_k'(R_k(\xi)) \cdot \ell(A_2)$. By using this inequality and the previous inequality we obtain

$$\begin{aligned}\ell(A_2) &= \ell(f_k(A_2)) + \frac{1}{k-1} \cdot \ell(f_{k-1}(A_0 \cup A_1)) \\ &< |f_k'(R_k(\xi))| \cdot \ell(A_2) + \frac{p-k+2}{p} \cdot \frac{p-2k+1}{k} \cdot \ell(A_2).\end{aligned}$$

We can cancel $\ell(A_2)$ and rewrite to obtain:

$$|f_k'(R_k(\xi))| > 1 - \frac{p-k+2}{p} \cdot \frac{p-2k+1}{k},$$

which is what we set out to prove. □

Corollary 2.4.6. *Let m be a positive integer, $b \in \left(\frac{m-1}{m+1}, 1\right)$ and $\xi \in \text{Arc}\,[\overline{\lambda_m}, \lambda_m]$ with $\xi \neq 1$ such that either of the following holds:*

(a) $m \geq 8$ and $|f_{2m}'(\xi)| \geq 1$;

(b) $m \geq 9$ and $|f_{2m+1}'(\xi)| \geq 1$.

Then the orbit of 1 *under the action of the semigroup generated by* $f_{\xi,m-3}$, $f_{\xi,m-2}$, $f_{\xi,m-1}$ *and* $f_{\xi,m}$ *is dense in an arc of* $\mathbb{S}$.

Proof. We will again assume that ξ lies in the upper half-plane. We apply Lemma 2.4.5 with $k = m - 1$ and $p = 2m$ for item (a) and $p = 2m + 1$ for item (b). If the first statement of that lemma holds we see that orbit of 1 under the action of f_{m-3}, f_{m-2} and f_{m-1} generates an arc in which case we are done. If we assume that the second statement holds we obtain

$$|f'_{m-1}(R_{m-1}(\xi))| > 1 - \frac{2m - (m-1) + 2}{2m} \cdot \frac{2m - 2(m-1) + 1}{m-1} > \frac{1}{2}$$

in the case where $p = 2m$ and

$$|f'_{m-1}(R_{m-1}(\xi))| > 1 - \frac{2m + 1 - (m-1) + 2}{2m+1} \cdot \frac{2m + 1 - 2(m-1) + 1}{m-1} > \frac{1}{2}$$

in the case where $p = 2m+1$. It follows that for $x \in \text{Arc}\,[R_{k-1}(\xi), R_k(\xi)]$ we obtain $1 > |f'_m(x)| > |f'_{m-1}(x)| > 1/2$. Therefore, with $A = \text{Arc}\,[R_{k-1}(\xi), R_k(\xi)]$, we get

$$\ell(f_m\,(A)) + \ell(f_{m-1}\,(A)) \geq \ell(A).$$

This, together with Corollary 2.4.2, implies that the orbit of 1 under the action of the semigroup generated by f_{m-1} and f_m is dense in A. □

2.5. Proof of Lemma 2.3.11 for Some Special Cases

The arguments of this section will be used to cover some left-over cases in the proof of Lemma 2.3.11 that are not directly covered by the results of the previous section.

2.5.1. Proof of Lemma 2.3.11 for powers of two

The following lemma will be used in the proof of Lemma 2.3.11 for those values of k for which either k or $k+1$ is a power of two, see the proof in Section 2.6 for details.

Lemma 2.5.1. *Let* $d \geq 2, k \geq 0$ *be integers,* $b \in \left(0, \frac{d-1}{d+1}\right] \cap \mathbb{Q}$, $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$ *and* $\xi \in \Lambda_{2^k}(b) \cap \mathbb{S}_{\mathbb{Q}}$ *with* $\xi \neq \pm 1$. *Suppose there is a tree in* $\mathcal{T}_{d+1}$ *with root degree at most* $d - (2^{k+1} - 1)$ *that implements the field* ξ. *Then the set of fields implemented by rooted trees in* $\mathcal{T}_{d+1}$ *is dense in* $\mathbb{S}$.

Proof. We will prove this by induction on k. For $k = 0$ the field ξ has to lie in $\text{Arc}(\lambda_1, \overline{\lambda_1}) \setminus \{-1\}$ and the root degree of the tree in $\mathcal{T}_{d+1}$ implementing ξ is at most $d - 1$. From Corollary 2.3.7 and Lemma 2.3.8, we have that $f_{\xi,1}$ is conjugate to an irrational rotation and thus the orbit of any initial point $z_0 \in \mathbb{S}$ is dense in $\mathbb{S}$. By Lemma 2.2.2, every element of the set $\{f^n_{\xi,1}(\lambda)\}_{n\geq 1}$ is the field implemented by a tree in $\mathcal{T}_{d+1}$, and hence we obtain the theorem for $k = 0$.

Now suppose that $k \geq 1$ and that we have proved the statement for $k-1$. If $b < (2^{k-1}-1)/(2^{k-1}+1)$, then we must have $k > 1$ and we can immediately apply the induction hypothesis with $\xi = \lambda$ and tree consisting of a single vertex. So, assume that $b \geq (2^{k-1}-1)/(2^{k-1}+1)$ and observe that the parameter $\lambda_{2^{k-1}} \in \mathbb{S}$ from Lemma 2.3.4 exists. It follows from Lemma 2.3.10 that there is $\sigma \in \mathbb{S}$ with $|f'_{2^k}(\sigma)| > 1$ and a set $\mathcal{R} = \{\zeta_n\}_{n\geq 1}$ accumulating on σ such that each ζ_n is implemented by a tree in $\mathcal{T}_{d+1}$ whose root degree is at most $d - (2^{k+1}-1) + 2^k = d - (2^k - 1)$. If $\mathcal{R}$ has a non-empty intersection with $\mathrm{Arc}\,(\lambda_{2^{k-1}}, \overline{\lambda_{2^{k-1}}}) \setminus \{-1\}$ we can apply the induction hypothesis to the tree corresponding to the field in this intersection. Therefore we assume that the elements of $\mathcal{R}$ accumulate on σ from inside $\mathrm{Arc}\,[\overline{\lambda_{2^{k-1}}}, \lambda_{2^{k-1}}]$. It follows that we can find two distinct elements $r_1, r_2 \in \mathcal{R}$ such that they both lie in either $\mathrm{Arc}\,(\overline{\lambda_{2^{k-1}}}, 1)$ or in $\mathrm{Arc}\,(1, \lambda_{2^{k-1}})$ and such that $|f'_{2^k}(r_i)| > 1$ for $i = 1, 2$. By Remark 2.3.5, we have $|f'_{2^k}(R_{2^{k-1}}(r_i))| > |f'_{2^k}(r_i)| > 1$ and thus we can apply Lemma 2.4.4 to conclude that the following set is dense in an arc A of the circle:

$$\mathcal{A} = \left\{(f_{r_{i_1},2^{k-1}} \circ \cdots \circ f_{r_{i_n},2^{k-1}})(1) : n \in \mathbb{Z}_{\geq 1} \text{ and } i_1, \ldots, i_n \in \{1, 2\}\right\}.$$

Since r_1, r_2 are implemented by trees in $\mathcal{T}_{d+1}$ whose root degrees are at most $d - (2^{k+1} - 1) + 2^k = d - (2^k - 1)$, by Lemma 2.2.2, every element of $\mathcal{A}$ is implemented by a tree in $\mathcal{T}_{d+1}$ whose root degree is bounded by $d - (2^k - 1) + 2^{k-1} = d - (2^{k-1} - 1) \leq d$. We have seen that (2.2) implies that for $b \leq (d-1)/(d+1)$ it holds that $|f'_d(z)| > 1$ for all $z \in \mathbb{S} \setminus \{1\}$. This implies that there is some $N \in \mathbb{Z}_{\geq 1}$ such that $f^N_d(A) = \mathbb{S}$. It follows that the set $\{f^N_{\lambda,d}(a) : a \in \mathcal{A}\}$ is dense in $\mathbb{S}$, finishing the proof since every element of this set corresponds to the field of a tree in $\mathcal{T}_{d+1}$ (using again Lemma 2.2.2). $\square$

2.5.2. Proof of Lemma 2.3.11 for small cases

In this section, we give the main lemma needed to cover certain small cases of Lemma 2.3.11. Interestingly, the proof uses a Cantor-style construction, explained in detail in the next subsection.

Near-arithmetic progressions

Let $\alpha \in (0, 1)$ and define the maps from the unit interval to itself given by $\phi_0(x) = \alpha x$ and $\phi_1(x) = \alpha x + (1 - \alpha)$. Let $\Omega = \cup_{n=0}^{\infty}\{0, 1\}^n$ be the set of finite binary sequences. For $\omega \in \Omega$ we let $|\omega|$ denote the length of ω and for $\omega_1, \omega_2 \in \Omega$ we let $\omega_1 \oplus \omega_2 \in \Omega$ denote the concatenation of the two sequences. For $\omega \in \Omega$ of the form $(\omega^1, \ldots, \omega^n)$ and two maps f_0, f_1 we let $f_\omega = f_{\omega^1} \circ \cdots \circ f_{\omega^n}$ and if $|\omega| = 0$ we let f_ω denote the identity map. The properties of the semigroup generated by ϕ_0 and ϕ_1 for certain parameters α are a topic that has been studied extensively. For $\alpha \in (0, \frac{1}{2})$ the set

$$\mathcal{C}_\alpha = \bigcap_{n=0}^{\infty} \bigcup_{\substack{\omega \in \Omega \\ |\omega| = n}} \phi_\omega([0,1])$$

is a Cantor set, with $\mathcal{C}_{1/3}$ being the Cantor ternary set. We will not use the properties of Cantor sets, so we do not define them. First we state some easily provable properties of this semigroup to describe a construction that will help us to prove Lemma 2.3.11 for small cases of k.

Lemma 2.5.2. *Let $\omega \in \Omega$ and $\alpha \in (0,1)$. Then $\phi_\omega([0,1])$ is an interval of length $\alpha^{|\omega|}$, furthermore the intervals $\phi_{\omega\oplus(0)}([0,1])$ and $\phi_{\omega\oplus(1)}([0,1])$ are subintervals of $\phi_\omega([0,1])$ sharing the left and right boundary respectively.*

Proof. Because the derivative of ϕ_i is constantly equal to α for $i = 1, 2$ it follows that the length of $\phi_\omega([0,1])$ is $\alpha^{|\omega|}$. The maps ϕ_i are increasing and thus we can write $\phi_\omega([0,1]) = [\phi_\omega(0), \phi_\omega(1)]$ and also $\phi_{\omega\oplus(0)}([0,1]) = [\phi_{\omega\oplus(0)}(0), \phi_{\omega\oplus(0)}(1)] = [\phi_\omega(0), \phi_{\omega\oplus(0)}(1)]$. Therefore the left boundaries of $\phi_\omega([0,1])$ and $\phi_{\omega\oplus(0)}([0,1])$ are equal. The length of the latter interval is $\alpha^{|\omega|+1}$, which is less than the length of $\phi_\omega([0,1])$ and thus $\phi_{\omega\oplus(0)}([0,1])$ is indeed contained in $\phi_\omega([0,1])$. The stated property of $\phi_{\omega\oplus(1)}([0,1])$ follows completely analogously. □

For two sets $A, B \subseteq \mathbb{R}$ we will let $A + B = \{a + b : a \in A, b \in B\}$. A famous property of the Cantor ternary set is that $\mathcal{C}_{1/3} + \mathcal{C}_{1/3} = [0, 2]$. More generally one can show that $\mathcal{C}_\alpha + \mathcal{C}_\alpha = [0, 2]$ for all $\alpha \in [\frac{1}{3}, 1)$. In [MO94] an overview is given of the possible structures of $\mathcal{C}_{\alpha_1} + \mathcal{C}_{\alpha_2}$ for pairs of $\alpha_1, \alpha_2 \in (0, 1)$. Similar methods to those used in [MO94] can be used to show the following.

Lemma 2.5.3. *Let $\alpha \in [\frac{1}{3}, 1)$ and $\epsilon > 0$. Then there are sequences $\omega_1, \omega_2, \omega_3 \in \Omega$ such that for all triples p_1, p_2, p_3 with $p_i \in \phi_{\omega_i}([0,1])$*

$$\left| \frac{p_2 - p_1}{p_3 - p_2} - 1 \right| < \epsilon. \tag{2.11}$$

Proof. First assume that $\alpha \in [\frac{1}{2}, 1)$. Then $\phi_0([0,1]) \cup \phi_1([0,1]) = [0,1]$. It follows from Lemma 2.4.1 that for any $\delta > 0$ there are elements ω_1, ω_2 and ω_3 in Ω such that

$$\phi_{\omega_1}([0,1]) \subseteq [0, \delta], \quad \phi_{\omega_2}([0,1]) \subseteq [1/2 - \delta, 1/2 + \delta] \quad \text{and} \quad \phi_{\omega_3}([0,1]) \subseteq [1 - \delta, 1].$$

By choosing δ small enough we can guarantee the inequality in (2.11).

Assume now that $\alpha \in [\frac{1}{3}, \frac{1}{2})$. We will first show that if there are $\omega_i \in \Omega$ with $|\omega_i| = n$ and $q_i \in \phi_{\omega_i}([0,1])$ for $i = 1, 2, 3$ such that $q_1 + q_3 = 2q_2$, then there are choices of indices $k_i \in \{0, 1\}$ such that there exist $\tilde{q}_i \in \phi_{\omega_i \oplus (k_i)}([0,1])$ for which $\tilde{q}_1 + \tilde{q}_3 = 2\tilde{q}_2$. Suppose that we are given such ω_i and q_i. Let $I_i = \phi_{\omega_i}([0,1])$ and $I_i^k = \phi_{\omega_i \oplus (k)}([0,1])$ for $i = 1, 2, 3$ and $k = 0, 1$. We will show that

$$I_1 + I_3 = (I_1^0 + I_3^0) \cup (I_1^1 + I_3^0) \cup (I_1^0 + I_3^1). \tag{2.12}$$

Let a_1 and a_3 be the left boundary of I_1 and I_3 respectively. Because $|\omega_1| = |\omega_3| = n$ it follows that $I_1 = [a_1, a_1 + \alpha^n]$ and $I_3 = [a_3, a_3 + \alpha^n]$ and thus $I_1 + I_3 = [a_1 + a_3, a_1 + a_3 + 2\alpha^n]$, which we can denote as $a_1 + a_3 + \alpha^n \cdot [0, 2]$. Now

$$\begin{aligned}
I_1^0 + I_3^0 &= (a_1 + \alpha^n \cdot [0, \alpha]) + (a_3 + \alpha^n \cdot [0, \alpha]) = a_1 + a_3 + \alpha^n \cdot [0, 2\alpha] \\
I_1^1 + I_3^0 &= (a_1 + \alpha^n \cdot [1 - \alpha, 1]) + (a_3 + \alpha^n \cdot [0, \alpha]) = a_1 + a_3 + \alpha^n \cdot [1 - \alpha, 1 + \alpha] \\
I_1^1 + I_3^1 &= (a_1 + \alpha^n \cdot [1 - \alpha, 1]) + (a_3 + \alpha^n \cdot [1 - \alpha, 1]) = a_1 + a_3 + \alpha^n \cdot [2 - 2\alpha, 2].
\end{aligned}$$

Because $\alpha \in [\frac{1}{3}, 1)$ it follows that

$$[0, 2] = [0, 2\alpha] \cup [1 - \alpha, 1 + \alpha] \cup [2 - 2\alpha, 2],$$

thus showing (2.12). Because there are $q_i \in I_i$ such that $q_1 + q_3 = 2q_2$ we know that $I_1 + I_3$ is not disjoint from $2I_2$. These two intervals have the same length and thus at least one of the boundary points of $2I_2$ lies in $I_1 + I_3$. Therefore there is a $k_2 \in \{0, 1\}$ such that $2I_2^{k_2}$ is not disjoint from $I_1 + I_3$ because the intervals I_2^0 and I_2^1 contain the respective boundary points of I_2. This means that $2I_2^{k_2}$ is not disjoint from $(I_1^0 + I_3^0) \cup (I_1^1 + I_3^0) \cup (I_1^0 + I_3^1)$ and thus there are also choices of $k_1, k_3 \in \{0, 1\}$ such that $I_1^{k_1} + I_3^{k_3}$ is not disjoint from $2I_2^{k_2}$. It follows that there are $\tilde{q}_i \in I_i^{k_i}$ such that $\tilde{q}_1 + \tilde{q}_3 = 2\tilde{q}_2$.

Let $\omega_1 = (0, 0)$, $\omega_2 = (0, 1)$ and $\omega_3 = (1, 0)$. Note that $0 \in \phi_{\omega_1}([0,1])$ and $1 - \alpha \in \phi_{\omega_3}([0,1]) = [1 - \alpha, 1 - \alpha + \alpha^2]$. Furthermore it can be checked, using the fact that $\alpha \in [\frac{1}{3}, \frac{1}{2})$, that $(1 - \alpha)/2 \in \phi_{\omega_2}([0,1]) = [\alpha - \alpha^2, \alpha]$ and thus there are $q_i \in \phi_{\omega_i}([0,1])$ such that $q_1 + q_3 - 2q_2 = 0$. From the previous considerations it follows that there are $\tilde{\omega}_i \in \Omega$ of arbitrary length such that there are $\tilde{q}_i \in \phi_{\omega_i \oplus \tilde{\omega}_i}([0,1])$ for which $\tilde{q}_1 + \tilde{q}_3 - 2\tilde{q}_2 = 0$. See Figure 2.1 for an illustration of the construction described in this proof. By taking the length of $\tilde{\omega}_i$ large enough, the lengths of the intervals can be made arbitrarily small and thus we can guarantee that

$$|p_3 - p_2| \cdot \left| \frac{p_2 - p_1}{p_3 - p_2} - 1 \right| = |p_1 + p_3 - 2p_2| < \epsilon \cdot (1 - 2\alpha)$$

for all triples $p_i \in \phi_{\omega_i \oplus \tilde{\omega}_i}([0,1])$. Because $\phi_{\omega_i \oplus \tilde{\omega}_i}([0,1]) \subseteq \phi_{\omega_i}([0,1])$, we conclude that $p_3 - p_2$ is at least $1 - 2\alpha$. The inequality in (2.11) follows. $\square$

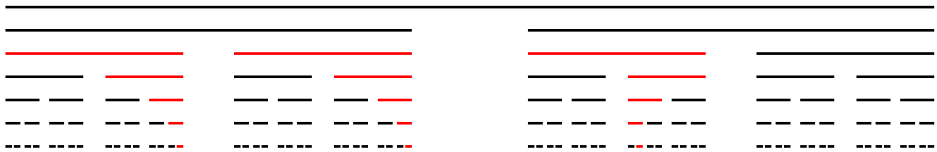

Figure 2.1: An illustration of the union of $\phi_\omega([0,1])$, where $\omega \in \Omega$ runs over all sequences of length n for $n = 0, 1, \ldots, 6$ for $\alpha = 7/16$. At each level, starting at level two, three red intervals are highlighted containing elements q_1, q_2 and q_3 respectively such that $q_1 + q_3 = 2q_2$.

Lemma 2.5.4. *Let $\alpha \in [\frac{1}{3}, 1)$, $\epsilon > 0$ and f_0, f_1 differentiable maps from $[0,1]$ to itself with fixed points 0 and 1 respectively. Then there is a constant $\delta > 0$ such that if $|f_i'(x) - \alpha| < \delta$ for $i = 0,1$ and all $x \in [0,1]$ then there are $\omega_1, \omega_2, \omega_3 \in \Omega$ such that for all triples p_1, p_2, p_3 with $p_i \in f_{\omega_i}([0,1])$ it holds that*

$$\left| \frac{p_2 - p_1}{p_3 - p_2} - 1 \right| < \epsilon. \tag{2.13}$$

Proof. Suppose that $|f_i'(x) - \alpha| < \delta$ for $i = 0, 1$ and all $x \in [0,1]$. For any $x \in [0,1]$ we can write

$$f_0(x) = \int_0^x f_0'(t)dt \quad \text{and} \quad f_1(x) = 1 - \int_x^1 f_1'(t)dt.$$

We show inductively that for all $x \in [0,1]$ and $\omega \in \Omega$ we have $|f_\omega(x) - \phi_\omega(x)| \leq |\omega| \cdot \delta$. When $|\omega| = 0$ the statement is clear, so we suppose that $|\omega| > 0$. Assume that the first entry of ω is a 0 so we write $\omega = (0) \oplus \omega'$ for some $\omega' \in \Omega$ with $|\omega| = |\omega'| + 1$. Let $x \in [0,1]$, we assume that we have shown that $|f_{\omega'}(x) - \phi_{\omega'}(x)| < \delta \cdot |\omega'|$. We denote $f_{\omega'}(x)$ by y and $\phi_{\omega'}(x)$ by $y + r$, where $|r| \leq \delta \cdot |\omega'|$. Now

$$\begin{aligned} |f_\omega(x) - \phi_\omega(x)| = |f_0(y) - \phi_0(y + r)| &= \left| \int_0^y f_0'(t)dt - \alpha \cdot (y + r) \right| \\ &= \left| \int_0^y (f_0'(t) - \alpha)\, dt - \alpha r \right| \leq \int_0^y |f_0'(t) - \alpha| dt + \alpha |r| \\ &\leq y\delta + \alpha\delta|\omega'| < \delta(|\omega'| + 1) = \delta|\omega|. \end{aligned}$$

If the first entry of ω is a 1 the calculation is analogous.

Let $\omega_1, \omega_2, \omega_3 \in \Omega$ such that for all triples p_1, p_2, p_3 with $p_i \in \phi_{\omega_i}([0,1])$ it holds that $\left|\frac{p_2 - p_1}{p_3 - p_2} - 1\right| < \epsilon/2$. These choices of ω_i exist by Lemma 2.5.3. For this inequality to hold it must be the case that $\phi_{\omega_2}([0,1]) \cap \phi_{\omega_3}([0,1]) = \emptyset$ and thus, since the map $(p_1, p_2, p_3) \to (p_2 - p_1)/(p_3 - p_2)$ is continuous in all points where $p_2 \neq p_3$, we can find three open intervals I_1, I_2, I_3 with $\phi_{\omega_i}([0,1]) \subseteq I_i$ such that for all triples $q_i \in I_i$ we have

$$\left| \frac{q_2 - q_1}{q_3 - q_2} - 1 \right| < \epsilon.$$

We showed that by making δ small enough we obtain bounds on the difference between $f_\omega(x)$ and $\phi_\omega(x)$ uniformly over all $x \in [0,1]$ and ω of bounded length. Therefore we can make δ sufficiently small such that $f_{\omega_i}([0,1]) \subset I_i$ for $i \in \{1,2,3\}$, which is enough to conclude the statement of the lemma. □

Corollary 2.5.5. *Let $\alpha \in [\frac{1}{3}, 1)$, $\epsilon > 0$. There is $\delta > 0$ such that the following holds for any closed circular arc $A \subseteq \mathbb{S}^1$ and two maps $f_0, f_1 : A \to A$ with the respective endpoints of A as fixed points with the property that $||f_i'(z)| - \alpha| < \delta$ for all $z \in A$.*

There exist $\omega_1, \omega_2, \omega_3 \in \Omega$ such that for all triples p_i with $p_i \in f_{\omega_i}(A)$ we have that $\mathrm{Arc}[p_1, p_2]$ and $\mathrm{Arc}[p_2, p_3]$ are subsets of A satisfying

$$\left| \frac{\ell(\mathrm{Arc}[p_1, p_2])}{\ell(\mathrm{Arc}[p_2, p_3])} - 1 \right| < \epsilon.$$

We are now ready to prove the following lemma.

Lemma 2.5.6. *Let $d \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{Z}_{\geq 1}$, $b \in \left[\frac{k-1}{k+1}, \frac{d-1}{d+1}\right]$ with $b \neq 0$ and $\lambda \in \mathbb{S}$. Let $\xi \in \mathbb{S} \setminus \{-1\}$ with $|f_{3k}'(\xi)| > 1$. Let $\{\xi_n\}_{n \geq 1}$ be a sequence in $\mathbb{S}$ converging to ξ and not equal to ξ such that for all positive integers n there is a rooted tree T_n in $\mathcal{T}_{d+1}$, with root degree $m \leq d - 2k$ implementing the field ξ_n. Then at least one of the following is true.*

1. *The set of fields implemented by rooted trees in $\mathcal{T}_{d+1}$ is dense in $\mathbb{S}$.*
2. *Given $\epsilon > 0$, there is a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $m + k$ that implements the field $r \in \mathrm{Arc}\,(\lambda_k, \overline{\lambda_k}) \setminus \{-1\}$ with $|f_{3k}'(r)| > |f_{3k}'(\xi)| - \epsilon$.*

Proof. We distinguish the following three cases.

(i) $\xi \in \mathrm{Arc}(\lambda_k, \overline{\lambda_k})$.

(ii) $\xi \in \mathrm{Arc}[\overline{\lambda_k}, \lambda_k]$ and $R_k(\xi) \in \mathrm{Arc}(\lambda_k, \overline{\lambda_k})$.

(iii) $\xi \in \mathrm{Arc}[\overline{\lambda_k}, \lambda_k]$ and $R_k(\xi) \in \mathrm{Arc}[\overline{\lambda_k}, \lambda_k]$.

First suppose we are in case (i). Then, since $\xi_n \to \xi$ and thus $f_{3k}'(\xi_n) \to f_{3k}'(\xi)$, given ϵ, there is an integer n such that $\xi_n \in \mathrm{Arc}(\lambda_k, \overline{\lambda_k}) \setminus \{-1\}$ and $|f_{3k}'(\xi_n)| >$

$|f'_{3k}(\xi)|-\epsilon$. The rooted tree T_n satisfies the requirements of statement (2) of the lemma.

To prove the lemma for cases (ii) and (iii) we define the following set

$$\mathcal{R} = \left\{ f^N_{\xi_n,k}(\xi_n) : n, N \geq 1 \right\}.$$

By repeatedly applying Lemma 2.2.2 we see that every element of $\mathcal{R}$ corresponds to the field implemented by a rooted tree in $\mathcal{T}_{d+1}$ whose root degree is at most $m+k$. The following limits follow from continuity

$$\lim_{N\to\infty}\lim_{n\to\infty} f^N_{\xi_n,k}(\xi_n) = \lim_{N\to\infty} f^N_{\xi,k}(\xi) = R_k(\xi).$$

Therefore $R_k(\xi)$ is an accumulation of $\mathcal{R}$ and in fact there is a sequence $\{\zeta_n\}_{n\geq 1}$ of elements in $\mathcal{R}$ converging to $R_k(\xi)$ but not equal to $R_k(\xi)$. If we are in case (ii), by Remark 2.3.5 we can take ζ_n sufficiently close to $R_k(\xi)$ so that $\zeta_n \in \mathrm{Arc}\,(\lambda_k, \overline{\lambda_k}) \setminus \{-1\}$. Since $|f'_{3k}(R_k(\xi))| > |f'_{3k}(\xi)|$ by Remark 2.3.5, we can further ensure that $|f'_{3k}(\zeta_n)| > |f'_{3k}(\xi)|$. The corresponding tree with field ζ_n satisfies the condition of statement (2) of the lemma.

Suppose now we are in case (iii) and suppose first that $R_k(\xi) \in \{\overline{\lambda_k}, \lambda_k\}$. If a subsequence (ζ_n) converges to $R_k(\xi)$ along the arc $\mathrm{Arc}(\lambda_k, \overline{\lambda_k})$, we obtain a $\zeta_n \in \mathrm{Arc}(\lambda_k, \overline{\lambda_k})$ and by the same reasoning as in the previous case we can conclude that statement (2) of the lemma holds. So we can assume that for large enough n all ζ_n lie in $\mathrm{Arc}(\overline{\lambda_k}, \lambda_k)$. In this case we find that for sufficiently high n the elements ζ_n get arbitrarily close to either λ_k or $\overline{\lambda_k}$ and thus $|f'_k(R_k(\zeta_n))|$ gets arbitrarily close to 1. It follows that we can find n_1 and n_2 such that ζ_{n_1} and ζ_{n_2} lie in the same half plane and such that $|f'_{2k}(R_k(\zeta_{n_i}))| > 1$ for $i = 1, 2$. It follows then from Lemma 2.4.4 that, if we let $g_0 = f_{\zeta_{n_1},k}$ and $g_1 = f_{\zeta_{n_2},k}$, the set

$$\mathcal{R}_1 = \{g_\omega(1) : \omega \in \Omega, |\omega| \geq 1\}$$

is dense in an arc $A \subseteq S$. By applying Lemma 2.2.2 we observe that every $r \in \mathcal{R}_1$ corresponds to the field implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $m + 2k \leq d$. Because the tree consisting of a single vertex implements the field λ we can apply Lemma 2.2.2 to see that every element in the set

$$\mathcal{R}_2 = \{f^n_{\lambda,d}(r) : r \in \mathcal{R}_1, n \geq 1\}$$

corresponds to the field implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most d. Because b is chosen such that $|f'_d(z)| > 1$ for all $z \in \mathbb{S} - \{1\}$ we find that $f^N_{\lambda,d}(A) = \mathbb{S}$ for a sufficiently large N and thus $\mathcal{R}_2$ is dense in $\mathbb{S}$, which shows that in this case statement (1) of the lemma holds.

Finally we assume that $R_k(\xi) \in \mathrm{Arc}(\overline{\lambda_k}, \lambda_k)$. We may assume that ξ lies in the upper half-plane. Let $\alpha = |f'_k(R_k(\xi))|$. It follows from the fact that $R_k(\xi) \in$

$\mathrm{Arc}(\xi, \lambda_k)$ that $\alpha \in (1/3, 1)$. Let $\epsilon_1, \epsilon_2 > 0$ be two reals whose value will be determined later. Let δ be the constant obtained from applying Corollary 2.5.5 to α and $\epsilon = \epsilon_1$. Now choose n_1, n_2 such that ξ_{n_1}, ξ_{n_2} have the following properties.

(a) ξ_{n_1} and ξ_{n_2} lie in the upper half-plane, $\mathrm{Arg}(\xi_{n_1}) < \mathrm{Arg}(\xi_{n_2})$, $\mathrm{Arc}\,[R_k(\xi_{n_1}), R_k(\xi_{n_2})] \subseteq \mathrm{Arc}\,(1, \lambda_k)$ and $\mathrm{Arg}(R_k(R_k(\xi_{n_1}))) > \mathrm{Arg}(R_k(\xi))$.

(b) For all $z \in \mathrm{Arc}\,[R_k(\xi_{n_1}), R_k(\xi_{n_2})]$ we have $||f_k'(z)| - \alpha| < \delta$.

(c) For all $z_1, z_2 \in \mathrm{Arc}\,[R_k(\xi_{n_1}), R_k(\xi_{n_2})]$ we have $||R_k'(z_1)/R_k'(z_2)| - 1| < \epsilon_2$.

That it is possible to choose n_1, n_2 such that the first two properties hold follows from the fact that both R_k and the derivative of f_k are continuous on $\mathrm{Arc}\,[\overline{\lambda_k}, \lambda_k]$. The existence of n_1, n_2 satisfying the third property follows from the fact that the derivative of $z \mapsto R_k(z)$ is continuous and non-zero on $\mathrm{Arc}\,(1, \lambda_k)$.

Let $g_0 = f_{\xi_{n_1},k}$ and $g_1 = f_{\xi_{n_2},k}$. Since ξ_{n_1}, ξ_{n_2} are implemented by rooted trees in $\mathcal{T}_{d+1}$ with root degrees at most m, we have by Lemma 2.2.2 that, if r is implemented by a rooted tree in $\mathcal{T}_{d+1}$, then $g_i(r)$ is the field implemented by a tree in $\mathcal{T}_{d+1}$ and root degree $m + k \leq d$. Let $A = \mathrm{Arc}\,[R_k(\xi_{n_1}), R_k(\xi_{n_2})]$ and note that the maps g_0, g_1 have the respective endpoints of A as fixed points. Furthermore $||g_i'(z)| - \alpha| < \delta$ for all $z \in A$ and thus it follows from Corollary 2.5.5 that there is a triple $\omega_1, \omega_2, \omega_3 \in \Omega$ such that for all triples $p_i \in g_{\omega_i}(A)$ we have $\mathrm{Arg}(p_1) < \mathrm{Arg}(p_2) < \mathrm{Arg}(p_3)$ and

$$\left| \frac{\ell(\mathrm{Arc}[p_1, p_2])}{\ell(\mathrm{Arc}[p_2, p_3])} - 1 \right| < \epsilon_1.$$

The orbit of ξ_{n_2} under iteration of g_1 converges to $R_k(\xi_{n_2})$ approaching from an anti-clockwise direction and thus there is some number N such that if we let ω_N be the constant 1 sequence of length N that $g_{\omega_N}(\xi_{n_2}) \in A$. For $i = 1, 2, 3$ we define $\zeta_i = g_{\omega_i \oplus \omega_N}(\xi_{n_2})$ and note that each ζ_i is contained in the interval $(\overline{\lambda_k}, \lambda_k)$ and is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree $m+k$. Furthermore we have $\mathrm{Arg}(\zeta_1) < \mathrm{Arg}(\zeta_2) < \mathrm{Arg}(\zeta_3)$ and

$$\left| \frac{\ell(\mathrm{Arc}[\zeta_1, \zeta_2])}{\ell(\mathrm{Arc}[\zeta_2, \zeta_3])} - 1 \right| < \epsilon_1. \tag{2.14}$$

Let $h_i = f_{\zeta_i,k}$. Analogously to above, if r is implemented by a rooted tree in $\mathcal{T}_{d+1}$, then $h_i(r)$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $m+2k \leq d$. Redefine $A = \mathrm{Arc}\,[R_k(\zeta_1), R_k(\zeta_3)]$. We will show that we can choose ϵ_1 and ϵ_2 sufficiently small such that $A = h_1(A) \cup h_2(A) \cup h_3(A)$. To do this define $A_1 = \mathrm{Arc}\,[R_k(\zeta_1), R_k(\zeta_2)]$ and $A_2 = \mathrm{Arc}\,[R_k(\zeta_2), R_k(\zeta_3)]$. It follows from the mean value theorem that there are $x_i \in \mathrm{Arc}\,[\zeta_i, \zeta_{i+1}]$ such that $\ell(A_i) = |R_k'(x_i)| \cdot \ell(\mathrm{Arc}\,[\zeta_i, \zeta_{i+1}])$ for $i = 1, 2$. Because both x_1 and x_2 lie in $\mathrm{Arc}\,[R_k(\xi_{n_1}), R_k(\xi_{n_2})]$

it follows from property (c) above that we can write $|R_k'(x_1)/R_k'(x_2)| = 1 + r_2$ for some $r_2 \in \mathbb{R}$ with $|r_2| < \epsilon_2$. We use the bound in (2.14) to obtain the following inequality

$$\begin{aligned}\left|\frac{\ell(A_1)}{\ell(A_2)} - 1\right| &= \left|\frac{|R_k'(x_1)| \cdot \ell(\mathrm{Arc}\,[\zeta_1, \zeta_2])}{|R_k'(x_2)| \cdot \ell(\mathrm{Arc}\,[\zeta_2, \zeta_3])} - 1\right| = \left|(1 + r_2)\frac{\ell(\mathrm{Arc}\,[\zeta_1, \zeta_2])}{\ell(\mathrm{Arc}\,[\zeta_2, \zeta_3])} - 1\right| \\ &\leq |1 + r_2| \cdot \left|\frac{\ell(\mathrm{Arc}\,[\zeta_1, \zeta_2])}{\ell(\mathrm{Arc}\,[\zeta_2, \zeta_3])} - 1\right| + |r_2| < |1 + r_2| \cdot \epsilon_1 + |r_2| \\ &\leq \epsilon_1 + \epsilon_2 + \epsilon_1 \cdot \epsilon_2.\end{aligned}$$

Let $\epsilon_3 = \epsilon_1 + \epsilon_2 + \epsilon_1 \cdot \epsilon_2$ and note that ϵ_3 can be made arbitrarily small by choosing ϵ_1 and ϵ_2 sufficiently small. It follows that there is some $r_3 \in \mathbb{R}$ with $|r_3| < \epsilon_3$ such that $\ell(A_1) = (1 + r_3) \cdot \ell(A_2)$. Because $\mathrm{Arg}(R_k(\zeta_1)) > \mathrm{Arg}(R_k(R_k(\xi_{n_1}))) > \mathrm{Arg}(R_k(\xi))$ we find that $1 > |f_k'(z)| > \alpha$ for all $z \in A$ and thus $1 > |h_i'(z)| > \alpha$ for all $z \in A$ and $i = 1, 2, 3$. It follows that

$$\begin{aligned}\ell(h_1(A_1 \cup A_2)) + \ell(h_2(A_1)) &> \alpha\ (\ell(A_1) + \ell(A_2)) + \alpha\ell(A_1) = \alpha\ (2\ell(A_1) + \ell(A_2)) \\ &= \alpha\left(2 + \frac{1}{1 + r_3}\right)\ell(A_1) = \alpha\ \frac{3 + 2r_3}{1 + r_3} \cdot \ell(A_1).\end{aligned}$$

and

$$\begin{aligned}\ell(h_2(A_2)) + \ell(h_3(A_1 \cup A_2)) &> \alpha \cdot \ell(A_2) + \alpha \cdot (\ell(A_1) + \ell(A_2)) \\ &= \alpha \cdot (\ell(A_1) + 2\ell(A_2)) = \alpha \cdot ((1 + r_3) + 2)\,\ell(A_2) \\ &= \alpha \cdot (3 + r_3) \cdot \ell(A_2).\end{aligned}$$

Because $\alpha > 1/3$ we can choose ϵ_3 sufficiently small such that

$$\ell(h_1(A_1 \cup A_2)) + \ell(h_2(A_1)) > \ell(A_1) \quad \text{and} \quad \ell(h_2(A_2)) + \ell(h_3(A_1 \cup A_2)) > \ell(A_2).$$

Because $h_1(A_1 \cup A_2)$ and $h_2(A_1)$ share the respective endpoints of A_1 it follows that $A_1 \subseteq h_1(A_1 \cup A_2) \cup h_2(A_1)$. Similarly we find that $A_2 \subseteq h_3(A_1 \cup A_2) \cup h_2(A_2)$. It follows that $A = h_1(A) \cup h_2(A) \cup h_3(A)$. Finally let $s = h_3^N(1)$, where we have taken N sufficiently large such that $s \in A$, and consider

$$\mathcal{S} = \{(h_{i_1} \circ \cdots \circ h_{i_l})(s) : l \in \mathbb{Z}_{\geq 1} \text{ and } i_1, \ldots, i_l \in \{1, 2, 3\}\}.$$

It follows from Corollary 2.4.2 that $\mathcal{S}$ is a dense subset of A. Every $r \in \mathcal{S}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree $m + 2k \leq d$. Finally we let

$$\mathcal{S}_2 = \{f_{\lambda,d}^n(r) : r \in \mathcal{S}, n \geq 1\}$$

and we find, because $|f_d'(z)| > 1$ for all $z \in \mathbb{S} - \{1\}$, that $\mathcal{S}_2$ is dense in $\mathbb{S}$. Every $r \in \mathcal{S}_2$ is implemented by a tree in $\mathcal{T}_{d+1}$. This shows that in this case item (1) of the lemma holds. □

Lemma 2.5.7. *Suppose* $d \in \mathbb{Z}_{\geq 5}$*,* $b \in \left(0, \frac{d-1}{d+1}\right] \cap \mathbb{Q}$*,* $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$ *and* $\xi \in \Lambda_3(b) \cap \mathbb{S}_{\mathbb{Q}}$ *with* $\xi \neq \pm 1$*.*

Suppose there is a rooted tree in $\mathcal{T}_{d+1}$ *with root degree at most* $d-5$ *and field* ξ*. Then the set of fields implemented by rooted trees in* $\mathcal{T}_{d+1}$ *is dense in* $\mathbb{S}$*.*

Proof. It follows from Lemma 2.3.10 that there is $\sigma \in \mathbb{S}$ with $|f_3'(\sigma)| > 1$ together with a sequence $\{\zeta_n\}_{n\geq 1}$ accumulating on σ such that every ζ_n is the field implemented by a tree in $\mathcal{T}_{d+1}$ whose root degree is bounded by $(d-5)+3 = d-2$. We can now apply Lemma 2.5.6 with $k=1$. It follows that either the set of fields implemented by rooted trees in $\mathcal{T}_{d+1}$ is dense in $\mathbb{S}$, or there is a tree in $\mathcal{T}_{d+1}$ with root degree at most $(d-2)+1 = d-1$ and field $\zeta \in \mathrm{Arc}\,(\lambda_1, \overline{\lambda_1}) \setminus \{-1\}$. We conclude from Lemma 2.3.8 that $f_{\zeta,1}$ is conjugate to an irrational rotation and thus the orbit $\{f_{\zeta,1}^n(1)\}_{n\geq 1}$ is dense in $\mathbb{S}$. Every element in this orbit is implemented by a rooted tree in $\mathcal{T}_{d+1}$ and thus we are done. □

2.6. Proof of Lemma 2.3.11

We are now ready to prove Lemma 2.3.11, which we restate here for convenience.

Lemma 2.3.11. *Let* $k, d \in \mathbb{Z}_{\geq 2}$ *with* $k \leq d$*,* $b \in \left(\frac{d-2}{d}, \frac{d-1}{d+1}\right] \cap \mathbb{Q}$ *and* $\lambda \in \mathbb{S}_{\mathbb{Q}} \setminus \{\pm 1\}$*. Suppose there exists a rooted tree in* $\mathcal{T}_{d+1}$ *with root degree at most* $d-k$ *that implements a field* $\xi \neq 1$ *with the property that* $|f_k'(\xi)| \geq 1$ *and* $\xi \in \mathrm{Arc}\,[\overline{\lambda_{\lfloor k/2 \rfloor}}, \lambda_{\lfloor k/2 \rfloor}]$*. Then the set of fields implemented by trees in* $\mathcal{T}_{d+1}$ *is dense in* $\mathbb{S}$*.*

Proof. The proof consists of a careful case analysis. We give a seperate argument first for when k is a power of two and for when $k+1$ is a power of two, then for each value of k within the set $\{5, 6, 9, 10, 11, 12, 13, 14, 17\}$ and lastly we prove the statement for all other k.

We remark that in some cases we will show that the set of fields implemented by rooted trees in $\mathcal{T}_{d+1}$ is dense in an arc A of the circle. Since b is such that $|f_d'(z)| > 1$ for all $z \in \mathbb{S} \setminus \{1\}$ (see (2.2) of Lemma 2.3.4), it follows that for all arcs A there is an $N \geq 1$ such that $f_{\lambda,d}^N(A) = \mathbb{S}$. Density of fields in the whole unit circle therefore follows from density in A.

First suppose $k = 2^m$ is a power of two. In this case $\xi \in \mathrm{Arc}\,[\overline{\lambda_{2^{m-1}}}, \lambda_{2^{m-1}}] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - 2^m$ and with $|f_{2^m}'(\xi)| \geq 1$. Let $\xi_2 = f_{\xi,1}(\xi)$. By item (v) of Lemma 2.3.4, we have $\xi \in \mathrm{Arc}\,[\overline{\lambda_1}, \lambda_1] \setminus \{1\}$ and hence $\xi_2 \neq \xi$ by item (iv) of the same lemma. Moreover, by Lemma 2.2.2, ξ_2 is the field of a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - (2^m - 1)$. If $\xi_2 \in \mathrm{Arc}\,(\lambda_{2^{m-1}}, \overline{\lambda_{2^{m-1}}})$, then the desired result follows from Lemma 2.5.1. Otherwise $\xi, \xi_2 \in \mathrm{Arc}\,[\overline{\lambda_{2^{m-1}}}, \lambda_{2^{m-1}}]$ and the result follows from applying Lemma 2.4.4 to these two parameters.

Now suppose $k+1$ is a power of two, so $k = 2^{m+1}-1$ for $m \geq 1$. In this case $\xi \in \operatorname{Arc}[\overline{\lambda_{2^m-1}}, \lambda_{2^m-1}] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-(2^{m+1}-1)$ and with $|f'_{2^{m+1}-1}(\xi)| \geq 1$. If $\xi \in \operatorname{Arc}(\lambda_{2^m}, \overline{\lambda_{2^m}})$ the result follows from Lemma 2.5.1. Otherwise, if $\xi \in \operatorname{Arc}[\overline{\lambda_{2^m}}, \lambda_{2^m}]$, the result follows from Lemma 2.4.3.

We now continue with the list of individual cases.

$\mathbf{k=5}$: In this case $\xi \in \operatorname{Arc}[\overline{\lambda_2}, \lambda_2] \setminus \{1\}$ is the field of a rooted tree with root degree at most $d-5$ and with $|f'_5(\xi)| \geq 1$. If $\xi \in \operatorname{Arc}(\lambda_3, \overline{\lambda_3})$ the result follows from Lemma 2.5.7. Otherwise, if $\xi \in \operatorname{Arc}[\overline{\lambda_3}, \lambda_3]$, the result follows from Lemma 2.4.3.

$\mathbf{k=6}$: In this case $\xi \in \operatorname{Arc}[\overline{\lambda_3}, \lambda_3] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-6$ and with $|f'_6(\xi)| \geq 1$. Let $\xi_2 = f_{\xi,1}(\xi)$, which is the field of a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-5$. If $\xi_2 \in \operatorname{Arc}(\lambda_3, \overline{\lambda_3})$ then the result follows from Lemma 2.5.7. Otherwise $\xi, \xi_2 \in \operatorname{Arc}[\overline{\lambda_3}, \lambda_3]$ and the result follows from applying Lemma 2.4.4 to these two parameters.

$\mathbf{k=9}$: In this case $\xi \in \operatorname{Arc}[\overline{\lambda_4}, \lambda_4] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-9$ and with $|f'_9(\xi)| \geq 1$. Consider the orbit $\{f^n_{\xi,1}(\xi) : n \geq 1\}$. The elements of this orbit are implemented by trees in $\mathcal{T}_{d+1}$ with root degree at most $d-8$ and they accumulate on $R_1(\xi)$. Note that $|f'_9(R_1(\xi))| > 1$. It follows from Lemma 2.5.6 that we either obtain the desired density or we obtain a rooted tree with root degree at most $(d-8)+3 = d-5$ that implements a field in $\operatorname{Arc}(\lambda_3, \overline{\lambda_3})$. In this latter case the result follows from applying Lemma 2.5.7 to this tree.

$\mathbf{k=10}$: In this case $\xi \in \operatorname{Arc}[\overline{\lambda_5}, \lambda_5] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-10$ and with $|f'_{10}(\xi)| \geq 1$. Then it follows from Lemma 2.4.5 that either the orbit of 1 under the action of the semigroup generated by $f_{\xi,3}, f_{\xi,4}$ and $f_{\xi,5}$ is dense in an arc of $\mathbb{S}$, in which case the result follows. Or we can conclude that $|f'_5(R_5(\xi))| > \frac{43}{50}$. In that case we consider the orbit $\mathcal{R} = \{f^n_{\xi,5}(\xi) : n \geq 1\}$. This orbit accumulates on $R_5(\xi)$ and every element is implemented by a rooted tree with root degree at most $d-10+5 = d-5$. If $R_5(\xi) \in \operatorname{Arc}(\lambda_3, \overline{\lambda_3})$ then there are also fields $\zeta \in \mathcal{R}$ with $\zeta \in \operatorname{Arc}(\lambda_3, \overline{\lambda_3})$. In that case we can apply Lemma 2.5.7 to obtain density of the fields. Otherwise, if $R_5(\xi) \in \operatorname{Arc}[\overline{\lambda_3}, \lambda_3]$, then we can find $\zeta_1, \zeta_2 \in \mathcal{R}$ such that $\zeta_1, \zeta_2 \in \operatorname{Arc}[\overline{\lambda_3}, \lambda_3]$ are distinct, lie in the same half-plane and $|f'_5(\zeta_i)| > \frac{43}{50}$ for $i = 1, 2$. It follows that for both fields ζ_i we have

$$|f'_6(R_3(\zeta_i))| > |f'_6(\zeta_i)| = \frac{6}{5} \cdot |f'_5(\zeta_i)| > \frac{6}{5} \cdot \frac{43}{50} = \frac{129}{125} > 1.$$

Density of the fields now follows from applying Lemma 2.4.4 to ζ_1 and ζ_2.

k = 11 : In this case $\xi \in \text{Arc}\,[\overline{\lambda_5}, \lambda_5] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - 11$ and with $|f'_{11}(\xi)| \geq 1$. If $\xi \in \text{Arc}\,[\overline{\lambda_6}, \lambda_6]$ the result follows from Lemma 2.4.3. Otherwise, if $\xi \in \text{Arc}\,(\lambda_6, \overline{\lambda_6})$, we apply Lemma 2.3.10 to find a parameter $\sigma \in \mathbb{S}$ with $|f'_6(\sigma)| > 1$ together with a sequence of fields $\{\zeta_n\}_{n\geq 1}$ accumulating on σ such that every ζ_n is implemented by a rooted tree in $\mathcal{T}_{d+1}$ whose root degree is at most $d - 11 + 6 = d - 5$. If there is any $\zeta_n \in \text{Arc}\,(\lambda_3, \overline{\lambda_3})$ then density of the fields in the circle follows from Lemma 2.5.7. Otherwise the sequence accumulates on σ from inside $\text{Arc}\,[\overline{\lambda_3}, \lambda_3]$ and thus we can find $\zeta_{n_1}, \zeta_{n_2} \in \text{Arc}\,[\overline{\lambda_3}, \lambda_3]$ that are distinct, lie in the same half-plane and have the property that $|f'_6(\zeta_i)| > 1$ for $i = 1, 2$. The desired density now follows from applying Lemma 2.4.4 to ζ_1 and ζ_2.

k = 12 : In this case $\xi \in \text{Arc}\,[\overline{\lambda_6}, \lambda_6] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - 12$ and with $|f'_{12}(\xi)| \geq 1$. This case can be argued in a very similar way to the $k = 9$ case. Consider the orbit $\{f^n_{\xi,1}(\xi) : n \geq 1\}$. The elements of this orbit are fields of trees in $\mathcal{T}_{d+1}$ with root degree at most $d-11$ and they accumulate on $R_1(\xi)$. Note that $|f'_{12}(R_1(\xi))| > 1$. It follows from Lemma 2.5.6 that we either obtain the desired density or we obtain a rooted tree with root degree at most $(d - 11) + 4 = d - 7$ and field in $\text{Arc}\,(\lambda_4, \overline{\lambda_4})$. In this latter case the result follows from applying Lemma 2.5.1 to this tree.

k = 13 : In this case $\xi \in \text{Arc}\,[\overline{\lambda_6}, \lambda_6] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - 13$ and with $|f'_{13}(\xi)| \geq 1$. Then it follows from Lemma 2.4.5 that either the orbit of 1 under the action of the semigroup generated by $f_{\xi,4}, f_{\xi,5}$ and $f_{\xi,6}$ is dense in an arc of $\mathbb{S}$, in which case the result follows. Or we can conclude that $|f'_6(R_6(\xi))| > \frac{10}{13}$. In that case we consider the orbit $\mathcal{R} = \{f^n_{\xi,6}(\xi) : n \geq 1\}$. This orbit accumulates on $R_6(\xi)$ and every element is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-13+6 = d-7$. If $R_6(\xi) \in \text{Arc}\,(\lambda_4, \overline{\lambda_4})$ then there is also a field $\zeta \in \mathcal{R}$ with $\zeta \in \text{Arc}\,(\lambda_4, \overline{\lambda_4})$. In that case we can apply Lemma 2.5.1 to obtain density of the fields. Otherwise, if $R_6(\xi) \in \text{Arc}\,[\overline{\lambda_4}, \lambda_4]$, then we can find $\zeta_1, \zeta_2 \in \mathcal{R}$ such that $\zeta_1, \zeta_2 \in \text{Arc}\,[\overline{\lambda_4}, \lambda_4]$ are distinct, lie in the same half-plane and $|f'_6(\zeta_i)| > \frac{10}{13}$ for $i = 1, 2$. It follows that for both fields ζ_i we have

$$|f'_8(R_4(\zeta_i))| > |f'_8(\zeta_i)| = \frac{8}{6} \cdot |f'_6(\zeta_i)| > \frac{8}{6} \cdot \frac{10}{13} = \frac{40}{39} > 1.$$

Density of the fields now follows from applying Lemma 2.4.4 to ζ_1 and ζ_2.

k = 14 : In this case $\xi \in \text{Arc}\,[\overline{\lambda_7}, \lambda_7] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - 14$ and with $|f'_{14}(\xi)| \geq 1$. Then it follows from Lemma 2.4.5 that either the orbit of 1 under the action of the semigroup generated by $f_{\xi,5}, f_{\xi,6}$ and $f_{\xi,7}$ is dense in an arc of $\mathbb{S}$, in which case the result follows. Or we can conclude that $|f'_7(R_7(\xi))| > \frac{89}{98}$. In that case we consider the orbit $\mathcal{R} = \{f^n_{\xi,7}(\xi) : n \geq 1\}$. This orbit accumulates on $R_7(\xi)$ and every element is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-14+7 = d-7$. If $R_7(\xi) \in \mathrm{Arc}\,(\lambda_4, \overline{\lambda_4})$ then there is also a field $\zeta \in \mathcal{R}$ with $\zeta \in \mathrm{Arc}\,(\lambda_4, \overline{\lambda_4})$. In that case we can apply Lemma 2.5.1 to obtain density of the fields. Otherwise, if $R_7(\xi) \in \mathrm{Arc}\,[\overline{\lambda_4}, \lambda_4]$, then we can find $\zeta_1, \zeta_2 \in \mathcal{R}$ such that $\zeta_1, \zeta_2 \in \mathrm{Arc}\,[\overline{\lambda_4}, \lambda_4]$ are distinct, lie in the same half-plane and $|f_7'(\zeta_i)| > \frac{89}{98}$ for $i = 1, 2$. It follows that for both fields ζ_i we have

$$|f_8'(R_4(\zeta_i))| > |f_8'(\zeta_i)| = \frac{8}{7} \cdot |f_7'(\zeta_i)| > \frac{8}{7} \cdot \frac{89}{98} = \frac{356}{343} > 1.$$

Density of the fields now follows from applying Lemma 2.4.4 to ζ_1 and ζ_2.

$\mathbf{k = 17}$: In this case $\xi \in \mathrm{Arc}\,[\overline{\lambda_8}, \lambda_8] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d-17$ and with $|f_{17}'(\xi)| \geq 1$. If $\xi \in \mathrm{Arc}\,[\overline{\lambda_9}, \lambda_9]$ the result follows from Lemma 2.4.3, therefore we assume that $\xi \in \mathrm{Arc}\,(\lambda_9, \overline{\lambda_9})$. We apply Lemma 2.3.10 to find a parameter $\sigma \in \mathbb{S}$ with $|f_9'(\sigma)| > 1$ together with a sequence of fields $\{\zeta_n\}_{n\geq 1}$ accumulating on σ such that every ζ_n is implemented by a rooted tree in $\mathcal{T}_{d+1}$ whose root degree is at most $d - 17 + 9 = d - 8$. It follows from Lemma 2.5.6 that we either obtain the required density of fields or there is a tree in $\mathcal{T}_{d+1}$ whose root degree is bounded by $d - 5$ with field inside $\mathrm{Arc}\,(\lambda_3, \overline{\lambda_3})$. In the latter case the result follows from Lemma 2.5.7.

Finally we complete the proof for $k > 17$. In that case write $k = 2m$ if k is even and $k = 2m + 1$ if k is odd. Note that $m \geq 9$. We are then given that $\xi \in \mathrm{Arc}\,[\overline{\lambda_m}, \lambda_m] \setminus \{1\}$ is implemented by a rooted tree in $\mathcal{T}_{d+1}$ with root degree at most $d - k$ and with $|f_k'(\xi)| \geq 1$. It follows from Corollary 2.4.6 that the orbit of 1 under the action of the semigroup generated by $f_{\xi,m-3}, f_{\xi,m-2}, f_{\xi,m-1}$ and $f_{\xi,m}$ is dense in an arc of $\mathbb{S}$ from which our desired conclusion follows. This concludes the proof of Lemma 2.3.11. □

2.7. Fast implementation of fields

In this section, we bootstrap Theorem 2.2.4 to obtain fast algorithms for implementing fields which will be important in our reductions. For a number $\alpha = p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$, we use *size*(α) to denote the total number of bits needed to represent p, q, and we extend this to numbers in $\mathbb{C}_{\mathbb{Q}}$ by adding the sizes of the real and imaginary parts. For $\alpha_1, \ldots, \alpha_t \in \mathbb{C}_{\mathbb{Q}}$, we denote by $\mathrm{size}(\alpha_1, \ldots, \alpha_t)$ the total of the sizes of $\alpha_1, \ldots, \alpha_t$.

Lemma 2.7.1. *Fix an integer $\Delta \geq 3$, a rational number $b \in (0, 1)$ and $\lambda \in \mathbb{S}_{\mathbb{Q}}(\Delta - 1, b)$. Then, there is an algorithm, which on input $\hat{\lambda} \in \mathbb{S}_{\mathbb{Q}}$ and rational $\epsilon > 0$, returns in time poly$(\mathrm{size}(\hat{\lambda}, \epsilon))$ a rooted tree T in $\mathcal{T}_{\Delta}$ with root degree 1 that implements a field λ' such that $|\lambda' - \hat{\lambda}| \leq \epsilon$.*

Proof of Lemma 2.7.1. Let $d = \Delta - 1$. We start by setting up some parameters that will be useful.

Let λ_1 be as in Lemma 2.3.4. As $\tilde{\lambda}$ approaches λ_1 from inside $\mathrm{Arc}\,(1, \lambda_1)$ we know that $R_1(\tilde{\lambda})$ approaches $R_1(\lambda_1)$. Since $|f_1'(R_1(\lambda_1))| = 1$ there must be $\tilde{\lambda} \in \mathrm{Arc}\,(1, \lambda_1)$ such that $|f_1'(R_1(\xi))| \in (\frac{1}{2}, 1)$ for all $\xi \in \mathrm{Arc}\,(\tilde{\lambda}, \lambda_1)$. By Theorem 2.2.4, there exist trees T_1, T_2 in $\mathcal{T}_{d+1}$ with root degree 1 and fields $\xi_1, \xi_2 \in \mathrm{Arc}\,(\tilde{\lambda}, \lambda_1) \cap \mathbb{S}_{\mathbb{Q}}$ with $\mathrm{Arg}(\xi_1) < \mathrm{Arg}(\xi_2)$. Because the map $\xi \mapsto R_1(\xi)$ is orientation preserving with nonzero derivative we have $\mathrm{Arg}(R_1(\xi_1)) < \mathrm{Arg}(R_1(\xi_2))$. For $i \in \{1, 2\}$, the fixed point $R_1(\xi_i)$ is a solution to the quadratic equation $\xi_i(z + b) = z(bz + 1)$, and hence we can approximate it with any desired rational precision $\tau > 0$ in time $poly(\mathrm{size}(\tau))$.

Let $I = \mathrm{Arc}(R_1(\xi_1), R_1(\xi_2))$ and note that this arc is contained in the upper half-plane. We will show that the arc I gets mapped onto $\mathbb{S}$ in a fixed number of applications of $f_{\lambda,d}$. The idea of the algorithm is then to find a small enough neighborhood of a point in I that gets mapped close to the field that we are trying to (approximately) implement. Then we use the fact that we are able to quickly and accurately approach any value inside I using $f_{\xi_1,1}$ and $f_{\xi_2,1}$. This algorithm is very similar to the proof of Lemma 2.4.1.

We now show that I gets mapped onto $\mathbb{S}$ in a fixed number of applications of $f_{\lambda,d}$. We first consider the case that $b \in (0, \frac{d-1}{d+1}]$. Let $C_1 = |f_{\lambda,d}'(1)| = d\frac{1-b}{1+b}$ and let $C_2 = |f_{\lambda,d}'(-1)| = d\frac{1+b}{1-b}$. Note that C_1 and C_2 are both greater than one and that for any $z \in \mathbb{S}$ the inequality $C_1 \le |f_{\lambda,d}'(z)| \le C_2$ holds (cf. item (i) of Lemma 2.3.4). This means that for any circular arc J and integer n we get

$$C_1^n \cdot \ell(J) \le \ell(f_{\lambda,d}^n(J)) \le C_2^n \cdot \ell(J). \tag{2.15}$$

From this, we deduce that $f_{\lambda,d}^N(I) = \mathbb{S}$, where $N = \lceil \frac{\log(2\pi/\ell(I))}{\log(C_1)} \rceil$.

Next, in case $b \in (\frac{d-1}{d+1}, 1)$, we recall the conformal metric μ from the proof of Lemma 2.3.9. Let us denote the length of a circular arc J with respect to this metric by $\mathrm{length}(J)$ and denote $c = \mathrm{length}(\mathbb{S})$. Since there exists a constant $\kappa > 1$ such that $f_{d,\lambda}$ is uniformly expanding on $\mathbb{S}$ with a factor κ with respect to this metric, it follows that $f_{\lambda,d}^N(I) = \mathbb{S}$, where $N = \lceil \frac{\log(c/\mathrm{length}(I))}{\log(\kappa)} \rceil$. Note that the right-hand side of (2.15) is also valid for $b \in (\frac{d-1}{d+1}, 1)$ (with C_2 defined in the same way).

Let $x_0, \ldots, x_m$ be points such that the clockwise arcs between x_{i-1} and x_i form a partition of I with $x_0 = R_1(\xi_1)$, $x_m = R_1(\xi_2)$ and chosen so that $x_1, \ldots, x_{m-1} \in \mathbb{S}_{\mathbb{Q}}$ and the length of an arc between two subsequent points is less than $2\pi/C_2^N$. In this way we ensure that these arcs are not mapped onto the whole circle by N applications of $f_{\lambda,d}$ and thus each arc is bijectively mapped to an arc on the unit circle by $f_{\lambda,d}^N$.

We now describe an algorithm that, on input $\hat{\lambda} \in \mathbb{S}_{\mathbb{Q}}$ and rational $\epsilon > 0$, yields in *poly*(size($\hat{\lambda}, \epsilon$)) a rooted tree $\hat{T}$ in $\mathcal{T}_{d+1}$ with $\mathcal{O}(\log(\epsilon^{-1}))$ vertices whose field has distance at most ϵ from $\hat{\lambda}$; we will account later for the degree of the root. We assume for convenience that $\epsilon \ll \ell(I)$.

The first step of the algorithm is to find $i \in \{1, \ldots, m\}$ such that $\hat{\lambda} \in \mathrm{Arc}\,[f^N_{\lambda,d}(x_{i-1}), f^N_{\lambda,d}(x_i)]$. We know that such an arc must exist because I is mapped surjectively onto $\mathbb{S}$ by $f^N_{\lambda,d}$ and, since $f^N_{\lambda,d}(z)$ is a rational function of z with fixed degree, we can find i in time *poly*(size($\hat{\lambda}$)). Now we consider the bijective map

$$f^N_{\lambda,d} : \mathrm{Arc}\,[x_{i-1}, x_i] \to \mathrm{Arc}\,[f^N_{\lambda,d}(x_{i-1}), f^N_{\lambda,d}(x_i)].$$

Analogously, with $n = \lceil \log_{3/2}(\ell(\mathrm{Arc}\,[x_{i-1}, x_i]) \cdot C_2^N/\epsilon) \rceil$ applications of $f^N_{\lambda,d}$, we can determine using binary search in time *poly*(size($\hat{\lambda}, \epsilon$)) an arc $J \subseteq \mathrm{Arc}\,[x_{i-1}, x_i]$ with endpoints in $\mathbb{S}_{\mathbb{Q}}$ such that $\hat{\lambda} \in f^N_{\lambda,d}(J)$ and whose length satisfies

$$3^{-n} \cdot \ell(\mathrm{Arc}\,[x_{i-1}, x_i]) \leq \ell(J) \leq (2/3)^n \cdot \ell(\mathrm{Arc}\,[x_{i-1}, x_i]) \leq \epsilon/C_2^N$$

Note that the length of J is bounded below by $C_3 \cdot \epsilon^5$, where C_3 is a constant independent of $\hat{\lambda}$ or ϵ. It follows from (2.15) that $\ell(f^N_{\lambda,d}(J)) \leq \epsilon$, which means that the arc J is mapped by $f^N_{\lambda,d}$ to an arc of length at most ϵ, that includes $\hat{\lambda}$. We will next show how to construct in *poly*(size($\hat{\lambda}, \epsilon$)) a rooted tree T in $\mathcal{T}_{d+1}$ with $s = \mathcal{O}(\log(\epsilon^{-1}))$ vertices that implements a field $w \in J$. Then, using Lemma 2.2.2,[6] we obtain a rooted tree $\hat{T}$ with $(d^N - 1)/(d-1) + d^N s$ vertices that implements the field $\lambda' = f^N_{\lambda,d}(w)$ with $|\lambda' - \hat{\lambda}| \leq \epsilon$.

To construct T, we first fix some constants. Let $C_4 = |f_1'(R_1(\xi_1))|$ and $C_5 = |f_1'(R_1(\xi_2))|$ and note that $C_4, C_5 \in (\frac{1}{2}, 1)$. We also have $C_4 \leq |f_1'(z)| \leq C_5$ for all $z \in I$. It follows that $f_{\xi_2,1}(I) = \mathrm{Arc}\,[f_{\xi_2,1}(R_1(\xi_1)), R_1(\xi_2)]$ is contained in I and its length is strictly bigger than $\ell(I)/2$. Furthermore it follows that $f^{-1}_{\xi_1,1}(\mathrm{Arc}\,[R_1(\xi_1), f_{\xi_2,1}(R_1(\xi_1))]) = \mathrm{Arc}\,[R_1(\xi_1), f^{-1}_{\xi_1}(f_{\xi_2}(R_1(\xi_1)))]$ is strictly contained inside I. Let $J_0 = J$ and for $k \geq 0$, as long as $f_{\xi_2,1}(R_1(\xi_1)) \notin J_k$, define

$$J_{k+1} = \begin{cases} f^{-1}_{\xi_1,1}(J_k) & \text{if } J_k \subset \mathrm{Arc}\,[R_1(\xi_1), f_{\xi_2,1}(R_1(\xi_1))] \\ f^{-1}_{\xi_2,1}(J_k) & \text{if } J_k \subset \mathrm{Arc}\,[f_{\xi_2,1}(R_1(\xi_1))), R_1(\xi_2)]. \end{cases}$$

We have that $J_k \subseteq I$ for every k and $\ell(J_k) \geq C_5^{-k} \cdot \ell(J_0) \geq C_3 \cdot C_5^{-k} \cdot \epsilon^5$. Because $C_5 < 1$, we deduce that there is $N_1 \geq 0$ such that $f_{\xi_2,1}(R_1(\xi_1)) \in J_{N_1}$ where N_1 is bounded above by

$$\left\lceil \frac{\log(C_3 \cdot \epsilon^5/\ell(I))}{\log(C_5)} \right\rceil = \mathcal{O}(\log(\epsilon^{-1})).$$

[6] Lemma 2.2.2 describes how to construct a tree of size $s \cdot d + 1$ with field $f_{\lambda,d}(z)$ from a tree of size s and field z. Repeating this construction N times yields the construction of $\hat{T}$ from T.

Let $i_1, \ldots, i_{N_1}$ be the sequence of indices such that $f_{\xi_{i_k}}(J_k) = J_{k-1}$ and note that these can be computed in $poly(\text{size}(\hat{\lambda}, \epsilon))$ time. Let $K = f^{-1}_{\xi_2,1}(J_{N_1})$. We see that $R_1(\xi_1) \in K$ and

$$\left(f_{\xi_{i_1},1} \circ \cdots \circ f_{\xi_{i_{N_1}},1} \circ f_{\xi_2,1}\right)(K) = J.$$

Furthermore, because the maps $f^{-1}_{\xi_i,1}$ are expanding on I, we find $\ell(K) \geq \ell(J) \geq C_3 \cdot \epsilon^5$. This means that there is an arc of length at least $\frac{1}{2} \cdot C_3 \cdot \epsilon^5$ extending from $R_1(\xi_1)$, going either clockwise or counterclockwise, contained in K. In the case that such a clockwise arc exists, i.e. $\text{Arc}[R_1(\xi_1) \cdot e^{-i\frac{1}{2}C_3\epsilon^5}, R_1(\xi_1)] \subseteq K$, we see that, because R_1 is an attracting fixed point of f_{ξ_1}, there is some N_2, specified below, such that $f^{N_2}_{\xi_1,1}(\xi_1) \in K$. Using that for integers n we have

$$\begin{aligned}\ell(\text{Arc}\,[f^n_{\xi_1,1}(\xi_1), R_1(\xi_1)]) &= \ell(f^n_{\xi_1,1}(\text{Arc}\,[\xi_1, R_1(\xi_1)])) \\ &\leq C_4^n \cdot \ell(\text{Arc}\,[\xi_1, R_1(\xi_1)]) < C_4^n \cdot 2\pi,\end{aligned}$$

we see that it suffices to take $N_2 = \left\lceil \frac{\log(C_3 \cdot \epsilon^5/(4\pi))}{\log(C_4)} \right\rceil = \mathcal{O}(\log(\epsilon^{-1}))$. In the case that such a clockwise arc does not exist, we find that a counterclockwise arc of length $\frac{1}{2} \cdot C_3 \cdot \epsilon^5$ is contained in K. Note that there is some integer N_c independent of $\hat{\lambda}$ and ϵ such that $f^{N_c}_{\xi_2,1}(\xi_1) \in I$. The same analysis as above shows that then $(f^{N_2}_{\xi_1,1} \circ f^{N_c}_{\xi_2,1})(\xi_1) \in K$. We let N_3 be equal to zero if a clockwise arc of sufficient length is contained in K and otherwise we let $N_3 = N_c$. We conclude that

$$\left(f^N_{\lambda,d} \circ f_{\xi_{i_1},1} \circ \cdots \circ f_{\xi_{i_{N_1}},1} \circ f_{\xi_2,1} \circ f^{N_2}_{\xi_1,1} \circ f^{N_3}_{\xi_2,1}\right)(\xi_1) \tag{2.16}$$

has a distance at most ϵ from $\hat{\lambda}$. By repeatedly applying the constructions laid out in Lemma 2.2.2 (cf. Footnote 6), we conclude that we can construct a tree T in $\mathcal{T}_{d+1}$ whose field is given by the value in (2.16) and with $\mathcal{O}(\log(\epsilon^{-1}))$ vertices.

This finishes the description of the algorithm, modulo that the root of the tree we constructed has degree d. To obtain a rooted tree with root degree 1, we run the algorithm described on input $f^{-1}_{\lambda,1}(\hat{\lambda})$ and $\epsilon \cdot \frac{d-1}{d+1}$ to obtain a rooted tree with root degree d and field ζ with $|f^{-1}_{\lambda,1}(\hat{\lambda}) - \zeta| < \epsilon \cdot \frac{d-1}{d+1}$. Attaching one new vertex by an edge to this root yields a rooted tree with root degree 1 and field $f_{\lambda,1}(\zeta)$ which satisfies, using Item i of Lemma 2.3.4, that

$$|\hat{\lambda} - f_{\lambda,1}(\zeta)| \leq |f^{-1}_{\lambda,1}(\hat{\lambda}) - \zeta| \cdot \max_{z \in S} |f'_{\lambda,1}(z)| < \epsilon \cdot \frac{d-1}{d+1} \cdot \frac{d+1}{d-1} = \epsilon,$$

as wanted. This finishes the proof of Lemma 2.7.1. $\square$

2.8. Reduction

In this section, we prove our inapproximability results. Throughout this section, we use $\mathcal{G}_\Delta$ to denote the set of all graphs with maximum degree at most Δ. We start in Section 2.8.1 with some preliminaries that will be used in our proofs, Section 2.8.2 gives the main reduction, and we show how to use this in Section 2.8.3 to conclude the proof of Theorem 2.1.1.

2.8.1. Preliminaries

We will use the following lemma from [PR20].

Lemma 2.8.1 ([PR20]). *Let $\Delta \geq 3$ be an integer and let $\lambda \in \mathbb{S}_{\mathbb{Q}}$ with $\lambda \neq -1$. Then there exists $\eta = \eta(\Delta, \lambda) > 1$ such that, for all $b \in (1/\eta, \eta)$, for all graphs $G \in \mathcal{G}_\Delta$, it holds that $Z_G(\lambda, b) \neq 0$.*

For a graph G and vertices u, v in G, let $Z_{G,\pm u,\pm v}(\lambda, b)$ denote the contribution to the partition function when u, v are assigned the spins $\pm$, respectively. For a configuration σ on G, we use $w_{G,\sigma}(\lambda, b)$ to denote the weight $\lambda^{|n_+(\sigma)|} b^{\delta(\sigma)}$ of σ. We will use the following observation.

Lemma 2.8.2. *Let $\lambda \in \mathbb{S}$ and $b \in \mathbb{R}$. Then, for an arbitrary graph $G = (V_G, E_G)$ and vertices u, v of G it holds that*

$$Z_{G,+u,+v}(\lambda, b) = \lambda^{|V(G)|}\, \overline{Z_{G,-u,-v}(\lambda, b)}, \quad Z_{G,+u,-v}(\lambda, b) = \lambda^{|V(G)|}\, \overline{Z_{G,-u,+v}(\lambda, b)}.$$

Proof. For an assignment $\sigma : V_G \to \{+, -\}$, let $\bar{\sigma} : V_G \to \{+, -\}$ be the assignment obtained by interchanging the assignment of +'s with -'s. Then

$$w_{G,\bar{\sigma}}(\lambda, b) = \lambda^{|n_+(\bar{\sigma})|} b^{\delta(\bar{\sigma})} = \lambda^{|V_G| - |n_+(\sigma)|} b^{\delta(\sigma)} = \lambda^{|V(G)|}\overline{w_{G,\sigma}(\lambda, b)}.$$

The result follows by summing over the relevant σ for each of $Z_{G,+u,+v}(\lambda, b)$ and $Z_{G,+u,-v}(\lambda, b)$. □

The following lemma will be useful in general for handling rational points on the circle. Ideally, we would like to describe a number on $\mathbb{S}$ by a rational angle, but this may not correspond to a rational cartesian point which would complicate computations. However, rational points are dense on the circle and we can compute one arbitrarily close to a given angle as follows.

Lemma 2.8.3. *Given a rational angle $\theta \in [0, 2\pi)$ and $\epsilon \in (0, 1)$, there exists a number $\hat{\theta}$ such that $|\theta - \hat{\theta}| < \epsilon$ and $\cos\hat{\theta}, \sin\hat{\theta} \in \mathbb{Q}$ are rational numbers of size at most poly*$(\mathrm{size}(\theta, \epsilon))$. *Furthermore, we can compute* $\cos\hat{\theta}$ *and* $\sin\hat{\theta}$ *in time poly*$(\mathrm{size}(\theta, \epsilon))$.

Proof. By symmetry, we may assume that $\theta \in [0, \pi/4]$. Given θ, take a rational approximation r of $\tan(\theta/2)$ such that $|\tan(\theta/2) - r| < \epsilon/2$. We claim that $\hat{\theta} = 2\arctan(r)$ has the desired properties.

Write s, c, t respectively for $\sin\hat{\theta}, \cos\hat{\theta}, \tan\hat{\theta}$. Using the tan double angle formula we have $s/c = t = 2r/(1-r^2)$. We also know that $s^2 + c^2 = 1$. Solving these simultaneously gives that $s = 2r/(1+r^2)$ and $c = (1-r^2)/(1+r^2)$, which are both rational since r is rational.

Also writing $f(x) = 2\arctan(x)$ for $x \in [0,1]$, note that $f'(x) = 2/(1+x^2) \in [1,2]$ for $x \in [0,1]$. Hence $|f(x) - f(y)| \le 2|x-y|$ for $x, y \in [0,1]$ and so $|\theta - \theta'| < \epsilon$.

Finally we can compute r in *poly*(size(θ, ϵ)) using a series expansion of tan from which we can compute s and c from the formulas above. □

Finally, we will use the following well-known lemma for continued-fraction approximation.

Lemma 2.8.4 ([Sch86, Corollary 6.3a])**.** *There is a poly-time algorithm which, on input a rational number α and integer $K \ge 1$, decides whether there exists a rational number p/q with $1 \le q \le K$ and $|\alpha - (p/q)| < 1/2K^2$, and if so, finds this (unique) rational number.*

2.8.2. The reduction

To prove Theorem 2.1.1, we will show how to use a poly-time algorithm for $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ to compute exactly $Z_G(\lambda, \hat{b})$ on graphs G of maximum degree three for some appropriate value of $\hat{b}$ that we next specify.

Let $\eta = \eta(3, \lambda) > 1$ be as in Lemma 2.8.1, so that

$$Z_G(\lambda, b') \neq 0 \text{ for all } b' \in (1/\eta, \eta) \text{ and } G \in \mathcal{G}_3. \tag{2.17}$$

For $k = 2, 3, \ldots$, let P_k be the path with k vertices whose endpoints are labeled u_k, v_k and all vertex activities are equal to 1. Then, it is not hard to see that

$$\begin{bmatrix} Z_{P_k,+u_k,+v_k}(1,b) & Z_{P_k,+u_k,-v_k}(1,b) \\ Z_{P_k,-u_k,+v_k}(1,b) & Z_{P_k,-u_k,-v_k}(1,b) \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}^{k-1}. \tag{2.18}$$

Clearly, for all k it holds that

$$Z_{P_k,+u_k,+v_k}(1,b) = Z_{P_k,-u_k,-v_k}(1,b) \text{ and } \frac{Z_{P_k,+u_k,-v_k}(1,b)}{Z_{P_k,-u_k,-v_k}(1,b)} = \frac{Z_{P_k,-u_k,+v_k}(1,b)}{Z_{P_k,-u_k,-v_k}(1,b)}. \tag{2.19}$$

We let b_k denote this last ratio. Using (2.18), we have that there exists k such that

$$1/\eta < \hat{b} = b_k < \eta. \tag{2.20}$$

By the choice of k and (2.17), we conclude that

$$Z_G(\lambda, \hat{b}) \neq 0 \text{ for all } G \in \mathcal{G}_3. \tag{2.21}$$

The main step in the reduction is captured by the following lemma.

Lemma 2.8.5. *Let $\Delta \geq 3$ be an integer, $b \in (0,1)$ be a rational and let $\lambda \in \mathbb{S}_{\mathbb{Q}}(\Delta-1, b)$ Let $K = 1.001$ and $\rho = \pi/40$. Assume that a poly-time algorithm exists for either $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ or $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$. Then there exists a poly-time algorithm that, on input a graph $G \in \mathcal{G}_3$ and an edge $e = \{u, v\}$ of G, outputs the value of the ratio*

$$R_{G,e} = \frac{\hat{b}^2 z_{++} + \hat{b}(z_{+-} + z_{-+}) + z_{--}}{\hat{b}^2 z_{--} + \hat{b}(z_{+-} + z_{-+}) + z_{++}}, \quad \textit{where} \quad z_{\pm\pm} := Z_{G\backslash e, \pm u, \pm v}(\lambda, \hat{b}).$$

The algorithm also outputs the value of the ratio $R'_{G,e} = z_{--}/z_{++}$, provided that $z_{++} \neq 0$.

Remark 2.8.6. As it will be shown in the proof of Lemma 2.8.5, the ratio $R_{G,e}$ is well-defined for all graphs $G \in \mathcal{G}_3$ and edges e in G using the zero-free region in Lemma 2.8.1 and the choice of $\hat{b}$. It is harder to show that $R'_{G,e}$ is well-defined (we cannot use Lemma 2.8.1 directly) and hence the need for the assumption that $z_{++} \neq 0$ in Lemma 2.8.5.

Proof. Suppose that $G = (V, E)$ with $n = |V|$ and $m = |E|$. Let

$$\begin{aligned} r &= \hat{b}^2 z_{++} + \hat{b}(z_{+-} + z_{-+}) + z_{--}, \quad r' = (\hat{b}^2 - 1)^2 z_{--}, \\ t &= \hat{b}^2 z_{--} + \hat{b}(z_{+-} + z_{-+}) + z_{++}, \quad t' = (\hat{b}^2 - 1)^2 z_{++}. \end{aligned} \tag{2.22}$$

We first show that $r, t \neq 0$. Consider the graph $H = (V_H, E_H)$ obtained from G by subdividing edge e, i.e., we remove edge $e = \{u, v\}$ and then add a new vertex s which is connected to both u, v. Note that H is obtained from $G\backslash e$ by adding the edges $\{s, u\}, \{s, v\}$, so it is not hard to see that

$$Z_H(\lambda, \hat{b}) = \lambda t + r.$$

Note that H is a graph of maximum degree Δ and we have $Z_H(\lambda, \hat{b}) \neq 0$ from (2.21). Moreover, from Lemma 2.8.2, we have $r = \lambda^n \bar{t}$. Combining these, we obtain that $r, t \neq 0$. By assumption, we also have that $t' \neq 0$.

We will show how to compute the ratios $R_{\mathsf{goal}} = -\frac{r}{t}$ and $R'_{\mathsf{goal}} = -\frac{r'}{t'}$ (note that these are well-defined since $t, t' \neq 0$). By Lemma 2.8.2, we have that $r = \lambda^n \bar{t}$ and $r' = \lambda^n \overline{t'}$, so $R_{\mathsf{goal}}, R'_{\mathsf{goal}} \in \mathbb{S}_{\mathbb{Q}}$. In fact, letting p, p', p'', q be integers such that $\hat{b} = p/q$ and $\lambda = (p' + \mathrm{i}p'')/q$, then we have that $R_{\mathsf{goal}}, R'_{\mathsf{goal}} \in \mathcal{R} \cap \mathbb{S}_{\mathbb{Q}}$, where

$$\mathcal{R} = \left\{ \frac{P + \mathrm{i}Q}{P' + \mathrm{i}Q'} \mid P, Q, P', Q' \in \{-M, \ldots, 0, \ldots, M\} \right\}$$

and
$$M := 2^n|p|^m(|p'|+|p''|)^n q^{m+n}.$$
Let $\epsilon = 1/(10M)^{16}$. Note that for any two distinct numbers $z, z' \in \mathcal{R}$ it holds that $|z-z'| \geq 10\epsilon$, so if we manage to produce $\hat{R}, \hat{R}' \in \mathbb{S}_{\mathbb{Q}}$ with *poly*(n) size so that $|R_{\mathsf{goal}} - \hat{R}| \leq \epsilon$ and $|R'_{\mathsf{goal}} - \hat{R}'| \leq \epsilon$, we can in fact compute R_{goal} and R'_{goal} in time $poly(n, \text{size}(\epsilon)) = poly(n)$.[7]

We first focus on how to compute $\hat{R} \in \mathbb{S}_{\mathbb{Q}}$ so that $|R_{\mathsf{goal}} - \hat{R}| \leq \epsilon$. At this point, it will be helpful to represent complex numbers on the unit circle $\mathbb{S}$ with their arguments. Let $\theta_{\mathsf{goal}} = \mathrm{Arg}(R_{\mathsf{goal}})$ and $g(\theta) := t\mathrm{e}^{\mathrm{i}\theta} + r$. Note that

$$\begin{aligned} |g(\theta)| = |g(\theta) - g(\theta_{\mathsf{goal}})| = |t||\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}\theta_{\mathsf{goal}}}| = 2|t|\big|\sin((\theta-\theta_{\mathsf{goal}})/2)\big|,\\ \mathrm{Arg}(g(\theta)) = (\theta - \theta_{\mathsf{goal}})/2 + \mathrm{Arg}(t) \mod 2\pi, \end{aligned} \tag{2.23}$$

the latter provided $\theta \neq \theta_{\mathsf{goal}}$.

We will compute in *poly*(n) time a rational $\hat{\theta}$ such that $|\hat{\theta} - \theta_{\mathsf{goal}}| \leq \epsilon/2$, yielding the desired $\hat{R}$ (via Lemma 2.8.3).

Let $\tau = 1/500$ and $\kappa = \epsilon/10^3$. We will show that a poly-time algorithm for $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ can be used to compute, for every rational θ, a positive number $\hat{g}_\theta$ in time $poly(n, \text{size}(\theta))$ such that, whenever $|\theta - a| > \kappa$ for every $a \in \arg(R_{\mathsf{goal}})$, it holds that

$$(1-\tau)|g(\theta)| \leq \hat{g}_\theta \leq (1+\tau)|g(\theta)|. \tag{2.24}$$

When $|\theta - a| \leq \kappa$ for some $a \in \arg(R_{\mathsf{goal}})$ there is no guarantee on the value of $\hat{g}_\theta$. Similarly, we will show that a poly-time algorithm for $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ can be used to compute, for every rational θ, a positive number $\hat{a}_\theta$ in time $poly(n, \text{size}(\theta))$ such that, whenever $|\theta - a| \geq \kappa$ for every $a \in \arg(R_{\mathsf{goal}})$, it holds that

$$|\mathrm{Arg}(g(\theta)) - \hat{a}_\theta| \leq 2\rho = \pi/20. \tag{2.25}$$

Using these, we compute the desired $\hat{\theta}$ via binary search following similar techniques as in [GG17, GJ14, BGGv20], though in our case the details are slightly different because we have to work on the unit circle. For the norm, we will utilise that $|g(\theta)|$ is increasing in the interval $[\theta_{\mathsf{goal}}, \theta_{\mathsf{goal}}+\pi]$ and decreasing in the interval

[7]We give briefly the details for R_{goal}, the details for R'_{goal} are similar. For $r \in \mathbb{N}$ let $\mathbb{Q}_r$ denote the set of rationals with denominator between 1 and r. Since $R_{\mathsf{goal}} \in \mathcal{R} \cap \mathbb{S}_{\mathbb{Q}}$ and $\hat{R} \in \mathbb{S}_{\mathbb{Q}}$ we have that there exist $\alpha, \beta \in \mathbb{Q}_{2M^2}$ and $\hat{\alpha}, \hat{\beta} \in \mathbb{Q}$ such that $R_{\mathsf{goal}} = \alpha + \mathrm{i}\beta$ and $\hat{R} = \hat{\alpha} + \mathrm{i}\hat{\beta}$. From $|R_{\mathsf{goal}} - \hat{R}| \leq \epsilon$, we have $|\alpha - \hat{\alpha}|, |\beta - \hat{\beta}| \leq \epsilon$. By Lemma 2.8.4 (applied to $\hat{\alpha}, \hat{\beta}$ and $K = 2M^2$), in poly(n) time, we can compute rationals $\alpha', \beta' \in \mathbb{Q}_{2M^2}$ such that $|\hat{\alpha} - \alpha'|, |\hat{\beta} - \beta'| \leq 1/(8M^4)$ and hence $|\alpha - \alpha'|, |\beta - \beta'| \leq \epsilon + 1/(8M^4) \leq 1/(4M^4)$. Now, for distinct $\gamma, \delta \in \mathbb{Q}_{2M^2}$ we have that $|\gamma - \delta| \geq 1/(2M^2)$, so it must be that $\alpha = \alpha'$ and $\beta = \beta'$, completing the computation of R_{goal}.

$[\theta_{\mathsf{goal}} - \pi, \theta_{\mathsf{goal}}]$, whereas for the argument we will utilise that $\mathrm{Arg}(g(\theta))$ changes abruptly around θ_{goal} (roughly by π). In particular, we proceed as follows.

Algorithm for $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ **(Step 1):** We first find an interval of length $< 2\pi/3$ with rational endpoints containing θ_{goal} in *poly*(n) time. For $j = 0, \ldots, 18$ let $\theta_j = j/3$, $g_j = |g(\theta_j)|$ and $\hat{g}_j = \hat{g}_{\theta_j}$; note that the $\hat{g}_j$'s can be computed in *poly*(n) time. For convenience, extend these definitions by setting $\theta_{19h+j} = \theta_j + 2h\pi$, $g_{19h+j} = g_j$ and $\hat{g}_{19h+j} = \hat{g}_j$ for every integer h and $j = 0, \ldots, 18$. Note that for all j we have that $1/3 \geq |\theta_{j+1} - \theta_j| \geq 1/4 > \pi/15$.

Consider an index $j \in \{0, \ldots, 18\}$ such that $\arg(R_{\mathsf{goal}})$ does not intersect with the intervals $[\theta_j - \kappa, \theta_{j+1} + \kappa]$ and $[\theta_j - \pi, \theta_{j+1} - \pi]$. Then, we have that

$$(1-\tau)g_j \leq \hat{g}_j \leq (1+\tau)g_j, \quad (1-\tau)g_{j+1} \leq \hat{g}_{j+1} \leq (1+\tau)g_{j+1}. \tag{2.26}$$

We claim that $g_{j+1} - g_j$ has the same sign as $\hat{g}_{j+1} - \hat{g}_j$. To see this, assume without loss of generality $g_{j+1} - g_j > 0$, the other possibility follows in a similar way. Observe that we must have $\theta_j, \theta_{j+1} \in (\theta_{\mathsf{goal}}, \theta_{\mathsf{goal}} + \pi)$, as $\theta \mapsto \sin(\theta/2 - \theta_{\mathsf{goal}}/2)$ is increasing on $(\theta_{\mathsf{goal}}, \theta_{\mathsf{goal}} + \pi)$ and so

$$\begin{aligned} g_{j+1} - g_j = |g(\theta_{j+1})| - |g(\theta_j)| &\geq 2|t| \min_{\phi \in [0, \pi/2 - \pi/30)]} [\sin(\phi + \pi/30) - \sin\phi] \\ &\geq 2|t| \big[\sin(\pi/2) - \sin(\pi/2 - \pi/30)\big] \geq |t|/100. \end{aligned} \tag{2.27}$$

On the other hand, if $\hat{g}_{j+1} - \hat{g}_j < 0$, from (2.26) we have $(1-\tau)g_{j+1} - (1+\tau)g_j < 0$. This gives $g_{j+1} - g_j \leq \tau(g_{j+1} + g_j) \leq 2\tau|t|$, a contradiction to the above.

Let j^* be such that $\theta_{\mathsf{goal}} \in [\theta_{j^*}, \theta_{j^*+1})$. From (2.23), the sequence g_j is decreasing till j^* and increasing after $j^* + 1$. From the claim above, the sequence $\hat{g}_j$ must therefore be decreasing for indices j in $[j^* - 8, j^* - 1]$ and increasing for indices $[j^* + 2, j^* + 9]$. Therefore, from the values of $\hat{g}_j$'s we can find $\hat{j}$ so that $\theta_{\mathsf{goal}} \in [\theta_{\hat{j}-3}, \theta_{\hat{j}+3}]$. By enlarging slightly the interval $[\theta_{\hat{j}-3}, \theta_{\hat{j}+3}]$, we obtain the desired interval of length $< 2\pi/3$ with rational endpoints.

Algorithm for $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ **(Step 2):** Given an interval $[\theta_1, \theta_2]$ with rational endpoints containing θ_{goal} with $|\theta_1 - \theta_2| = \ell$ and $\ell \in (100\kappa, 2\pi/3)$, we show how to find in *poly*$(n, \mathrm{size}(\theta_1, \theta_2))$ time an interval with rational endpoints that is a factor of $1/2$ smaller in length and also contains θ_{goal}. The analysis will be similar to step 1.

For $j = 0, \ldots, 19$ define $\phi_j = \theta_1 + (\theta_2 - \theta_1)j/19$ and let $g_j = |g(\phi_j)|$ and $\hat{g}_j = \hat{g}_{\phi_j}$. Since $\theta_{\mathsf{goal}} \in [\theta_1, \theta_2]$ and $|\theta_1 - \theta_2| = \ell$, for any $\theta \in [\theta_1, \theta_2]$ we have $|g(\theta)| \leq 2|t|\sin(\ell/2) \leq \ell|t|$. In particular we have $g_j \leq \ell|t|$ for all j.

Moreover, for an index j such that $\theta_{\mathsf{goal}} \notin [\phi_j, \phi_{j+1}]$ we claim that $g_{j+1} - g_j$ has the same sign as $\hat{g}_{j+1} - \hat{g}_j$. To prove the claim assume $g_{j+1} - g_j \geq 0$, so $\phi_{j+1}, \phi_j \geq \theta_{\mathsf{goal}}$, the other possibility follows in a similar way. The derivative of

$|g(\theta)|$ in the interval $[\theta_{\mathsf{goal}}, \theta_{\mathsf{goal}} + \ell]$ is bounded below by $|t|\cos(\ell/2) \geq |t|/2$, so by the mean value theorem we have that

$$g_{j+1} - g_j \geq \frac{|t|}{2}(\phi_{j+1} - \phi_j) \geq |t|\ell/50.$$

On the other hand if $\hat{g}_{j+1} - \hat{g}_j < 0$ then as before we have $(1-\tau)g_{j+1} - (1+\tau)g_j < 0$, which implies $g_{j+1} - g_j \leq \tau(g_{j+1} + g_j) \leq 2\tau\ell|t|$, a contradiction to above. This proves the claim.

Using the claim we can conclude just as we did in step 1 and find an index $\hat{j}$ so that $\theta_{\mathsf{goal}} \in [\phi_{\hat{j}-3}, \phi_{\hat{j}+3}]$, giving the desired interval.

Algorithm for $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$**:** Given a rational endpoint θ_1 and a rational length $\ell \in (100\kappa, \frac{63}{10}]$ such that θ_{goal} lies in the interval $[\theta_1, \theta_2]$ for some $\theta_2 \leq \theta_1 + \ell$, we show how to find in $poly(n, \mathrm{size}(\theta_1, \ell))$ time a rational endpoint θ_1' and a rational length ℓ' such that $\ell' \leq \ell/4$ and $\theta_{\mathsf{goal}} \in [\theta_1', \theta_2']$ for some $\theta_2' \leq \theta_1' + \ell'$.

For $j = 0, \ldots, 25$ define $\phi_j = \theta_1 + \ell j/26$ and let $a_j = \mathrm{Arg}(g(\phi_j))$, $\hat{a}_j = \hat{a}_{\phi_j}$. For convenience, extend these definitions by setting $\phi_{26h+j} = \phi_j$, $a_{26h+j} = a_j$ and $\hat{a}_{26h+j} = \hat{a}_j$ for every integer h and $j = 0, \ldots, 25$. For indices j, j', let

$$D_{j,j'} = \min\{|a_{j'} - a_j|, 2\pi - |a_{j'} - a_j|\} \text{ and } \widehat{D}_{j,j'} = \min\{|\hat{a}_{j'} - \hat{a}_j|, 2\pi - |\hat{a}_{j'} - \hat{a}_j|\}.$$

Consider an index j such that $\theta_{\mathsf{goal}} \notin [\phi_j - \kappa, \phi_{j+1} + \kappa]$. Then, we have that $D_{j,j+1} = |\phi_{j+1} - \phi_j|/2 \leq \pi/10$ and hence $\widehat{D}_{j,j+1} \leq \pi/5$. On the other hand, for an index j such that $\theta_{\mathsf{goal}} \in [\phi_j, \phi_{j+1}]$ we have that $D_{j-1,j+1} = \pi - |\phi_{j+1} - \phi_{j-1}|/2 \geq 4\pi/5$ and similarly $D_{j,j+2} \geq 4\pi/5$. Therefore, at least one of $\widehat{D}_{j-1,j+1} \geq 3\pi/5$, $\widehat{D}_{j,j+2} \geq 3\pi/5$ must hold. Therefore, using the $\hat{a}_j$'s, we can find an index $\hat{j}$ so that $\theta_{\mathsf{goal}} \in [\phi_{\hat{j}-2}, \phi_{\hat{j}+2}]$, giving the desired interval.

By repeating the above, we conclude that, using a poly-time algorithm for either the problem $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ or $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$, we can compute in $poly(n)$ time a rational $\hat{\theta}$ such that $|\hat{\theta} - \theta| \leq 400\kappa \leq \epsilon/2$, yielding the desired $\hat{R}$ (via Lemma 2.8.3). We thus focus on proving that for a rational θ we can obtain in time $poly(n, \mathrm{size}(\theta))$ values $\hat{g}_\theta, \hat{a}_\theta$ satisfying (2.24) and (2.25), respectively.

Let $\epsilon_2 = \kappa\epsilon/10^5$, $\epsilon_1 := \epsilon_2/\big(2^{4n}(2\hat{b})^{2m}\big)$, $\epsilon_0 = \epsilon_1/(k4^k)$. By Lemmas 2.7.1 and 2.8.3, for a rational number ϕ, we can construct in time $poly(n, \mathrm{size}(\phi))$ a rooted tree T_ϕ in $\mathcal{T}_\Delta$ with root x_ϕ that has degree 1 and implements a field λ_ϕ such that $|\lambda_\phi - \mathrm{e}^{\mathrm{i}\phi}| \leq \epsilon_0$. For convenience, let

$$Q^{\pm}_\phi := Z_{T_\phi, \pm x_\phi}(\lambda, b) \text{ and note that } \left|\frac{Q^+_\phi}{Q^-_\phi} - \mathrm{e}^{\mathrm{i}\phi}\right| \leq \epsilon_0. \tag{2.28}$$

Let T_θ, T_0 be the trees obtained for $\phi = \theta, 0$ and note that that T_θ, T_0 implement the vertex activities $\mathrm{e}^{\mathrm{i}\theta}, 1$ respectively (with precision ϵ_0).

Recall that P_k is the path with k vertices and endpoints u_k, v_k. We denote by V_{P_k} the set of its vertices. Let P_{k,T_0} be the tree obtained from P_k by attaching $k-2$ disjoint copies of the graph T_0 to the internal vertices of the path, i.e., for $i = 1, \dots, k-2$, identify the root x_0 of the i-th copy of T_0 with the i-th internal vertex of the path. For convenience, let

$$A_{\pm\pm} := Z_{P_{k,T_0}, \pm u_k, \pm v_k}(\lambda, b). \tag{2.29}$$

Recall that $H = (V_H, E_H)$ denotes the graph obtained by subdividing edge e of G. Let $H_\theta \in \mathcal{G}_\Delta$ be the graph obtained from H by replacing every edge $\{x, y\}$ of H by a distinct copy of P_{k,T_0} (identifying x with u_k and y with v_k) and attaching the tree T_θ on the vertex s of H (identifying s with the root x_θ). Effectively, the construction of H_θ is so that the Ising model on H_θ with edge activities equal to b and vertex activities equal to λ corresponds to an Ising model on H with edge activities equal to $\hat{b}$, and vertex activities equal to λ apart from that of vertex s which is set to $\mathrm{e}^{\mathrm{i}\theta}$. In this latter model, the contribution to the partition function from configurations where s is set to + is given by t and the contribution to the partition function from configurations where s is set to - is given by r, where t, r are as in (2.22). Based on this, we will soon show that

$$\Big| \frac{Z_{H_\theta}(\lambda, b)}{Q_\theta^{-}(A_{++})^{m+1}} - g(\theta) \Big| \leq \epsilon_2. \tag{2.30}$$

From (2.30), we obtain the desired approximations $\hat{g}_\theta, \hat{a}_\theta$ that satisfy (2.24), (2.25) respectively, as follows. First, observe that $|g(\theta)| \geq |t|\kappa/2 \geq 10\epsilon_2/\tau$ since $|\theta - a| \geq \kappa$ for every $a \in \arg(R_{\mathsf{goal}})$. Second, T_θ and P_{k,T_0} are trees of size $poly(n, \mathrm{size}(\theta))$, so we can compute Q_θ^{-} and A_{++} in time $poly(n, \mathrm{size}(\theta))$. Using a poly-time algorithm for $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$, we can compute $\hat{Z}_\theta$ in time $poly(n, \mathrm{size}(\theta))$ which is within a factor of $1 \pm \tau$ from $|Z_{H_\theta}(\lambda, b)|$, thus yielding $\hat{g}_\theta = \frac{\hat{Z}_\theta}{|Q_\theta^{-}|\,|A_{++}|^{m+1}}$ that satisfies (2.24). Similarly, using a poly-time algorithm for $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$, we can compute $\hat{A}_\theta$ in time $poly(n, \mathrm{size}(\theta))$ which is within distance ρ from $\mathrm{Arg}(Z_{H_\theta}(\lambda, b))$. Noting that the argument α of $\frac{Z_{H_\theta}(\lambda,b)}{Q_\theta^{-}(A_{++})^{m+1}} - g(\theta)$ satisfies $\sin(\alpha) \leq \epsilon_2/g(\theta)$, from which it follows that $\alpha \leq \rho$. Hence $\hat{a}_\theta = \hat{A}_\theta - \mathrm{Arg}(Q_\theta^{-}) - (m+1)\mathrm{Arg}(A_{++}) \pmod{2\pi}$ satisfies (2.25).

It remains to prove (2.30). We first claim that

$$\Big| \frac{A_{\pm\pm}}{(Q_0^{-})^{k-2}} - Z_{P_k, \pm u_k, \pm v_k}(1, b) \Big| \leq \frac{\epsilon_1}{4} Z_{P_k, \pm u_k, \pm v_k}(1, b). \tag{2.31}$$

Indeed, for a fixed $\sigma : V_{P_k} \to \{+, -\}$, the aggregate contribution to $Z_{P_{k,T_0}}(1, b)$ from configurations on P_{k,T_0} that agree with σ on V_{P_k} is $(Q_0^{+})^{n_+(\sigma)}(Q_0^{-})^{n_-(\sigma)} w_{P_k,\sigma}(1, b)$ where $n_\pm(\sigma)$ is the number of internal vertices in P_k that have spin $\pm$ under

σ: so (2.31) follows from aggregating over the relevant σ and observing that[8] $\big|\frac{(Q_0^{+})^j}{(Q_0^{-})^j}-1\big| \leq k\epsilon_0$ for all $j=0,\dots,k$. From (2.19) and (2.31), it follows that $A_{\pm,\pm}\neq 0$ and

$$\Big|\frac{A_{-+}}{A_{++}}-\hat{b}\Big|\leq \epsilon_1,\quad \Big|\frac{A_{--}}{A_{++}}-1\Big|\leq \epsilon_1. \tag{2.32}$$

Now, for $\sigma: V_H\to\{+,-\}$ with $\sigma(s)=+$, let W_σ^{+} be the aggregate weight of configurations on H_θ that agree with σ on $V(H)$. Define analogously W_σ^{-}. Then, we have that

$$W_\sigma^{\pm}=Q_\theta^{\pm}(A_{++})^{m_{++}(\sigma)}(A_{+-})^{m_{+-}(\sigma)}(A_{--})^{m_{--}(\sigma)},$$

where $m_{++}(\sigma), m_{+-}(\sigma), m_{--}(\sigma)$ denote the number edges of E_H whose endpoints are assigned $++, +-, --$, respectively. Since the total number of edges in E_H is $m+1$, we obtain

$$\Big|\frac{W_\sigma^{+}}{Q_\theta^{-}(A_{++})^{m+1}}-\mathrm{e}^{\mathrm{i}\theta}\,w_{H,\sigma}(\lambda,\hat{b})\Big|\leq \epsilon_2/10^n,\quad \Big|\frac{W_\sigma^{-}}{Q_\theta^{-}(A_{++})^{m+1}}-w_{H,\sigma}(\lambda,\hat{b})\Big|\leq \epsilon_2/10^n. \tag{2.33}$$

Observe also that the quantities t, r, as defined in (2.22), are such that

$$t=\sum_{\sigma:V_H\to\{+,-\};\sigma(s)=+} w_{H,\sigma}(\lambda,\hat{b}) \text{ and } r=\sum_{\sigma:V_H\to\{+,-\};\sigma(s)=-} w_{H,\sigma}(\lambda,\hat{b}),$$

so summing (2.33) over all σ gives (2.30). This concludes the proof of (2.30) and hence completes the computation of R_{goal} in $poly(n)$ time.

The computation of R'_{goal} is completely analogous, once we establish an analogue of (2.30). In particular, let H' be the graph obtained from H by removing vertex s and adding the vertices u', v', s' and the edges $\{u,u'\},\{u',s'\},\{s',v'\},\{v',v\}$; note that H' is obtained from G by replacing the edge e by a path with three vertices. We construct H'_θ from H' as above, with a minor twist: we replace every edge $\{x,y\}$ of H' with a distinct copy of P_{k,T_0} (identifying x with u_k and y with v_k), we attach the rooted tree T_θ on the vertex s' of H' (identifying s' with the root x_θ), and we attach two distinct copies of the rooted tree T_π on the vertices u', v' of H' (identifying u', v' with the corresponding roots x_π in the two copies of T_π). Note the use of the tree T_π in the construction of H' which, analogously[9] to (2.28), implements the field $\mathrm{e}^{\mathrm{i}\pi}=-1$ (with precision ϵ_0). Effectively, the construction of H'_θ is so that the Ising model on H'_θ with edge activities equal to b and vertex activities equal to λ corresponds to an Ising model on H' with edge activities equal to $\hat{b}$, and vertex activities equal to λ apart from those

[8] Here, and in the follow-up estimates, we use that for complex numbers $c_1,\dots,c_i$ and $d_1,\dots,d_i$ it holds that $\big|\prod_{j=1}^i c_j-\prod_{j=1}^i d_j\big|\leq \sum_{j=1}^i |c_j-d_j|\prod_{j'=1}^{j-1}|c_j|\prod_{j'=j+1}^{i}|d_j|$.

[9] Even though π is irrational, it holds that $\mathrm{e}^{\mathrm{i}\pi}=-1$ and we can therefore construct T_π satisfying (2.28) for $\phi=\pi$ using Lemma 2.7.1.

of u', s', v' which are set to $-1, \mathrm{e}^{\mathrm{i}\theta}, -1$, respectively. In this latter model, the contribution to the partition function from configurations where s' is set to + is given by $t' = (\hat{b}^2 - 1)^2 z_{++}$ and the contribution to the partition function from configurations where s' is set to - is given by $r' = (\hat{b}^2 - 1)^2 z_{--}$. Based on this, we obtain similarly to above, the following analogue of (2.30):

$$\Big|\frac{Z_{H'_\theta}(\lambda, b)}{Q^-_\theta (Q^-_\pi)^2 (A_{++})^{m+2}} - \big(t'\mathrm{e}^{\mathrm{i}\theta} + r'\big)\Big| \leq \epsilon_2. \tag{2.34}$$

Having (2.34) at hand, the computation of R'_{goal} can be carried out using exactly the same procedure as for R_{goal}. This finishes the proof of Lemma 2.8.5. □

2.8.3. Proof of the main theorem

We are now ready to finish the proof of Theorem 2.1.1, which we restate here for convenience.

Theorem 2.1.1. *Let $\Delta \geq 3$ be an integer and let $K = 1.001$ and $\rho = \pi/40$.*

(a) Let $b \in \big(0, \frac{\Delta-2}{\Delta}\big]$ be a rational, and $\lambda \in \mathbb{S}_{\mathbb{Q}}$ such that $\lambda \neq \pm 1$. Then the problems $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard.

(b) Let $b \in \big(\frac{\Delta-2}{\Delta}, 1\big)$ be a rational. Then the collection of complex numbers $\lambda \in \mathbb{S}_{\mathbb{Q}}$ for which $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard is dense in the arc $\mathbb{S} \setminus I(\theta_b)$.

Proof of Theorem 2.1.1. Let $b \in (0, 1)$ be a rational number and let $\lambda \in \mathbb{S}_{\mathbb{Q}}(\Delta - 1, b)$. By Theorem 2.2.4 it suffices to show that $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ and $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$ are #P-hard. To prove the #P-hardness for these problems, we will show that, assuming a poly-time algorithm for either $\#\mathsf{IsingNorm}(\lambda, b, \Delta, K)$ or $\#\mathsf{IsingArg}(\lambda, b, \Delta, \rho)$, on input of a graph $G \in \mathcal{G}_3$ we can compute $Z_G(\lambda, \hat{b})$ in poly-time, which is #P-hard by [KC16, Theorem 1.1]. In fact, it suffices to compute in poly-time, for an arbitrary edge e of G, the ratio $\frac{Z_G(\lambda,\hat{b})}{Z_{G\setminus e}(\lambda,\hat{b})}$ since then we can compute $Z_G(\lambda, \hat{b})$ using a telescoping product over the edges of the graph G.

So fix an arbitrary edge $e = \{u, v\}$ of G and let $z_{\pm\pm} := Z_{G,\pm u,\pm v}(\lambda, \hat{b})$. The ratio $r^* := \frac{Z_G(\lambda,\hat{b})}{Z_{G\setminus e}(\lambda,\hat{b})}$ is well-defined since, by the choice of $\hat{b}$, we have $Z_{G\setminus e}(\lambda, \hat{b}) \neq 0$ (cf. (2.20) and (2.21)). Moreover, we can express r^* using the $z_{\pm\pm}$'s as follows:

$$r^* = \frac{z_{++} + z_{--} + \hat{b}(z_{+-} + z_{-+})}{z_{++} + z_{--} + z_{+-} + z_{-+}}.$$

We will compute r^* using Lemma 2.8.5. Namely, by Lemma 2.8.5, we can compute in poly-time the value of the ratio

$$r = R_{G,e} = \frac{A^2 z_{++} + AB(z_{+-} + z_{-+}) + B^2 z_{--}}{A^2 z_{--} + AC(z_{+-} + z_{-+}) + C^2 z_{++}}, \quad \text{where} \quad \begin{array}{l} A := \hat{b} \\ B := 1 \\ C := 1 \end{array}. \tag{2.35}$$

Let G' be the graph obtained from $G \setminus e$ by adding two new vertices u', v' and adding the edges $\{u, u'\}, \{u', v'\}, \{v', v\}$. We next apply Lemma 2.8.5 to the graph G' with the edge $e' = \{u', v'\}$. We first express $Z_{G'\setminus e', \pm u', \pm v'}(\lambda, \hat{b})$ in terms of the $z_{\pm\pm}$'s. We have

$$\begin{aligned} Z_{G'\setminus e', +u', +v'}(\lambda, \hat{b}) &= \lambda^2\big(z_{++} + \hat{b}(z_{+-} + z_{-+}) + \hat{b}^2 z_{--}\big), \\ Z_{G'\setminus e', +u', -v'}(\lambda, \hat{b}) &= \lambda\big(\hat{b} z_{++} + z_{+-} + \hat{b}^2 z_{-+} + \hat{b} z_{--}\big), \\ Z_{G'\setminus e', -u', +v'}(\lambda, \hat{b}) &= \lambda\big(\hat{b} z_{++} + \hat{b}^2 z_{+-} + z_{-+} + \hat{b} z_{--}\big), \\ Z_{G'\setminus e', -u', -v'}(\lambda, \hat{b}) &= \hat{b}^2 z_{++} + \hat{b}(z_{+-} + z_{-+}) + z_{--}. \end{aligned}$$

Then, by Lemma 2.8.5, we can compute in poly-time the value of the ratio

$$r' = R_{G',e'} = \frac{(A')^2 z_{++} + A'B'(z_{+-} + z_{-+}) + (B')^2 z_{--}}{(A')^2 z_{--} + A'C'(z_{+-} + z_{-+}) + (C')^2 z_{++}}, \quad \text{where} \quad \begin{array}{l} A' := \hat{b}(\lambda + 1) \\ B' := 1 + \hat{b}^2\lambda \\ C' := \hat{b}^2 + \lambda \end{array}. \tag{2.36}$$

We are now in a position to complete the computation of r^*. We first show how to decide in poly-time whether $z_{++} = 0$. We claim that

$$z_{++} = 0 \Longleftrightarrow r = B/C \text{ and } r' = B'/C'. \tag{2.37}$$

Indeed, if $z_{++} = 0$, then $z_{--} = 0$ from Lemma 2.8.2, and therefore from (2.35), (2.36) we have that $r = B/C$ and $r' = B'/C'$. Conversely, using that $A^2 \neq BC$ and $(A')^2 \neq B'C'$, we have that

$$r = B/C \Longrightarrow C z_{++} = B z_{--}, \qquad r' = B'/C' \Longrightarrow C' z_{++} = B' z_{--},$$

which together imply that $z_{++} = 0$.

Using (2.37) we can decide in poly-time whether $z_{++} = 0$. If so, by Lemma 2.8.2, we have $z_{--} = 0$ and hence $r^* = \hat{b}$. So, assume $z_{++} \neq 0$, and hence $z_{--} \neq 0$ in what follows. We claim that

$$z_{+-} + z_{-+} = 0 \Longleftrightarrow r = \frac{A^2 z_{++} + B^2 z_{--}}{A^2 z_{--} + C^2 z_{++}}, r' = \frac{(A')^2 z_{++} + (B')^2 z_{--}}{(A')^2 z_{--} + (C')^2 z_{++}}. \tag{2.38}$$

The forward direction is again trivial. For the backward direction, we have

$$r = \frac{A^2 z_{++} + B^2 z_{--}}{A^2 z_{--} + C^2 z_{++}} \Longrightarrow C z_{++} = B z_{--} \text{ or } z_{+-} + z_{-+} = 0,$$
$$r' = \frac{(A')^2 z_{++} + (B')^2 z_{--}}{(A')^2 z_{--} + (C')^2 z_{++}} \Longrightarrow C' z_{++} = B' z_{--} \text{ or } z_{+-} + z_{-+} = 0.$$

Since $z_{++}, z_{--} \neq 0$, we therefore obtain that $z_{+-} + z_{-+} = 0$, proving (2.38).

Note that we can decide the right-hand of (2.38) in poly-time using the value of the ratio $r'' = z_{--}/z_{++}$ from the second part of Lemma 2.8.5. If it turns out that $z_{+-} + z_{-+} = 0$, then $r^* = 1$ and we are done. Otherwise, we can use the values of r and r'' to compute the ratios $\frac{z_{++}}{z_{+-}+z_{-+}}, \frac{z_{--}}{z_{+-}+z_{-+}}$, which we can then use to compute r^*.

This completes the computation of the ratio r^*, and therefore the proof of Theorem 2.1.1. □

2.9. Equivalence for $\lambda = -1$ with Approximately Counting Perfect Matchings

In this section, we show that for $\lambda = -1$, the problem of approximating the partition of the ferromagnetic Ising model on graphs of maximum degree Δ is equivalent to the problem #PerfectMatchings, the problem of approximately counting perfect matchings on general graphs. The proof follows the technique in [GJ08], where the case of negative b but $\lambda = 1$ was considered; here however, we need to rework the relevant ingredients. The main such ingredient is the following "high-temperature" expansion formula for $\lambda = -1$.

Lemma 2.9.1. *Let $\lambda = -1$ and $b \neq -1$ be an arbitrary number. Then, for any graph $G = (V, E)$,*

$$Z_G(\lambda, b) = (-2)^{|V|}\Big(\frac{1+b}{2}\Big)^{|E|} \sum_{S \subseteq E;\, S\ odd} \Big(\frac{1-b}{1+b}\Big)^{|S|},$$

where the sum is over $S \subseteq E$ such that every vertex $v \in V$ has odd degree in the subgraph (V, S).

Proof. Let $G = (V, E)$ be a graph. For a set $S \subseteq E$ and a vertex $v \in V$, we let $d_v(S)$ denote the degree of v in the subgraph (V, S).

For the purposes of this proof, it will be convenient to view configurations of the Ising model on G as vectors in $\{\pm1\}^V$. Now, for a configuration $\sigma \in \{\pm1\}^V$ we use the notation $n_+(\sigma)$ to denote the number of vertices with spin $+1$. Observe that $n_+(\sigma) = \frac{1}{2}\big(|V| + \sum_{v \in V} \sigma_v\big)$ and that for an edge $e = (u, v)$, we have $b^{\mathbf{1}\{\sigma_u \neq \sigma_v\}} = \frac{1+b}{2}\big(1 + \frac{1-b}{1+b}\sigma_u\sigma_v\big) = \rho\big(1 + \nu\sigma_u\sigma_v\big)$, where for convenience we set $\rho := \frac{1+b}{2}$ and $\nu := \frac{1-b}{1+b}$. So, using that $\mathrm{i}^2 = -1$,

$$\begin{aligned} Z_G(\lambda, b) &= \rho^{|E|} \sum_{\sigma \in \{\pm 1\}^V} \lambda^{n_+(\sigma)} \prod_{e=(u,v) \in E} (1 + \nu\sigma_u\sigma_v) \\ &= \rho^{|E|} \sum_{\sigma \to \{\pm 1\}^V} \lambda^{n_+(\sigma)} \sum_{S \subseteq E} \nu^{|S|} \prod_{v \in V} (\sigma_v)^{d_v(S)} \\ &= \mathrm{i}^{|V|} \rho^{|E|} \sum_{S \subseteq E} \nu^{|S|} \sum_{\sigma \in \{\pm 1\}^V} \prod_{v \in V} \mathrm{i}^{\sigma_v} (\sigma_v)^{d_v(S)}. \end{aligned}$$

The latter sum is equal to $\prod_{v \in V} \sum_{\sigma_v \in \{\pm 1\}} \mathrm{i}^{\sigma_v} (\sigma_v)^{d_v(S)}$, which equals 0 if $d_v(S)$ is even, and 2i otherwise. Plugging this in the expression above yields the lemma. □

Now we are ready to show the main theorem for this section. For counting problems A, B we use the notion of AP-reductions, see [DGGJ04]. Roughly, we have that $A \leq_{\mathsf{AP}} B$ if an FPRAS for B can be converted to an FPRAS for A, and $A \equiv_{\mathsf{AP}} B$ if both $A \leq_{\mathsf{AP}} B$ and $B \leq_{\mathsf{AP}} A$ hold.

Theorem 2.9.2. *Let $\lambda = -1$ and $b \in (0, 1)$ be a rational. Then, for any connected graph G, we have $Z_G(\lambda, b) > 0$ if G has an even number of vertices and $Z_G(\lambda, b) = 0$, otherwise.*

Moreover, for all integers $\Delta \geq 3$, we have that #IsingNorm$(\lambda, b, \Delta) \equiv_{\mathsf{AP}}$ #PerfectMatchings.

Proof. The statement about the sign of $Z_G(\lambda, b)$ follows from Lemma 2.9.1, and the fact that every connected graph with an even number of vertices has a spanning subgraph where every vertex has odd degree. We thus focus on proving the AP-equivalence.

#PerfectMatchings $\leq_{\mathsf{AP}}$ #IsingNorm(λ, b, Δ). It is well-known that the problem of approximating the number of perfect matchings on general graphs is AP-equivalent to the same problem on graphs of maximum degree 3, see, e.g., [GLLZ19, Lemma 28]. So, let $G = (V_G, E_G)$ be a graph of maximum degree 3, with $n = |V_G|$ and $m = |E_G|$, and let $\mathcal{M}$ be the set of perfect matchings of G. Since we can check whether a graph has a perfect matching in polynomial time, we may further assume that $|\mathcal{M}| > 0$ and in particular that n is even. Let $\epsilon \in (0, 1)$ be the desired relative error that we want to approximate $|\mathcal{M}|$.

Analogously to (2.18) and (2.19), for $k = 1 + 2\lceil \frac{m^2 + \ln(1/\epsilon)}{-\ln(1-b)} \rceil$, let $P_k = (V_k, E_k)$ be the path with k vertices whose endpoints are labeled u_k, v_k and $P_k^* = (V_k^*, E_k^*)$ be the graph obtained from P_k by attaching a vertex z_i to the i-th internal vertex

w_i of P_k, for $i = 1, \dots, k-2$. Let $A_{\pm,\pm} := Z_{P_k^*,\pm u_k,\pm v_k}(\lambda, b)$. Then, it is not hard to see that[10]

$$\begin{bmatrix} A_{++} & -A_{+-} \\ -A_{-+} & A_{--} \end{bmatrix} = (1-b)^{k-2} \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}^{k-1}$$

and so

$$A_{++} = A_{--} = \tfrac{1}{2}\big((1+b)^{k-1} + (1-b)^{k-1}\big)(1-b)^{k-2}, \text{ and}$$
$$A_{+-} = A_{-+} = \tfrac{1}{2}\big((1-b)^{k-1} - (1+b)^{k-1}\big)(1-b)^{k-2}.$$

We next set

$$b_k := -\frac{A_{+-}}{A_{++}} = -\frac{A_{-+}}{A_{--}}, \quad \text{and observe that} \quad 1-(1-b)^{k-1} < b_k < 1. \tag{2.39}$$

Let $H = (V_H, E_H)$ be an instance of $\#\mathsf{IsingNorm}(\lambda, b, \Delta)$ obtained from G by replacing every edge $e = (u, v)$ of G with a distinct copy of P_k^*, identifying the endpoints u, v with u_k, v_k, respectively. Then, we claim that

$$Z_H(\lambda, b) = (A_{++})^m Z_G(\lambda, b_k). \tag{2.40}$$

Indeed, for a configuration $\sigma : V_G \to \{+,-\}$, let $\Omega_{H,\sigma} = \{\sigma' : V_H \to \{+,-\} \mid \sigma'_{V_G} = \sigma\}$ be the configurations on H which agree with σ on V_G, and $Z_{H,\sigma}(\lambda, b)$ be the contribution to $Z_H(\lambda, b)$ from configurations in $\Omega_{H,\sigma}$. Then, we have

$$Z_{H,\sigma}(\lambda, b) = \lambda^{|n_+(\sigma)|} \prod_{e=(u,v)\in E_G} (-1)^{\mathbf{1}_{\sigma_u \neq \sigma_v}} A_{\sigma_u \sigma_v} = (A_{++})^m \lambda^{|n_+(\sigma)|} b_k^{\delta(\sigma)},$$

proving (2.40). Note, from Lemma 2.9.1 we have that

$$Z_G(\lambda, b_k) = 2^n \Big(\frac{1+b_k}{2}\Big)^m \sum_{S\subseteq E;\, S \text{ odd}} \Big(\frac{1-b_k}{1+b_k}\Big)^{|S|}. \tag{2.41}$$

Perfect matchings in G are in 1-1 correspondence with odd sets $S \subseteq E$ with $|S| = n/2$. Moreover, for any other odd set $S \subseteq E$ we have $|S| > n/2 + 1$, and hence, using also (2.40), we obtain

$$\left| \frac{Z_H(\lambda, b)}{(A_{++})^m 2^n \big(\frac{1+b_k}{2}\big)^m \big(\frac{1-b_k}{1+b_k}\big)^{n/2}} - |\mathcal{M}| \right| \le 2^m \Big(\frac{1-b_k}{1+b_k}\Big) \le \epsilon |\mathcal{M}|.$$

[10]Here, the key observation is that for a configuration $\tau : V_k \to \{+,-\}$, the aggregate weight of configurations $\sigma : V_k^* \to \{+,-\}$ with $\sigma_{V_k} = \tau$ is $(-1)^{\mathbf{1}\{\tau_{u_k} \neq \tau_{v_k}\}}(1-b)^{k-2} w_{P_k,\tau}(1,b)$. Indeed, if $\tau(w_i) = +$ then the contribution of the edge (w_i, z_i) and the external field on z_i is $b + \lambda = b - 1$, whereas if $\tau(w_i) = -$ the contribution is $1 + b\lambda = 1 - b$. This, combined with the factor $\lambda^{n_+(\tau)}$ coming from the external fields on V_k, gives the factor $(-1)^{\mathbf{1}\{\tau_{u_k} \neq \tau_{v_k}\}}(1-b)^{k-2}$ above; the remaining contribution is just the weight of τ on P_k when the external field of all vertices on P_k is equal to 1.

Using therefore an FPRAS for $\#\mathsf{IsingNorm}(\lambda, b, \Delta)$, we can approximate $Z_H(\lambda, b)$ within relative error ϵ in time $poly(n, 1/\epsilon)$, and compute therefore $|\mathcal{M}|$ within relative error ϵ, finishing the AP-reduction.

$\#\mathsf{IsingNorm}(\lambda, b, \Delta) \leq_{\mathsf{AP}} \#\mathsf{PerfectMatchings}$. We first consider the case $\Delta = 3$. Let $G = (V, E)$ be a graph of maximum degree $\Delta = 3$ that is input to $\#\mathsf{IsingNorm}(\lambda, b, \Delta)$, and set $n = |V|$, $m = |E|$. We may assume that n is even, since otherwise we can output 0 for the partition function. By Lemma 2.9.1 we have that

$$Z_G(\lambda, b) = 2^n \Big(\frac{1+b}{2}\Big)^m \sum_{S \subseteq E;\, S\, \mathrm{odd}} \Big(\frac{1-b}{1+b}\Big)^{|S|}. \tag{2.42}$$

To formulate this in terms of perfect matchings, we construct a graph $G' = (V', E')$ as follows, resembling the construction in [Fis66]. For $v \in V$, let d_v be the degree of v in G. For a vertex $v \in V$, if $d_v = 3$, replace v with a triangle of vertices $T_v = \{v_1, v_2, v_3\}$; otherwise, keep v in G' as well and let for convenience $T_v = \{v\}$. For every edge $(u, v) \in E$, add an edge in G' between a vertex in T_u and T_w so that G' has maximum degree 3; note that edges of G that are not incident to degree-3 vertices belong to G' as well. We call internal all edges of G' whose endpoints belong to some T_v and external all other edges of G'. Note that an edge e of G maps to an external edge $\mathrm{ex}(e)$ of G' bijectively, under the natural mapping. We use $\mathrm{ex}(G')$ to denote the external edges of G'.

For $v \in V$, observe that any perfect matching in G' must contain exactly one external edge incident to a vertex in T_v if $|T_v| = 1$, and two or three edges if $|T_v| = 3$, either one internal and one external, or three external, respectively. Based on this, we have that a perfect matching M' in G' maps bijectively to an odd subset S of G, by adding an edge e of G to S iff $\mathrm{ex}(e) \in M'$. Therefore, with $\mathcal{M}'$ denoting the set of perfect matchings in G', we can rewrite (2.42) as

$$Z_G(\lambda, b) = 2^n \Big(\frac{1+b}{2}\Big)^m \sum_{M' \in \mathcal{M}'} \Big(\frac{1-b}{1+b}\Big)^{|M' \cap \mathrm{ex}(G')|}.$$

Let $n' = |V'| \leq 3n$ and $m' = |E'|$. Let p, q be positive integers with $\gcd(p, q) = 1$ such that $\frac{p}{q} = \frac{1-b}{1+b}$. Let G'' be the multigraph obtained from G' by replacing every external edge $e = (u, v)$ with p parallel edges connecting u to a new vertex w_e, q parallel edges connecting w_e to a new vertex z_e, and an edge between z_e and v; note, internal edges of G' are left intact. Let $\mathcal{M}'$ and $\mathcal{M}''$ be the set of perfect matchings of G' and G'' Then, there is a one-to-many correspondence between perfect matchings $M' \in \mathcal{M}'$ in G' and perfect matchings $M'' \in \mathcal{M}''$, where an internal edge e is matched in M' iff e is matched in M'', while an external edge $e = (u, v)$ is matched in M' iff (z_e, v) is matched in M''. Note that, for an external edge e and a perfect matching M'' of G'', if (z_e, v) belongs to

M'' then u must be matched by one of the p parallel edges connecting u to w_e, whereas if (z_e, v) does not belong to M'', w_e and z_e must be matched by one of the q parallel edges connecting u to w_e; it follows that

$$|\mathcal{M}''| = \sum_{M \in \mathcal{M}'} p^{|M \cap \mathrm{ex}(G')|} q^{m - |M \cap \mathrm{ex}(G')|}.$$

Finally, if we let G''' be the graph obtained from G'' by replacing every edge of G'' with a path of length 3, we have that the set of perfect matchings $\mathcal{M}'''$ off G''' is in 1-1 correspondence with $\mathcal{M}''$, and we see that $2^n\left(\frac{1+b}{2}\right)^m q^m |\mathcal{M}'''|$ equals $Z_G(\lambda, b)$, completing the AP-reduction for $\Delta = 3$.

To handle the case $\Delta \geq 4$, it suffices to show that $\#\mathsf{IsingNorm}(\lambda, b, \Delta) \leq_{\mathsf{AP}} \#\mathsf{IsingNorm}(\lambda, b, 3)$ since AP-reductions are transitive, see [DGGJ04]. Let $G = (V, E)$ be a graph of maximum degree Δ, and set $n = |V|$. Let $V_{\leq 3} = \{v \in V \mid d_v \leq 3\}$ be the set of vertices in G with degree ≤ 3, and $V_{>3}$ be the set of the remaining vertices.

Construct a graph $G' = (V', E')$ from G by replacing every vertex $v \in V$ with $d_v = t \geq 4$, with a path of $2t - 1$ vertices if t is odd and of $2t - 3$ vertices if t is even. We partition the vertices on the path into two sets T_v, T'_v according to their parity, so that the endpoints of the path belong to T_v; note that $|T_v| = t$ if t is odd, while $|T_v| = t - 1$ if t is even. We keep vertices $v \in V_{\leq 3}$ in G', and for such vertices, we let for convenience $T_v = \{v\}$. Then, for every edge $(u, v) \in E$, we add an edge in G' between a vertex in T_u and T_v so that, in the end, G' has maximum degree 3 and, further, for vertices $v \in V_{>3}$ with d_v even, exactly one endpoint of the path on $T_v \cup T'_v$ has degree 3 in G' (and the other has degree two). As before, we call an edge in G' internal if both of its endpoints lie within a set T_v for some $v \in V$, and external otherwise.

The key observation is that the aggregate contribution to $Z_{G'}(\lambda, b)$ from configurations on G' where, for some $v \in V$, the vertices in T_v do not get the same spin is zero.[11] For a configuration σ on G, let $\Omega_{G',\sigma}$ be the set of configurations on G' such that all vertices in T_v get the spin σ_v and let $Z_{G',\sigma}(\lambda, b)$ be their aggregate contribution to $Z_{G'}(\lambda, b)$, so that, from the observation above, we have

$$Z_{G'}(\lambda, b) = \sum_{\sigma : V \to \{\texttt{+}, \texttt{-}\}} Z_{G',\sigma}(\lambda, b).$$

For a configuration $\sigma : V \to \{\texttt{+}, \texttt{-}\}$, external edges and the external fields on $V_{\leq 3}$ contribute to $Z_{G',\sigma}(\lambda, b)$ a factor of $\lambda^{|n_{+}(\sigma) \cap V_{\leq 3}|} b^{|\delta_G(\sigma)|}$. For $v \in V_{>3}$ with $\sigma_v = \texttt{+}$, the edges in $T_v \cup T'_v$ and the external fields on $T_v \cup T'_v$ contribute to $Z_{G',\sigma}(\lambda, b)$ a factor of $-(1 - b^2)^{|T_v|}$, and a factor of $(1 - b^2)^{|T_v|}$ if $\sigma_v = \texttt{-}$. It follows that

[11] This follows by observing that for a path with two edges, the aggregate weight of configurations where the endpoints of the path have different spins is equal to 0 (using that $\lambda = -1$).

$Z_{G',\sigma}(\lambda, b) = (1-b^2)^{|T|}\lambda^{|n_+(\sigma)|}w_{G,\sigma}(\lambda, b)$ where $T = \cup_{v\in V; d_v\geq 4}|T_v|$. It follows that

$$Z_{G'}(\lambda, b) = (1-b^2)^{|T|}Z_G(\lambda, b),$$

therefore completing the AP-reduction, since by construction G' is a graph of maximum degree 3.

This finishes the proof of Theorem 2.9.2. □

Chapter 3

Zeros, chaotic ratios and the computational complexity of approximating the independence polynomial

3.1. Introduction

The remainder of the thesis will concern the independence polynomial, which, for a graph $G = (V, E)$, is defined by

$$Z_G(\lambda) = \sum_{I \subseteq V} \lambda^{|I|},$$

where the sum is taken over all *independent* subsets I of the vertex set V. Recall that I is said to be independent if no two vertices in I are connected by an edge.

Exact computation of the independence polynomial for large graphs is not feasible for most values of λ; it is a #P-hard problem[1], cf. [Rot96, Gre00, Vad01]. A question that has received significant interest is for which $\lambda \in \mathbb{C}$ there exist polynomial time algorithms that approximate $Z_G(\lambda)$ up to small multiplicative constants. See e.g. [Wei06, SS14, Bar16, PR17, BGGv20, ALG20] and the references therein.

We recall that the absence of complex zeros implies the existence of efficient algorithms for this computational problem (see also Section 1.3). More formally, on the maximal simply connected open set containing the origin on which the independence polynomial does not vanish for all graphs of a given maximum degree Δ there exists an efficient algorithm for approximating the independence polynomial [Bar16, PR17]. Let us denote this maximal 'zero-free' set by $\mathcal{U}_\Delta$. For real values of λ in the complement of the closure of $\mathcal{U}_\Delta$ approximating the partition function is computationally hard [SS05, SS14, PR19, BGGv20]. In other words, the absence/presence of complex zeros near the real axis marks a

[1]The complexity class #P may be seen as the counting version of the complexity class NP. See Section 1.2 or [Val79, Jer03, AB09] for further background.

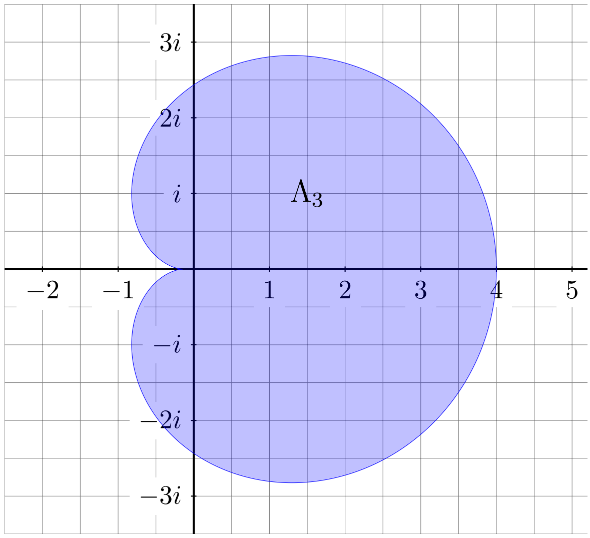

Figure 3.1: The cardioid Λ_3.

transition in the computational complexity of approximating the independence polynomial of graphs of bounded degree Δ for real values of λ. The transition point for positive λ coincides with the phase transition for the hard-core model on the Cayley tree of degree Δ.

A natural question is whether a similar phenomenon manifests itself for non-real λ. Bezaková, Galanis, Goldberg and Štefankovič [BGGv20] made an important contribution towards solving this question by showing that for any integer $\Delta \geq 3$ and non-real λ outside a certain cardioid[2], Λ_Δ, approximation of the independence polynomial for graphs of bounded degree at most Δ is computationally hard (in fact #P-hard). See Figure 3.1 for a picture of Λ_3 and Definition 3.2.5 for the definition of Λ_Δ.

Earlier it was shown by Peters and Regts [PR19] that zeros of the independence polynomial of graphs of maximum degree at most Δ accumulate on the entire boundary of Λ_Δ. In particular the 'zero-free' set $\mathcal{U}_\Delta$ is contained in the cardioid; their intersections with the real axis in fact coincide [SS05, PR19]. In the next chapter we will show that Λ_Δ does contain zeros of the independence polynomial of graphs of bounded degree Δ. This in particular indicates that the result of [BGGv20] does not fully answer the question how zeros relate to

[2]Although the domain Λ_Δ resembles a cardioid, it is formally not a cardioid. However, as discussed in Section 3.7, it plays an analogous role as the Main Cardioid of the Mandelbrot set, justifying the use of the term cardioid.

computational hardness for non-real λ.

As is done for the Ising model in the previous chapter, we will solve this question for the independence polynomial by directly relating, for any fixed integer $\Delta \geq 3$, the zeros for the family of graphs of maximum degree at most Δ to the parameters where approximating evaluations of the independence polynomial is computationally hard. Again the result is obtained by studying a natural family of rational maps associated to this family of graphs, using techniques and ideas from complex dynamics. We show that 'chaotic behaviour' of this family is equivalent to the presence of zeros, and implies computational hardness.

3.1.1. Occupation ratios

Given $\Delta \in \mathbb{Z}_{\geq 2}$, we define $\mathcal{G}_\Delta$ as the collection of finite simple rooted graphs (G, v) such that the maximum degree of G is at most Δ. For $i \in \{1, \ldots, \Delta\}$ we define $\mathcal{G}_\Delta^i = \{(G, v) \in \mathcal{G}_\Delta : \deg(v) \leq i\}$. The *occupation ratio*, or *ratio* for short, of a rooted graph (G, v) is defined by the rational function

$$R_{G,v}(\lambda) := \frac{Z_{G,v}^{\text{in}}(\lambda)}{Z_{G,v}^{\text{out}}(\lambda)},$$

where "in" means that in the definition of $Z_G(\lambda)$ the sum is taken only over independent sets I that contain the marked point v, while "out" means that the independent sets do not contain v. The ratio is a very useful tool in studying the zeros of the independence polynomial, see Lemma 3.2.1 below, and has been key in several of the aforementioned works. The ratio is also relevant from a statistical physics perspective as it is closely related to the *free energy*.

When (G, v) is a rooted Cayley tree of depth $n-1$ and down-degree $d = \Delta - 1$, the ratio satisfies

$$R_{G,v}(\lambda) = f_{\lambda,d}^n(0),$$

where

$$f_{\lambda,d}(z) := \frac{\lambda}{(1+z)^d}$$

and throughout the chapter we write f^n for the n-th iterate of the map f.

In this context it is therefore natural to consider λ as the parameter which determines the orbit of the marked point 0. This type of setting is often studied in complex dynamical systems, where one is interested in the sets where the parameter λ is *active* or *passive*. A parameter λ_0 is said to be passive if the family of rational functions $\{\lambda \mapsto f_{\lambda,d}^n(0)\}$ is normal at λ_0, i.e. there exists a neighborhood such that every sequence in this family has a subsequence that converges uniformly. A parameter is active if it is not passive. The most well-known activity-locus is undoubtedly the boundary of the Mandelbrot set, where the iterates of the functions $z^2 + c$ are considered. Following this terminology we define the *activity-locus*, $\mathcal{A}_\Delta$, by

$$\mathcal{A}_\Delta := \{\lambda_0 \in \mathbb{C} \mid \{\lambda \mapsto R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}_\Delta\} \text{ is not locally normal at } \lambda_0\}.$$

Another notion of chaotic behaviour of the ratios appears in the proof of the result of Bezaková, Galanis, Goldberg and Šefankovič [BGGv20]. An important step towards proving #P-hardness is showing that for every non-real λ outside of the closed cardioid $\overline{\Lambda_\Delta}$ the set of values $\{R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$. Motivated by this we define

$$\mathcal{D}_\Delta := \{\lambda \in \mathbb{C} \mid \{R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}^1_\Delta\} \text{ is dense in } \hat{\mathbb{C}}\}$$

and refer to the closure of $\mathcal{D}_\Delta$ as the *density-locus*. We will prove it is equal to the activity-locus, thereby showing that these two notions of chaotic behaviour of the ratios are essentially equivalent.

3.1.2. Main result

To state the main result connecting the presence of zeros to computational hardness, we define the *zero-locus* as the closure of

$$\mathcal{Z}_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

We informally define the *#$\mathcal{P}$-locus* as the closure of the collection of λ for which approximating $Z_G(\lambda)$ is #P-hard for $G \in \mathcal{G}_\Delta$. See Section 3.1.3 below for a formal definition.

The main results of this chapter can now be stated succinctly as follows.

Theorem 3.1.1. *For any integer $\Delta \geq 3$ the zero-locus, the activity-locus and the density-locus are equal and contained in the #$\mathcal{P}$-locus. In other words:*

$$\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta = \overline{\mathcal{D}_\Delta} \subseteq \overline{\#\mathcal{P}_\Delta}.$$

We remark that the topological structure of the complement of the zero-locus is not yet understood. We have the following conjecture.

Conjecture 1. *For each integer $\Delta \geq 3$, the set $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ is connected.*

Should this conjecture be true, then by Proposition 3.4.3 below, we know that the maximal 'zero-free' set containing 0, $\mathcal{U}_\Delta$, equals the complement of the zero-locus. Since there exists a polynomial time algorithm [Bar16, PR17] for approximating the independence polynomial on $\mathcal{U}_\Delta$, this coupled with Theorem 3.1.1 would imply a complete understanding of the computational complexity of approximating the independence polynomial in terms of the location of the zeros as well as in terms of chaotic behaviour of the ratios.

3.1.3. Computational complexity

We formally state here the computational problems we are interested in. We denote by $\mathbb{Q}[i]$ the collection of complex numbers with rational real and imaginary part. Let $\lambda \in \mathbb{Q}[i]$, $\Delta \in \mathbb{N}$ and consider the following computational problems.

Name #Hard-CoreNorm(λ, Δ)

Input A graph G of maximum degree at most Δ.

Output If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number N such that $N/1.001 \leq |Z_G(\lambda)| \leq 1.001N$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

Name #Hard-CoreArg(λ, Δ)

Input A graph G of maximum degree at most Δ.

Output If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number A such that $|A - a| \leq \pi/3$ for some $a \in \arg(Z_G(\lambda))$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

We can now formally define the $\#\mathcal{P}$-locus, as the closure of the set,

$$\#\mathcal{P}_\Delta := \{\lambda \in \mathbb{Q}[i] : \text{ the problem \#Hard-CoreNorm}(\lambda, \Delta) \text{ is \#P-hard}\}.$$

We remark that in the definition of $\#\mathcal{P}_\Delta$ we could also replace #Hard-CoreNorm(λ, Δ) by #Hard-CoreArg(λ, Δ) without altering the validity of Theorem 3.1.1.

We moreover note that the constant 1.001 is rather arbitrary. It originates from [BGGv20]. As remarked there the constant can be replaced by any other constant. Let us quickly explain the idea. If say #Hard-CoreNorm(λ, Δ) is #P-hard, but there would be a polynomial time algorithm for the problem with 1.001 replaced by 1.001^2, then we could run this algorithm on the disjoint union of two copies of the same graph G obtaining an a 1.001^2 approximation to the norm of $Z_{G\cup G}(\lambda) = Z_G(\lambda)^2$. This would immediately give us a 1.001-approximation to the norm of $Z_G(\lambda)$. Since the number of vertices of $G \cup G$ is polynomial in the number of vertices of G, we would thus also get a polynomial time algorithm for the problem with constant 1.001.

Organization. After introducing preliminary definitions and results in Section 3.2, we treat the degree $\Delta = 2$ case in Section 3.3. While the equalities between different loci are different when $\Delta = 2$, the explicit descriptions of the zero- and activity-locus will be used in the higher degree cases.

In Section 3.4 we prove the equality of the zero-locus and the activity-locus. The inclusion of the latter in the former is actually an immediate consequence of

Montel's Theorem, and proved earlier in Corollary 3.2.12. We end that section by showing that connected components of the complement of the zero-locus are simply connected.

In Section 3.5 we prove the equality of the activity- and the density-locus, and in Section 3.6 we prove that the density-locus is contained in the $\#\mathcal{P}$-locus.

We end the chapter by discussing a special subclass of graphs: the finite Cayley trees of fixed down-degree $\Delta - 1$. In this setting classical results from complex dynamical systems can be used to obtain a precise description of the zero- and activity-locus. While there zeros do not lie in the activity-locus, the activity-locus equals the accumulation set of the zeros.

3.2. Preliminaries

In this section we collect some preliminary results and conventions that will be used in the remainder of the chapter. The results in this section are not necessarily new but often cannot be found in the literature in the exact way they are stated here. For convenience of the reader we include proofs, especially when the methods are similar to those used later in the thesis.

3.2.1. Ratios of graphs and trees.

Recall that for a rooted graph (G, v) the occupation ratio is defined as the following rational function in λ

$$R_{G,v}(\lambda) = \frac{Z^{\mathrm{in}}_{G,v}(\lambda)}{Z^{\mathrm{out}}_{G,v}(\lambda)}.$$

We note that $Z_G(\lambda) = Z^{\mathrm{in}}_{G,v}(\lambda) + Z^{\mathrm{out}}_{G,v}(\lambda)$, which implies that $Z_G(\lambda) = 0$ if and only if $R_{G,v}(\lambda) = -1$, unless $Z^{\mathrm{in}}_{G,v}(\lambda)$ and $Z^{\mathrm{out}}_{G,v}(\lambda)$ both vanish, in which case the value of the rational function $R_{G,v}(\lambda)$ may not equal -1. The next lemma will show that we can often ignore this difficulty.

We will write $G - v$ for the graph G with vertex v removed, and $G - N[v]$ for the graph with $N[v]$ removed, where $N[v] = \{u \in V(G) : \{u, v\} \in E(G)\} \cup \{v\}$ is the closed neighborhood of v. We observe that $Z^{out}_{G,v}(\lambda) = Z_{G-v}(\lambda)$, and similarly $Z^{in}_{G,v}(\lambda) = \lambda \cdot Z_{G-N[v]}(\lambda)$.

Lemma 3.2.1. *Let $\lambda \in \mathbb{C}^*$. The following three statements are equivalent.*

1. *There exists a graph G of maximum degree at most Δ for which $Z_G(\lambda) = 0$.*
2. *There exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) = -1$.*
3. *There exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$.*

Proof. Assume that (1) holds, then there is a graph G of maximum degree at most Δ for which $Z_G(\lambda) = 0$. Without loss of generality we can assume $G \in \mathcal{G}_\Delta$ satisfies $Z_G(\lambda) = 0$ and has a minimal number of vertices, i.e. for any graph $H \in \mathcal{G}_\Delta$ with $Z_H(\lambda) = 0$ we have $|V(G)| \leq |V(H)|$. For any vertex $v \in V(G)$ we have

$$0 = Z_G(\lambda) = Z^{in}_{G,v}(\lambda) + Z^{out}_{G,v}(\lambda).$$

As $|V(G-v)| < |V(G)|$ we have $Z^{out}_{G,v}(\lambda) = Z_{G-v}(\lambda) \neq 0$, which implies $R_{G,v}(\lambda) = -1$. Thus (2) holds. Trivially, if (2) holds then (3) also holds. To complete the proof we will assume (3) holds and show that (1) follows.

Assume there is a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$. If $R_{G,v}(\lambda) = -1$, we either have $Z^{out}_{G,v}(\lambda) = 0$, in which case (1) follows, or $Z^{out}_{G,v}(\lambda) \neq 0$, in which case $Z^{in}_{G,v}(\lambda) = -Z^{out}_{G,v}(\lambda)$ and (1) follows as well. If $R_{G,v}(\lambda) = \infty$ we have $Z^{out}_{G,v}(\lambda) = 0$. As $Z^{out}_{G,v}(\lambda) = Z_{G-v}(\lambda)$ we see (1) holds. The final case is $R_{G,v}(\lambda) = 0$, in which case we have $0 = Z^{in}_{G,v}(\lambda) = \lambda \cdot Z_{G-N[v]}(\lambda)$. Now as $\lambda \neq 0$, we must have $Z_{G-N[v]}(\lambda) = 0$, which concludes the proof. □

Note that for $\lambda = 0$ we have $R_{G,v}(\lambda) = 0$ and $Z_G(\lambda) = 1$ for any graph G and any vertex $v \in V(G)$. Hence for $\lambda = 0$, statements (1) and (2) in Lemma 3.2.1 are still equivalent, while statement (3) is not equivalent to (1) or (2).

The following result due to Bencs [Ben18] will play an important role in this chapter.

Theorem 3.2.2. *Let $(G, v) \in \mathcal{G}^i_\Delta$ be a rooted connected graph. Then there is a rooted tree $(T, u) \in \mathcal{G}^i_\Delta$ and induced graphs $G_1, \ldots, G_k$ of G such that*

(i) $Z_T = Z_G \prod_{i=1}^k Z_{G_i}$,

(ii) $R_{G,v} = R_{T,u}$.

The following result is a consequence.

Lemma 3.2.3. *Let $\lambda \in \mathbb{C}$ and $(G, v) \in \mathcal{G}_\Delta$ with $Z_G(\lambda) = 0$. Then there is a rooted tree $(T, u) \in \mathcal{G}^1_\Delta$ such that $Z_T(\lambda) = 0$ and $R_{T,u}(\lambda) = -1$.*

Proof. Note that for any graph G we have $Z_G(0) = 1$, so we can assume $\lambda \neq 0$. By Lemma 3.2.1 there exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ such that $R_{G,v}(\lambda) = -1$. By Theorem 3.2.2 (i) we see there is a rooted tree $(T, u) \in \mathcal{G}_\Delta$ with $Z_T(\lambda) = 0$. It follows there is a tree $\tilde{T}$ of maximum degree Δ with a minimal number of vertices such that $Z_{\tilde{T}}(\lambda) = 0$. For $\tilde{T}$ and any vertex $v \in V(\tilde{T})$ we have $R_{\tilde{T},v}(\lambda) = -1$. The lemma follows by choosing v to be a leaf of $\tilde{T}$. □

At a later stage we will need to worry about the degree of the root vertex in our definition of the activity- and density-locus. We therefore introduce some definitions to facilitate their discussion.

Fix an integer $\Delta \geq 2$ throughout. For $i = 1, \ldots, \Delta$ we denote the family of ratios with root degree at most i by

$$\mathcal{R}^i_\Delta := \{R_{G,v} \mid (G,v) \in \mathcal{G}^i_\Delta\}.$$

We just write $\mathcal{R}_\Delta$ instead of $\mathcal{R}^\Delta_\Delta$. For a given $\lambda \in \mathbb{C}$, we denote the set of values of these ratios by

$$\mathcal{R}^i_\Delta(\lambda) := \{R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}^i_\Delta\}.$$

Then we define $\mathcal{A}^i_\Delta$ to be the collection of λ_0 at which the family $\mathcal{R}^i_\Delta$ is not normal. We just write $\mathcal{A}_\Delta$ instead of $\mathcal{A}^\Delta_\Delta$. Finally, we introduce $\mathcal{D}^i_\Delta$ to be the collection of λ for which the set $\mathcal{R}^i_\Delta(\lambda)$ is dense in $\mathbb{C}$. Note that we denote $\mathcal{D}^1_\Delta$ by $\mathcal{D}_\Delta$ (as opposed to the above convention).

3.2.2. Graph manipulations and definition of the cardioid

The recursion formula given in the following lemma is well known.

Lemma 3.2.4. *Let $T = (V, E)$ be a tree and v a vertex of T. Suppose v is connected to $d \geq 1$ other vertices $u_1, \ldots, u_d$. Denote T_s for the tree that is the connected component of $T - v$ containing u_s. Then we have*

$$R_{T,v}(\lambda) = \frac{\lambda}{\prod_{s=1}^{d}(1 + R_{T_s,u_s}(\lambda))}. \tag{3.1}$$

Proof. We have

$$\frac{Z^{in}_{T,v}(\lambda)}{Z^{out}_{T,v}(\lambda)} = \lambda \frac{Z_{T-N[v]}(\lambda)}{Z_{T-v}(\lambda)} = \lambda \prod_{s=1}^{d} \frac{Z^{out}_{T_s,u_s}(\lambda)}{Z_{T_s}(\lambda)} = \lambda \prod_{s=1}^{d} \frac{Z^{out}_{T_s,u_s}(\lambda)}{Z^{out}_{T_s,u_s}(\lambda) + Z^{in}_{T_s,u_s}(\lambda)}, \tag{3.2}$$

where in the second equality we use that the partition function of a graph factors into the partition functions of its connected components. By dividing for each $s \in \{1, \ldots, d\}$ the denominator and enumerator of the right hand side of equation (3.2) by $Z^{out}_{T_s,u_s}(\lambda)$ we obtain the desired formula. □

This lemma implies the claim from the introduction that the ratios of Cayley trees are given by iterating $f_{\lambda,d}(z) = \frac{\lambda}{(1+z)^d}$. We refer to Section 3.7 for an in-depth discussion of Cayley trees and their associated dynamics.

Definition 3.2.5. Define the cardioid Λ_Δ as the closure of the set of parameters λ for which $f_{\lambda,d}$ has an attracting fixed point.

Note that $0 \in \Lambda_\Delta$. It can be shown, see Section 2.1 in [PR19], that

$$\Lambda_\Delta = \left\{ \frac{z}{(1-z)^\Delta} \mid |z| \leq \frac{1}{\Delta - 1} \right\}.$$

Taking $z = \frac{-1}{\Delta-1}$, we observe that

$$\lambda^*(\Delta) = \frac{-(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$$

is the intersection point of Λ_Δ with the negative real line.

Let $G = (V, E)$ be a graph and let (G_i, v_i) be rooted graphs for $i \in V$. We refer to the graph obtained from G and the G_i by identifying each vertex $i \in V$ with v_i as *implementing* the G_i in G; see Figure 3.2. The next lemmas describe the effect on the ratios for various choices of G and G_i.

Lemma 3.2.6. *Let P_n denote the path on n vertices. Let (G_i, v_i) be rooted graphs for $i \in \{1, \ldots, n\}$ and denote $\mu_i(\lambda) = R_{G_i,v_i}(\lambda)$. Let $\tilde{P}_n$ be the graph obtained by implementing the G_i in P_n. Then*

$$R_{\tilde{P}_n,v_n}(\lambda) = (f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

where $f_\mu(z) = \frac{\mu}{1+z}$.

Proof. We use induction on n. For $n = 1$, by definition we have $R_{G_1,v_1}(\lambda) = \mu_1(\lambda) = f_{\mu_1(\lambda)}(0)$. As $\tilde{P}_1 = G_1$, we have $R_{\tilde{P}_1,v_1}(\lambda) = R_{G_1,v_1}(\lambda)$. The base case follows.

Suppose the statement holds for some $n \geq 1$. The vertex v_{n+1} has 1 neighbor that is part of the path P_n. Let us denote that neighbor as v_n. It follows that

$$\begin{aligned} R_{\tilde{P}_{n+1},v_{n+1}}(\lambda) &= \frac{Z^{in}_{\tilde{P}_{n+1},v_{n+1}}(\lambda)}{Z^{out}_{\tilde{P}_{n+1},v_{n+1}}(\lambda)} = \frac{Z^{in}_{G_{n+1},v_{n+1}}(\lambda)}{Z^{out}_{G_{n+1},v_{n+1}}(\lambda)} \cdot \frac{Z^{out}_{\tilde{P}_n,v_n}(\lambda)}{Z_{\tilde{P}_n,v_n}(\lambda)} \\ &= R_{G_{n+1},v_{n+1}}(\lambda) \cdot \frac{Z^{out}_{\tilde{P}_n,v_n}(\lambda)}{Z^{out}_{\tilde{P}_n,v_n}(\lambda) + Z^{in}_{\tilde{P}_n,v_n}(\lambda)} = \frac{R_{G_{n+1},v_{n+1}}(\lambda)}{1 + R_{\tilde{P}_n,v_n}(\lambda)} \\ &= f_{\mu_{n+1}(\lambda)}(R_{\tilde{P}_n,v_n}(\lambda)), \end{aligned}$$

where in the second equality we use that the partition function of a graph factors into the partition functions of its connected components.

By the induction hypothesis we have

$$R_{\tilde{P}_n,v_n}(\lambda) = (f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

from which it follows that

$$R_{\tilde{P}_{n+1},v_{n+1}}(\lambda) = (f_{\mu_{n+1}(\lambda)} \circ f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

completing the proof. $\square$

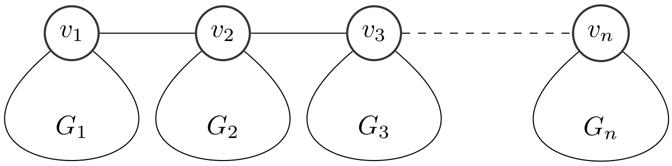

Figure 3.2: The graph $\tilde{P}_n$ in Lemma 3.2.6

Remark 3.2.7. Note that if the graphs G_i in Lemma 3.2.6 are all of maximum degree Δ and the roots v_i have degree at most $\Delta - 2$ for $i \in \{2, \ldots, n-1\}$ and at most degree $\Delta - 1$ for $i \in \{1, n\}$, then the graph $\tilde{P}_n$ is also of maximum degree Δ.

Lemma 3.2.8. *Let $G = (V, E)$ be a graph and denote $n = |V|$. Let (H, v) be a rooted graph. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be obtained from G by implementing n copies of (H, v) in G. Then for any $w \in V$ we have*

$$\frac{Z_{\tilde{G},w}(\lambda)}{(Z^{out}_{H,v}(\lambda))^n} = Z_{G,w}(R_{H,u}(\lambda)) \tag{3.3}$$

and

$$R_{\tilde{G},w}(\lambda) = R_{G,w}(R_{H,v}(\lambda)). \tag{3.4}$$

Proof. We have

$$\frac{Z^{in}_{\tilde{G},w}(\lambda)}{(Z^{out}_{H,v}(\lambda))^n} = \frac{\sum_{\substack{I \in \mathcal{I}(G) \\ w \in I}} Z^{in}_{H,v}(\lambda)^{|I|} Z^{out}_{H,v}(\lambda)^{n-|I|}}{(Z^{out}_{H,v}(\lambda))^n} = Z^{in}_{G,w}(R_{H,v}(\lambda)) \tag{3.5}$$

and

$$\frac{Z^{out}_{\tilde{G},w}(\lambda)}{(Z^{out}_{H,v}(\lambda))^n} = \frac{\sum_{\substack{I \in \mathcal{I}(G) \\ w \notin I}} Z^{in}_{H,v}(\lambda)^{|I|} Z^{out}_{H,v}(\lambda)^{n-|I|}}{(Z^{out}_{H,v}(\lambda))^n} = Z^{out}_{G,w}(R_{H,v}(\lambda)). \tag{3.6}$$

Equality (3.3) follows from equalities (3.5) and (3.6) noting that for any graph W and any vertex u of W we have $Z_W(\lambda) = Z^{in}_{W,u}(\lambda) + Z^{out}_{W,u}(\lambda)$. Equality (3.4) follows from equalities (3.5) and (3.6) and the definition of the ratio. □

We will also need the following slight variation on Lemma 3.2.4.

Lemma 3.2.9. *Let (G_1, v_1) and (G_2, v_2) be rooted graphs, and define the rooted graph $(\tilde{G}, \tilde{v})$ by identifying the roots v_1 and v_2. Then*

$$R_{\tilde{G},\tilde{v}}(\lambda) = \lambda^{-1} \cdot R_{G_1,v_1}(\lambda) \cdot R_{G_2,v_2}(\lambda).$$

Proof. We compute

$$R_{\tilde{G},\tilde{v}}(\lambda) = \frac{Z^{in}_{\tilde{G},\tilde{v}}(\lambda)}{Z^{out}_{\tilde{G},\tilde{v}}(\lambda)} = \frac{Z^{in}_{G_1,v_1}(\lambda) \cdot Z^{in}_{G_2,v_2}(\lambda) \cdot \lambda^{-1}}{Z^{out}_{G_1,v_1}(\lambda) \cdot Z^{out}_{G_2,v_2}(\lambda)} = \lambda^{-1} \cdot R_{G_1,v_1}(\lambda) \cdot R_{G_2,v_2}(\lambda).$$

□

3.2.3. The Shearer region

Denote the open disk around 0 with radius $\frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ by B_Δ. This region, also known as the Shearer region, is the maximal open disk centered around 0 that is zero free for the independence polynomial of graphs of maximum degree Δ [SS05, She85]. We will show the Shearer region is also disjoint from the activity-locus and the density-locus, which will later be used to deal with the $\lambda = 0$ case in the proof of the main theorem.

Lemma 3.2.10. *Let $\Delta \geq 2$ be an integer. Then B_Δ is disjoint from the activity-locus, the zero-locus and the density-locus, i.e., we have $B_\Delta \cap \overline{\mathcal{D}_\Delta} = B_\Delta \cap \mathcal{A}_\Delta = B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset$.*

Proof. We claim that for any rooted graph $(G, v) \in \mathcal{G}_\Delta$ and any $\lambda \in B_\Delta$ we have

$$|R_{G,v}(\lambda)| < \begin{cases} \frac{1}{\Delta} & \text{if } \deg(v) \leq \Delta - 1, \\ \frac{1}{\Delta-1} & \text{otherwise.} \end{cases}$$

By Theorem 3.2.2 we can equivalently work with rooted trees $(T, v) \in \mathcal{G}_\Delta$ instead of rooted graphs.

We will prove the claim by induction on the number of vertices of T. If $|V(T)| = 1$, we have $\deg(v) = 0$ and therefore $R_{T,v}(\lambda) = \lambda$. The claim then follows as $\frac{(\Delta-1)^{(\Delta-1)}}{\Delta^\Delta} < \frac{1}{\Delta}$ for all $\Delta \geq 2$. Suppose the claim holds for all rooted trees $(T, v) \in \mathcal{G}_\Delta$ with $|V(T)| \leq n$ for some $n \geq 1$. Let $(\tilde{T}, \tilde{v}) \in \mathcal{G}_\Delta$ be a rooted tree with $n + 1$ vertices. Denote the d children of $\tilde{v}$ as $u_1, \ldots, u_d$ and denote (T_i, u_i) for the rooted subtree of $\tilde{T}$ with root u_i. By Lemma 3.2.4 we have

$$R_{\tilde{T},\tilde{v}}(\lambda) = \frac{\lambda}{\prod_{i=1}^d (1 + R_{T_i,u_i}(\lambda))}.$$

We note that each (T_i, u_i) has at most n vertices, hence the induction hypothesis applies. Furthermore in T_i we have $\deg(u_i) \leq \Delta - 1$ as $\tilde{T}$ has maximum degree at most Δ. Thus we see

$$|R_{\tilde{T},\tilde{v}}(\lambda)| = \frac{|\lambda|}{\prod_{i=1}^{d}|1+R_{T_i,u_i}(\lambda)|} \leq \frac{|\lambda|}{\prod_{i=1}^{d}(1-|R_{T_i,u_i}(\lambda)|)}$$
$$< \frac{|\lambda|}{(1-\frac{1}{\Delta})^d} = \frac{\Delta^d|\lambda|}{(1-\Delta)^d} < \frac{(\Delta-1)^{\Delta-1-d}}{\Delta^{\Delta-d}}.$$

Now if $d \leq \Delta-1$, we see $\frac{(\Delta-1)^{\Delta-1-d}}{\Delta^{\Delta-d}} < \frac{1}{\Delta}$ hence the claim follows for that case. If $d = \Delta$ we have $\frac{(\Delta-1)^{\Delta-1-d}}{\Delta^{\Delta-d}} = \frac{1}{\Delta-1}$, which proves the claim.

It follows from the claim above that the family of ratios $\mathcal{R}_\Delta$ maps B_Δ into the open unit disk, for all $\Delta \geq 2$. So clearly $B_\Delta \cap \mathcal{D}_\Delta = \emptyset$. As B_Δ is open, we have $B_\Delta \cap \overline{\mathcal{D}_\Delta} = \emptyset$.

By Montel's Theorem the family $\mathcal{R}_\Delta$ is normal on B_Δ, so $B_\Delta \cap \mathcal{A}_\Delta = \emptyset$. We showed for all rooted graphs $(G,v) \in \mathcal{G}_\Delta$ that $|R_{G,v}(\lambda)| < \frac{1}{\Delta-1} \leq 1$, hence the ratio will never equal -1. For $\lambda \neq 0$, we see by Lemma 3.2.1 that $\lambda \notin \mathcal{Z}_\Delta$. For $\lambda = 0$ we note that $Z_G(0) = 1$ for any graph $G \in \mathcal{G}_\Delta$. It follows that $B_\Delta \cap \mathcal{Z}_\Delta = \emptyset$. Again, as B_Δ is open, we have $B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset$. This completes the proof. □

Remark 3.2.11. We note that on the negative real line the Shearer region agrees with the interior of the cardioid, i.e we have $\mathbb{R}_{\leq 0} \cap B_\Delta = \mathbb{R}_{\leq 0} \cap \operatorname{int} \Lambda_\Delta$ for all integers $\Delta \geq 3$.

Lemmas 3.2.10 and 3.2.1 together imply one of the inclusions in the main result.

Corollary 3.2.12. *For all $\Delta \geq 2$ the activity-locus is contained in the zero-locus, i.e. $\mathcal{A}_\Delta \subseteq \overline{\mathcal{Z}_\Delta}$.*

Proof. Equivalently, we want to show that for any $\lambda \in \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ the family $\mathcal{R}_\Delta$ is normal at λ. By Lemma 3.2.10 this is the case for $\lambda = 0$ and thus we assume that $\lambda \neq 0$. Take a sufficiently small neighborhood U around λ such that $0 \notin U$ and $U \cap \overline{\mathcal{Z}_\Delta} = \emptyset$. It follows from Lemma 3.2.1 that the family $\mathcal{R}_\Delta$ avoids $\{-1, 0, \infty\}$ for all $\lambda' \in U$. Hence by Montel's Theorem the family is normal on U. □

3.3. Graphs with maximum degree at most two

In this section we will deal with graphs of maximum degree at most two, in other words graphs for which each component is a path or a cycle. We will show that

$$\overline{\mathcal{Z}_2} = \mathcal{A}_2 = (-\infty, -1/4] \quad \text{and} \quad \mathcal{D}_2^1 = \mathcal{D}_2^2 = \emptyset.$$

An explicit description of $\mathcal{Z}_2$ was already known [HL72, SS05]; we provide a new proof for the sake of completeness.

Note that this is in contrast to the situation for $\Delta \geq 3$ as stated in Theorem 3.1.1. It follows from Lemma 3.2.3 that $\mathcal{Z}_2$ is equal to the set of λ for which there is a $(T, v) \in \mathcal{G}_2^1$, with T a tree, such that $R_{T,v}(\lambda) = -1$. The collection $\mathcal{G}_2^1$ consists of rooted graphs where the component containing the root is a path rooted at an endpoint. Let (P_n, v_n) denote a path on n vertices rooted at an endpoint v_n. If we let $f_\lambda(z) = \lambda/(1+z)$ then it follows from Lemma 3.2.6 that $R_{P_n,v_n}(\lambda) = f_\lambda^n(0)$. For fixed λ the map f_λ is a Möbius transformation and therefore we first review some properties of Möbius transformations. The following section is similar to Section 2.3.2, where we discussed properties of function belonging to the same path construction in the Ising model.

3.3.1. Möbius transformations

Everything that is outlined in this section can for example be found in [Bea95, Section 4.3]. Let $\mathcal{M}$ denote the group of Möbius transformations with composition as group operation and let $\mathrm{GL}_2(\mathbb{C})$ denote the group of 2×2 invertible matrices with complex entries. The following map is a surjective group homomorphism.

$$\Phi : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d}\right).$$

For any $g \in \mathcal{M}$ take an element $A \in \Phi^{-1}(\{g\})$ and define $\mathrm{tr}^2(g) = \mathrm{tr}(A)^2/\det(A)$. This quantity does not depend on the choice of A and thus tr^2 is a well-defined function on $\mathcal{M}$. We say that elements $f, g \in \mathcal{M}$ are conjugate if there exists an $h \in \mathcal{M}$ such that $f = h \circ g \circ h^{-1}$.

Lemma 3.3.1 ([Bea95, Theorem 4.3.4]). *Let $f, g \in \mathcal{M}$ not equal to the identity. The maps f, g are conjugate if and only if $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$. It follows that g is conjugate to*

- *a rotation $z \mapsto e^{i\theta} \cdot z$ for some $\theta \in (0, \pi]$ if and only if $\mathrm{tr}^2(g) \in [0, 4)$;*
- *the translation $z \mapsto z + 1$ if and only if $\mathrm{tr}^2(g) = 4$;*
- *a multiplication $z \mapsto \xi \cdot z$ for some $\xi \in \mathbb{C}^*$ with $|\xi| < 1$ if and only if $\mathrm{tr}^2(g) \in \mathbb{C} \setminus [0, 4]$.*

The map g is said to be elliptic, parabolic or loxodromic in these three cases respectively.

Observe that if $f = h \circ g \circ h^{-1}$ then $f^n = h \circ g^n \circ h^{-1}$. It follows that, to understand the dynamical behaviour of a Möbius transformation g, it is enough to understand the dynamical behaviour of any element in the conjugacy class of g. If g is loxodromic then it has two distinct fixed points in $\widehat{\mathbb{C}}$, one of which is attracting and the other is repelling. Under iteration of g the orbit of every initial value except for the repelling fixed point converges to the attracting fixed point. If g is parabolic then g has a unique fixed point, and under iteration of g all orbits converge to this fixed point. If g is elliptic then g is conjugate to a rotation $z \mapsto e^{i\theta} \cdot z$. We say that g is conjugate to a *rational rotation* if θ is a rational multiple of π and otherwise we say that g is conjugate to an *irrational rotation*. If g is conjugate to a rational rotation there is a positive integer n such that g^n is equal to the identity. If g is conjugate to an irrational rotation it has two fixed points, say p, q, and $\widehat{\mathbb{C}} \setminus \{p, q\}$ is foliated by generalized circles on which g acts conjugately to an irrational rotation.

We end this subsection by classifying f_λ in terms of its parameter.

Lemma 3.3.2. *The Möbius transformation f_λ is*

- *elliptic if $\lambda \in (-\infty, -1/4)$;*
- *parabolic if $\lambda = -1/4$;*
- *loxodromic if $\lambda \in \mathbb{C}^* \setminus (-\infty, -1/4]$.*

Proof. This follows from Lemma 3.3.1 and the fact that $\mathrm{tr}^2(f_\lambda) = -1/\lambda$. □

3.3.2. Determining the zero and activity-locus

In this subsection we we will show that both $\overline{\mathcal{Z}_2}$ and $\mathcal{A}_2^1$ are equal to $(-\infty, -1/4]$. By definition we have $\mathcal{A}_2^1 \subseteq \mathcal{A}_2$ and by Corollary 3.2.12 we have $\mathcal{A}_2 \subseteq \overline{\mathcal{Z}_2}$, hence it will follow that $\mathcal{A}_2$ is equal to $(-\infty, -1/4]$ as well.

Lemma 3.3.3. *Zeros of Z_G for graphs $G \in \mathcal{G}_2$ form a dense subset of the interval $(-\infty, -1/4)$, hence $\overline{\mathcal{Z}_2} = (-\infty, -1/4]$.*

Proof. We claim that $\lambda \in \mathcal{Z}_2$ if and only if f_λ is conjugate to a rational rotation.

First suppose that $\lambda \in \mathcal{Z}_2$. Then, by Lemma 3.2.3, there is an $n \geq 1$ such that for the path on n vertices P_n rooted at the endpoint v_n we have $R_{P_n, v_n}(\lambda) = -1$ and thus $f_\lambda^n(0) = -1$. Because $f_\lambda^2(-1) = 0$ regardless of the value of λ we obtain that $f_\lambda^{n+2}(0) = 0$. This means that 0 is a periodic point of f_λ of period strictly larger than 1. This can only occur if f_λ is conjugate to a rational rotation, as is explained in Section 3.3.1.

Suppose that f_λ is conjugate to a rational rotation. Note that this implies that λ is not equal to zero. Take the smallest positive integer n such that f_λ^n is equal to the identity and thus specifically $f_\lambda^n(0) = 0$. Note that $f_\lambda(0) = \lambda$ and $f_\lambda^2(0) = \lambda/(1 + \lambda)$ and thus $n \geq 3$. Since $f_\lambda^{-2}(0) = -1$ we obtain that $R_{P_{n-2}, v_{n-2}}(\lambda) = f_\lambda^{n-2}(0) = -1$. It follows from the proof of Lemma 3.2.1 that λ is a root of $Z_{P_{n-2}}$.

Parameters λ for which f_λ is conjugate to a rational rotation lie dense in the set of parameters for which f_λ is conjugate to any rotation. It follows from Lemma 3.3.2 that $\overline{\mathcal{Z}_2} = (-\infty, -1/4]$. □

We remark that tr^2 of the map that sends z to $e^{i\theta} \cdot z$ is equal to $2(1+\cos(\theta))$. By comparing this to the value of $\mathrm{tr}^2(f_\lambda)$ it follows from the previous proof that

$$\mathcal{Z}_2 = \left\{ \frac{-1}{2(1+\cos(t\pi))} : t \in (0,1) \cap \mathbb{Q} \right\}.$$

We will now prove the final lemma needed to determine $\mathcal{A}_2$.

Lemma 3.3.4. *The family* $\mathcal{R}_2^1$ *is not normal around any* $\lambda \in (-\infty, -1/4]$*, i.e.* $(-\infty, -1/4] \subseteq \mathcal{A}_2^1$.

Proof. Recall that

$$\mathcal{R}_2^1 = \{R_{P_n, v_n} : n \geq 1\} = \{\lambda \mapsto f_\lambda^n(0) : n \geq 1\}.$$

Take a $\lambda_0 \in (-\infty, -1/4]$ and suppose for the sake of contradiction that there exists a neighborhood U of λ_0 on which $\mathcal{R}_2^1$ is normal. We take U connected and sufficiently small so that it does not contain 0, and 0 is not a fixed point of f_λ for any $\lambda \in U$. Because $\mathcal{R}_2^1$ is normal on U there exists a subsequence of $\{R_{P_n,v_n}\}_{n\geq 1}$ that converges locally uniformly to a holomorphic function $F : U \to \widehat{\mathbb{C}}$. For $\lambda \in U \setminus (-\infty, -1/4]$ the map f_λ is loxodromic, hence $f_\lambda^n(0)$ converges to the attracting fixed point of f_λ as n goes to infinity. This means that for $\lambda \in U \setminus (-\infty, -1/4]$ we have $f_\lambda(F(\lambda)) = F(\lambda)$. Because $U \setminus (-\infty, -1/4]$ is non-empty and open in U it follows from the identity theorem for holomorphic functions that the equality $f_\lambda(F(\lambda)) = F(\lambda)$ must hold for all $\lambda \in U$. The set U contains a parameter λ_1 for which f_{λ_1} is elliptic. The value 0 is not a fixed point of f_{λ_1} and thus the distance of $f_{\lambda_1}^n(0)$ to both of the fixed points of f_{λ_1} is uniformly bounded below for all n by a positive constant. This means that no subsequence of $\{R_{P_n,v_n}(\lambda_1)\}_{n\geq 1}$ can converge to the fixed point $F(\lambda_1)$. We conclude that $\mathcal{R}_2^1$ is not normal at λ_0. □

It follows from the previous two lemmas and Corollary 3.2.12 that

$$(-\infty, -1/4] \subseteq \mathcal{A}_2^1 \subseteq \mathcal{A}_2^2 \subseteq \overline{\mathcal{Z}_2} = (-\infty, -1/4].$$

Therefore we can conclude that both $\mathcal{A}_2$ and $\overline{\mathcal{Z}_2}$ are equal to $(-\infty, -1/4]$.

3.3.3. Determining the density-locus.

Recall that for $\lambda \in \mathbb{C}$ we defined $\mathcal{R}_\Delta^i(\lambda) = \{R_{G,v}(\lambda) : (G, v) \in \mathcal{G}_\Delta^i\}$. Subsequently we defined $\mathcal{D}_\Delta^i$ as the set consisting of those λ for which $\mathcal{R}_\Delta^i(\lambda)$ is dense in $\widehat{\mathbb{C}}$. It is thus clear that $\mathcal{D}_2^1 \subseteq \mathcal{D}_2^2$. To conclude the section we show the following.

Lemma 3.3.5. *There is no $\lambda_0 \in \mathbb{C}$ for which $\mathcal{R}_2^2(\lambda_0)$ is dense in $\widehat{\mathbb{C}}$, i.e. $\mathcal{D}_2^2 = \emptyset$.*

Proof. It follows from Theorem 3.2.2 that

$$\mathcal{R}_2^2(\lambda) = \{R_{T,v}(\lambda) : (T,v) \in \mathcal{G}_2^2 \text{ with } T \text{ a tree}\}.$$

A rooted tree $(T,v) \in \mathcal{G}_2^2$ can be viewed as a vertex v onto which two rooted paths (P_n, v_n) and (P_m, v_m) are attached for $n, m \geq 0$. It follows from Lemma 3.2.4 and Lemma 3.2.6 that

$$R_{T,v}(\lambda) = \lambda \cdot \frac{1}{1+f_\lambda^n(0)} \cdot \frac{1}{1+f_\lambda^m(0)} = \frac{1}{\lambda} \cdot f_\lambda^{n+1}(0) \cdot f_\lambda^{m+1}(0).$$

For a specific λ_0 the right-hand side of this equality is not defined if $f_{\lambda_0}^{n+1}(0)$ and $f_{\lambda_0}^{m+1}(0)$ take on the values 0 and ∞. Recall that if $f_{\lambda_0}^{n+1}(0) = \infty$, then $f_{\lambda_0}^n(0) = -1$, which implies that $\lambda_0 \in \mathcal{Z}_2$. If this is the case then Lemma 3.3.3 implies that λ_0 is real, and thus $\mathcal{R}_2^2(\lambda_0)$ is contained in $\mathbb{R} \cup \{\infty\}$ and is not dense in $\widehat{\mathbb{C}}$.

Assume that λ_0 is not real. In this case we have the equality

$$\mathcal{R}_2^2(\lambda_0) = \left\{\frac{1}{\lambda_0} \cdot f_{\lambda_0}^{n+1}(0) \cdot f_{\lambda_0}^{m+1}(0) : n, m \geq 0\right\}.$$

The map f_{λ_0} is loxodromic, hence the orbit of 0 converges to an attracting fixed point without passing through ∞. Note that $f_{\lambda_0}(\infty) = 0$, therefore ∞ is not the attracting fixed point, and thus there is a positive bound $B \in \mathbb{R}_{>0}$ such that $|f_{\lambda_0}^n(0)| < B$ for all n. It follows that

$$\left|\frac{1}{\lambda_0} \cdot f_{\lambda_0}^{n+1}(0) \cdot f_{\lambda_0}^{m+1}(0)\right| < \frac{B^2}{|\lambda_0|}$$

for all n, m, and thus $\mathcal{R}_2^2(\lambda_0)$ is bounded and in particular not dense in $\widehat{\mathbb{C}}$. □

3.4. Equality of the zero-locus and the activity-locus for $\Delta \geq 3$

In this section we prove the equalities $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^2 = \cdots = \mathcal{A}_\Delta^\Delta = \overline{\mathcal{Z}_\Delta}$ for $\Delta \geq 3$, thereby proving that the activity-locus is equal to the zero-locus. The strategy is similar to the one employed in the $\Delta = 2$ case. By definition we have $\mathcal{A}_\Delta^1 \subseteq \mathcal{A}_\Delta^2 \subseteq \cdots \subseteq \mathcal{A}_\Delta^\Delta$. We will first show that $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^2 = \cdots = \mathcal{A}_\Delta^{\Delta-1}$ and subsequently we will show that $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}_\Delta^{\Delta-1}$. Then Corollary 3.2.12, which states that $\mathcal{A}_\Delta^\Delta \subseteq \overline{\mathcal{Z}_\Delta}$, is enough to arrive at our desired conclusion.

Lemma 3.4.1. *The family $\mathcal{R}^1_\Delta$ is normal at $\lambda_0 \in \mathbb{C}$ if and only if $\mathcal{R}^{\Delta-1}_\Delta$ is normal at λ_0, and hence $\mathcal{A}^1_\Delta = \mathcal{A}^{\Delta-1}_\Delta$.*

Proof. Recall that

$$\mathcal{R}^i_\Delta := \{R_{G,v} : (G,v) \in \mathcal{G}^i_\Delta\}$$

and thus $\mathcal{R}^1_\Delta \subseteq \mathcal{R}^{\Delta-1}_\Delta$. It follows that if $\mathcal{R}^{\Delta-1}_\Delta$ is normal at λ_0 then the same holds for $\mathcal{R}^1_\Delta$.

To show the other direction, assume that $\mathcal{R}^1_\Delta$ is normal at λ_0. Note that the family $\mathcal{R}^\Delta_\Delta$ is normal at 0 by Lemma 3.2.10, hence we can assume $\lambda_0 \neq 0$. As $\mathcal{R}^1_\Delta$ is normal at λ_0, there is a neighborhood U of λ_0 on which $\mathcal{R}^1_\Delta$ is a normal family. We can take U such that $0 \notin U$. We will show that $\mathcal{R}^{\Delta-1}_\Delta$ is also a normal family on U. To that effect take a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1} \subseteq \mathcal{G}^{\Delta-1}_\Delta$. Construct the rooted graphs $(\hat{G}_n, \hat{v}_n) \in \mathcal{G}^1_\Delta$ by attaching a root $\hat{v}_n$ to the root v_n of G_n by a single edge. By assumption the sequence $\{R_{\hat{G}_n,\hat{v}_n}\}_{n\geq 1}$ has a subsequence that converges locally uniformly to a function $H : U \to \widehat{\mathbb{C}}$. Let $I \subseteq \mathbb{N}$ be the indices belonging to this subsequence. By Lemma 3.2.6 we have $R_{\hat{G}_n,\hat{v}_n}(\lambda) = f_\lambda(R_{G_n,v_n}(\lambda))$ for every $\lambda \in U$. Because U does not contain 0, the Möbius transformation f_λ is invertible for every $\lambda \in U$. Therefore for these λ we have

$$\lim_{\substack{n\to\infty \\ n\in I}} R_{G_n,v_n}(\lambda) = \lim_{\substack{n\to\infty \\ n\in I}} f_\lambda^{-1}(f_\lambda(R_{G_n,v_n}(\lambda))) = \lim_{\substack{n\to\infty \\ n\in I}} f_\lambda^{-1}(R_{\hat{G}_n,\hat{v}_n}(\lambda)) = f_\lambda^{-1}(H(\lambda)).$$

Because the map $U \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that sends (λ, z) to $f_\lambda^{-1}(z)$ is continuous, we can conclude that this limit converges locally uniformly on U. Therefore we have shown that the sequence $\{R_{G_n,v_n}\}_{n\geq 1}$ has a subsequence that converges locally uniformly to the holomorphic function $\lambda \mapsto f_\lambda^{-1}(H(\lambda))$, and thus $\mathcal{R}^{\Delta-1}_\Delta$ is normal at λ_0. □

Proposition 3.4.2. *Let $\Delta \geq 3$. Then $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}^{\Delta-1}_\Delta$, and hence the zero-locus is contained in the activity-locus.*

Proof. Let us assume $\lambda \in \overline{\mathcal{Z}_\Delta}$. Then for any open neighborhood V of λ there is a $\lambda_0 \in V$ for which $Z_G(\lambda_0) = 0$ for some $G \in \mathcal{G}_\Delta$. We will prove that the family $\mathcal{R}^{\Delta-1}_\Delta$ cannot be normal on V.

By Lemma 3.2.3 there is a a rooted tree $(T,u) \in \mathcal{G}^1_\Delta$ for which $R_{T,u}(\lambda_0) = -1$. Consider the rooted trees (T_n, v) obtained by implementing a copy of (T,u) in every vertex of the rooted paths (P_n, v). It follows from Lemma 3.2.8 that

$$R_{T_n,v} = R_{P_n,v} \circ R_{T,u}.$$

We note that in T_n the root v has degree $2 \leq \Delta - 1$. Furthermore $R_{T,u}$ maps a neighborhood of λ_0 holomorphically to a neighborhood of -1, since $R_{T,u}$ is not constantly equal to -1. Lemma 3.3.4 states that the family $\{R_{P_n,v}\}_{n>0}$ is not normal at -1 and thus it follows that $\{R_{T_n,v}\}_{n>0}$ is not normal at λ_0. □

Summarising we have the following relations between sets

$$\mathcal{A}^1_\Delta \overset{(1)}{=} \mathcal{A}^{\Delta-1}_\Delta \subseteq \mathcal{A}^\Delta_\Delta \overset{(2)}{\subseteq} \overline{\mathcal{Z}_\Delta} \overset{(3)}{\subseteq} \mathcal{A}^{\Delta-1}_\Delta,$$

where equality (1) is due to Lemma 3.4.1, inclusion (2) is due to Corollary 3.2.12 and inclusion (3) is due to Proposition 3.4.2. It follows that $\mathcal{A}^1_\Delta = \ldots = \mathcal{A}^\Delta_\Delta$, and $\mathcal{A}_\Delta = \overline{\mathcal{Z}_\Delta}$ for all $\Delta \geq 2$.

3.4.1. The complement of the zero-locus

As an application of the equality of the zero-locus and the activity-locus, we show here that each component of the complement of the zero-locus is simply connected. We recall from the introduction that this implies that if the complement of the zero-locus is connected (as we conjecture in Conjecture 1), then our main result gives a complete understanding of the complexity of approximately computing the independence polynomial.

Proposition 3.4.3. *Let $\Delta \geq 2$ be an integer. Any connected component of the complement of the zero-locus, $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$, is simply connected.*

Proof. For $\Delta = 2$ the statement follows directly by the exact characterization of the closure of the zero-locus. We will therefore assume that $\Delta \geq 3$.

Let γ be a simple closed curve contained in the complement of the zero-locus, $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$. It is sufficient to prove that the interior of γ, which we will denote by V, is zero free. Let us suppose for the sake of contradiction that this is not the case.

Let T be a minimal tree for which $Z_T(\lambda_0) = 0$ for some $\lambda_0 \in V$. Let v be a leaf of T. Since $|T|$ is chosen to be minimal it follows that $R_{T,v}(\lambda_0) = -1$. Denote the neighbor of v in T by w. By minimality of T it also follows that $R_{T-v,w}(\lambda) \neq -1$ for any $\lambda \in \overline{V}$.

Note that V is necessarily bounded as it is a subset of the cardioid Λ_Δ. Hence by compactness of $\overline{V}$ it follows that $R_{T-v,w}$ is bounded away from -1 on V. Since

$$R_{T,v}(\lambda) = \frac{\lambda}{1 + R_{T-v,w}(\lambda)}$$

it follows that $R_{T,v}$ is bounded on $\overline{V}$. By the Open Mapping Theorem for holomorphic functions it follows that there must be a $\lambda_1 \in \partial V = \gamma$ for which

$$R_{T,v}(\lambda_1) \in (-\infty, -1).$$

Use Lemma 3.2.6 to implement the rooted tree (T, v) in the paths P_n to obtain a sequence of rooted graphs $\{(G_n, u_n)\}_{n\geq 1}$ with

$$R_{G_n,u_n} = R_{P_n,u_n} \circ R_{T,v}.$$

Since $R_{T,v}(\lambda_1) \in (-\infty, -1)$, which is contained in the half-line where the family $\{\lambda \mapsto R_{P_n,u}(\lambda)\}_{n\in\mathbb{N}}$ is not normal, it follows that the family of ratios $\{R_{G_n,u_n}\}$ is not normal at λ_1. This contradicts the assumption that γ is contained in $\mathbb{C}\setminus\overline{\mathcal{Z}_\Delta}$, by the equivalence of the activity-locus and the zero-locus. □

3.5. Equality of the density-locus and the activity-locus for $\Delta \geq 3$

We first show that the inclusion $\overline{\mathcal{D}_\Delta} \subseteq \mathcal{A}_\Delta$ holds. Note that as $\mathcal{A}_\Delta$ is closed, it suffices to show $\mathcal{D}_\Delta \subseteq \mathcal{A}_\Delta$.

Theorem 3.5.1. *The density-locus is contained in the activity-locus. More precisely, we have $\mathcal{D}_\Delta \subseteq \mathcal{A}_\Delta$ for all $\Delta \geq 3$.*

Remark 3.5.2. Recall the remarkable Proposition 6 in [BGGv20], in which it is shown that non-real $\lambda \in \mathbb{Q}[i]$ outside the cardioid Λ_Δ are contained in the density-locus. As a consequence Theorem 3.5.1 implies that $\mathcal{Z}_\Delta$ is dense in the complement of the cardioid.

We will prove Theorem 3.5.1 by assuming there is a $\lambda_0 \in \mathcal{D}_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$ which will yield a contradiction. In order to do this we state and prove three helpful lemmas.

Lemma 3.5.3. *Let $\lambda_0 \in \mathbb{C}\setminus\mathcal{A}_\Delta$. Assume the family $\mathcal{R}_\Delta$ is normal on some open neighborhood U of λ_0 and that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to -1 for a sequence $\{(G_n, v_n)\}_{n\geq 1}$ of rooted graphs from $\mathcal{G}_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to -1 locally uniformly on U.*

Proof. It follows from the conclusion of Section 3.4, i.e. $\mathcal{A}_\Delta = \overline{\mathcal{Z}_\Delta}$, that $Z_G(\lambda) \neq 0$ for all $\lambda \in U$ and $G \in \mathcal{G}_\Delta$. Suppose $\{R_{G_n,v_n}\}_{n\geq 1}$ does not converge to -1 locally uniformly on U. Then, after taking a subsequence if necessary, we may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly on U to a non-constant holomorphic function f. Clearly $f(\lambda_0) = -1$. Since zeros of holomorphic functions are isolated there exists $\varepsilon > 0$ so that $\overline{B(\lambda_0, \varepsilon)} \subset U$ and such that

$$\delta := \inf_{\lambda\in\partial B(\lambda_0,\varepsilon)} |f(\lambda) + 1| > 0.$$

Let n be sufficiently large so that $|R_{G_n,v_n} - f| < \delta$ uniformly on $\overline{B(\lambda_0, \varepsilon)}$. Then

$$|(R_{G_n,v_n}(\lambda) + 1) - (f(\lambda) + 1)| < \delta < |f(\lambda) + 1| + |R_{G_n,v_n}(\lambda) + 1|$$

for all $\lambda \in \partial B(\lambda_0, \varepsilon)$. By Rouché's theorem there exists $\lambda_1 \in B(\lambda_0, \varepsilon)$ for which $R_{G_n,v_n}(\lambda_1) = -1$. By Lemma 3.2.1 it follows λ_1 is a zero of the independence polynomial Z_G for some graph G of maximum degree at most Δ, which is a contradiction as we assumed $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta = \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$. □

Lemma 3.5.4. *Let $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta$. Assume the family $\mathcal{R}_\Delta$ is normal on some open neighborhood U of λ_0 and that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to $\mu \leq -\frac{1}{4}$ for a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to μ locally uniformly on U.*

Proof. If this is not the case then, as in the previous lemma, we may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly to a non-constant holomorphic function f with $f(\lambda_0) = \mu$. By Rouché's theorem we can find $\lambda_1 \in U$ and n sufficiently large so that $R_{G_n,v_n}(\lambda_1) = \mu$, by the same argument as in the previous lemma.

Consider the family of rooted graphs $\{(\widetilde{G}_k, w_k)\}$ obtained by implementing (G_n, v_n) in every vertex of the rooted paths (P_k, w_k), where P_k is the path with k vertices and w_k is one of its extreme vertices. Since v_n has degree 1, the graph $\widetilde{G}_k$ has maximum degree at most Δ. Hence by Lemma 3.2.8 we have

$$R_{\widetilde{G}_k,w_k} = R_{P_k,w_k} \circ R_{G_n,v_n}$$

By Lemma 3.3.4 the family $\{R_{P_k,w_k}\}_{k\geq 1}$ is non-normal at μ, and therefore the family $\{R_{\widetilde{G}_k,w_k}\}$ is non-normal at λ_1, contradicting the fact that the family $\mathcal{R}_\Delta$ is normal on U. □

Lemma 3.5.5. *Assume there is a $\lambda_0 \in \mathcal{D}_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$. Denote U for an open neighborhood of λ_0 on which the family $\mathcal{R}_\Delta$ is normal. Assume furthermore that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to $\mu \in \mathbb{R}$ for a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$ in $\mathcal{G}^{\Delta-1}_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to μ locally uniformly on U.*

Proof. If $\mu = -1$ the result follows by Lemma 3.5.3. We may therefore assume that $\mu \neq -1$. Recall that we denote $f_\lambda(z) = \lambda/(1 + z)$. We will show for each $\mu \in \mathbb{R}$ there exists $\mu_1, \mu_2, \mu_3 \leq -\frac{1}{4}$ so that

$$f_{\mu_m} \circ \cdots \circ f_{\mu_1}(\mu) = -1,$$

for some $m \leq 3$. We distinguish between different cases:

1. $\mu \geq -3/4$. Take $\mu_1 = -1 - \mu \leq -1/4$; one can check $f_{\mu_1}(\mu) = -1$.

2. $\mu < -1$. Take $\mu_1 = -1/4$ and $\mu_2 = 1 - f_{\mu_1}(\mu)$, then $f_{\mu_1}(\mu) > 0 > -3/4$ and so $\mu_2 \leq -1/4$. One can check $f_{\mu_2} \circ f_{\mu_1}(\mu) = -1$.

3. $-1 < \mu < -3/4$. Take $\mu_1 = \mu_2 = -1/4$ and $\mu_3 = 1 - f_{\mu_2}(f_{\mu_1}(\mu))$, then $f_{\mu_1}(\mu) < -1$ so we see $\mu_3 \leq -1/4$. One can check $f_{\mu_3} \circ f_{\mu_2} \circ f_{\mu_1}(\mu) = -1$.

We may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly on U to a holomorphic function f with $f(\lambda_0) = \mu$. We want to show f is constant on U. Since the set $\{R_{G,v}(\lambda_0) : (G,v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$ by assumption, we can choose sequences of rooted graphs $\{(G^i_n, v^i_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ so that $\{R_{G^i_n,v^i_n}(\lambda_0)\}_{n\geq 1}$ converges to μ_i for each $i = 1, \dots, m$. By Lemma 3.5.4 every sequence $\{R_{G^i_n,v^i_n}\}_{n\geq 1}$ converges locally uniformly on U to the constant function μ_i for each i.

Consider for each $n \geq 1$ the rooted graph $(\widetilde{G}_n, v^m_n)$ obtained by implementing the rooted graphs (G_n, v_n), $(G^1_n, v^1_n), \dots, (G^m_n, v^m_n)$ on the vertices of the path P_{m+1} of length m. Note that $\widetilde{G}_n$ has maximum degree at most Δ.

It follows from Lemma 3.2.6 that

$$R_{\widetilde{G}_n,v^m_n}(\lambda) = f_{R_{G^m_n,v^m_n}(\lambda)} \circ \cdots \circ f_{R_{G^1_n,v^1_n}(\lambda)} \circ R_{G_n,v_n}(\lambda).$$

By our choice of the μ_i the sequence of ratios $\{R_{\widetilde{G}_n,v^m_n}(\lambda_0)\}_{n\geq 1}$ converges to $f_{\mu_m} \circ \cdots \circ f_{\mu_1}(\mu) = -1$. Hence by Lemma 3.5.3 the sequence of ratios $\{R_{\widetilde{G}_n,v^m_n}\}_{n\geq 1}$ converges locally uniformly to the constant function -1. Furthermore the sequence of ratios $\{R_{\widetilde{G}_n,v^m_n}\}_{n\geq 1}$ converges to the function $F := f_{\mu_m} \circ \cdots \circ f_{\mu_1} \circ f$. As $f_{\mu_m} \circ \cdots \circ f_{\mu_1}(z)$ is a non-constant holomorphic function and $F = -1$ on U, it follows that f is constant on U, as desired. □

We are now ready to prove Theorem 3.5.1.

Proof of Theorem 3.5.1. Assume for the purpose of contradiction that there exists $\lambda_0 \in \mathcal{D}^1_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$. We note that by Lemma 3.2.10 we know $\lambda_0 \neq 0$. Throughout the proof denote U for an open neighborhood of λ_0 on which the family $\mathcal{R}_\Delta$ is normal; we may assume $0 \notin U$ by taking U small enough. Assume first that λ_0 is not purely imaginary. Consider the real number $c = \frac{-|\lambda_0|^2}{2\mathrm{Re}\,\lambda_0}$ and notice that

$$\frac{\lambda_0^2}{\lambda_0 + c} = 2\mathrm{Re}\,\lambda_0 \in \mathbb{R}.$$

Choose two sequences of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$, $\{(H_n, w_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ so that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ and $\{R_{H_n,w_n}(\lambda_0)\}_{n\geq 1}$ converge to respectively 1 and c. By Lemma 3.5.5 we must have that these sequence of ratios converge locally uniformly on U to the respective constants 1 and c.

Consider the sequence of graphs $\widetilde{G}_{n\geq 1}$ constructed by merging v_n and w_n and by then connecting this vertex to a vertex $\widetilde{v}_n$.

It follows from Lemma 3.2.6 and Lemma 3.2.9 for all $\lambda \in U$ that

$$R_{\widetilde{G}_n,\widetilde{v}_n}(\lambda) = \frac{\lambda}{1 + \lambda^{-1} R_{G_n,v_n}(\lambda)\, R_{H_n,w_n}(\lambda)},$$

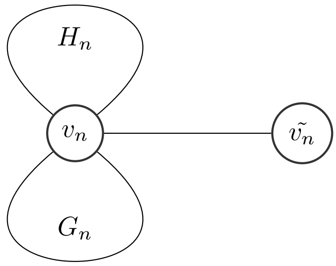

Figure 3.3: The rooted graph $\widetilde{G}_n$ in the proof of Theorem 3.5.1

where we use $0 \notin U$. Therefore the sequence of holomorphic functions $\{R_{\widetilde{G}_n,\widetilde{v}_n}\}_{n\geq 1}$ converges locally uniformly on U to the function $f(\lambda) = \frac{\lambda^2}{\lambda+c}$ as $n \to \infty$. Note that f is not a constant function and that $f(\lambda_0) \in \mathbb{R}$, contradicting Lemma 3.5.5. This contradiction completes the proof for when λ_0 is not purely imaginary.

Assume instead that λ_0 is purely imaginary and let $(G, v) \in \mathcal{G}^1_\Delta$ so that $R_{G,v}(\lambda_0)$ is not purely imaginary. For $c \in \mathbb{R}$, to be determined later, choose again two sequences of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$, $\{(H_n, w_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ such that sequences $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ and $\{R_{H_n,w_n}(\lambda_0)\}_{n\geq 1}$ converge to 1 and c respectively. Define for each $n \geq 1$, $(\tilde{G_n}, \widetilde{v_n})$ as above and let (K_n, v_n) be the rooted graph obtained from the disjoint union of $(\tilde{G_n}, \widetilde{v_n})$ and (G, v) by identifying the vertex $\widetilde{v}_n$ with v. It follows from Lemma 3.2.6 and Lemma 3.2.9 for $\lambda \in U$ that

$$R_{K_n,v_n}(\lambda) = \frac{R_{G,v}(\lambda)}{1 + \lambda^{-1} R_{G_n,v_n}(\lambda)\, R_{H_n,w_n}(\lambda)},$$

where we use $0 \notin U$. Thus in order to apply the same argument as before we require $c \in \mathbb{R}$ for which

$$\frac{\lambda_0 \cdot R_{G,v}(\lambda_0)}{\lambda_0 + c} \in \mathbb{R}.$$

It is clear that such a real number c exists, hence the identical argument leads to the desired contradiction. □

We will now show the other inclusion $\mathcal{A}_\Delta \subseteq \overline{D_\Delta}$ also holds for all $\Delta \geq 3$. We first show the inclusion holds for non-real parameters $\lambda \in \mathcal{A}_\Delta$.

Theorem 3.5.6. *Let $\Delta \geq 3$ and suppose that the family $\mathcal{R}_\Delta$ is not normal in any neighborhood of $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Then there exists λ_1 arbitrarily close to λ_0 for which the set $\{R_{G,v}(\lambda_1) : (G, v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$.*

Proof. Because $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$ there exists λ_2 arbitrarily close to λ_0 for which there is a graph G of maximum degree at most Δ such that $Z_G(\lambda_2) = 0$. We claim that we can assume $\lambda_2 \notin \mathbb{R}$. This is clear if $\lambda_0 \notin \mathbb{R}$. Moreover, if $\lambda_0 \in \mathbb{R}$ then λ_0 is a strictly positive real number. Because $Z_G(x) > 0$ for any positive real number x, it follows that λ_2 is necessarily not real as long as it is sufficiently close to λ_0.

By Lemma 3.2.3 there is a rooted tree $(T, v) \in \mathcal{G}^1_\Delta$ such that $R_{T,v}(\lambda_2) = -1$. Since the rational function $\lambda \mapsto R_{T,v}(\lambda)$ is non-constant, it is an open map. The image of a neighborhood of λ_2 therefore contains a small open real interval around -1. Recall that Lemma 3.3.2 states that for $\mu \in (-\infty, -1/4)$ the map $f_\mu : z \mapsto \mu/(1+z)$ is conjugate to a rotation $w \mapsto e^{i\theta} \cdot w$. Furthermore, by comparing tr^2 of both maps, it is not hard to see that those parameters μ for which f_μ is conjugate to an irrational rotation lie dense in $(-\infty, -1/4)$. Therefore we can choose a $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ arbitrarily close to λ_2 such that for $\mu := R_{T,v}(\lambda_1)$ the map f_μ is conjugate to an irrational rotation. From now on μ is fixed to be this value.

Let p, q be the two fixed points of the transformation f_μ. In Section 3.3.1 we explained that $\hat{\mathbb{C}} \setminus \{p, q\}$ is foliated by generalized circles invariant under f_μ, and on which f_μ acts conjugate to an irrational rotation. We denote the generalized circle through z by C_z, and write C_q and C_p for $\{q\}$ and $\{p\}$ respectively. The map $z \mapsto C_z$ is continuous as a map from $\hat{\mathbb{C}}$ to the space $\{K \subseteq \hat{\mathbb{C}} : K \text{ compact}\}$ equipped with the Hausdorff metric.

Our goal is to show that $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in $\hat{\mathbb{C}}$. We first claim that if $w \in \mathcal{R}^1_\Delta(\lambda_1)$, then $\mathcal{R}^2_\Delta(\lambda_1) \cap C_w$ is dense in C_w.

To prove the claim, let $(H, u) \in \mathcal{G}^1_\Delta$ be a rooted graph such that $R_{H,u}(\lambda_1) = w$. Let $\tilde{G}_n$ as follows be obtained from the path P_{n+1} on $n+1$ vertices, labeled v_0 up to v_n, by implementing (H, u) at v_0 and the rooted tree (T, v) at the remaining n vertices of P_{n+1}; see Figure 3.4. Now by Lemma 3.2.6 we have

$$R_{\tilde{G}_n, v_n}(\lambda_1) = f^n_\mu(R_{H,u}(\lambda_1)) = f^n_\mu(w).$$

Observe that for each $n \geq 1$ we have $(\tilde{G}_n, v_n) \in \mathcal{G}^2_\Delta$. Because f_μ acts conjugately to an irrational rotation on C_w it follows that $\mathcal{R}^2_\Delta(\lambda_1) \cap C_w$ is dense in C_w.

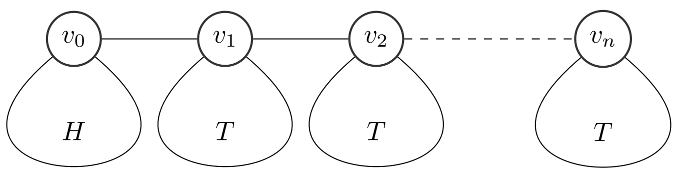

Figure 3.4: The graph $(\tilde{G}_n, v_n)$ in the proof of the claim

Because $\mu \in \mathcal{R}^1_\Delta(\lambda_1)$ and $C_\mu = \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ it follows from the claim that $\mathcal{R}^2_\Delta(\lambda_1) \cap \hat{\mathbb{R}}$ is dense in $\hat{\mathbb{R}}$. Observe that $f_{\lambda_1}(\hat{\mathbb{R}}) = \lambda_1 \cdot \hat{\mathbb{R}}$. So by attaching a vertex at the root with an edge, we obtain that $\mathcal{R}^1_\Delta(\lambda_1) \cap \lambda_1 \cdot \hat{\mathbb{R}}$ is dense in $\lambda_1 \cdot \hat{\mathbb{R}}$.

The set

$$U = \{z \in \hat{\mathbb{C}} : C_z \text{ intersects } \lambda_1 \cdot \hat{\mathbb{R}} \text{ transversely}\}$$

is an open set in $\hat{\mathbb{C}}$; see Figure 3.5. Because $\lambda_1 \notin \mathbb{R}$ we see that $C_{-1} = \hat{\mathbb{R}}$ intersects $\lambda_1 \cdot \hat{\mathbb{R}}$ transversely, and thus $-1 \in U$. The set U is contained in $\cup_{w \in \lambda_1 \cdot \hat{\mathbb{R}}} C_w$. Because $\mathcal{R}^1_\Delta(\lambda_1) \cap \lambda_1 \cdot \hat{\mathbb{R}}$ is dense in $\lambda_1 \cdot \hat{\mathbb{R}}$, it follows that $\cup_{w \in \mathcal{R}^1_\Delta(\lambda_1)} C_w$ is dense in U. From the claim we proved earlier, it follows $\mathcal{R}^2_\Delta(\lambda_1)$ is dense in U. Attaching a vertex to the root of a tree in $\mathcal{G}^2_\Delta$ with ratio r yields a rooted tree in $\mathcal{G}^1_\Delta$ with ratio $f_{\lambda_1}(r)$, and thus $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in the neighborhood $U_\infty := f_{\lambda_1}(U)$ of ∞.

For two rooted trees $(T_1, v_1) \in \mathcal{G}^1_\Delta$ and $(T_2, v_2) \in \mathcal{G}^2_\Delta$ with ratios r_1 and r_2 respectively we can define the rooted tree $(T_3, v_1) \in \mathcal{G}^2_\Delta$ by adding an edge between the roots of T_1 and T_2 and considering v_1 the root of the obtained tree. By Lemma 3.2.6 the ratio of (T_3, v_1) is given by

$$F(r_1, r_2) := f_{r_1}(r_2) = \frac{r_1}{1 + r_2}$$

under the assumption that this fraction is well defined, i.e., $(r_1, r_2) \notin \{(0, -1), (\infty, \infty)\}$. It is not hard to see that

$$F(U_\infty \times \hat{\mathbb{R}} \setminus \{(0, -1), (\infty, \infty)\}) = \hat{\mathbb{C}}.$$

Because $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in U_∞ and $\mathcal{R}^2_\Delta(\lambda_1)$ is dense in $\hat{\mathbb{R}}$ it follows that $\mathcal{R}^2_\Delta(\lambda_1)$ is dense in $\hat{\mathbb{C}}$. We finally conclude that $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in $f_{\lambda_1}(\hat{\mathbb{C}}) = \hat{\mathbb{C}}$. □

We can now finally prove the inclusion $\mathcal{A}_\Delta \subseteq \overline{\mathcal{D}_\Delta}$ building on Proposition 6 of [BGGv20] to deal with the real parameters $\lambda \in \mathcal{A}_\Delta$.

Theorem 3.5.7. *Let $\Delta \geq 3$. Then the activity locus is contained in the density locus, i.e. $\mathcal{A}_\Delta \subseteq \overline{\mathcal{D}_\Delta}$.*

Proof. Let $\lambda_0 \in \mathcal{A}_\Delta$. If $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $\lambda_0 \in \overline{\mathcal{D}_\Delta}$ follows from Theorem 3.5.6.

We know $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$. By Remark 3.2.11 we know that

$$\overline{\mathcal{Z}_\Delta} \cap \mathbb{R}_{\leq 0} = \mathbb{R}_{\leq 0} \setminus \operatorname{int}(\Lambda_\Delta)$$

for all $\Delta \geq 3$. Proposition 6 of [BGGv20] implies that $\mathbb{C} \setminus (\mathbb{R} \cup \Lambda_\Delta) \subseteq \overline{\mathcal{D}_\Delta}$ for $\Delta \geq 3$. Hence it follows that $\mathbb{R}_{\leq 0} \setminus \operatorname{int}(\Lambda_\Delta) \subseteq \overline{\mathcal{D}_\Delta}$, which completes the proof. □

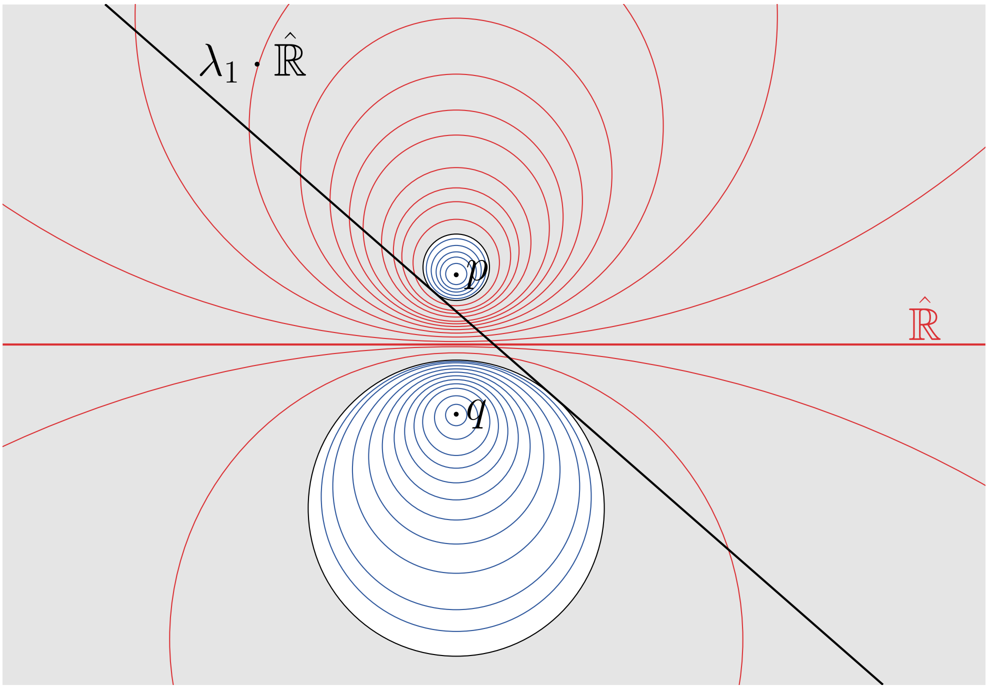

Figure 3.5: The generalized circles $\lambda_1 \cdot \hat{\mathbb{R}}$ and $\hat{\mathbb{R}}$ intersect in the points 0 and ∞. A region around 0 is drawn. The open set U is shaded in gray. Examples of generalized circles C_w that intersect $\lambda_1 \cdot \hat{\mathbb{R}}$ transversely are drawn in red, while examples of circles that do not intersect $\lambda_1 \cdot \hat{\mathbb{R}}$ are drawn in blue.

3.6. Density implies #P-hardness

In this section we will show that the density-locus is contained in the $\#\mathcal{P}$-locus. To prove the main result we will need to to show 'exponential' density for ratios of a specific family of trees: we need to get ε-close to a given point $P \in \mathbb{Q}[i]$ with ratios of trees of size at most $O(\log(1/\varepsilon) + \text{size}(P))$. Here $\text{size}(P)$ denotes the sum of the bit sizes of the real and imaginary part of P. Moreover, we denote for rational $\varepsilon > 0$ by $\text{size}(\varepsilon, P)$ the sum of the the bit size of ε and $\text{size}(P)$.

Let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Then the Möbius transformation f_{λ_0} is loxodromic (cf. Section 3.3.1) and hence has a repelling fixed point, which we denote by z_0.

Let

$$A := \{z \in \mathbb{C} : 2\pi/3 - 0.01 < \arg z < 2\pi/3, 1/17 < |z| < 1/16\}. \tag{3.7}$$

Let $\mathcal{T} = \{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G}_1, \overline{v}_1), \ldots, (\overline{G}_M, \overline{v}_M)\}$ be a family of rooted trees and U an open disk containing z_0. The pair $(\mathcal{T}, U)$ is called a *fast implementer* for λ_0 if the ratios $\mu_i := R_{G_i,v_i}(\lambda_0)$ and $\chi_i := R_{\overline{G}_i,\overline{v}_i}(\lambda_0)$ are such that the maps $g_i := f_{\mu_i} \circ f_{\chi_i}$ are loxodromic and satisfy

1. the attracting fixed point z_i of g_i lies in U for all i,
2. $\overline{U} \subseteq \cup_{i=1}^{M} g_i(U)$,
3. $g_i'(z) \in A$ for all i and all $z \in \overline{U}$,

and the disk U is such that

1. $\overline{U} \subset f_{\lambda_0}(U)$,
2. $\overline{U}$ does not contain the attracting fixed point of f_{λ_0},
3. U has three rational points on its boundary.

We have the following results concerning fast implementers.

Lemma 3.6.1. *Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in \mathcal{D}_\Delta$. Then there exists a fast implementer $(\mathcal{T}, U)$ for λ_0.*

Lemma 3.6.2. *Let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and assume that there exists a fast implementer $(\{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G}_1, \overline{v_1}), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$ for λ_0. Then, given $P \in \mathbb{C}$ and $\epsilon > 0$ there exists an algorithm that yields a sequence of ratios*

$$w_1, \ldots, w_K \in \{\lambda_0\} \cup \bigcup_{i=1}^{M} \{\mu_i := R_{G_i,v_i}(\lambda_0), \chi_i := R_{\overline{G}_i,\overline{v_i}}(\lambda_0)\}$$

such that $|(f_{w_K} \circ \cdots \circ f_{w_1})(0) - P| < \epsilon$, $w_K = \lambda_0$ and $K = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$. If $\lambda_0 \in \mathbb{Q}[i]$ and the input parameters P, ϵ are also in $\mathbb{Q}[i]$ then the algorithm runs in poly$(\text{size}(P, \epsilon))$ time.

We provide proofs for these lemmas in the next subsection, but first we collect some consequences.

Corollary 3.6.3. *Let $\Delta \geq 3$ be an integer. The set $\mathcal{D}_\Delta$ is an open set.*

Proof. Let $\lambda_0 \in \mathcal{D}_\Delta$. Let $(\mathcal{T}, U)$ be a fast implementer for λ_0, as is guaranteed to exist by Lemma 3.6.1. For λ nearby λ_0 we still have that the repelling fixed point of f_λ is contained in U, its attracting fixed point does not lie in $\overline{U}$ and $\overline{U} \subset f_\lambda(U)$. In other words $(\mathcal{T}, U)$ is a fast implementer for λ. Therefore, applying the algorithm of Lemma 3.6.2 to λ, we obtain that the collection of values $\{R_{G,v}(\lambda) \mid (G, v) \in \mathcal{G}_\Delta^1\}$ is dense in $\hat{\mathbb{C}}$ and hence $\lambda \in \mathcal{D}_\Delta$. □

For our next corollary we first need a result about the set

$$\mathcal{E}_\Delta := \{\lambda \in \mathbb{Q}[i] \mid Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

Lemma 3.6.4. *Let $\Delta \geq 3$ be an integer. Then the collection $\mathcal{E}_\Delta$ is contained in the set*

$$\left\{(a+ib)^{-1} \mid a, b \in \mathbb{Z},\ 0 < \sqrt{a^2+b^2} \leq \frac{\Delta^\Delta}{(\Delta-1)^{\Delta-1}}\right\}.$$

Proof. Let $\lambda \in \mathcal{E}_\Delta$. Then there exists a graph $G = (V, E)$ such that $1/\lambda$ is a root of $P(z) := z^{|V|} Z_G(1/z)$. Now P is a monic polynomial and therefore $1/\lambda \in \mathbb{Z}[i]$ (since $\mathbb{Z}[i]$ is integrally closed by Gauss's lemma). We also know that $|1/\lambda| \leq \frac{\Delta^\Delta}{(\Delta-1)^{\Delta-1}}$ by Lemma 3.2.10. This proves the lemma. □

Corollary 3.6.5. *Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in (\mathcal{D}_\Delta \cap \mathbb{Q}[i]) \setminus \mathcal{E}_\Delta$. Then given $P \in \mathbb{Q}[i]$ and rational $\varepsilon > 0$ there exists an algorithm that generates a rooted tree (T, v) such that $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$ and $Z^{out}_{T,v}(\lambda_0) \neq 0$, and outputs $Z^{in}_{T,v}(\lambda_0)$ and $Z^{out}_{T,v}(\lambda_0)$ in time bounded by poly*(size(ε, P))*.*

Proof. We first perform a brute force but constant time computation to obtain a fast implementer $(\{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G}_1, \overline{v_1}), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$ for λ_0. Denote for $i = 1, \ldots, M$, $\mu_i := R_{G_i, v_i}(\lambda_0)$ and $\chi_i := R_{\overline{G}_i, \overline{v_i}}(\lambda_0)$ and $g_i := f_{\mu_i} \circ f_{\chi_i}$.

The algorithm of Lemma 3.6.2 applied to P now returns in time *poly*(size(ε, P)) a sequence of ratios $\omega_1 \ldots, \omega_K \in \{\lambda_0\} \cup \bigcup_{i=1}^M \{\mu_i, \chi_i\}$ that, by Lemma 3.2.6, correspond to the implementation of the trees G_i and $\overline{G}_i$ on a path with $K = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon))$ vertices. The resulting rooted tree (T, v) has maximum degree at most Δ and root degree 1 and satisfies $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$. Denote the rooted tree corresponding to the sequence $\omega_1, \ldots \omega_i$ by (T_i, u_i). Then (T_{i+1}, u_{i+1}) is obtained from (T_i, u_i) by adding the edge $\{v_{i+1}, u_i\}$ to T_i and gluing a rooted tree $(H, v) \in \{K_1, (G_j, v_j), (\overline{G_j}, v_j) \mid j = 1, \ldots, M\}$ to u_{i+1} (here K_1 denotes a single vertex.) We then have

$$\left(Z^{\text{in}}_{T_{i+1}, u_{i+1}}(\lambda_0), Z^{\text{out}}_{T_{i+1}, u_{i+1}}(\lambda_0)\right) = \left(Z^{\text{in}}_{H,v}(\lambda_0) Z^{\text{out}}_{T_i, u_i}(\lambda_0), Z^{\text{out}}_{H,v}(\lambda_0) Z_{T_i}(\lambda_0)\right). \tag{3.8}$$

Note that (3.8) describes a simple recurrence to compute $Z^{\text{in}}_{T,v}(\lambda_0)$ and $Z^{\text{out}}_{T,v}(\lambda_0)$ in time linear in the number of vertices of T.

Finally, we remark that $Z^{\text{out}}_{T,v}(\lambda_0) \neq 0$ since $\lambda_0 \notin \mathcal{E}_\Delta$ by assumption. □

We can now prove the desired inclusion of the density-locus in the #$\mathcal{P}$-locus.

Theorem 3.6.6. *For any integer $\Delta \geq 3$ the density-locus $\overline{\mathcal{D}_\Delta}$ is contained in the $\#\mathcal{P}$-locus $\overline{\#\mathcal{P}_\Delta}$.*

Proof. We will show that for any $\lambda_0 \in (\mathcal{D}_\Delta \cap \mathbb{Q}[i]) \setminus \mathcal{E}_\Delta$ the computational problem #Hard-CoreNorm(λ_0, Δ) is #P-hard. Since $\mathcal{D}_\Delta$ is an open set and $\mathcal{E}_\Delta$ is finite, this implies the theorem.

This in fact follows directly from the work of [BGGv20]. Let us briefly indicate why. In [BGGv20, Section 6] the authors show that a polynomial time algorithm for #Hard-CoreNorm(λ_0, Δ) combined with the statement of Corollary 3.6.5 for λ_0 yields an algorithm that on input of a graph G of maximum degree at most Δ exactly computes $Z_G(1)$, the number of independent sets of G, in polynomial time in the number of vertices of G. (The algorithm is obtained by cleverly utilizing Corollary 3.6.5 for suitable choices of P and gluing combinations of the obtained trees to G and applying the assumed algorithm for #Hard-CoreNorm(λ_0, Δ) to the resulting graph.) Since determining $Z_G(1)$ is a known #P-complete problem, this implies that #Hard-CoreNorm(λ_0, Δ) is #P-hard. □

We note that our result does not allow us to say anything about the complexity of #Hard-CoreNorm(λ_0, Δ) for $\lambda_0 \in \partial(\mathcal{D}_\Delta) \cap \mathbb{Q}[i]$. For example, for $\lambda_0 \in \partial(\mathcal{D}_\Delta) \cap \mathbb{Q}_{\leq 0}$ it follows from [BGGv20] that the problem #Hard-CoreNorm(λ, Δ) is #P-hard. For $\lambda \in \mathbb{Q}$ such that $\lambda \geq \lambda_c(\Delta) := \frac{(\Delta-1)^{\Delta-1)}}{(\Delta-2)^\Delta}$ we know from [BGGv20] that $\lambda \in \partial(\mathcal{D}_\Delta)$, while the complexity of #Hard-CoreNorm$(\lambda_c(\Delta), \Delta)$ is unknown. For $\lambda > \lambda_c(\Delta)$ the problem Hard-CoreNorm(λ, Δ) is only known to be NP-hard [SS14], and unlikely to be #P-hard cf. [BGGv20].

3.6.1. Proofs of Lemma 3.6.1 and Lemma 3.6.2

The next lemma directly implies Lemma 3.6.1.

Lemma 3.6.7. *Given $z_0 \in \mathbb{C} \setminus \{-1, 0\}$, a dense subset D of $\mathbb{C}^*$ and a non-empty open subset A of the unit disk $\mathbb{D}$ then there exists a finite set of tuples $\{(\mu_i, \chi_i)\}_{i=1}^M \subset D \times D$ and an arbitrarily small open disk $U \subseteq \mathbb{C}$ containing z_0 such that the maps $g_i := f_{\mu_i} \circ f_{\chi_i}$ are loxodromic Möbius transformations and*

1. *the attracting fixed point z_i of g_i lies in U for all i,*
2. *$\overline{U} \subseteq \cup_{i=1}^M g_i(U)$,*
3. *$g_i'(z) \in A$ for all i and all $z \in \overline{U}$.*

Proof. We denote $g_{\mu,\chi} = f_\mu \circ f_\chi$ throughout this proof. Note $g_{\mu,\chi}$ is a Möbius transformation for $\mu, \chi \neq 0$. Without loss of generality assume that A is bounded away from 0. Take $\alpha \in A$ such that $\alpha \neq \frac{z_0}{z_0+1}$. Note that $\chi_0 = \frac{(z_0+1)^2\alpha}{z_0-(z_0+1)\alpha}$ and

$\mu_0 = \frac{z_0(z_0+\chi_0+1)}{z_0+1}$ are nonzero and well defined as $z_0 \neq -1, 0$ and $\alpha \neq \frac{z_0}{z_0+1}, 0$. Furthermore we have $g_{\mu_0,\chi_0}(z_0) = z_0$ and $g'_{\mu_0,\chi_0}(z_0) = \alpha$.

Define $F : (\mathbb{C}^*)^2 \times \mathbb{C} \to \hat{\mathbb{C}}$ as $F(\mu, \chi, z) = g_{\mu,\chi}(z) - z$. Now as $\frac{\partial F}{\partial z}(\mu_0, \chi_0, z_0) = \alpha - 1 \neq 0$, the implicit function theorem gives an open neighborhood W of (μ_0, χ_0) and a holomorphic function $h : W \to \mathbb{C}$ with $h(\mu_0, \chi_0) = z_0$ and $F(\mu, \chi, h(\mu, \chi)) = 0$ for all $(\mu, \chi) \in W$. As h is a non-constant holomorphic map, it is an open map and so $h(W)$ is an open neighborhood of z_0.

Let $B \subseteq A$ be an open set in $\mathbb{C}$ with $\alpha \in B$ and $\overline{B} \subseteq A$. Denote $H(\mu, \chi, z) = \frac{\partial g_{\mu,\chi}}{\partial z}(z) = \frac{\mu\chi}{(1+z+\chi)^2}$ and note that H is continuous as a function on $\mathbb{C}^3 \setminus \{(\mu, \chi, z) : \chi + z + 1 = 0\}$. It follows there is an open neighborhood C of z_0 such that we have $H(\mu_0, \chi_0, z) \in B$ for all $z \in C$. We have $\{(\mu_0, \chi_0)\} \times \overline{C} \subseteq H^{-1}(\overline{B}) \subseteq H^{-1}(A)$. As $H^{-1}(A)$ is an open subset of $\mathbb{C}^3 \setminus \{(\mu, \chi, z) : \chi + z + 1 = 0\}$ containing the set $\{(\mu_0, \chi_0)\} \times \overline{C}$, by a compactness argument it follows that $H^{-1}(A)$ contains a set of the form $L \times C$ for some open neighborhood L of the point (μ_0, χ_0). Hence the set $Y := L \cap W \cap h^{-1}(C)$ is an open neighborhood of (μ_0, χ_0) and so $h(Y)$ is an open neighborhood of z_0.

Take $U \subset h(Y)$ an open disk containing z_0, such that $\overline{U} \subset h(Y)$. Note that we can take U arbitrarily small. By construction, we have for all $(\mu, \chi) \in Y$ that $g'_{\mu,\chi}(z) \in A$ for all $z \in \overline{U}$. Furthermore, we have $F(\mu, \chi, h(\mu, \chi)) = 0$, so $h(\mu, \chi)$ is the attracting fixed point of $g_{\mu,\chi}$. Note $D \times D$ is dense in $h^{-1}(U)$, hence the fixed points of $g_{\mu,\chi}$ for $(\mu, \chi) \in h^{-1}(U) \cap (D \times D)$ lie dense in U. There is a uniform lower bound on the diameters of the disks $g_{\mu,\chi}(U)$ for $(\mu, \chi) \in h^{-1}(U)$, because $g'_{\mu,\chi}(z) \in A$ for all $z \in U$ and A is bounded away from 0. Therefore

$$\left\{g_{\mu,\chi}(U) : (\mu, \chi) \in h^{-1}(U) \cap (D \times D)\right\}$$

is an open cover of $\overline{U}$. As $\overline{U}$ is compact, there is a finite set of tuples $\{(\mu_i, \chi_i)\}_{i=1}^M \subseteq h^{-1}(U) \cap (D \times D)$ such that $\overline{U} \subseteq \cup_{i=1}^M g_{\mu_i,\chi_i}(U)$. We thus found the desired set of tuples in $D \times D$ and the open disk U containing z_0. □

We next focus on proving Lemma 3.6.2. To this end let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and let $(\{(G_1, v_1), \ldots, (G_m, v_m), (\overline{G}_1, \overline{v_1}), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$ be a fast implementer for λ_0. We fix these throughout this section. We denote the repelling fixed point of f_{λ_0} by z_0 and we denote for $i = 1, \ldots, M$, $\mu_i := R_{G_i,v_i}$, $\chi_i := R_{\overline{G}_i,\overline{v_i}}$ and $g_i := f_{\mu_i} \circ f_{\chi_i}$. We distinguish between the case that P is close to the attracting fixed point of f_{λ_0} and the case that it is not. In the first case the algorithm is much simpler.

Let a be the attracting fixed point of f_{λ_0}. Because $f_{\lambda_0}(\infty) = 0$ we observe that ∞ is not a fixed point and thus $a \in \mathbb{C}$. Suppose that $|P - a| \leq \epsilon/2$. Choose $\delta > 0$ for which there is a constant $\eta < 1$ such that $|f'_{\lambda_0}(z)| < \eta$ for all $z \in B(a, \delta)$. The point 0 is not a fixed point of f_{λ_0} because $f_{\lambda_0}(0) = \lambda_0 \neq 0$ and thus $f^n_{\lambda_0}(0)$ converges to a as $n \to \infty$. It follows that there is a constant N_0 such that $f_{\lambda_0}^{N_0}(0) \in B(a,\delta)$. Note that the value of N_0 does not depend on the input parameters. Now let $N_\epsilon = \max\{\lceil \log_\eta(\frac{\epsilon}{2\delta})\rceil, 0\} + 1$. Then for any $w \in B(a,\delta)$ we have

$$|f_{\lambda_0}^{N_\epsilon}(w) - a| < \eta^{N_\epsilon}|w-a| < \epsilon/2$$

and thus for $K = N_0 + N_\epsilon$ we have

$$|f_{\lambda_0}^{K}(0) - P| \leq |f_{\lambda_0}^{N_\epsilon}(f_{\lambda_0}^{N}(0)) - a| + |a-P| < \epsilon/2 + \epsilon/2 < \epsilon.$$

Because $K = \mathcal{O}(\log(1/\epsilon))$ this describes the algorithm when $|P-a| \leq \epsilon/2$.

The case that $|P-a| > \epsilon/2$ is more involved and we will describe the algorithm as a sequence of simpler subroutines. Just as in Lemma 3.6.7 let z_i denote the attracting fixed point of g_i. We will first show that, given a parameter Q that is at most distance ϵ away from some z_i, we only have to apply g_i to the starting value 0 an $\mathcal{O}(\log(1/\epsilon))$ number of times to get ϵ close to Q. Morally this should be true because after a fixed number of steps the orbit of 0 converges exponentially quickly to z_i and because z_i is close to Q the orbit should also get close to Q. The only way that this reasoning could be incorrect is if z_i and Q are almost ϵ apart and the orbit of 0 converges to z_i from the wrong direction. An example of this is given by the red orbit in Figure 3.6. This is the reason that we required $g_i'(z_i)$ to have an argument close to $2\pi/3$ in which case the above reasoning is correct as the green orbit in Figure 3.6 demonstrates. In the following proof most time is spent on formalizing this argument.

Lemma 3.6.8. *There exists an algorithm that, given $\epsilon > 0$, $Q \in \mathbb{C}$ and $i \in \{1,\ldots,M\}$ such that $|Q - z_i| < \epsilon$, yields an integer K such that $|g_i^K(0) - Q| < \epsilon$, where $K = \mathcal{O}(\log(1/\epsilon))$. If $\lambda_0 \in \mathbb{Q}[i]$ and the input parameters Q, ϵ lie in $\mathbb{Q}[i]$ then the algorithm runs in poly*$(\text{size}(Q,\epsilon))$ *time.*

Proof. Let δ be such that $B(z_i,\delta) \subseteq U$ and let $\epsilon' = \min\{\epsilon/2, \delta\}$. Note that $g_i(0) = \frac{\mu_i}{1+\chi_i} \neq 0$ and thus 0 is not a fixed point of g_i. Because z_i is the attracting fixed point of g_i we can find (in a similar way as described above) a positive integer $\tilde{K}$ that is $\mathcal{O}(\log(1/\epsilon')) = \mathcal{O}(\log(1/\epsilon))$ such that $|g_i^{\tilde{K}}(0) - z_i| < \epsilon'$. If $|Q - z_i| \leq \epsilon/2$ we take $K = \tilde{K}$ because

$$|g_i^{\tilde{K}}(0) - Q| \leq |g_i^{\tilde{K}}(0) - z_i| + |Q - z_i| < \epsilon' + \epsilon/2 \leq \epsilon.$$

So from now on we assume that $|Q - z_i| > \epsilon/2$. Define the following sector S of $B(z_i, \epsilon')$

$$S = \{z_i + \xi \cdot \left(\frac{Q - z_i}{|Q - z_i|}\right) : |\xi| < \epsilon', -\pi/3 \leq \arg(\xi) \leq \pi/3\}.$$

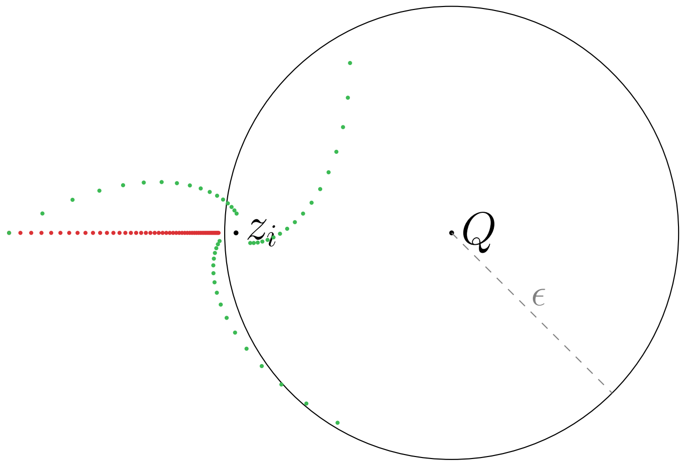

Figure 3.6: An example of two orbits with the same initial value converging to z_i under iteration of two different maps. For the red orbit the derivative at z_i is real. For the green orbit the derivative at z_i has the same magnitude, while its argument is a little less than $2\pi/3$.

We claim that $S \subseteq B(Q, \epsilon)$. To show this note that

$$\left| Q - \left[z_i + \xi \cdot \left(\frac{Q - z_i}{|Q - z_i|} \right) \right] \right| = |Q - z_i| \cdot \left| 1 - \frac{\xi}{|Q - z_i|} \right| < \epsilon \cdot \left| 1 - \frac{\xi}{|Q - z_i|} \right|.$$

If ξ satisfies the conditions in the definition of S the complex number $\xi/|Q - z_i|$ has its argument between $-\pi/3$ and $\pi/3$. Furthermore, because $|\xi| < \epsilon/2$ and $|Q - z_i| > \epsilon/2$, its norm is bounded above by 1. It follows that the norm of $1 - \xi/|Q - z_i|$ is at most 1. Indeed, because $|1 - re^{i\phi}|^2 = 1 + r^2 - 2r\cos(\phi)$, the statement $|1 - re^{i\phi}| \leq 1$ is equivalent to $r = 0$ or $r \leq 2\cos(\phi)$, and the latter is satisfied for all $0 \leq r \leq 1$ and $-\pi/3 \leq \phi \leq \pi/3$. The claim follows.

We now claim that for $w \in B(z_i, \epsilon')$ the intersection of $\{w, g_i(w), g_i^2(w), g_i^3(w)\}$ with S is not empty. Note that because $\epsilon' \leq \delta$ we have that $B(z_i, \epsilon') \subseteq U$ and thus $g'(w) \in A$ for every $w \in B(z_i, \epsilon')$. It follows that applying g_i to w has the effect of rotating around z_i with an angle strictly between $2\pi/3 - 0.01$ and $2\pi/3$ and contracting towards z_i. Therefore applying g_i to w three times has the effect of rotating w a little less than a full circle around z_i, with steps that are strictly less than $2\pi/3$ radians. Because the internal angle of the sector S is $2\pi/3$ the orbit $w, g_i(w), g_i^2(w), g_i^3(w)$ cannot miss S.

To summarize the algorithm, define ϵ' and determine $\tilde{K}$ such that $g_i^{\tilde{K}}(0) \in B(z_i, \epsilon')$. Then determine a $j \in \{0, 1, 2, 3\}$ such that $|g_i^{\tilde{K}+j}(0) - Q| < \epsilon$. We have shown that there exists at least one such j. The output of the algorithm is $K = \tilde{K} + j$.

$\square$

We shall now describe an algorithm that does the following. Given a disk D of radius ϵ inside U, it returns an index i, a disk $\tilde{D}$ of radius at least ϵ containing z_i and a sequence of indices $j_1, \ldots, j_K$ such that $(g_{j_1} \circ \cdots \circ g_{j_{K-1}})(\tilde{D}) \subseteq D$. To describe the computational complexity of this algorithm, we need a finite way to represent disks in the complex plane. A pleasant way for our purposes is to represent an open disk D by three distinct points P_1, P_2, P_3 on its boundary. This is an unambiguous way to represent a disk because three different points on a circle uniquely determine that circle. If $P_1, P_2, P_3 \in \mathbb{Q}[i]$ we say that the disk D is rational and that $\text{size}(D) = \text{size}(P_1) + \text{size}(P_2) + \text{size}(P_3)$.

Recall that a Möbius transformation maps generalized circles (circles and straight lines) to generalized circles. In what follows, we will apply Möbius transformations to disks in the complex plane. We shall make sure that the image of the disks involved is always again a disk in the complex plane and not the complement of a disk or a half-plane as it could in general be. Therefore, if D is a disk represented by P_1, P_2 and P_3 and g is one of the Möbius transformations, then $g(D)$ will be a disk represented by $g(P_1), g(P_2)$ and $g(P_3)$. Note that if D is rational and g has rational coefficients then $g(D)$ is again rational. The Möbius transformations that we will apply come from a fixed finite set and thus there is a fixed constant C for which $\text{size}(g(D)) \leq C \cdot \text{size}(D)$.

Let us denote $P_j = x_j + iy_j$ for $j \in \{1, 2, 3\}$. The center $c_D = x + yi$ of the disk D is known as the circumcenter of the triangle with vertices P_1, P_2 and P_3. The coordinates of c_D can be calculated using complicated looking but easily derivable formulas

$$x = \frac{(x_1^2 + y_1^2)(y_2 - y_3) + (x_2^2 + y_2^2)(y_3 - y_1) + (x_3^2 + y_3^2)(y_1 - y_2)}{2(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))},$$
$$y = \frac{(x_1^2 + y_1^2)(x_3 - x_2) + (x_2^2 + y_2^2)(x_1 - x_3) + (x_3^2 + y_3^2)(x_2 - x_1)}{2(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))}.$$

We note that if D is rational, then c_D is rational and can be computed in time linear in $\text{size}(D)$. We can also decide whether a given point $Q \in \mathbb{Q}[i]$ lies in a given rational disk D in time linear in $\text{size}(Q)$ and $\text{size}(D)$.

We next need a lemma concerning a geometric construction involving disks.

Lemma 3.6.9. *There exists an algorithm that, given two disks A, B in the complex plane for which the center of A is contained in B and B is not contained in A, returns a disk D contained in both A and B, such that the area of D is at least $1/128$ times that of A. Furthermore, if A and B are rational then D is*

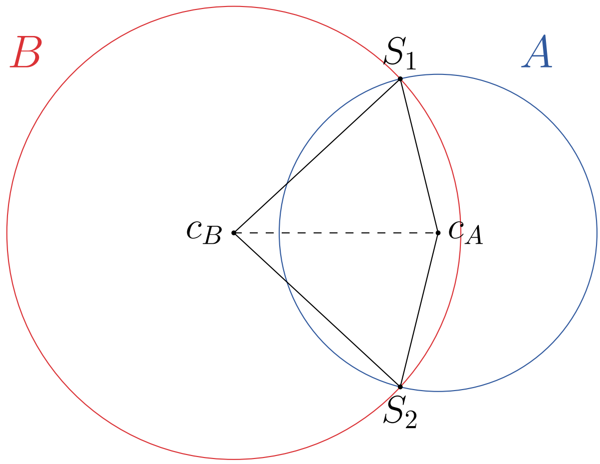

Figure 3.7: -

rational and both the running time of the algorithm and size(D) *are bounded by a fixed constant times* size(A, B).

Proof. For a disk D we denote its center by c_D and its radius by r_D and recall that if D is rational then c_D is rational and can be computed efficiently. For two distinct points P, Q on the boundary of D we denote the closed counterclockwise arc from P to Q by $\mathrm{Arc}_D(P, Q)$ and we denote the sector given by the convex hull of $\mathrm{Arc}_D(P, Q)$ and c_D by $\mathrm{Sec}_D(P, Q)$. We note that the internal angle of both $\mathrm{Arc}_D(P, Q)$ and $\mathrm{Sec}_D(P, Q)$ is given by the arclength of $\mathrm{Arc}_D(P, Q)$ divided by r_D. We claim that either a sector of A whose internal angle is greater than $2\pi/3$ is contained in the closure of B or a sector of B whose internal angle is greater than $2\pi/3$ is contained in the closure of A.

If the boundaries of A and B either do not intersect or intersect in one point then A is contained in B and the claim is obvious. Otherwise let S_1, S_2 be the two intersection points such that $\mathrm{Arc}_A(S_1, S_2)$ is contained in B and thus $\mathrm{Arc}_B(S_2, S_1)$ is contained in A, see Figure 3.7. Consider the quadrilateral $\square c_B S_1 c_A S_2$ and suppose for the sake of contradiction that the internal angles at both c_A and c_B are at most $2\pi/3$, then the sum of the internal angles at S_1 and S_2 is at least $2\pi/3$ and since they are equal by symmetry the internal angle at S_1 is at least $\pi/3$. By then considering the triangle $\triangle c_B S_1 c_A$ it should follow that $|c_B - c_A| \geq |c_B - S_1| = r_B$, which contradicts the assumption that c_A is contained in B. We therefore find that either the angle $\angle S_2 c_B S_1$ or $\angle S_2 c_A S_1$ is at least $2\pi/3$. If the latter is the case then both $\mathrm{Arc}_A(S_1, S_2)$ and c_A are contained in the closure of B and thus the same is true for $\mathrm{Sec}_A(S_1, S_2)$. If the angle $\angle S_2 c_A S_1$

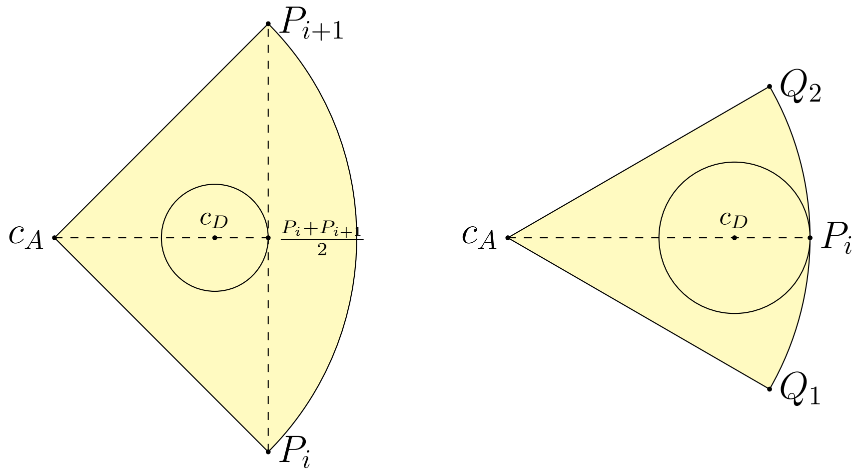

Figure 3.8: -

is less than $2\pi/3$, then $\angle S_2 c_B S_1$ is at least $2\pi/3$. It follows that $\angle c_A c_B S_1$ is the largest internal angle of the triangle $\triangle c_B S_1 c_A$ and thus $|c_A - c_B| < |c_A - S_1| = r_A$, from which it follows that c_B is contained in A. So in this case $\mathrm{Sec}_B\,(S_2, S_1)$ is contained in A.

For the algorithm we do not need to know whether a large sector of B is contained in A or vice versa. Assume for simplicity that a sector S of A with internal angle at least $2\pi/3$ is contained in B. Take S to be as large as possible. In the case that A is contained in B we let $S = A$. Let P_0 denote one of the given (rational) points on the boundary of A. Now for $i = 0, 1, 2$ inductively define P_{i+1} as P_i rotated around c_A with an angle of $\pi/2$. Calculating these points is computationally easy because $P_{i+1} = c_A + i(P_i - c_A)$. Now one of the following is guaranteed to be the case.

1. Two consecutive points P_i and P_{i+1} are contained in S.

2. There is a unique index i such that $P_i \in S$.

Determining which of the two cases is true is easy since checking membership of S is equivalent to checking membership of B. In the first case we note that $\mathrm{Sec}_A\,(P_i, P_{i+1})$ is contained in both the closures of A and B. Now let $R = (P_i + P_{i+1})/2$ and $c_D = (c_A + 3R)/4$ and let D be the disk with center c_D and the point R on the boundary; see Figure 3.8. It can be checked that D is now contained in $\mathrm{Sec}_A\,(P_i, P_{i+1})$ and its area is $1/32$ that of A.

In the second case note that the arc $\mathrm{Arc}_A\,(Q_1, Q_2)$ containing P_i such that the internal angle of both $\mathrm{Arc}_A(Q_1, P_i)$ and $\mathrm{Arc}_A(P_i, Q_2)$ is $\pi/6$ contained in S, otherwise, since the internal angle of S is at least $2\pi/3$, S has to contain two consecutive points P_i and P_{i+1}. Now let D be the disk with center $(c_A + 3P_i)/4$ containing P_i on its boundary; see Figure 3.8. It can be checked that D is contained in $\mathrm{Sec}_A(Q_1, Q_2)$ and its area is $1/16$ that of A.

The algorithm above is only guaranteed to successfully return a disk contained in both A and B if a large sector of A is contained in B. Therefore we have to run the algorithm described above (and let it fail if neither of the two described cases is true) and run the same algorithm with the roles of A and B reversed. If both instances of the algorithm return a disk, say D_1 and D_2, then at least one of them is contained in both A and B but the other one might not be. So in this case we have to run one final check to see which one of the two disks is indeed contained in both A and B (which is computationally easy). If they both are we can return either D_1 or D_2.

In conclusion we obtain a disk D contained in both A and B that is either at least $1/32$ of the area of A or $1/32$ of the area of B. Because the area of B is at least $1/4$ that of A (otherwise $r_B < r_A/2$ and $c_A \in B$ would imply $B \subseteq A$), we can conclude that the area of D is at least $(1/4) \cdot (1/32) = 1/128$ that of A.

□

Lemma 3.6.10. *There exists an algorithm that, given a disk $D_1 \subseteq U$ with radius ϵ, returns an index $i \in \{1, \ldots, M\}$, a sequence of indices $j_1, \ldots, j_K \in \{1, \ldots, M\}$ and a disk D_K such that $z_i \in D_K$, the radius of D_K is at least ϵ and $(g_{j_1} \circ \cdots \circ g_{j_{K-1}})(D_K) \subseteq D_1$. Furthermore, $K = \mathcal{O}(\log(1/\epsilon))$. If $\lambda_0 \in \mathbb{Q}[i]$ and D_1 and ϵ are both rational then D_K is rational and the algorithm runs in poly$(\mathrm{size}(D_1, \epsilon))$ time.*

Proof. For every index i let $U_i = g_i(U)$. Recall that we took U as a rational disk. Because the derivative of g_i is bounded on U, the image U_i is again a disk in the complex plane. If $\lambda_0 \in \mathbb{Q}[i]$, the coefficients of g_i are also in $\mathbb{Q}[i]$ and then U_i is rational. The point z_i is fixed for g_i and contained in U, therefore also contained in U_i. We describe a procedure to generate a sequence of disks $\{D_n\}_{n \geq 1}$, starting with the given disk D_1. The sequence is defined in such a way such that $D_n \subseteq U$ for all n, which is, by assumption, the case for D_1.

Suppose we have arrived at disk $D_n \subseteq U$. Check whether there is any index $i \in \{1, \ldots, M\}$ such that $z_i \in D_n$; if there is stop the procedure and let $K = n$. Otherwise, let m_n be the center of D_n and determine an index $j_n \in \{1, \ldots, M\}$ such that $m_n \in U_{j_n}$. Such an index must exist because $m_n \in U$ and the disks $U_1, \ldots, U_M$ cover U. Because the center of D_n lies in U_{j_n} but U_{j_n} is not contained in D_n (z_{j_n} does not lie in D_n) we can use Lemma 3.6.9 to generate a disk $\tilde{D}_n$ that is contained in both D_n and U_{j_n} whose area is at least $1/128$ times that of D_n and which can be assumed to be rational if D_n is. Now we define $D_{n+1} = g_{j_n}^{-1}(\tilde{D}_n)$.

Because $\tilde{D}_n \subseteq U_{j_n}$ the disk D_{n+1} lies in U and because $\tilde{D}_n \subseteq D_n$ the disk $g_{j_n}(D_{n+1})$ lies in D_n. Furthermore, by the properties of the fast implementer, $g_{j_n}^{-1}$ is expanding the norm on U_{j_n} with a factor at least 16, the area of D_{n+1} is at least $16^2 \cdot (1/128) = 2$ times that of D_n. This means that the area of D_n grows exponentially with n and thus, because the area of U is fixed, the procedure will terminate after $K = \mathcal{O}(\log(1/\epsilon))$ steps. Note that indeed $z_i \in D_K$ for some i, the radius of D_K is at least that of D_1 and $(g_{j_1} \circ \cdots \circ j_{K-1})(D_K) \subseteq D_1$. □

Recall that we had defined a and z_0 to be the attracting and repelling fixed point of f_{λ_0} respectively. We have already described the algorithm in Lemma 3.6.2 when P is near a. What follows is the final lemma needed to describe the algorithm when P is not near a.

Lemma 3.6.11. *There exists a fixed positive constant c and an algorithm that, given $P \in \mathbb{C}$ and $\epsilon > 0$ such that $|P - a| \geq \epsilon/2$, yields a disk $D \subseteq U$ and a positive integer K with $f_{\lambda_0}^K(D) \subseteq B(P, \epsilon)$, such that the radius of D is at least $c \cdot \min(\epsilon, \epsilon/|P|^2)$ and $K = \mathcal{O}(\log(1/\epsilon))$. If both λ_0 and the input parameters are in $\mathbb{Q}[i]$ then D is also rational and both* size(D) *and the running time of the algorithm is polynomial in* size(P, ϵ).

Proof. Let V be a compact neighborhood of a such that $|f'_{\lambda_0}(z)| < \xi < 1$ for some constant ξ for all $z \in V$. We first claim that there is an integer N such that the complement of $f_{\lambda_0}^N(U)$ is contained in V. To show this let $U_n = f_{\lambda_0}^n(U)$. Recall that we assumed that $\overline{U} = \overline{U_0} \subseteq U_1$ and thus inductively $\overline{U_n} \subseteq U_{n+1}$. Under iteration of $f_{\lambda_0}^{-1}$ every initial point that is not a converges to z_0 and thus eventually lands in U. Therefore,

$$\bigcup_{n=1}^{\infty} U_n = \widehat{\mathbb{C}} \setminus \{a\}.$$

For n large enough the point ∞ is contained in U_n and from then on the sequence $(U_n)^c$ consists of nested disks, containing a, whose radii must necessarily converge to 0, proving that there is an N such that $(U_N)^c$ is contained in V. Note that N does not depend on the input parameters. Let D_0 be the interior of $(U_N)^c$, this is a rational disk whose size also does not depend on the input, and let $D_i = f_{\lambda_0}^i(D_0)$ for $i \in \{1, 2, 3\}$. Let $\delta > 0$ be a constant smaller than the minimum distance between points on the boundary of D_i and D_{i-1}. From now on we will assume that $\epsilon < \delta$. Finally let $h = f_{\lambda_0}^{-(N+3)}$.

If P lies outside D_2, then let $\tilde{D}$ be the disk of radius ϵ represented by $P + \epsilon, P + i\epsilon$ and $P - \epsilon$. Note that $\tilde{D}$ lies outside D_3 and thus $D = h(\tilde{D}) \subset U$. Because the derivative of a Möbius transformation of the form $z \mapsto (az+b)/(cz+d)$ is $z \mapsto \frac{ad-bc}{(cz+d)^2}$ there is a constant c_1 such that the radius of D is at least $c_1 \cdot \min(\epsilon, \epsilon/|P|^2)$. In this case D and $K = N + 3$ are the output of the algorithm.

If P lies inside D_2 we determine N_0 such that for $P_{N_0} := f_\lambda^{-N_0}(P)$ we have $P_{N_0} \in D_1$ and $P_{N_0} \notin D_2$. Because $|P - a| \geq \epsilon/2$ and $D_1 \subset V$ we find that $N_0 = \mathcal{O}(\log(1/\epsilon))$. Let $\tilde{D}$ be the disk of radius ϵ represented by $P_{N_0} + \epsilon, P_{N_0} + i\epsilon$ and $P_{N_0} - \epsilon$. Note that again $\tilde{D}$ lies outside D_3 and thus $D = h(\tilde{D}) \subseteq U$. Furthermore, because $\tilde{D} \subset D_0 \subset V$ and f_{λ_0} is attracting on V it follows that $f_{\lambda_0}^{N_0}(\tilde{D}) \subseteq B(P, \epsilon)$. Finally, if we let c_2 be the minimum of $|h'(z)|$ for $z \in D_0$, we find that the radius of D is at least $c_2 \cdot \epsilon$. So in this case the output is the disk D together with $K = N_0 + N + 3$.

□

We are now ready to complete the proof of Lemma 3.6.2.

Proof of Lemma 3.6.2. Recall we had defined a to be the attracting fixed point of f_{λ_0} and that we already described the algorithm in the case that $|P - a| < \epsilon/2$, therefore we assume that $|P - a| \geq \epsilon/2$.

It follows from Lemma 3.6.11 that we can generate a disk $D_1 \subseteq U$ of radius $r = \mathcal{O}(\min\{\epsilon, \epsilon/|P|^2\})$ and size polynomial in size(P, ϵ) together with a positive integer K_1 that is $\mathcal{O}(\log(1/\epsilon))$ such that $f_{\lambda_0}^{K_1}(D_1)$ is contained in $B(P, \epsilon)$. From Lemma 3.6.10 it follows that we can find an index $i \in \{1, \ldots, M\}$, a sequence of indices $j_1, \ldots, j_{K_2}$ and a disk D_2 such that $z_i \in D_2$, the radius of D_2 is at least r, its size is polynomial in size(r, D_1), which is again polynomial in size(ϵ, P), and such that

$$(g_{j_1} \circ \cdots \circ g_{j_{K_2}})(D_2) \subseteq D_1.$$

Furthermore $K_2 = \mathcal{O}(\log(1/r)) = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$. Finally let Q be the center of D_2 and note that size(Q) is polynomial in size(D_2). Then, because $|Q - z_i| < r$, it follows from Lemma 3.6.8 that we can generate a K_3 such that $g_i^{K_3}(0) \in D_2$, where $K_3 = \mathcal{O}(\log(1/r)) = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$. Concluding, we find that

$$(f_{\lambda_0}^{K_1} \circ g_{j_1} \circ \cdots \circ g_{j_{K_2}} \circ g_i^{K_3})(0) \in B(P, \epsilon).$$

Furthermore, adding the running times of the individual algorithms, we find that the final algorithm runs in *poly*(size(P, ϵ)) time. □

3.7. Activity and zeros for Cayley trees

For fixed $\Delta \geq 2$ notions such as the activity-locus and the zero sets can be considered for subcollections of $\mathcal{G}_\Delta$. Particularly interesting subcollections from a physical viewpoint are given by subgraphs of regular lattices. However, it is notoriously difficult to rigorously deduce the properties for such collections.

A much simpler collection of rooted graphs in $\mathcal{G}_\Delta$ is given by finite Cayley trees, and we will describe the properties of those in this section. The trees are uniquely determined by the conditions that every leaf has fixed distance n to the root vertex v, and every non-leaf has down-degree $d = \Delta - 1$. The root vertex therefore has degree d, while every other non-leaf has degree Δ. We denote the Cayley tree of depth n by T_n, and its root by v_n.

As an immediate consequence of Lemma 3.2.4 we obtain

$$R_{T_n,v_n}(\lambda) = f_{\lambda,d}(R_{T_{n-1},v_{n-1}}(\lambda)),$$

where $f_{\lambda,d}(z) = \lambda/(1+z)^d$. Since the ratio of a single point is given by $\lambda = f_{\lambda,d}(0)$, it follows inductively that

$$R_{T_n,v_n}(\lambda) = f_{\lambda,d}^{n+1}(0).$$

In fact, since

$$Z_{T_n,v_n}^{out}(\lambda) = \left(Z_{T_{n-1},v_{n-1}}(\lambda)\right)^d$$

and

$$Z_{T_n,v_n}^{in}(\lambda) = \lambda\left(Z_{T_{n-1},v_{n-1}}^{out}(\lambda)\right)^d$$

it follows by induction on n that for $\lambda \in \mathbb{C}^*$ the polynomials $Z_{T_n,v_n}^{in}(\lambda)$ and $Z_{T_n,v_n}^{out}(\lambda)$ cannot vanish simultaneously. For $\lambda \in \mathbb{C}^*$ it follows that $Z_{T_n}(\lambda) = 0$ if and only if $R_{T_n,v_n}(\lambda) = f_{\lambda,d}^n(0) = -1$.

In what follows we deduce properties of the zeros of $Z_{T_n}(\lambda)$ and the activity-locus of $f_{\lambda,d}^n(0)$ from well-known results in the field of holomorphic dynamical systems, occasionally adapting the proofs to our setting. We refer the reader to the standard references [Mil06, CG93].

Observe that $f_{\lambda,d}(-1) = \infty$ and $f_{\lambda,d}(\infty) = 0$, and $f_{\lambda,d}'(-1) = f_{\lambda,d}'(\infty) = 0$. Thus if $f_{\lambda_0,d}^n(0) = -1$ for some λ_0 and n, then 0 is an attracting periodic cycle of period $n+2$. This cycle is stable under perturbations of λ_0, i.e. the attracting cycle persists and in fact varies holomorphically for nearby parameters $\lambda \sim \lambda_0$ by the implicit function theorem.

Recall that every attracting cycle attracts the orbit of a critical point. But $f_{\lambda,d}$ has only one critical orbit: the orbit of -1, ∞ and 0. Thus whenever $f_{\lambda,d}$ has an attracting cycle, the orbit $f_{\lambda,d}^n(0)$ converges to the attracting cycle. In fact, the convergence is uniform in a neighborhood of the parameter λ_0, hence λ_0 cannot lie in the activity-locus. The situation is therefore fundamentally different from the setting where the whole family of graphs $\mathcal{G}_\Delta$ is considered, as there λ_0 must lie in the activity-locus. The following however does hold:

Proposition 3.7.1. *The activity-locus of the family $\{T_n, v_n\}$ equals the collection of accumulation points of the zeros of the collection $\{Z_{T_n}\}$.*

Proof. If there are no zeros in a neighborhood of some λ_0, then the family $\{R_{T_n,v_n}\}$ avoids the values $0, -1$ and ∞, and is normal by Montel's Theorem.

Suppose on the other hand that λ_0 is an accumulation point of zeros $\lambda_1, \lambda_2, \ldots$. Let $n_1, n_2, \ldots$ be the minimal integers for which $f^{n_i}(\lambda_i) = -1$. Since for fixed n the zeros of Z_{T_n} are isolated, we may assume that $n_i \to \infty$ and (n_i) is strictly increasing.

When for a parameter λ the rational function f has an attracting periodic cycle, the unique critical orbit $\{f^n(0)\}_{n \geq 1}$ must converge to this periodic orbit. Since attracting periodic cycles are stable, i.e. they persist under small changes of the parameter λ, such parameters lie in a passivity component, i.e., a maximal connected open subset where the family $\{\lambda \mapsto f^n(0)\}$ is normal. The passivity component agrees exactly with the connected component where the attracting periodic cycle persists, since by [MnSS83] the parameter must become active when the periodic cycle becomes neutral.

Thus, λ_i lies in a connected component of the open set where the family $\{\lambda \mapsto f^n(0)\}$ is normal, and associated to this component is the unique period $n_i + 2$. Since the sequence $\{n_i\}_{i \geq 1}$ is strictly increasing, the parameters λ_i must all lie in distinct connected components. It follows that the limit parameter λ_0 cannot lie in an open component where the family is normal, and therefore λ_0 must be an active parameter. □

The activity-locus for Cayley trees of down degree $d = 2, 3$ and 4 is illustrated in Figure 3.9. Each of these diagrams represents the spherical derivative of the function $\lambda \mapsto f^{120}_{\lambda,d}(0)$.

It follows from Proposition 3.7.1 above, plus the observation that zeros do not lie in the activity-locus for the Cayley tree setting, that the Cayley tree activity-locus never has interior. On the other hand, it follows from the universality of the Mandelbrot set, a result due to McMullen [McM00], that the activity-locus must contain a quasiconformal image of the Mandelbrot set of some degree. Therefore by Shishikura's result [Shi98] the Hausdorff dimension of the activity-locus is equal to 2 for any $d \geq 2$.

It follows from the proof of Proposition 3.7.1 that the complement of the activity-locus consists of infinitely many connected components. Each λ for which $f_{\lambda,d}$ has an attracting periodic cycle lies in such a passive component, a so-called *hyperbolic* component associated to the period k. Whether all connected components are hyperbolic is an open question, which is conjectured to hold for quadratic polynomials.

For any down-degree d there are two special connected components that can easily be identified. The unbounded component is always a hyperbolic component of period 2. For degree 2 this is the complement of the closed disk of radius 4. For down-degrees 3 and 4 the boundary has respectively 1 and 2 singular points.

For each down-degree $d = \Delta - 1$ there is a single hyperbolic component of period 1, which naturally contains the parameter $\lambda = 0$ and equals the cardioid Λ_Δ.

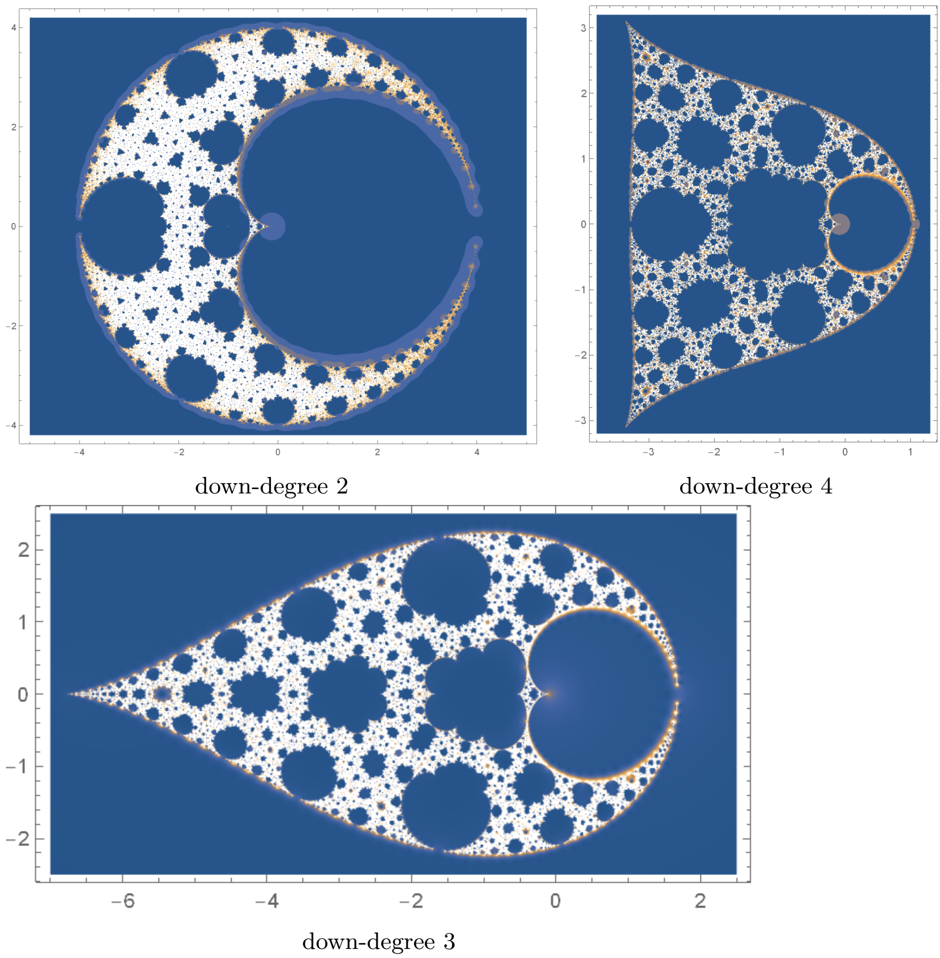

Figure 3.9: The activity-locus of Cayley trees for down-degrees $2, 3$ and 4. For each pixel the spherical derivative of the occupation ratio is computed for the Cayley tree of depth 120. Pixels for which this derivative is sufficiently large are depicted in white, suggesting that the corresponding parameter λ lies approximately on the activity locus.

Apart from these two special hyperbolic components, any hyperbolic component contains a unique zero of the partition function, i.e. a unique parameter λ for which $f_{\lambda,d}^n(\lambda) = -1$ for some $n \in \mathbb{N}$. Since $f_{\lambda,d}^2(-1) = 0$ and $f_{\lambda,d}^{-2}(0) = \{-1\}$, these are exactly the parameters $\lambda \in \mathbb{C} \setminus \{0\}$ for which the unique critical orbit $\{f_{\lambda,d}^i(0)\}_{i \in \mathbb{N}}$ is periodic, i.e. for which $f_{\lambda,d}$ is super-attracting.

For the family $p_c(z) = z^2 + c$ the fact that every hyperbolic component of the Mandelbrot set contains a unique super-attracting parameter is a consequence of the Multiplier Theorem, due to Douady-Hubbard and Sullivan, see [Dou83].

Let us recall this fundamental result in the field. Let H be a hyperbolic component of the Mandelbrot set, say of period n. For every parameter $c \in H$ there exist an attracting periodic cycle $a_0, a_1, \ldots, a_n = a_0$. The multiplier $h(c) = (f_c^n)'(a_0)$ is independent from the choice of a_n, and gives a holomorphic map from H to the unit disk.

Theorem 3.7.2 (Multiplier Theorem)**.** *For every hyperbolic component H the map $c \mapsto h(c)$ gives a conformal bijection from H to the unit disk.*

The proof of the Multiplier Theorem can be found in [CG93], Theorem 2.1 on page 133, and can be applied almost directly to our setting. We present a high-level discussion to outline how the proof adapts to our setting.

Let H be a hyperbolic component of period at least 3. One easily sees that $h(\lambda)$, the multiplier of the attracting periodic cycle of $f_{\lambda,d}$ is a holomorphic and surjective map from the hyperbolic component H to the unit disk $\mathbb{D}$, hence is a branched covering. Let Z be the set of super-attracting parameters in H, i.e. $Z = h^{-1}(0)$. If it can be shown that $h : H \setminus Z \to \mathbb{D} \setminus \{0\}$ is a covering map, it follows from the Riemann-Hurwitz Theorem that $\mathrm{card}(Z) = 1$.

Thus, it needs to be shown that h is locally invertible near parameters $\lambda_0 \in H \setminus Z$. Write $\eta_0 = h(\lambda_0) \in \mathbb{D}$, and consider values of η near η_0. Following the proof of the Multiplier Theorem one applies quasiconformal surgery by modifying the ellipse field near the attracting periodic cycle in order to obtain attracting periodic cycles with multipliers η. Using the dynamics the ellipse field can be extended to the full basin of the attracting cycle, obtaining an invariant ellipse field that is invariant under the map $f_{\lambda_0,d}$. The ellipse field corresponds to a Beltrami coefficient, which can be extended to the entire Riemann sphere by setting it equal to 0 outside of the basin of attraction. The Measurable Riemann Mapping Theorem gives a holomorphic family of quasiconformal maps φ_η, with φ_{η_0} the identity. By composing with suitable Möbius transformations we can guarantee that the points $-1, \infty$ and 0 are fixed under all φ_η.

Since each ellipse field is invariant under $f_{\lambda_0,d}$, conjugating $f_{\lambda,d}$ by φ_η yields a holomorphic family of self-maps of the Riemann sphere $g_{\eta,d}$, which are necessarily rational functions of the same degree d. In fact, since each φ_η fixes the points $-1, \infty$ and 0, each rational function $f_{\eta,d}$ must send -1 to ∞ and ∞ to 0, each with local degree d. It follows that the rational function $g_{\eta,d}$ must be of the form

$$g_{\eta,d}(z) = \frac{\lambda(\eta)}{(1+z)^d}.$$

It follows that $\lambda(\eta)$ gives a local inverse of the multiplier function h, completing this step of the proof. This step guarantees that there exists a unique zero in each hyperbolic component of period at least 3, which equals the super-attracting *center* of the hyperbolic component. The proof of the Multiplier Theorem in our setting can be concluded by analyzing the local degree near the center. We have therefore obtained the following description of the zeros of the Cayley trees:

Corollary 3.7.3. *Every $\lambda \in \mathbb{C}$ for which $Z_{T_n}(\lambda) = 0$ for some $n \in \mathbb{N}$ is the center of a hyperbolic component of the complement of the activity locus. On the other hand: apart from the two special hyperbolic components, the unbounded component and the component containing 0, for each center λ of a hyperbolic component there exists an $n \in \mathbb{N}$ for which $Z_{T_n}(\lambda) = 0$. As a consequence zero-parameters are isolated.*

Chapter 4

Cayley trees do not determine the maximal zero-free locus of the independence polynomial

4.1. Introduction

This chapter continues with the independence polynomial, which, for a simple graph $G = (V, E)$, is defined as

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V \\ \text{independent}}} \lambda^{|I|}. \tag{4.1}$$

As in the previous chapter we let $\mathcal{G}_\Delta$ be the set of graphs of maximum degree at most Δ and we let $\mathcal{Z}_\Delta$ denote the union of the complex zeros of Z_G for $G \in \mathcal{G}_\Delta$, i.e.

$$\mathcal{Z}_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

We briefly restate why we are interested in $\mathcal{Z}_\Delta$ and recall some relevant results.

Patel and Regts [PR17] showed that a zero-free domain containing the point 0 gives rise to a polynomial time algorithm for approximating Z_G in that region. Their work is based on the interpolation method developed by Barvinok (see e.g. his book [Bar16]). Results on zero-free regions regarding both the univariate independence polynomial as stated in (4.1) and its multivariate generalization can be found in [SS05], [Bar16], [PR19] and [BC18]. We let $U_\Delta(0)$ denote the largest possible zero-free domain for $\mathcal{G}_\Delta$ containing the origin, i.e. $U_\Delta(0)$ is the connected component containing 0 of the complement of the closure of $\mathcal{Z}_\Delta$. The following result is one of the main motivators for this chapter.

Theorem 4.1.1. *Let $\Delta \in \mathbb{Z}_{\geq 3}$ and define $\lambda^*(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ and $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}$. There exists a zero-free complex domain containing the real interval $(-\lambda^*(\Delta), \lambda_c(\Delta))$, i.e. $(-\lambda^*(\Delta), \lambda_c(\Delta)) \subseteq U_\Delta(0)$.*

The disk of radius $\lambda^*(\Delta)$ is known as the Shearer region and is zero-free (see [She85, SS05] or Section 3.2.3). The fact that $[0, \lambda_c(\Delta)) \subseteq U_\Delta(0)$ was first conjectured by Sokal [Sok01] and is proved in [PR19].

The result of Patel and Regts can be interpreted as saying that zero-freeness implies that approximation is easy. In the previous chapter we showed that an inverse statement also holds. Namely, we saw that parameters λ for which approximating $Z_G(\lambda)$ is #P-Hard lie dense in $\overline{\mathcal{Z}_\Delta}$ (see Theorem 3.1.1). The following follows from Proposition 6 in [BGGv20] together with Theorem 3.1.1.

Theorem 4.1.2. *Let $\Delta \in \mathbb{Z}_{\geq 3}$ and define*

$$\Lambda_\Delta = \left\{ \frac{-\alpha \cdot (\Delta - 1)^{\Delta - 1}}{(\Delta - 1 + \alpha)^\Delta} : |\alpha| < 1 \right\}. \tag{4.2}$$

Then zeros lie dense outside Λ_Δ, i.e. $\overline{\mathcal{Z}_\Delta}$ contains the complement of Λ_Δ.

Finite Cayley trees are rooted trees such that every leaf has the same distance to the root and every other vertex has down degree $\Delta - 1$. Proposition 2.1 of [PR19] states that Λ_Δ is zero-free for Cayley trees, while their zeros accumulate everywhere on the boundary of Λ_Δ (see also Section 3.7). Because $\Lambda_\Delta \cap \mathbb{R} = (-\lambda^*(\Delta), \lambda_c(\Delta))$ we see that $U_\Delta(0) \cap \mathbb{R} = (-\lambda^*(\Delta), \lambda_c(\Delta))$. Moreover, zeros of Cayley trees accumulate at both real boundary points and thus they are in a certain sense extremal at the real boundary of $U_\Delta(0)$. One could hope that Cayley trees are extremal at the whole boundary of $U_\Delta(0)$ and thus that $U_\Delta(0) = \Lambda_\Delta$. This would, in a certain sense, give a full dichotomy of the complexity of approximating $Z_G(\lambda)$ for bounded degree graphs in terms of the location of λ in the complex plane. In this chapter we show that Cayley trees are, in general, not extremal,[1] i.e. we prove the following.

Theorem 4.1.3. *For $\Delta \in \{3, \ldots, 9\}$ there exist $\lambda \in \Lambda_\Delta$ with $G \in \mathcal{G}_\Delta$ such that $Z_G(\lambda) = 0$.*

We will define a region V_Δ for which we get the inclusions $U_\Delta(0) \subseteq V_\Delta \subseteq \Lambda_\Delta$, and we will show that the latter inclusion is strict for $3 \leq \Delta \leq 9$. The definition of V_Δ is given in Section 4.5. The other sections are dedicated to the proof of Theorem 4.1.3.

The main tool used in this chapter comes from the area of complex dynamics that concerns the analysis of stable parameters of families of rational maps.

[1] This was first claimed by Juan Rivera-Letelier and Daniel Štefankovič in personal communication.

4.2. Setup and strategy

In this section we give the main definitions and results that we will use to prove Theorem 4.1.3. We will also outline the general strategy that the proof follows. We start by defining the occupation ratio of a rooted tree and we analyze some of its properties. Most definitions in the following subsection appear in [PR19] and are inspired by [Wei06]. They can also be found in Section 3.2.

4.2.1. Iteration of occupation ratios of rooted trees

Let $G = (V, E)$ denote a simple graph. For any $v \in V$ we define *the closed neighborhood* $N[v]$ *of* v as

$$N[v] = \{u \in V : \{u, v\} \in E\} \cup \{v\}.$$

If $S \subseteq V$ we denote by $G[S]$ the subgraph of G induced by the vertices in S. We denote the subgraph induced by the complement of S, i.e., $G[V\backslash S]$, by $G\backslash S$. Finally, for any $v \in V$ we denote $G\backslash\{v\}$ by $G-v$. By considering independent sets containing v and not containing v separately we obtain the following recurrence relation of independence polynomials

$$Z_G(\lambda) = \lambda \cdot Z_{G\backslash N[v]}(\lambda) + Z_{G-v}(\lambda).$$

If $Z_{G-v}(\lambda) \neq 0$, we define the *occupation ratio at* v as

$$R_{G,v}(\lambda) = \frac{Z_G(\lambda)}{Z_{G-v}(\lambda)} - 1 = \frac{\lambda \cdot Z_{G\backslash N[v]}(\lambda)}{Z_{G-v}(\lambda)}.$$

We observe that for those λ with $Z_{G-v}(\lambda) \neq 0$ we have that $Z_G(\lambda) = 0$ if and only if $R_{G,v}(\lambda) = -1$. Now suppose that T is a tree with root vertex v and $\lambda \in \mathbb{C}$ such that $Z_T(\lambda) \neq 0$ and $Z_{T-v}(\lambda) \neq 0$. Define for $d \in \mathbb{Z}_{\geq 1}$ the larger tree $\tilde{T}$ with a root vertex $\tilde{v}$ such that $\tilde{v}$ is attached to d copies of T at their respective root vertices. Then

$$R_{\tilde{T},\tilde{v}}(\lambda) = \lambda \cdot \left(\frac{Z_{T-v}(\lambda)}{Z_T(\lambda)}\right)^d = \lambda \cdot \frac{1}{(1 + R_{T,v}(\lambda))^d}. \tag{4.3}$$

So if we define

$$f_{\lambda,d}(z) = \frac{\lambda}{(1+z)^d},$$

we find that $R_{\tilde{T},\tilde{v}} = f_{\lambda,d}(R_{T,v}(\lambda))$. The occupation ratio of a graph consisting of a single point is equal to λ. Therefore, to understand whether λ can occur as a zero of the independence polynomial of a finite Cayley tree with branching number

Δ it suffices to determine whether -1 appears in the orbit of λ under the map $f_{\lambda,\Delta-1}$. This analysis is done in section 3.7. Instead of iterating with a single map we will consider iteration by a pattern of different maps $f_{\lambda,d_1},\ldots,f_{\lambda,d_k}$, periodically applied. Effectively we will analyze the roots of the independence polynomials of trees whose down degree is regular at every level.

4.2.2. The rational semigroups H_Δ

In the rest of this chapter we will usually drop the subscript λ from $f_{\lambda,d}$ and write f_d unless we want to stress a specific parameter λ. For $\Delta \in \mathbb{Z}_{\geq 3}$ we define the rational semigroup H_Δ as the semigroup generated by $f_1,\ldots f_{\Delta-1}$, i.e,

$$H_\Delta = \langle f_1,\ldots,f_{\Delta-1}\rangle .$$

This semigroup consists of families of rational maps with the following property.

Lemma 4.2.1. *Let $g \in H_\Delta$. If for some $\lambda \in \mathbb{C}$ we have $g_\lambda(0) = -1$. Then there exists a tree $T \in \mathcal{G}_\Delta$ with $Z_T(\lambda) = 0$.*

Proof. We can write $g = f_{d_n} \circ \cdots \circ f_{d_1}$. Let k be the smallest positive integer such that $f_{\lambda,d_k} \circ \cdots \circ f_{\lambda,d_1}(0) = -1$. If $k = 1$, then $\lambda = f_{\lambda,d_1}(0) = -1$ and thus the statement is true since the independence polynomial of the graph consisting of one vertex is $\lambda + 1$.

If $k \geq 2$ then $\lambda \neq -1$. We let T_0 correspond to the empty graph and T_1 to the graph consisting of one root vertex v_1. Furthermore, we define for $m \in \{2,\ldots,k\}$ the rooted tree T_m as a root v_m connected to d_m copies of T_{m-1} by their respective root vertices. Note that in this way $T_m \in \mathcal{G}_\Delta$ for all m. Also observe that $Z_{T_m - v_m}(\lambda) = \left(Z_{T_{m-1}}(\lambda)\right)^{d_m}$ and $Z_{T_m \setminus N[v_m]}(\lambda) = \left(Z_{T_{m-2}}(\lambda)\right)^{d_m \cdot d_{m-1}}$. We will prove the following by induction. For $m \in \{2,\ldots,k\}$ we have that

$$Z_{T_l}(\lambda) \neq 0 \text{ for } 0 \leq l < m \quad \text{ and } \quad R_{T_m,v_m} = \left(f_{\lambda,d_m} \circ \cdots \circ f_{\lambda,d_1}\right)(0). \tag{4.4}$$

For $m = 2$ we have that $Z_{T_0}(\lambda) = 1 \neq 0$ and $Z_{T_1}(\lambda) = 1 + \lambda \neq 0$. As a result we find that $Z_{T_2 - v_2}(\lambda)$ and $Z_{T_2 \setminus N[v_2]}(\lambda)$ are not zero since they are powers of $Z_{T_1}(\lambda)$ and $Z_{T_0}(\lambda)$ respectively. It follows that we can use (4.3) to calculate the occupation ratio of T_2 at v_2 by

$$R_{T_2,v_2}(\lambda) = f_{\lambda,d_2}(R_{T_1,v_1}) = f_{\lambda,d_2}(\lambda) = \left(f_{\lambda,d_2} \circ f_{\lambda,d_1}\right)(0).$$

Now suppose that the statement in (4.4) is true for all values less than a certain $m > 2$. Then we know that $Z_{T_{m-1} - v_{m-1}}(\lambda) \neq 0$ and that $R_{T_{m-1},v_{m-1}}(\lambda) \neq -1$, which implies that $Z_{T_{m-1}}(\lambda) \neq 0$. This again implies that

$$R_{T_m,v_m}(\lambda) = f_{\lambda,d_m}(R_{T_{m-1},v_{m-1}}(\lambda)) = \left(f_{\lambda,d_m} \circ \cdots \circ f_{\lambda,d_1}\right)(0).$$

This proves the statement in (4.4). Finally we can conclude that $R_{T_k,v_k}(\lambda) = -1$, while $Z_{T_k - v_k}(\lambda) \neq 0$. This implies that $Z_{T_k}(\lambda) = 0$, which concludes the proof since $T_k \in \mathcal{G}_\Delta$. □

4.2.3. Stable parameters of rational maps

This section contains the relevant results from the area of complex dynamics. The primary object of study is that of the stable parameters of a holomorphic family of rational maps. The basis for this section is Chapter 4 of [McM94]. The result that we will state follows from the λ-Lemma by Mañé, Sad and Sullivan [MnSS83].

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere and let $\Omega \subseteq \mathbb{C}$ denote a complex domain. We define a *holomorphic family of rational maps*, parameterized Ω, as a holomorphic map $f : \Omega \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with the property that for every $\lambda \in \Omega$ the map $z \mapsto f(\lambda, z)$ is a rational map. The first argument of f is thought of as a parameter and the map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}} : z \mapsto f(\lambda, z)$ is often denoted by f_λ. Note that the elements of H_Δ are holomorphic families of rational maps with respect to any complex domain. We will use the following definition to state the subsequent theorem.

Definition 4.2.2. Let f be a holomorphic family of rational maps and let $\lambda_0 \in \Omega$. We call a periodic point z of f_{λ_0} with period n *persistently indifferent* if there exists a neighborhood U of λ_0 and a holomorphic map $w : U \to \widehat{\mathbb{C}}$ such that

$$w(\lambda_0) = z, \quad f_\lambda^n(w(\lambda)) = w(\lambda) \quad \text{and} \quad |(f_\lambda^n)'(w(\lambda))| = 1$$

for all $\lambda \in U$.

Theorem 4.2.3 (Part of Theorem 4.2 in [McM94]). *Let f be a holomorphic family of rational maps parameterized by Ω. And suppose there exist holomorphic maps $c_i : \Omega \to \widehat{\mathbb{C}}$ parameterizing the critical points of f. Let $\lambda_0 \in \Omega$ and suppose that for all i the families of maps given by*

$$\mathcal{F}_i = \{\lambda \mapsto f_\lambda^n(c_i(\lambda))\}_{n \geq 1}$$

are normal at λ_0. Then there is a neighborhood U of λ_0 such that for all $\lambda \in U$ every periodic point of f_λ is either attracting, repelling or persistently indifferent.

Our strategy will be to show that there are $g \in H_\Delta$ with $\lambda_0 \in \Lambda_\Delta$ such that g_{λ_0} has an indifferent fixed point that is not persistent. Note that we do not lose generality by considering only fixed points instead of periodic points since $g \in H_\Delta$ implies that $g^N \in H_\Delta$ for any $N \in \mathbb{Z}_{\geq 1}$. Then we will be able to use non-normality of one of the critical points to show that arbitrarily close to λ_0 we can find λ for which we can derive a function $\tilde{g} \in H_\Delta$ with $\tilde{g}_\lambda(0) = -1$. Subsequently we will use Lemma 4.2.1 to prove Theorem 4.1.3. This will be made more precise in the next two sections.

4.3. Properties of the maps in H_Δ

4.3.1. The critical points

To apply Theorem 4.2.3 we need an understanding of the behaviour of the critical points of the elements of H_Δ. The following lemma states that for all $g \in H_\Delta$ the critical points move locally holomorphically near all but finitely many λ.

Lemma 4.3.1. *Let $g \in H_\Delta$ with $g = f_{d_k} \circ \cdots \circ f_{d_1}$. Let $\lambda_0 \in \mathbb{C} - \{0\}$ be a parameter such that there are no indices i, j with $1 \leq i < j \leq k$ with*

$$\left(f_{\lambda_0,d_j} \circ \cdots \circ f_{\lambda_0,d_i}\right)(0) = -1. \tag{4.5}$$

Then there exists a neighborhood of λ_0 on which the critical points of g can be parameterized by holomorphic maps.

Proof. For any $\lambda \in \mathbb{C} - \{0\}$ and $d \geq 2$ the critical points of $f_{\lambda,d}$ are -1 and ∞. Therefore the critical points of g_λ are given by points z for which there is some $i \in \{2, \ldots, k\}$ with $d_i \geq 2$ and

$$\left(f_{\lambda,d_{i-1}} \circ \cdots \circ f_{\lambda,d_1}\right)(z) \in \{-1, \infty\},$$

possibly together with $-1, \infty$ if $d_1 \geq 2$. Since for any d and nonzero λ we have that $f_{\lambda,d}(z) = \infty$ if and only if $z = -1$, we can write the critical points of g_λ as $X_\lambda = Y_\lambda \cup E$, where

$$Y_\lambda = \bigcup_{\substack{1 \leq i < k: \\ d_{i+1} \geq 2 \text{ or } d_{i+2} \geq 2}} \{z : f_{\lambda,d_i} \circ \cdots \circ f_{\lambda,d_1}(z) = -1\}$$

and $E \subseteq \{-1, \infty\}$ with $\infty \in E$ only if $d_1 \geq 2$ and $-1 \in E$ only if $d_1 \geq 2$ or $d_2 \geq 2$. Clearly the critical points in E move holomorphically around any neighborhood of λ_0 not containing 0 since they do not depend on the parameter λ. We will show that we can find a neighborhood of λ_0 on which the elements of Y can also be parameterized by holomorphic functions. Note that, since $f_{\lambda,d}(\infty) = 0$, it follows from the assumption in (4.5) that $-1, \infty \notin Y_{\lambda_0}$. The Implicit Function Theorem guarantees that the elements of Y move holomorphically near λ_0 if for all l and z_0, where z_0 is a solution to

$$\left(f_{\lambda_0,d_l} \circ \cdots \circ f_{\lambda_0,d_1}\right)(z_0) = -1,$$

we have that

$$\left(f_{\lambda_0,d_l} \circ \cdots \circ f_{\lambda_0,d_1}\right)'(z_0) \notin \{0, \infty\}. \tag{4.6}$$

To show that this is the case we first calculate that

$$f'_{\lambda,d}(z) = -\frac{d}{1+z} \cdot f_{\lambda,d}(z),$$

for all λ, d. We denote for all $i > 0$ the ith element of the orbit of z_0 by z_i, i.e.,

$$z_i = (f_{\lambda_0,d_i} \circ \cdots \circ f_{\lambda_0,d_1})(z_0).$$

Now we can write

$$(f_{\lambda_0,d_l} \circ \cdots \circ f_{\lambda_0,d_1})'(z_0) = \prod_{i=1}^{l} -\frac{d_i \cdot z_i}{1 + z_{i-1}}.$$

The assumption of the lemma now guarantees that $\{-1, \infty, 0\} \cap \{z_0, \ldots, z_{l-1}\} = \emptyset$ and since $z_l = -1$, we can conclude that the equation in (4.6) holds. The lemma now follows from an application of the Implicit Function Theorem. $\square$

Remark 4.3.2. Note that it follows from the proof that if c is a holomorphic map parameterizing a critical point of $g = f_{d_k} \circ \cdots \circ f_{d_1}$ on a domain Ω that either c is constantly -1 or ∞ on Ω, or there is some index l such that the holomorphic map

$$\lambda \mapsto (f_{\lambda,d_l} \circ \cdots \circ f_{\lambda,d_1})(c(\lambda))$$

is constantly -1. As -1 gets mapped to 0 in two applications of any two maps f_d, independent of the degree of the individual maps and of λ, we get that there must be some sequence of indices $d_{i_1}, \ldots, d_{i_t} \in \{d_1, \ldots, d_k\}$ such that

$$g_\lambda^3(c(\lambda)) = \left(f_{\lambda,d_{i_t}} \circ \cdots \circ f_{\lambda,d_{i_1}}\right)(0)$$

for all $\lambda \in \Omega$.

4.3.2. The indifferent fixed points

To show that there are $g \in H_\Delta$ with $\lambda_0 \in \Lambda_\Delta$ such that g_{λ_0} has an indifferent fixed point that is not persistent we first show that there do not exist $g \in H_\Delta$ and $\lambda_0 \in \mathbb{C}$ such that g_{λ_0} has a persistently indifferent fixed point. The argument relies on the following fact.

Lemma 4.3.3. *Let f be a holomorphic family of rational maps parameterized by a domain Ω. Suppose that $\lambda_0 \in \Omega$ is a parameter such that f_{λ_0} has a persistently indifferent fixed point. Suppose also that on Ω we can write*

$$f_\lambda(z) = \frac{p(\lambda, z)}{q(\lambda, z)}, \tag{4.7}$$

with $p, q \in \mathbb{C}[\lambda, z]$. Then the holomorphic family of rational maps p/q, where the parameter plane is now taken to be the whole complex plane, has an indifferent fixed point for all but finitely many parameters $\lambda \in \mathbb{C}$.

The proof of this lemma is elementary and can be found in Section 4.6.

Any $g \in H_\Delta$ can be written in the form displayed in (4.7). A consequence of Lemma 4.3.3 is now that if we can find a region of parameters for which some $g \in H_\Delta$ has no indifferent fixed points, then we can conclude that g has no persistently indifferent fixed points for any parameter λ. We will prove that this is the case for all $g \in H_\Delta$ by describing the fixed points of g_λ for λ near 0. These results are found in the next two lemmas.

Lemma 4.3.4. *Let $g \in H_\Delta$ and $\lambda \in \mathbb{C}$ with $|\lambda| < \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$. Then g_λ has an attracting fixed point.*

Proof. Write $g = f_{d_k} \circ \cdots \circ f_{d_1}$ and let B be a open disc of radius $\frac{1}{\Delta}$ centered around zero. Then for any $d \in \{1, \ldots, \Delta-1\}$ and $z \in \overline{B}$ we have

$$|f_{d,\lambda}(z)| = \frac{|\lambda|}{|(1+z)|^d} \leq \frac{|\lambda|}{(1-|z|)^{\Delta-1}} < \frac{\frac{(\Delta-1)^{\Delta-1}}{(\Delta)^\Delta}}{\left(1-\frac{1}{\Delta}\right)^{\Delta-1}} = \frac{1}{\Delta}.$$

This means that B gets mapped strictly into itself by all the maps $f_{d_i,\lambda}$ and thus also by g_λ. By the Schwarz–Pick theorem g_λ is a strict contraction on B with respect to the Poincaré metric. It follows from the Brouwer fixed-point theorem that g_λ has a fixed point in B, which by the contraction property must necessarily be unique. The strict contraction moreover guarantees that all orbits in B converge uniformly to this fixed point, which therefore must be attracting. □

Note that it follows from the proof that the disc of radius $(\Delta-1)^{\Delta-1}/\Delta^\Delta$ is a zero-free region of Z_T for all trees $T \in \mathcal{G}_\Delta$ where the down degree is regular at every level. Scott and Sokal show in [SS05, Cor. 5.7] that this region remains zero-free for Z_G for all $G \in \mathcal{G}_\Delta$, even in the multivariate case (see also [She85]). It turns out that we can also describe the repelling fixed points of elements in H_Δ for all parameters inside this region.

Lemma 4.3.5. *Let $g \in H_\Delta$ and write $g = f_{d_k} \circ \cdots \circ f_{d_1}$. Let $\lambda \in \mathbb{C} - \{0\}$ with $|\lambda| < \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$. Then g_λ has $d_1 \cdots d_k$ distinct repelling fixed points.*

Proof. For this proof denote g_λ by g. Let B be an open disc of radius $\frac{\Delta-1}{\Delta}$ centered around -1. Let $d \in \{1, \ldots, \Delta-1\}$ and let $h(z) = \lambda/z^d$. Since $\overline{B}$ does not intersect the positive real axis, we find that the inverse image $h^{-1}(B)$ consists of d disjoint domains $V_1, \ldots, V_d$ such that for each i the map $h|_{V_i} : V_i \to B$ is a biholomorphism. Denote the inverse branches as $h_1^{-1}, \ldots, h_d^{-1}$. Then for all $z \in B$ all i we have

$$\left|h_i^{-1}(z)\right| = \left(\frac{|\lambda|}{|z|}\right)^{1/d} < \left(\frac{\frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}}{1-\frac{\Delta-1}{\Delta}}\right)^{1/d} = \left(\frac{\Delta-1}{\Delta}\right)^{(\Delta-1)/d} \leq \frac{\Delta-1}{\Delta}.$$

The inverse branches of $f_{\lambda,d}$ on B are given by $z \mapsto h_i^{-1}(z) - 1$. If we denote $h_i^{-1}(B) - 1$ by U_i, we see that $f_{\lambda,d}^{-1}(B) = U_1 \cup \cdots \cup U_d$, $U_i \subsetneq B$ and $U_i \cap U_j = \emptyset$ for all i, j with $i \neq j$. Furthermore, $f_{\lambda,d}|_{U_i}$ is a biholomorphism for all i. By composition, we find that there are $d_1 \cdots d_k$ inverse branches of g on B, denoted by $g_1^{-1}, \ldots, g_{d_1 \ldots, d_k}^{-1}$ with pairwise disjoint domains $W_1 \ldots, W_{d_1 \cdots d_k} \subsetneq B$ such that $g_i^{-1} : B \to W_i$ is a biholomorphism for all i. We can show g_i^{-1} has an attracting fixed point in W_i by using the same argument as in Lemma 4.3.4. Namely, because W_i is a strict subset of B we can use the Brouwer fixed-point theorem to conclude that g_i^{-1} has a fixed point inside W_i. It then follows from the Schwarz–Pick theorem that this fixed point is unique and attracting. This attracting fixed point of g_i^{-1} is a repelling fixed point for g. Since every subset W_i contains such a point, we find that there are $d_1 \cdots d_k$ distinct repelling fixed points inside B. □

The previous three lemmas combined imply the following result.

Corollary 4.3.6. *Let $g \in H_\Delta$ be parameterized by some domain Ω and let $\lambda_0 \in \Omega$ such that g_{λ_0} has an indifferent fixed point. Then this fixed point is not persistently indifferent.*

Proof. Write $g = f_{d_k} \circ \cdots \circ f_{d_1}$. The map g_λ is rational and of degree at most $d_1 \cdots d_k$ for all $\lambda \in \mathbb{C}$. It follows that g_λ can have at most $d_1 \cdots d_k + 1$ distinct fixed points. The previous two lemmas imply that for all λ inside a punctured disc of radius $\frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ the map g_λ has one attracting fixed point and $d_1 \cdots d_k$ repelling fixed points and thus cannot have an indifferent fixed point. The conclusion that g_λ cannot have a persistently indifferent fixed point for any $\lambda \in \mathbb{C}$ follows from Lemma 4.3.3. □

4.4. Proof of the main theorem

In this section we provide a proof for Theorem 4.1.3. The essential idea is contained in the following lemma.

Lemma 4.4.1. *Let $g \in H_\Delta$ not of the form f_1^N and $\lambda_0 \in \mathbb{C}$ such that g_{λ_0} has an indifferent fixed point. Then for every neighborhood U of λ_0 there exists a $\lambda \in U$ and a tree $T \in \mathcal{G}_\Delta$ such that $Z_T(\lambda) = 0$.*

Proof. Write $g = f_{d_k} \circ \cdots \circ f_{d_1}$. If there are indices i, j with $1 \leq i < j \leq k$ such that $\left(f_{\lambda_0,d_j} \circ \cdots \circ f_{\lambda_0,d_i}\right)(0) = -1$, then we can apply Lemma 4.2.1 on $f_{d_j} \circ \cdots \circ f_{d_i}$ to find that there is a tree $T \in \mathcal{G}_\Delta$ such that $Z_T(\lambda_0) = 0$, so in this case the statement is true. If these indices do not exist, then we apply Lemma 4.3.1 to get a domain Ω containing λ_0 on which the critical points of g can be parameterized by holomorphic maps. Note that, since g is not of the form f_1^N, g has critical points. By Corollary 4.3.6, the indifferent fixed point of g_{λ_0}

is not persistently indifferent and thus we can conclude from Theorem 4.2.3 that there is at least one marked critical point c such that the family defined by

$$\{\lambda \mapsto g_\lambda^n(c(\lambda))\}_{n\geq 1}$$

is not normal around λ_0. From Remark 4.3.2 it follows that there is some $h \in H_\Delta$ such that

$$\{\lambda \mapsto (g_\lambda^n \circ h_\lambda)(0)\}_{n\geq 1}$$

is not normal around λ_0. Montel's Theorem now guarantees that in any neighborhood U of λ_0 there is a $\lambda \in U \cap \Omega$ and an $N \in \mathbb{Z}_{\geq 3}$ such that $(g_\lambda^N \circ h_\lambda)(0) \in \{0, \infty, -1\}$. If $(g_\lambda^N \circ h_\lambda)(0) = -1$ we can directly apply Lemma 4.2.1 to guarantee the existence of a tree $T \in \mathcal{G}_\Delta$ with $Z_T(\lambda) = 0$. Otherwise, we remark that, since we have chosen $N \geq 3$, we can write $g_\lambda^N \circ h_\lambda = f_{\lambda,\tilde{d}_2} \circ f_{\lambda,\tilde{d}_1} \circ \tilde{g}_\lambda$ for some $\tilde{g} \in H_\Delta$ and $\tilde{d}_1, \tilde{d}_2 \in \mathbb{Z}_{\geq 1}$. We find that $(f_{\lambda,\tilde{d}_2} \circ f_{\lambda,\tilde{d}_1} \circ \tilde{g}_\lambda)(0) = \infty$ implies $(f_{\lambda,\tilde{d}_1} \circ \tilde{g}_\lambda)(0) = -1$ and $(f_{\lambda,\tilde{d}_2} \circ f_{\lambda,\tilde{d}_1} \circ \tilde{g}_\lambda)(0) = 0$ implies $\tilde{g}_\lambda(0) = -1$. In these cases we apply Lemma 4.2.1 to the respective maps to obtain the result. □

The remainder of the proof of Theorem 4.1.3 now consists of providing explicit examples of nontrivial $g \in H_\Delta$ with a parameter $\lambda \in \Lambda_\Delta$ such that g_λ has an indifferent fixed point for each $\Delta \in \{3, \ldots, 9\}$. For $\Delta = 3$ the degree is sufficiently small that we can accurately calculate all such parameters for low degree $g \in H_3$, see Figure 4.1. It is immediately clear that there are many parameters that lie inside Λ_3.

For larger Δ it quickly becomes intractable to calculate images as in Figure 4.1, but it remains possible, given some $g \in H_\Delta$, to accurately calculate those parameters λ for which g_λ has a parabolic fixed point of some given multiplier. Numerical approximations for such parameters inside Λ_Δ where the multiplier is 1 are given in Table 4.1. These results prove Theorem 4.1.3. An explanation of how these numerical values were obtained can be found in Section 4.7.

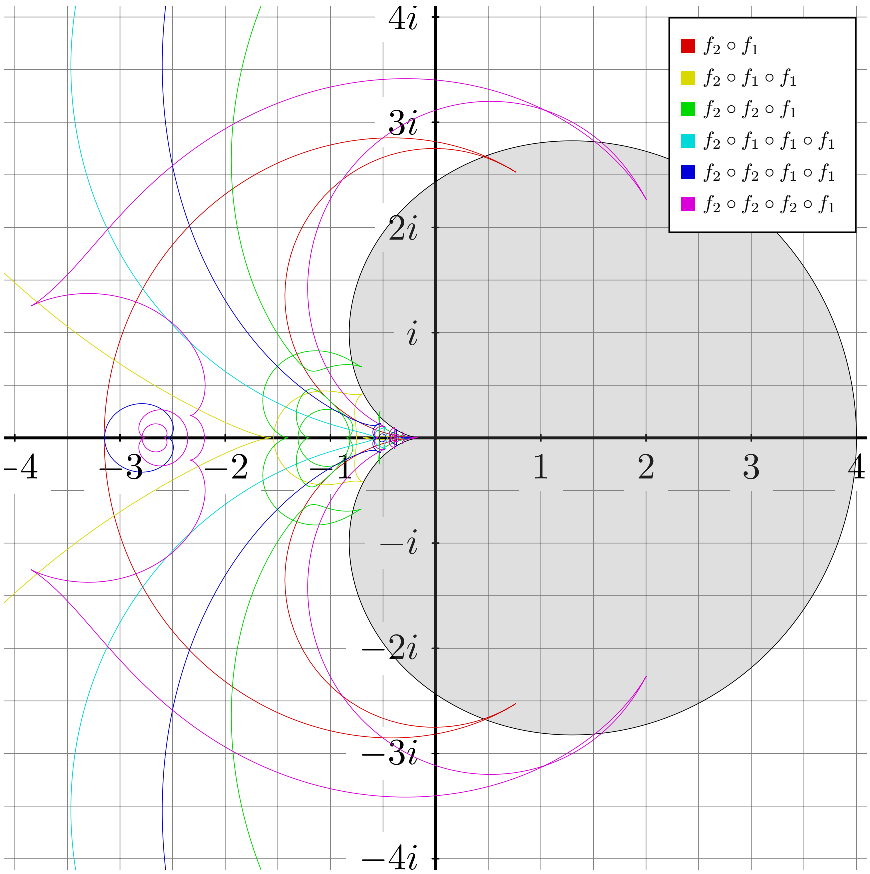

Figure 4.1: The shaded area is Λ_3. Those parameters λ for which g_λ has an indifferent fixed point for different $g \in H_3$ are colored according to the legend.

Table 4.1: Given for each $\Delta \in \{3, \ldots, 9\}$ is a $g \in H_\Delta$ together with an approximation of a $\lambda \in \Lambda_\Delta$ such that g_λ has a fixed point z with $g'_\lambda(z) = 1$. An approximation is given for a value of α and $|\alpha|$, where α is a solution to $\lambda = \frac{-\alpha \cdot (\Delta-1)^{\Delta-1}}{(\Delta-1+\alpha)^\Delta}$ of least absolute value. This value being less than 1 confirms that $\lambda \in \Lambda_\Delta$, see (4.2). See Section 4.7 for an explanation of how these numerical values were obtained.

Δ	g	λ	α	$\|\alpha\|$
3	$f_2 \circ f_1$	$0.76247 + 2.52537\,i$	$-0.913430 - 0.343301\,i$	0.97581
4	$f_3 \circ f_1$	$0.377257 + 1.217961\,i$	$-0.899723 - 0.436161\,i$	0.99987
5	$f_4 \circ f_4 \circ f_1$	$-0.248040 + 0.176140\,i$	$-0.282909 - 0.944616\,i$	0.98607
6	$f_5 \circ f_5 \circ f_1$	$-0.196570 + 0.146650\,i$	$-0.266454 - 0.960007\,i$	0.99630
7	$f_6 \circ f_6 \circ f_2$	$-0.156046 + 0.148986\,i$	$-0.310450 - 0.927735\,i$	0.97830
8	$f_7 \circ f_7 \circ f_2$	$0.132762 + 0.127288\,i$	$-0.296582 - 0.938323\,i$	0.98408
9	$f_8 \circ f_8 \circ f_2$	$-0.115875 + 0.110901\,i$	$-0.285520 - 0.947592\,i$	0.98967

4.5. Concluding remarks

It follows from Lemma 4.4.1 that the set of roots of Z_G for all $G \in \mathcal{G}_\Delta$ accumulates at the boundary of V_Δ, where

$$V_\Delta = \left\{\lambda : g_\lambda \text{ has exactly 1 attracting fixed point for all } g \in H_\Delta \text{ with } g \neq f_1^N\right\}.$$

Recall that we defined $U_\Delta(0)$ to be the largest domain containing 0 that is zero-free for all $G \in \mathcal{G}_\Delta$. In Section 4.3.2 we showed that parameters λ with $|\lambda| < (\Delta-1)^{\Delta-1}/\Delta^\Delta$ lie in V_Δ. It follows that $U_\Delta(0) \subseteq V_\Delta$. Recall that Λ_Δ is exactly the region consisting of those λ for which $f_{\lambda,\Delta-1}$ has an attracting fixed point and thus we can write

$$U_\Delta(0) \subseteq V_\Delta \subseteq \Lambda_\Delta,$$

where the last inclusion was shown to be strict for $3 \leq \Delta \leq 9$ in this chapter. Very recently it was shown in [BHR22], using techniques from this chapter, that this inclusion is in fact strict for $\Delta \leq 45$. Two obvious questions that remain open are whether $V_\Delta \neq \Lambda_\Delta$ for all $\Delta \geq 46$ and whether $U_\Delta(0) = V_\Delta$ for any Δ.

In the next chapter we will analyse what happens if we let Δ go to infinity. We will show that the rescaled regions $\Delta \cdot U_\Delta(0)$ and $\Delta \cdot \Lambda_\Delta$ converge to limit domains U_∞ and Λ_∞ and that $U_\infty \subsetneq \Lambda_\infty$ with the boundary of U_∞ only intersecting the boundary of Λ_∞ on the real line. Moreover, we will explicitly identify a certain section of ∂U_∞ that is given by the limit of points in $\Delta \cdot V_\Delta$ (see Section 5.5.2).

4.6. Proof of Lemma 4.3.3

The proof that we present here is algebraic rather than analytic in nature. We view $\mathbb{C}[\lambda, z]$ as a subring of the ring $\mathbb{C}(\lambda)[z]$. This ring is Euclidian, so in particular it is a unique factorization domain. Therefore we can state the following simple lemma.

Lemma 4.6.1. *Let $p, q \in \mathbb{C}[\lambda, z]$ be coprime in $\mathbb{C}(\lambda)[z]$. Then there are only finitely many $\lambda \in \mathbb{C}$ such that the polynomials $p(\lambda, z)$ and $q(\lambda, z)$ viewed as elements of $\mathbb{C}[z]$ have common roots.*

Proof. Since p, q are coprime in the Euclidian domain $\mathbb{C}(\lambda)[z]$, there exist elements $u, v \in \mathbb{C}(\lambda)[z]$ such that $u \cdot p + v \cdot q = 1$. There exists an element $w \in \mathbb{C}[\lambda]$ such that the coefficients of $w \cdot u$ and $w \cdot v$ are elements of $\mathbb{C}[\lambda]$. It follows that for all λ, z we have the following equality of polynomials

$$w(\lambda)u(\lambda, z) \cdot p(\lambda, z) + w(\lambda)v(\lambda, z) \cdot q(\lambda, z) = w(\lambda).$$

We deduce now that if there is some pair (λ_0, z_0) that is both a root of p and q, then λ_0 is a root of w. Since w has only finitely many roots, we deduce that only finitely many such λ can exist. □

Before we prove Lemma 4.3.3, we recall some properties of the algebraic construction called the *resultant*. Namely, if k is a field and $f, g \in k[x]$, then the resultant of f and g, denoted by $\mathrm{Res}_x(f, g)$, is an integer polynomial in the coefficients of f and g with the property that $\mathrm{Res}_x(f, g) = 0$ if and only if f, g have a common factor in $k[x]$. One can read about the theory of resultants in many introductory texts on computational algebraic geometry, see e.g. [CLO07, Chapter 3, §5]. We now present a proof of Lemma 4.3.3.

Proof of Lemma 4.3.3. We can assume that p, q are coprime in $\mathbb{C}(\lambda)[z]$. Let U be a neighborhood of λ_0 together with a map $w : U \to \widehat{\mathbb{C}}$ that has the properties described in Definition 4.2.2. We can assume that the map w avoids ∞. The holomorphic map

$$\lambda \mapsto f'_\lambda(w(\lambda))$$

is an open map with constant absolute value and is thus constant on U, say equal to α with $|\alpha| = 1$. Note that we can write

$$f'_\lambda(z) = \frac{s(\lambda, z)}{t(\lambda, z)},$$

with $s, t \in \mathbb{C}[\lambda, z]$ coprime in $\mathbb{C}(\lambda)[z]$. Define the following polynomials

$$l(\lambda, z) = p(\lambda, z) - z \cdot q(\lambda, z) \quad \text{and} \quad m(\lambda, z) = s(\lambda, z) - \alpha \cdot t(\lambda, z).$$

It follows from Lemma 4.6.1 that for all but finitely many λ we have that $f_\lambda(z) = z$ if and only if $l(\lambda, z) = 0$ and similarly for all but finitely many λ we have $f'_\lambda(z) = \alpha$ if and only if $m(\lambda, z) = 0$. Consider the polynomial

$$R(\lambda) = \mathrm{Res}_z(l, m).$$

Note that $R(\lambda) \in \mathbb{C}[\lambda]$. Since for all but finitely many $\lambda \in U$ the polynomials $m(\lambda, z)$ and $l(\lambda, z)$ have a common root, namely $w(\lambda)$, we find that $R(\lambda)$ has infinitely many roots and is constantly 0 as a result. This means that for all $\lambda \in \mathbb{C}$ the polynomials $m(\lambda, z)$ and $l(\lambda, z)$ have a common root. This again means that for all but finitely many λ there is some $z \in \mathbb{C}$ such that $f_\lambda(z) = z$ and $f'_\lambda(z) = \alpha$, where we now consider f_λ to be defined for every complex parameter λ. This concludes the proof. □

4.7. Remarks on calculations

This section is dedicated to an explanation of how the numerical values in Table 4.1 were obtained. Let $g \in H_\Delta$ and write $g = f_{d_n} \circ \cdots \circ f_{d_1}$ as is done in the second column of Table 4.1. The goal is to accurately find an approximation of those λ for which g_λ has a parabolic fixed point, i.e., those λ for which there exists a z such that $g_\lambda(z) = z$ and $g'_\lambda(z) = 1$.

For $d \in \mathbb{Z}_{\geq 1}$ define the following map from $\mathbb{C}[\lambda, z] \times \mathbb{C}[\lambda, z]$ to itself:

$$F_d(p(\lambda, z), q(\lambda, z)) = \left(\lambda \cdot q(\lambda, z)^d, (p(\lambda, z) + q(\lambda, z))^d\right).$$

Let $p_0(\lambda, z) = z$ and $q_0(\lambda, z) = 1$ and define for $i = 1, \ldots, n$ the polynomials $(p_i, q_i) = F_{d_i}(p_{i-1}, q_{i-1})$. We see that in $\mathbb{C}(\lambda, z)$ the elements $g_\lambda(z)$ and $p_n(\lambda, z)/q_n(\lambda, z)$ are equal. Define the following two polynomials

$$\begin{aligned} P_1(\lambda, z) &= p_n(\lambda, z) - z \cdot q_n(\lambda, z) \\ P_2(\lambda, z) &= q_n(\lambda, z) \cdot \frac{\partial p_n}{\partial z}(\lambda, z) - p_n(\lambda, z) \cdot \frac{\partial q_n}{\partial z}(\lambda, z) - q_n(\lambda, z)^2. \end{aligned}$$

The polynomials P_1 and P_2 are defined in such a way that if $(\lambda_0, z_0) \in \mathbb{C}^2$ is a zero for both polynomials with the additional property that $q_n(\lambda_0, z_0) \neq 0$, then z_0 is a parabolic fixed point of g_{λ_0}.

We claim that if (λ_0, z_0) is such that $P_1(\lambda_0, z_0) = q_n(\lambda_0, z_0) = 0$, then $\lambda_0 = 0$. To show this suppose that $\lambda_0 \neq 0$. We can deduce from the fact that both P_1 and q_n are zero at (λ_0, z_0) that also $p_n(\lambda_0, z_0) = 0$. From the relation $p_n(\lambda_0, z_0) = \lambda_0 \cdot q_{n-1}(\lambda_0, z_0)^{d_n}$ and the fact that $\lambda_0 \neq 0$ we deduce that $q_{n-1}(\lambda_0, z_0) = 0$. Finally it follows from the relation $q_n(\lambda_0, z_0) = (p_{n-1}(\lambda_0, z_0) + q_{n-1}(\lambda_0, z_0))^{d_n}$ that $p_{n-1}(\lambda_0, z_0) = 0$. We see that $p_n(\lambda_0, z_0) = q_n(\lambda_0, z_0) = 0$ implies that

$p_{n-1}(\lambda_0, z_0) = q_{n-1}(\lambda_0, z_0) = 0$. Proceeding inductively we should find that $q_0(\lambda_0, z_0) = 0$, but this is a contradiction because $q_0(\lambda_0, z_0) = 1$.

We have shown that, to find a parameter λ_0 such that g_{λ_0} has a parabolic fixed point, it is enough to find a $\lambda_0 \neq 0$ such that $P_1(\lambda_0, z)$ and $P_2(\lambda_0, z)$, viewed as polynomials in z, share a common root. To find such parameters we use the resultant, as we did in Section 4.6. We define the following polynomial in λ

$$R(\lambda) = \mathrm{Res}_z(P_1(\lambda, z), P_2(\lambda, z)).$$

A property of the resultant is that if λ_0 is a root of R then either the polynomials $P_1(\lambda_0, z)$ and $P_2(\lambda_0, z)$ share a common root or the degree of $P_1(\lambda_0, z)$ is lower than the degree of $P_1(\lambda, z)$ for generic λ. So, if we let $c(\lambda)$ denote the leading coefficient of $P_1(\lambda, z)$ as polynomial in $(\mathbb{Z}[\lambda])[z]$, we conclude that the parameters that we are looking for are $\lambda_0 \neq 0$ such that $R(\lambda_0) = 0$ and $c(\lambda_0) \neq 0$. We are now ready to describe the procedure that was used to obtain the results of Table 4.1.

- For a given Δ and $g \in H_\Delta$ the polynomials $R(\lambda)$ and $c(\lambda)$ were calculated using Mathematica, which contains a built-in function to calculate resultants symbolically. For the last row in Table 4.1 this calculation took several hours.

- Factors of λ and common factors with $c(\lambda)$ were removed from $R(\lambda)$ to obtain a polynomial $\tilde{R}(\lambda)$.

- Mathematica was used to calculate the roots of $\tilde{R}(\lambda)$ numerically with the settings set such that the returned values are accurate up to 20 decimal places.

- These approximations of roots λ_0 were used to numerically calculate the solutions α to the equation $\lambda_0 = \frac{-\alpha\cdot(\Delta-1)^{\Delta-1}}{(\Delta-1+\alpha)^{\Delta}}$. Certain combinations of Δ, g, λ_0 and α for which $|\alpha| < 1$ are recorded in Table 4.1.

Chapter 5

The limit of the zero locus of the independence polynomial for bounded degree graphs

5.1. Introduction

In this final chapter we continue exploring the closure of the complex zeros of the independence polynomial for bounded degree graphs. We again denote the independence polynomial of a simple graph $G = (V, E)$ by

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V: \\ I \text{ is independent}}} \lambda^{|I|}.$$

We will denote by $\mathcal{G}_d$ the class of all graphs with maximum degree at most $d + 1$. Note that the index is shifted by one in contrast with the previous two chapters. This is done to make some of the later calculations more readable. We let $\mathcal{Z}_d$ denote the set of complex values that can occur as a zero of the independence polynomial of such graphs. Furthermore, we let $\mathcal{U}_d$ denote the complement of the closure of $\mathcal{Z}_d$, i.e. the maximal open set that is zero-free for all graphs with maximum degree at most $d + 1$. We denote the connected component of $\mathcal{U}_d$ containing 0 by $\mathcal{U}_d(0)$.

Recall that the understanding of $\mathcal{U}_d$ and $\mathcal{U}_d(0)$ is related to understanding the complexity of approximating the partition function for different parameters inside the complex plane within the class of bounded degree graphs (see Section 4.1 for a more detailed exposition). For finite $d \geq 2$ many interesting questions remain open. One of the more tantalizing questions is whether or not $\mathcal{U}_d$ is connected, or equivalently, whether $\mathcal{U}_d = \mathcal{U}_d(0)$. It is known that $\mathcal{U}_d$ is contained in the following domain

$$\mathcal{C}_d = \left\{ \frac{-d^d u}{(u+d)^{d+1}} : |u| < 1 \right\},$$

see Theorem 4.1.2. The notation has changed because the letter Λ is reserved for something different in this chapter.

In the previous chapter it was shown that equality of the sets $\mathcal{U}_d$ and $\mathcal{C}_d$ does not hold for small values of d: there are graphs in $\mathcal{G}_d$ for all $2 \leq d \leq 8$ with zeros inside $\mathcal{C}_d$ (see Theorem 4.1.3). While it is likely that these methods extend to arbitrarily high degrees, the proof involves computer calculations which can only run for relatively small degrees. This chapter deals with the opposite end of the spectrum, that is, we will let the degree bound go to infinity. The results in this chapter imply that for large degree d the sets $\mathcal{U}_d$ and $\mathcal{C}_d$ are also not equal.

Given this fact, a natural question to ask is whether the zero-parameters inside $\mathcal{C}_d$ converge to the boundary of the (properly rescaled) sets $\mathcal{C}_d$ as $d \to \infty$, thus disappearing in the limit. We will see that this is not the case. The sets $\mathcal{C}_d$ shrink down to the origin as $d \to \infty$, hence the sets $\mathcal{U}_d \subset \mathcal{C}_d$ do too. However, the rescaled sets $d \cdot \mathcal{C}_d$ converge (in terms of the Hausdorff distance) to a limit set $\mathcal{C}_\infty$, given by

$$\mathcal{C}_\infty = \{-ue^{-u} \ : \ |u| < 1\}, \tag{5.1}$$

see the illustration in Figure 5.1. This "infinite-degree cardioid" intersects the real axis in the open interval $(-e^{-1}, e)$.

In this chapter we describe the limit of the scaled zero-free regions $d \cdot \mathcal{U}_d$. We define a complex domain $\mathcal{U}_\infty$ (see Definition 5.2.16), show that it is the limit of the sets $d \cdot \mathcal{U}_d$, and determine some geometric properties of $\mathcal{U}_\infty$. Our main results are summarized in the following two theorems. The locations of their proofs within the chapter can be found at the end of the introduction.

Theorem 5.1.1. *The sets $d \cdot \mathcal{U}_d$ converge to $\mathcal{U}_\infty$ in terms of the Hausdorff distance.*

In other words, for any closed $K_1 \subseteq \mathcal{U}_\infty$ and any open $K_2 \supseteq \overline{\mathcal{U}_\infty}$ there exists a d_0 such that $K_1 \subseteq d \cdot \mathcal{U}_d \subseteq K_2$ for $d \geq d_0$.

Theorem 5.1.2.
1. *The set $\mathcal{U}_\infty$ is star-convex from 0, i.e. if $\Lambda \in \mathcal{U}_\infty$ then $c \cdot \Lambda \in \mathcal{U}_\infty$ for all $c \in [0, 1]$.*
2. *The boundaries of $\mathcal{U}_\infty$ and $\mathcal{C}_\infty$ intersect only in the two real parameters $-e^{-1}$ and e. In particular, the smaller set $\mathcal{U}_\infty$ avoids a neighborhood of any non-real boundary point of $\mathcal{C}_\infty$.*
3. *Near the positive real boundary point e the boundary of $\mathcal{U}_\infty$ is contained in an analytic curve Γ (given explicitly in Corollary 5.5.4) and its complex conjugate $\overline{\Gamma}$, and is in particular piece-wise analytic but not smooth.*

As an immediate consequence of Theorems 5.1.1 and 5.1.2 we obtain the following:

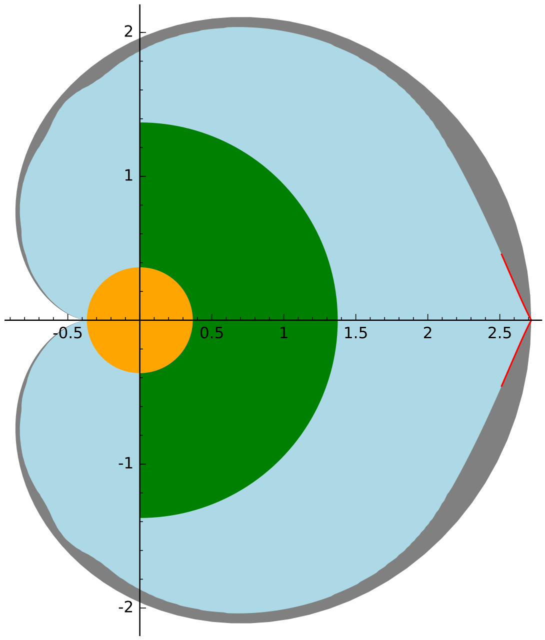

Figure 5.1: The cardioid $\mathcal{C}_\infty$, with known zero-free regions depicted in yellow and green, and the curve Γ depicted in red. The gray strip near the boundary of the cardioid is known not to be zero-free anywhere, i.e. it does not intersect $\mathcal{U}_\infty$. The algorithm used to compute the gray region relies upon Lemma 5.3.7.

Corollary 5.1.3. *For all sufficiently large degrees $d \in \mathbb{N}$ the set $\mathcal{U}_d$ is strictly smaller than $\mathcal{C}_d$.*

A different consequence concerns the connected component of $\mathcal{U}_d$ containing 0, previously denoted by $\mathcal{U}_d(0)$. While we cannot conclude that $\mathcal{U}_d$ is equal to $\mathcal{U}_d(0)$ for large d, we can conclude that the proportion of the area of $\mathcal{U}_d(0)$ inside $\mathcal{U}_d$ goes to 1 as d approaches infinity.

5.1.1. Organization

The organization of this chapter is as follows. In Section 5.2 we will cover background material and discuss known results regarding the sets $\mathcal{U}_d$; we moreover define the set $\mathcal{V}_\infty$ and its interior $\mathcal{U}_\infty$. In Section 5.3 we prove properties of these sets. Our main results rely on the analysis done in this section. Section 5.4 is dedicated to the proof of Theorem 5.1.1, and Section 5.5 to the proof of part (3) of Theorem 5.1.2.

Specifically: parts (1), (2) and (3) of Theorem 5.1.2 are proved in Lemma 5.2.17, Corollary 5.3.5 and Corollary 5.5.4. Theorem 5.1.1 is proved in Section 5.4.3.

5.2. Preliminaries

5.2.1. Known subsets of $\mathcal{U}_d$.

There is an extensive history of estimates on the zero-free regions for the independence polynomial for different classes of graphs. We discuss some of the most relevant results here, focusing only on the class of graphs $\mathcal{G}_d$ and the region $\mathcal{U}_d$.

Theorem 5.2.1 ([She85, SS05]). *For any $d \geq 2$ the disk centered at 0 with radius $d^d/(d+1)^{(d+1)}$ is contained in $\mathcal{U}_d$.*

Since zeros accumulate on the point $-d^d/(d+1)^{(d+1)}$, the radius of this so-called Shearer disk is sharp.

Theorem 5.2.2 ([BC18]). *For any $d \geq 2$ the semi-disk given by the intersection of the disk of radius $\frac{7}{8}\tan\left(\frac{\pi}{2d}\right)$ centered at 0 with the right half plane is contained in $\mathcal{U}_d$.*

Due to Theorem 5.1.1 we immediately obtain the following:

Corollary 5.2.3. *The Shearer disk $D_{e^{-1}}(0)$ and the semi-disk $D_{7\pi/16}(0) \cap \{Z : Re(Z) > 0\}$ are contained in $\mathcal{U}_\infty$.*

Since $\mathcal{U}_\infty$ is contained in $\mathcal{C}_\infty$, the radius of the Shearer disk $D_{e^{-1}}(0)$ is sharp.

Theorem 5.2.4 ([PR19]). *For any $d \geq 2$ the set $\mathcal{U}_d$ contains an open neighborhood of the real interval $[0, \frac{d^d}{(d-1)^{d+1}})$.*

As this statement does not give bounds on the size of the neighborhood of the real interval that are uniform after rescaling, we cannot deduce directly from Theorem 5.1.1 that $\mathcal{U}_\infty$ contains an open neighborhood of the real interval $[0, e)$. However, we can prove this result directly:

Theorem 5.2.5. *The set $\mathcal{U}_\infty$ contains an open neighborhood of the real interval $[0, e)$.*

In fact, parts (1) and (3) of Theorem 5.1.2 together imply that $\mathcal{U}_\infty$ contains a neighborhood of $[0, e)$.

5.2.2. From finite graphs to rational dynamics

In this section we will recall how the set of zeros of bounded degree graphs can be related to a semi-group generated by rational maps. Most results in this section will be stated without proofs. The proofs can be found in Chapter 3 or the references therein.

Definition 5.2.6. Given a rooted graph (G, v), write

$$Z^{in}_{G,v}(\lambda) = \sum_{I:\, v \in I} \lambda^{|I|} \quad \text{and} \quad Z^{out}_{G,v}(\lambda) = \sum_{I:\, v \notin I} \lambda^{|I|},$$

where both summations run over the independent vertex subsets $I \subset V(G)$. We moreover define the *occupation ration* of G at v as the rational function

$$R_{G,v}(\lambda) = \frac{Z^{in}_{G,v}(\lambda)}{Z^{out}_{G,v}(\lambda)}.$$

Since $Z_G(\lambda) = Z^{in}_{G,v}(\lambda) + Z^{out}_{G,v}(\lambda)$, an immediate consequence is that $Z_G(\lambda) = 0$ if and only either $R_{G,v}(\lambda) = -1$ or both $Z^{in}_{G,v}$ and $Z^{out}_{G,v}$ vanish at λ. While it is in fact possible that both $Z^{in}_{G,v}$ and $Z^{out}_{G,v}$ vanish simultaneously, we can eliminate this condition by considering the whole family of graphs $G \in \mathcal{G}_d$. To make this explicit we introduce some notation.

Let $\mathcal{G}^k_d$ denote the set of rooted graphs (G, v) such that the maximum degree of G is at most $d+1$ and the degree of the vertex v is at most k. We let

$$\mathcal{R}^k_d = \{R_{G,v} : (G, v) \in \mathcal{G}^k_d\}$$

denote the corresponding family of occupation ratios. For any λ in the Riemann sphere $\widehat{\mathbb{C}}$ we denote by $\mathcal{R}^k_d(\lambda)$ the set of all rational functions in $\mathcal{R}^k_d$ evaluated at λ. Thus $\mathcal{R}^k_d$ is a family of rational functions, and its values $\mathcal{R}^k_d(\lambda)$ are contained in $\widehat{\mathbb{C}}$. Recall that $\mathcal{Z}_d$ denotes the set of all zeros for graphs of degree at most $d+1$. The following lemma follows directly from Lemma 3.2.3.

Lemma 5.2.7. *Let $d \geq 2$, then $\lambda \in \mathcal{Z}_d$ if and only if $-1 \in \mathcal{R}^d_d(\lambda)$.*

It was shown in [SS05, Wei06, Ben18] that given a rooted graph $(G, v) \in \mathcal{G}^k_d$ there is a rooted tree $(T, u) \in \mathcal{G}^k_d$ such that $R_{G,v} = R_{T,u}$. This is especially useful because for a rooted tree (T, u) the ratio can be expressed recursively. Given $k \geq 1$ we define

$$F_\lambda(z_1, \ldots, z_k) = \frac{\lambda}{\prod_{j=1}^{k}(1 + z_j)}.$$

Note that the map F_λ is defined for any number of inputs.

Definition 5.2.8. Let $d \geq 2$ and $\lambda \in \mathbb{C}$. We recursively define a class of rational functions $\mathcal{F}_{d,\lambda}$ as follows:

1. the identity map $z \mapsto z$ is contained in $\mathcal{F}_{d,\lambda}$;
2. given any $1 \leq k \leq d$ and functions $f_1, \ldots, f_k \in \mathcal{F}_{d,\lambda}$, the rational function
$$z \mapsto F_\lambda(f_1(z), \ldots, f_k(z))$$
is also contained in $\mathcal{F}_{d,\lambda}$.

The following lemma follows for example from Lemma 3.2.4 and gives the relation between the rational semi-group $\mathcal{F}_{d,\lambda}$ parameterized by λ and the set of occupation ratios.

Lemma 5.2.9. *Let $d \geq 2$ and $f_\lambda \in \mathcal{F}_{d,\lambda}$ not equal to the identity. Write*
$$f_\lambda(z) = F_\lambda(f_1(z), \ldots, f_k(z)).$$
The rational map $\lambda \mapsto f_\lambda(0)$ is an element of $\mathcal{R}_d^k$, moreover the set of rational functions
$$\{\lambda \mapsto f_\lambda(0) : f_\lambda \in \mathcal{F}_{d,\lambda}\}$$
is equal to $\mathcal{R}_d^d$.

Definition 5.2.10. Let $d \geq 2$ and fix $\lambda \in \mathbb{C}$. We say that a set $Y \subset \mathbb{C} \setminus \{-1\}$ is invariant under the maps F_λ if for every $1 \leq k \leq d$ and every $z_1, \ldots, z_k \in Y$ we have
$$F_\lambda(z_1, \ldots, z_k) \in Y.$$

A direct consequence of lemmas 5.2.7 and 5.2.9 is the following:

Lemma 5.2.11. *Let $d \geq 2$ and $\lambda \in \mathbb{C}$. Suppose there exists a set $Y \subset \mathbb{C} \setminus \{-1\}$ containing 0 which is invariant under the maps F_λ. Then $Z_G(\lambda) \neq 0$ for all graphs $G \in \mathcal{G}_d$.*

An important technique used in many recent papers in the area is the construction of new graphs by *implementing* one rooted graph into another graph, i.e. by replacing each vertex of the latter graph by a copy of the former, where edges now connect corresponding copies at the root vertices. This technique adds the maximum degrees of the latter graph to the degree of the root vertex of the former, hence the sum of these two numbers should be at most $d + 1$ to remain in the class $\mathcal{G}_d$.

Lemma 5.2.12. *Let $d, k, m \in \mathbb{Z}_{\geq 1}$ such that $k + m \leq d$. If there is a rooted graph $(G, v) \in \mathcal{G}_d^k$ and a $\lambda_0 \in \widehat{\mathbb{C}}$ such that $R_{G,v}(\lambda_0) \in \overline{\mathcal{Z}_m}$, then $\lambda_0 \in \overline{\mathcal{Z}_d}$.*

Proof. Suppose $(H,u) \in \mathcal{G}_m^m$. We transform the graph H to $\tilde{H}$ by replacing each of its vertices by a copy of the rooted graph (G,v). On the level of ratios this has the effect of composition, that is $R_{\tilde{H},\tilde{u}}(\lambda) = R_{H,u}(R_{G,v}(\lambda))$, where $\tilde{u}$ in $\tilde{H}$ is the vertex corresponding u in the original graph H. A proof of this fact can be found for example in Lemma 3.2.8. Note that the maximum degree in $\tilde{H}$ is at most $\max(m+1+k, d+1) = d+1$ and the degree of $\tilde{u}$ is at most $m+k \leq d$ and thus $(\tilde{H},\tilde{u}) \in \mathcal{G}_d^d$. This is true for any graph $(H,u) \in \mathcal{G}_m^m$ and therefore we find that $\mathcal{R}_m^m(R_{G,v}(\lambda)) \subseteq \mathcal{R}_d^d(\lambda)$ for every $\lambda \in \widehat{\mathbb{C}}$.

Because $R_{G,v}$ is an open map and $R_{G,v}(\lambda_0) \in \overline{\mathcal{Z}_m}$ we can find a sequence $\{\lambda_n\}_{n\geq 1}$ converging to λ_0 such that $R_{G,v}(\lambda_n) \in \mathcal{Z}_m$ for all n. According to Lemma 5.2.7 this means that -1 is contained in $\mathcal{R}_m^m(R_{G,v}(\lambda_n))$ for all n. This thus means that $-1 \in \mathcal{R}_d^d(\lambda_n)$ for all n, which again by Lemma 5.2.7 implies that $\lambda_n \in \mathcal{Z}_d$. We conclude that $\lambda_0 \in \overline{\mathcal{Z}_d}$. □

5.2.3. From rational to transcendental dynamics

In this section we will define the “rescaled limit” of objects from the previous subsection. We start by formalizing the relationship between the cardioids $\mathcal{C}_d$ and their limit $\mathcal{C}_\infty$.

Remark 5.2.13. The cardioid $d \cdot \mathcal{C}_d$ contains all parameters Λ for which the function $Z \mapsto \Lambda/(1+Z/d)^d$ has an attracting fixed point. As $d \to \infty$ these rational functions converge locally uniformly to the entire functions $E_\Lambda(Z) = \Lambda \cdot e^{-Z}$, and the domains $d \cdot C_d$ converge in the Hausdorff distance to the domain $\mathcal{C}_\infty$. It follows that the set $\mathcal{C}_\infty$ coincides with the hyperbolic component of the family E_Λ, containing all parameters Λ for which E_Λ has an attracting fixed point.

We note that this is a particular case of the more general phenomenon described in [KK97]. In our setting the relationship between the hyperbolic components can also be computed explicitly as follows:

Suppose E_Λ has a fixed point p with derivative $E'_\Lambda(p) = \alpha$. Then, because $E_\Lambda(Z)/E'_\Lambda(Z) = -1$ we find that $p = -\alpha$ and thus $\Lambda = -\alpha e^{-\alpha}$. In the other direction, if $\Lambda = -\alpha e^{-\alpha}$, then $-\alpha$ is a fixed point of E_Λ with derivative α.

For any $n \in \mathbb{N}$ and $s_1, \ldots, s_n \geq 0$ with $\sum_{i=1}^n s_i \leq 1$, and for fixed $\Lambda \in \mathbb{C}$, define the function $E_{\Lambda,(s_1,\ldots,s_n)} : \mathbb{C}^n \to \mathbb{C}$ by

$$E_{\Lambda,(s_1,\ldots,s_n)}(Z_1,\ldots,Z_n) = \Lambda \cdot e^{-s_1 Z_1 - \cdots - s_n Z_n}.$$

We will denote $E_{\Lambda,(1)}$ by E_Λ. We define the set of functions G_Λ in the following way.

Definition 5.2.14. Let $\Lambda \in \mathbb{C}$. We recursively define a class of transcendental functions G_Λ as follows:

1. the identity map $Z \mapsto Z$ is contained in G_Λ;

2. for any $k \geq 1$, tuple $(s_1, \ldots, s_k) \in \mathbb{R}^k_{\geq 0}$ with $\sum_{m=1}^k s_m \leq 1$ and functions $g_1, \ldots, g_k \in G_\Lambda$ the function

$$Z \mapsto E_{\Lambda,(s_1,\ldots,s_k)}\left(g_1(Z), \ldots, g_k(Z)\right)$$

is also contained in G_Λ.

We define the set V_Λ as the orbit of 0 under the set of functions G_Λ, i.e. $V_\Lambda = \{g(0) : g \in G_\Lambda\}$. We define $\widehat{V}_\Lambda$ as the convex hull of V_Λ.

The following lemma follows directly from the definitions, but gathers some elementary properties of G_Λ, V_Λ and $\widehat{V}_\Lambda$ that will be used frequently.

Lemma 5.2.15. *Let $\Lambda \in \mathbb{C}$.*

1. *We have $E_\Lambda(\widehat{V}_\Lambda) = V_\Lambda \setminus \{0\}$.*

2. *If K is a convex set containing 0 that is forward invariant for E_Λ, then it is forward invariant for every map in G_Λ and $\widehat{V}_\Lambda \subseteq K$.*

Definition 5.2.16. We define the set $\mathcal{V}_\infty$ as the set consisting of those Λ for which V_Λ is bounded. We define $\mathcal{U}_\infty$ as its interior.

Lemma 5.2.17. *Part (1) of Theorem 5.1.2 holds, i.e. the set $\mathcal{U}_\infty$ is star-convex from 0.*

Proof. We will first show that $\mathcal{V}_\infty$ is star-convex from 0. To that end we suppose that $\Lambda \in \mathcal{V}_\infty$ and we will argue that $t \cdot \Lambda \in \mathcal{V}_\infty$ for all $t \in [0, 1]$. By definition $\widehat{V}_\Lambda$ is a bounded convex set containing 0 that is forward invariant for E_Λ. We find that

$$E_{t\cdot\Lambda}(\widehat{V}_\Lambda) = t \cdot E_\Lambda(\widehat{V}_\Lambda) \subseteq t \cdot \widehat{V}_\Lambda \subseteq \widehat{V}_\Lambda$$

and thus it follows from Lemma 5.2.15 that $\widehat{V}_{t\cdot\Lambda} \subseteq \widehat{V}_\Lambda$. Therefore $V_{t\cdot\Lambda}$ is bounded and thus $t \cdot \Lambda \in \mathcal{V}_\infty$. We conclude that $\mathcal{V}_\infty$ is star-convex from 0.

To conclude the same for its interior $\mathcal{U}_\infty$ we only need to show that $0 \in \mathcal{U}_\infty$. This follows from the fact that if $|\Lambda| < 1/e$ the unit disk $\mathbb{D}$ is forward invariant for E_Λ. Therefore it follows from Lemma 5.2.15 that $V_\Lambda \subseteq \mathbb{D}$ and thus $\Lambda \in \mathcal{V}_\infty$. This shows that the open disk of radius $1/e$ is contained in $\mathcal{U}_\infty$. $\square$

The next three lemmas form the bridge between the finite (but large degree) world and the transcendental one. First let us emphasize the relation between F_λ and E_Λ.

Lemma 5.2.18. *For each pair of compact subsets $K, L \subset \mathbb{C}$, $s_1, \ldots, s_j > 0$ with $\sum s_i \leq 1$, and each $\epsilon > 0$, there exists a $d_0 \in \mathbb{N}$ such that the following holds: For any $d \geq d_0$, and for any $Z_1, \ldots, Z_j \in K$ and $\Lambda \in L$ we have*

$$\left| d \cdot F_{\Lambda/d}\left(\underbrace{Z_1/d, \ldots, Z_1/d}_{p_1}, \ldots, \underbrace{Z_j/d, \ldots, Z_j/d}_{p_j}\right) - E_{\Lambda,(s_1,\ldots,s_j)}(Z_1, \ldots, Z_j)\right| < \epsilon,$$

where $p_i = \lfloor s_i d \rfloor$ for $1 \leq i \leq j$.

Proof. Observe that

$$d \cdot F_{\Lambda/d}\left(\underbrace{Z_1/d, \ldots, Z_1/d}_{p_1}, \ldots, \underbrace{Z_j/d, \ldots, Z_j/d}_{p_j}\right) = \Lambda \prod_{i=1}^{j} (1 + Z_i/d)^{-p_i}.$$

As d tends to infinity, the map $Z_i \mapsto (1 + Z_i/d)^{-p_i}$ uniformly converges to $Z_i \mapsto e^{-s_i Z_i}$ on K for all $1 \leq i \leq j$, thus their product also converges uniformly. The lemma now follows from the fact that the set L is bounded. □

Lemma 5.2.19. *For any $g_\Lambda \in G_\Lambda$ there exists a sequence of graphs $(G_d, v_d) \in \mathcal{G}_d^d$, such that $\Lambda \mapsto d \cdot R_{G_d,v_d}(\Lambda/d)$ converges locally uniformly to $\Lambda \mapsto g_\Lambda(0)$. Furthermore, if we can write $g_\Lambda(Z) = E_{\Lambda,(s_1,\ldots,s_j)}(g_1(Z), \ldots, g_j(Z))$ then we can choose the sequence such that*

$$\deg(v_d) \leq d \cdot \sum_{1 \leq i \leq j} s_i.$$

Proof. The proof relies upon the recursive definition of the family G_Λ, and uses induction on the number of compositions in the definition of the map g_λ. The statement is trivial for $Z \mapsto Z$. Let us assume that $g_\Lambda \neq \text{id}$. Then we can write $g_\Lambda(Z) = E_{\Lambda,(s_1,\ldots,s_j)}(g_{1,\Lambda}(Z), \ldots, g_{j,\Lambda}(Z))$ and by induction for each g_i there exists a sequence $f^{(i)}_{\lambda,d} \in \mathcal{F}_{d,\lambda}$, such that

$$\Lambda \mapsto d \cdot f^{(i)}_{\Lambda/d,d}(0)$$

converges locally uniformly to $\Lambda \mapsto g_{i,\Lambda}(0)$ for each $1 \leq i \leq j$. By the previous lemma we know that the composition

$$\Lambda \mapsto d \cdot F_{\Lambda/d}\left(\underbrace{f^{(1)}_{\Lambda/d,d}(0), \ldots, f^{(1)}_{\Lambda/d,d}(0)}_{p_1}, \ldots, \underbrace{f^{(j)}_{\Lambda/d,d}(0), \ldots, f^{(j)}_{\Lambda/d,d}(0)}_{p_j}\right)$$

converges locally uniformly to $\Lambda \mapsto g_\Lambda(0)$. Since $p_1 + \cdots + p_j \leq d(s_1 + \cdots + s_j)$ we can conclude by Lemma 5.2.9 the desired statement. □

Lemma 5.2.20. *Let K be convex bounded set containing* 0*, such that* $E_{\Lambda_0}(K)$ *is relatively compact in int*(K)*. Then there exists* $\varepsilon > 0$ *and* d_0*, such that* $B_\varepsilon(\Lambda_0) \subseteq d \cdot \mathcal{U}_d$ *for any* $d \geq d_0$*.*

Proof. For d large we can define a function $\delta_d : K \to \mathbb{C}$ such that $1/(1 + Z/d) = e^{-Z/d+\delta_d(Z)}$. Note that δ_d can be chosen such that $d \cdot \delta_d(Z)$ converges to 0 uniformly on K as d tends to infinity. Let $1 \leq k \leq d$ and let $(z_1, \ldots, z_k)$ be an arbitrary k-tuple in K/d and write $z_i = Z_i/d$. We find that

$$d \cdot F_{\Lambda/d}(Z_1/d, \cdots, Z_k/d) = \Lambda \prod_{i=1}^{k} \frac{1}{(1 + Z_i/d)} = \Lambda \prod_{i=1}^{k} e^{-Z_i/d+\delta_d(Z_i)}$$
$$= \frac{\Lambda}{\Lambda_0} \cdot E_{\Lambda_0}\left(\frac{1}{d}\sum_{i=1}^{k} Z_i\right) \cdot e^{\sum_{i=1}^{k} \delta_d(Z_i)}.$$

By choosing d large enough the term $e^{\sum_{i=1}^{k} \delta_d(Z_i)}$ can be bounded arbitrarily close to 1 independently of the chosen tuple. Furthermore, by taking Λ close to Λ_0, the same is true for the term Λ/Λ_0. Therefore there exists an $\epsilon > 0$ and a d_0 such that K/d is forward invariant under the maps $\mathcal{F}_{\Lambda/d}$ for $\Lambda \in B_\epsilon(\Lambda_0)$ and $d \geq d_0$. The statement thus follows from Lemma 5.2.11. □

5.3. The sets $\mathcal{V}_\infty$, $\mathcal{U}_\infty$ and V_Λ

5.3.1. Properties of $\mathcal{V}_\infty$, $\mathcal{U}_\infty$ and V_Λ

In this section we will mostly be concerned with parameters Λ that are not real, but for completeness we first show that

$$\mathcal{V}_\infty \cap \mathbb{R} = [-1/e, \infty).$$

If $\Lambda \geq 0$ then $E_\Lambda|_\mathbb{R}$ is a decreasing function. Furthermore $E_\Lambda(0) = \Lambda$ and $E_\Lambda(\Lambda) > 0$ and thus the (convex) interval $[0, \Lambda]$ is forward invariant for E_Λ. Therefore $\widehat{V}_\Lambda$ is contained in $[0, \Lambda]$ and in fact, because $\widehat{V}_\Lambda$ must contain both 0 and Λ, $\widehat{V}_\Lambda = [0, \Lambda]$. For $-e^{-1} \leq \Lambda < 0$ the map E_Λ has an attracting or a neutral fixed point $w_\Lambda < 0$. In this case the interval $(w_\Lambda, 0]$ is forwards invariant and thus it contains $\widehat{V}_\Lambda$. Because the orbit of 0 under iteration of E_Λ must converge to w_Λ we see that $\widehat{V}_\Lambda = (w_\Lambda, 0]$. Finally, if $\Lambda < -1/e$ the equation $E_\Lambda(x) = x$ has no real solutions and thus the orbit of 0, which is a decreasing sequence, must converge to $-\infty$. Therefore Λ is not contained in $\mathcal{V}_\infty$ in this case.

The statements in our main theorem concern the interior of $\mathcal{V}_\infty$, which we have denoted by $\mathcal{U}_\infty$. In the course of this chapter we will show that $\mathcal{V}_\infty$ is equal to the disjoint union of the closure of $\mathcal{U}_\infty$ and the interval (e, ∞).

Lemma 5.3.1. *If $\Lambda \notin \mathcal{V}_\infty \cup \mathbb{R}$ then $V_\Lambda = \mathbb{C}$.*

Proof. Take Λ to be nonreal and such that V_Λ is unbounded. Let us suppose that $\widehat{V}_\Lambda \neq \mathbb{C}$. The set $\widehat{V}_\Lambda$ is convex and contains the origin and thus it must avoid a sector of the form

$$S := \{Z \in \mathbb{C} \ : \ |Z| > R, \text{ and } \theta_1 < \arg(Z) < \theta_2\}.$$

Because $E_\Lambda(\widehat{V}_\Lambda) = V_\Lambda$ and V_Λ is unbounded we can choose $Z_0 \in \widehat{V}_\Lambda$ for which $|e^{-Z_0}|$ is arbitrarily large. We claim that we can choose this Z_0 such that in addition $|\mathrm{Im}(Z_0)|$ is arbitrarily large. Assuming this claim holds we obtain a contradiction: the curve $t \mapsto \Lambda e^{-tZ_0}$, for $t \in [0,1]$, must pass through the sector S, however this curve must also be contained in $\widehat{V}_\Lambda$, contradicting our earlier assumption.

It remains to prove the claim. Suppose that there is a sequence of points $\{Z_n\}_{n\geq 1}$ in $\widehat{V}_\Lambda$ with real parts converging to minus infinity, but bounded imaginary parts. Write $Z_n = a_n + ib_n$ and $\Lambda = x + iy$. Define a sequence $s_n \in [0,1]$ for which $e^{-s_n a_n}$ converges to infinity sufficiently slow such that $a_n + e^{-s_n a_n}(|x|+|y|)$ still converges to $-\infty$. Let $\epsilon = |y|/(2(|x|+|y|))$. Because the sequence s_n necessarily converges to 0 and the sequence b_n is bounded we find that $\cos(s_n b_n) > 1-\epsilon$ and $|\sin(s_n b_n)| < \epsilon$ for all n sufficiently large. Now let $W_n = Z_n + \Lambda e^{-s_n Z_n}$, then

$$\mathrm{Re}(W_n) = a_n + e^{-a_n s_n}(x\cos(b_n s_n) + y\sin(b_n s_n)) \leq a_n + e^{-a_n s_n}(|x|+|y|).$$

and

$$\begin{aligned} |\mathrm{Im}(W_n)| &= |b_n + e^{-a_n s_n}(y\cos(b_n s_n) - x\sin(b_n s_n))| \\ &\geq e^{-a_n s_n}(|y|(1-\epsilon) - |x|\epsilon) - |b_n| \\ &= \frac{|y|}{2} e^{-a_n s_n} - |b_n|. \end{aligned}$$

We see that $\mathrm{Re}(W_n) \to \infty$ and $|\mathrm{Im}(W_n)| \to \infty$. Since both Z_n and $\Lambda e^{-s_n Z_n}$ are contained in $\widehat{V}_\Lambda$, it follows that $W_n/2 \in \widehat{V}_\Lambda$. Working with a large enough point $W_n/2$ instead of Z_0 we obtain the contradiction described above. □

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map, then a point $p \in \mathbb{C}$ is periodic with period n if $f^{\circ n}(p) = p$ and $f^{\circ m}(p) \neq p$ for all $1 < m < n$. The derivative $\mu = (f^{\circ n})'(p)$ is called the multiplier of p and p is called attracting, neutral or repelling according to whether $|\mu| < 1$, $|\mu| = 1$ or $|\mu| > 1$ respectively.

Lemma 5.3.2. *Let $\Lambda \in \mathbb{C}\setminus\mathbb{R}$ and suppose that $K \neq \mathbb{C}$ is a convex set containing 0 for which $E_\Lambda(K) \subseteq K$. Then $\Lambda \in \mathcal{V}_\infty$, and every non-identity $g \in G_\Lambda$ has a fixed point p in the closure of K that is either attracting or neutral with multiplier 1. All other periodic points of g in the closure of K are repelling and lie on the boundary of K.*

Proof. Because K is convex and contains 0 it follows from Lemma 5.2.15 that $g(K) \subseteq K$ and $\widehat{V}_\Lambda \subseteq K$. Therefore, because $K \neq \mathbb{C}$, we can use Lemma 5.3.1 to conclude that $\widehat{V}_\Lambda$ is bounded and $\Lambda \in \mathcal{V}_\infty$.

The set $\widehat{V}_\Lambda$ contains the convex hull of the spiral given by the image of the interval $[0, 1]$ under the map $t \mapsto \Lambda e^{-t\Lambda}$. Because Λ is not real, this spiral is not a straight line segment, and thus both $\widehat{V}_\Lambda$ and K have a non-empty interior. The interior of K is a simply connected proper subset of $\mathbb{C}$ and thus conformally isomorphic to the open unit disk. The Denjoy–Wolff theorem states that either the action of g on $\mathrm{int}(K)$ is conjugate to a rotation or the orbit of any initial value in $\mathrm{int}(K)$ under repeated application of g converges to a fixed point in the closure of K (this point might be ∞). We will first argue that the former cannot be the case.

Suppose that the action of g is conjugate to a rotation on $\mathrm{int}(K)$. Then it should also be conjugate to a rotation on the invariant subset $\mathrm{int}(\widehat{V}_\Lambda)$ and in particular it should be a bijection on this subset. But, since $\widehat{V}_\Lambda$ is bounded and $g(\widehat{V}_\Lambda) \subseteq E_\Lambda(\widehat{V}_\Lambda)$, $g(\mathrm{int}(\widehat{V}_\Lambda))$ avoids a neighborhood of 0 and thus g cannot be a bijection.

We can conclude that every value in $\mathrm{int}(K)$ must converge to a point p in the closure of K. Note that p must be included in the closure of $\widehat{V}_\Lambda$ and thus cannot be equal to ∞. It follows that $\mathrm{int}(K)$ is contained in an invariant Fatou component U of the map g. There are five possible types of invariant Fatou components for transcendental maps (see e.g. [Mil06, Theorem 16.1 + Remark 16.6]). We have excluded U being either a Siegel disk or a Herman ring, since g exhibits attracting behaviour on U. The component also cannot be a Baker domain because $p \neq \infty$. The only two remaining cases are that U is the immediate basin for an attracting fixed point or for a petal of a parabolic fixed point with multiplier 1. There are no other periodic points of g on U and all other periodic points in the boundary of U are repelling.

□

Recall from Remark 5.2.13 that E_Λ has an attracting or a neutral fixed point if and only if Λ lies in the interior or the boundary of $\mathcal{C}_\infty$ respectively (see equation (5.1)). It follows from Lemma 5.3.2 that the nonreal elements of $\mathcal{V}_\infty$ are contained in $\overline{\mathcal{C}_\infty}$. Furthermore, because for any $\Lambda \in \partial\mathcal{C}_\infty \setminus \mathbb{R}$ the neutral fixed point of E_Λ does not have multiplier 1, we see that $\mathcal{V}_\infty \cap \partial\mathcal{C}_\infty = \{-e^{-1}, e\}$. To prove part (2) of Theorem 5.1.2 it is thus enough to show that $\mathcal{V}_\infty$ is closed.

Lemma 5.3.3. *The set $\mathcal{V}_\infty$ is closed.*

Proof. We will show that the complement is open. To that effect take a Λ_0 that is not in $\mathcal{V}_\infty$. If Λ_0 is real then Λ_0 lies outside the interval $[-e^{-1}, \infty)$ and thus there is a neighborhood of Λ_0 that does not intersect $\overline{\mathcal{C}_\infty}$. This neighborhood can also be chosen not to intersect with $\mathcal{V}_\infty$ because $\mathcal{V}_\infty$ is a subset of $\overline{\mathcal{C}_\infty} \cup [-1/e, \infty)$. We will henceforth assume that Λ_0 is not real.

Let p_{Λ_0} be a repelling periodic point of E_{Λ_0}. Write p_{Λ_0} as a sum of the following form

$$p_{\Lambda_0} = \sum_{i=1}^{n} s_i \cdot z_i,$$

where the s_i lie in $(0,1]$ with their sum being strictly less than one and the z_i are nonzero complex numbers that are not co-linear. According to Lemma 5.3.1 there exist $g_{i,\Lambda_0} \in G_{\Lambda_0}$ such that $g_{i,\Lambda_0}(0) = z_i$ for all i.

Recall that the values $g_{i,\Lambda}(0)$ depend continuously on the parameter Λ. For $\Lambda \in \mathbb{C}$ we define

$$A_\Lambda = \left\{ \sum_{i=1}^{n} t_i \cdot g_{i,\Lambda}(0) : t_i \in (0,1) \text{ with } \sum_{i=1}^{n} t_i < 1 \right\}.$$

The set A_{Λ_0} is an open neighborhood of p_{Λ_0}. Using the implicit function theorem we can extend the repelling periodic point p_{Λ_0} to a neighborhood of Λ_0 on which there exists a holomorphic function $\Lambda \mapsto p_\Lambda$ such that p_Λ is a repelling periodic point of E_Λ for all Λ. By continuity it follows that there is a neighborhood U of Λ_0 such that A_Λ is an open set containing p_Λ for all $\Lambda \in U$. Because $A_\Lambda \subseteq \text{int}(\widehat{V}_\Lambda)$ for all $\Lambda \in U$ it follows from Lemma 5.3.2 that these Λ cannot lie in $\mathcal{V}_\infty$. This concludes the proof. □

Remark 5.3.4. A shorter proof of Lemma 5.3.3 can be given using Lemma 5.3.7 below. To elaborate, suppose that $\Lambda_0 \notin \mathcal{V}_\infty$, in which case $V_{\Lambda_0} = \mathbb{C}$. Choose $g_{1,\Lambda}, g_{2,\Lambda}, g_{3,\Lambda} \in \mathcal{G}_\Lambda$ such that 0 lies in the interior of the convex hull of the three points $g_{i,\Lambda_0}(0)$. Since each $g_{i,\Lambda}(0)$ varies holomorphically with Λ, it follows that the same holds for Λ sufficiently close to Λ_0, which by Lemma 5.3.7 implies Lemma 5.3.3. In order to avoid a circular argument we have included the longer proof above.

Corollary 5.3.5. *Part (2) of Theorem 5.1.2 holds, i.e. there is a neighborhood of $\partial C_\infty \setminus \{-e^{-1}, e\}$ that does not intersect $\mathcal{V}_\infty$.*

Lemma 5.3.6. *Let $\Lambda \in \mathbb{C}$, $g \in G_\Lambda$ and K a convex set with non-empty interior such that $g(K) \subseteq K$. Then any periodic point of g on the boundary of K must have a positive real multiplier.*

Proof. Let $p \in \partial K$ be a periodic point of g of order k, let $I = (p, q]$ be an interval contained in K, and let H be a closed half-plane containing K with $p \in \partial H$. By invariance $g^{k \cdot n}(I) \subset H$ for all $n \in \mathbb{N}$, which implies that $(g^k)'(p) \geq 0$. □

Lemma 5.3.7. *If $\Lambda \in \mathcal{V}_\infty$ then $0 \in \partial \widehat{V}_\Lambda$.*

Proof. As we have seen, the statement is true for real Λ, so we assume that Λ is not real. In that case Λ lies in $\mathcal{C}_\infty$. This means that E_Λ has a unique attracting fixed point, which we will denote by w, and all other fixed points are repelling. Moreover the multiplier $E'_\Lambda(w)$ is not real. It follows from Lemmas 5.3.2 and 5.3.6 that w lies in the interior of $\widehat{V}_\Lambda$.

By the Riemann mapping theorem, there exists a comformal bijection $h : \mathbb{D} \to \mathrm{int}(\widehat{V}_\Lambda)$ with $h(0) = w$. For the sake of contradiction let us assume that $0 = h(q)$ for some $|q| < 1$. Without loss of generality we may assume that $q > 0$. Let $D_q(0)$ denote the open disk of radius q centered around 0. We claim that the image $T = h\left(\overline{D_q(0)}\right)$ is convex and E_Λ-invariant.

To see convexity, let us recall the classic result (see e.g. [Dur83, §2.5]) that a univalent image $f(\mathbb{D})$ is convex if and only if for any $z \in \mathbb{D}$:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0. \tag{5.2}$$

This equation is thus satisfied for $f = h$, and therefore must also be satisfied for the function $f_q(z) = h(q \cdot z)$. It follows that T is convex, being the closure of $f_q(\mathbb{D})$.

To see the forward invariance, write $H = h^{-1} \circ E_\Lambda \circ h$. Clearly H maps the disk into itself and

$$H(0) = h^{-1}(E_\Lambda(h(0))) = h^{-1}(E_\Lambda(w)) = h^{-1}(w) = 0.$$

Therefore the Schwarz Lemma gives

$$H(D_q(0)) \subset D_q(0),$$

and thus $E_\Lambda(T) \subset T$. The set T is therefore a forward invariant convex proper subset of $\widehat{V}_\Lambda$ containing 0. It follows from Lemma 5.2.15 that T should contain $\widehat{V}_\Lambda$, which is clearly a contradiction. □

In Figure 5.1, the gray region depicting the set $\mathcal{C}_\infty \setminus \mathcal{V}_\infty$ was computed by checking whether the condition $0 \in \partial\widehat{V}_\Lambda$ was violated for an approximation of the set $\widehat{V}_\Lambda$.

Corollary 5.3.8. *The family* $\{\Lambda \mapsto g_\Lambda(0) : g_\Lambda \in G_\Lambda\}$ *is normal on* $\mathcal{U}_\infty$.

Proof. For any $\Lambda \in \mathcal{U}_\infty$ the set $\widehat{V}_\Lambda$ cannot contain any point z of the form $k \cdot i$ for $k \geq 2\pi$ because otherwise $\widehat{V}_\Lambda$ would contain the convex hull of the circle parameterized by $t \mapsto \Lambda \cdot e^{-t \cdot z}$ contradicting Lemma 5.3.7. The family in question is thus normal by Montel's theorem. □

Lemma 5.3.9. *Let $\Lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Suppose there is a non-identity map $g \in G_{\Lambda_0}$ and a g-invariant convex set K containing 0 whose closure contains a neutral fixed point of g. Then $\Lambda_0 \notin \mathcal{U}_\infty$.*

Proof. Recall that g depends holomorphically on the parameter Λ. We consider the family of functions $\mathcal{F} = \{\Lambda \mapsto g_\Lambda^n(0) : n \geq 1\}$. Corollary 5.3.8 states that $\mathcal{F}$ is normal on $\mathcal{U}_\infty$. For $|\Lambda| < 1/e$ the image of the unit disk $\mathbb{D}$ under the map $Z \mapsto E_\Lambda(Z)$ is relatively compact in $\mathbb{D}$. Because $\mathbb{D}$ is convex and contains 0 the same is true for $Z \mapsto g_\Lambda(Z)$. Therefore the orbit $\{g_\Lambda^n(0)\}_{n\geq 1}$ converges to an attracting fixed point of g_Λ. Because $\mathcal{F}$ is normal on $\mathcal{U}_\infty$ and $\mathcal{U}_\infty$ is connected containing the disk of radius $1/e$ it follows that the orbit $\{g_\Lambda^n(0)\}_{n\geq 1}$ must converge to a fixed point p_Λ of g_Λ on the whole of $\mathcal{U}_\infty$. Furthermore the map $\Lambda \mapsto p_\Lambda$ is holomorphic.

For every Λ the point p_Λ lies in the closure of $\widehat{V}_\Lambda$. It follows then from Lemma 5.3.2 that if $|g'_\Lambda(p_\Lambda)| = 1$ then $g'_\Lambda(p_\Lambda)$ is exactly equal to 1. But because $\Lambda \mapsto g'_\Lambda(p_\Lambda)$ is holomorphic and thus, if it is not constant, it is an open map. This cannot occur on $\mathcal{U}_\infty$. Therefore p_Λ is an attracting fixed point for all $\Lambda \in \mathcal{U}_\infty$.

If Λ_0 were an element of $\mathcal{U}_\infty$ then p_{Λ_0} would be an attracting fixed point of g_{Λ_0} contained in K. Lemma 5.3.2 states that g_{Λ_0} cannot contain any other attracting or neutral fixed points in the closure of K, which contradicts the assumption of the lemma that the closure of K contains a neutral fixed point of g. □

5.4. Proof of Theorem 5.1.1

In this section we will prove our first main result, which stated that for any closed $K_1 \subseteq \mathcal{U}_\infty$ and any open $K_2 \supseteq \overline{\mathcal{U}_\infty}$ there exists a d_0 such that $K_1 \subseteq d \cdot \mathcal{U}_d \subseteq K_2$ for $d \geq d_0$.

The strategy to prove that K_1/d is eventually zero-free is to show that for any $\Lambda \in \mathcal{U}_\infty$ there exists a convex set K_Λ that is strictly invariant for E_Λ, i.e. the closure of $E_\Lambda(K)$ is contained in the interior of K. This allows us to apply Lemma 5.2.20 to find that there is a neighborhood X_Λ of Λ such that X_Λ/d is eventually zero-free. Because K_1 is compact it follows that K_1/d is eventually zero-free.

The strategy to prove that K_2/d eventually contains $\mathcal{U}_d$ is to show that for $\Lambda \notin K_2$ we can find elements $g_\Lambda \in G_\Lambda$ such that $|g_\Lambda(0)|$ is very large. We can then use Lemma 5.2.19 to find a sequence of ratios R_d of finite graphs with degree at most d and root degree at most $\frac{d}{2}$ such that $|R_d(\Lambda/d)|$ is very large compared to the cardioid $\mathcal{C}_{\lfloor \frac{d}{2} \rfloor}$. Because zeros are dense outside the cardioid this will allow us to conclude, using Lemma 5.2.12, that a neighborhood of Λ is eventually contained in the rescaled set of zeros $d \cdot \overline{\mathcal{Z}_d}$.

5.4.1. Strictly forward invariant regions for nonreal $\Lambda \in \mathcal{U}_\infty$

Before we prove the next lemma let us first describe a number of general properties of the set $\widehat{V}_\Lambda$ for $\Lambda \in \mathcal{V}_\infty \setminus \mathbb{R}$. Because $\widehat{V}_\Lambda$ is bounded there are points $m, M \in \partial \widehat{V}_\Lambda$ whose imaginary parts are respectively minimal and maximal inside the closure of $\widehat{V}_\Lambda$. The argument of $E_\Lambda(z)$ only depends on the imaginary part of z and decreases as $\mathrm{Im}(z)$ increases. This means that the argument of any element in $E_\Lambda(\widehat{V}_\Lambda)$ lies between the arguments of $E_\Lambda(M)$ and $E_\Lambda(m)$. It follows that $\partial \widehat{V}_\Lambda$ must contain the straight line segments between 0 and both $E_\Lambda(M)$ and $E_\Lambda(m)$. Therefore $\partial \widehat{V}_\Lambda$ always contains two radial arc segments, which we will denote by I_1, I_2. The arguments of all other elements in the closure of $\widehat{V}$ are strictly between the arguments of I_1 and I_2; see Figure 5.2 for an example.

Lemma 5.4.1. *For $\Lambda \in \mathcal{U}_\infty \setminus \mathbb{R}$ there is a bounded convex set K_Λ containing 0 that is strictly invariant under the map E_Λ.*

Proof. Because Λ lies in $\mathcal{U}_\infty$, which is an open set, there is a real number $t > 1$ such that $t \cdot \Lambda \in \mathcal{U}_\infty$. Let $t' = \frac{1+t}{2}$, and define

$$\mathbf{V} := (1/t') \cdot \widehat{V}_{t\cdot\Lambda}.$$

Observe that

$$E_\Lambda(\mathbf{V}) \subset E_\Lambda(\widehat{V}_{t\cdot\Lambda}) = \frac{1}{t} E_{t\cdot\Lambda}(\widehat{V}_{t\cdot\Lambda}) \subset \frac{1}{t}\widehat{V}_{t\cdot\Lambda} = \frac{t'}{t}\mathbf{V}.$$

If $\mathbf{V}$ is strictly invariant under E_Λ we can take $K_\Lambda = \mathbf{V}$, therefore we will assume that there is an element $q \in \partial E_\Lambda(\mathbf{V}) \cap \partial \mathbf{V}$. The element $\frac{t}{t'}q$ lies in $\overline{\mathbf{V}}$ and thus, by convexity of $\mathbf{V}$, the whole straight line segment between $\frac{t}{t'}q$ and 0 lies in $\overline{\mathbf{V}}$. This line segment passes through q, which lies on the boundary of $\mathbf{V}$, and thus the whole line segment is contained in $\partial \mathbf{V}$. It follows that q is contained in one of the two radial arcs I_1, I_2 of $\widehat{V}_{t\cdot\Lambda}$ as described above.

Let $p \in \mathbf{V}$ such that $E_\Lambda(p) = q$. Both sets $\mathbf{V}$ and $\widehat{V}_{t\cdot\Lambda}$ are forward invariant under E_Λ. The point q lies in the boundary of both sets and thus p must lie in the boundary of both sets because E_Λ is an open map. We have $\partial \mathbf{V} \cap \partial \widehat{V}_{t\cdot\Lambda} = (I_1 \cup I_2) \cap \partial \mathbf{V}$ and thus $p \in (I_1 \cup I_2) \cap \partial \mathbf{V}$.

We claim that p is real. Clearly this is true if $p = 0$ and thus we may assume that p is nonzero. Because $p \in (I_1 \cup I_2) \cap \partial \mathbf{V}$ there is an ϵ such that $r \cdot p \in I_1 \cup I_2$ for $1 - \epsilon < r < 1 + \epsilon$. Consider the curve $\gamma : (1-\epsilon, 1+\epsilon) \to \mathbb{C}$ given by $r \mapsto E_\Lambda(r \cdot p)$. The image of this curve is completely contained in the closure of $\widehat{V}_{t\cdot\Lambda}$ and thus cannot cross the radial arc through $q = \gamma(1)$ transversely. This means that $\gamma'(1)$ must be a real multiple of q. Because $\gamma'(1) = -pq$ it follows that p must be real.

We can conclude that any element of $\partial E_\Lambda(\mathbf{V}) \cap \partial \mathbf{V}$ lies on the radial ray through Λ. This ray must also contain one of the radial arc segments (say I_1). We suppose without loss of generality that the argument of I_1 is maximal within $\mathbf{V}$. Because $E_\Lambda(\mathbf{V}) \subseteq \frac{t'}{t}\mathbf{V}$ and $E_\Lambda(\mathbf{V})$ avoids a neighborhood of 0, we see that $\partial E_\Lambda(\mathbf{V}) \cap \partial \mathbf{V}$ avoids neighborhoods of both endpoints of $\overline{\mathbf{V}} \cap I_1$.

Let $re^{i\alpha}$ be the element of maximum modulus in $\overline{\mathbf{V}} \cap I_1$. Define for $\epsilon > 0$ the set $\mathcal{K}_\epsilon$ as the convex hull of $\mathbf{V}$ and $re^{i(\alpha+\epsilon)}$. Observe that for sufficiently small ϵ the set $\mathbf{V}$ gets mapped strictly inside $\mathcal{K}_\epsilon$ by E_Λ. In fact, the distance between $E_\Lambda(\mathbf{V})$ and the boundary of $\mathcal{K}_\epsilon$ is of order ϵ as ϵ goes to 0. To conclude the proof we will show that for sufficiently small ϵ the set $\mathcal{K}_\epsilon$ gets mapped strictly inside itself by E_Λ. It is sufficient to show that the newly added boundary $\partial\mathcal{K}_\epsilon \setminus \partial\mathbf{V}$ is mapped strictly inside $\mathcal{K}_\epsilon$.

In a neighborhood of 0 the elements of $\partial\mathcal{K}_\epsilon \setminus \partial\mathbf{V}$ are all of the form $ze^{i\epsilon}$ with $z \in I_1 \cap \overline{\mathbf{V}}$. Observe that $E_\Lambda(ze^{i\epsilon}) = E_\Lambda(z) - i\epsilon z E_\Lambda(z) + \mathcal{O}(\epsilon^2)$. Because the distance between $E_\Lambda(\mathbf{V})$ and $\partial\mathcal{K}_\epsilon$ is of order ϵ we can choose a fixed neighborhood U of 0 such that $U \cap \partial\mathcal{K}_\epsilon$ gets mapped strictly inside $\mathcal{K}_\epsilon$ for sufficiently small ϵ. Finally, we observe that $\partial\mathcal{K}_\epsilon \setminus (U \cup \partial\mathbf{V})$ converges to a part of the boundary of $\mathbf{V}$ that avoids a neighborhood of the real line. This part of the boundary of $\mathbf{V}$ gets mapped strictly inside $\mathbf{V}$. We can therefore conclude that, for sufficiently small ϵ, $\partial\mathcal{K}_\epsilon \setminus (U \cup \partial\mathbf{V})$ gets mapped strictly inside $\mathbf{V} \subseteq \mathcal{K}_\epsilon$. We can thus take $K_\Lambda = \mathcal{K}_\epsilon$ for sufficiently small ϵ. □

5.4.2. Strictly forward invariant regions for real $\Lambda \in \mathcal{U}_\infty$

When $\Lambda \in (-1/e, 1/e)$ it is easy to see that the image of the open unit disk $\mathbb{D}$ under the map E_Λ is relatively compact in $\mathbb{D}$. This takes care of all $\Lambda \in \mathcal{U}_\infty \cap \mathbb{R}_{\leq 0}$. For $\Lambda > 0$ we will use the same strategy as that in [PR19], namely to find a change of coordinates which transforms the map to a strict contraction on the positive real axis. Using the contraction we will be able to construct a convex forward invariant set.

The map E_Λ sends $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$. Define the following bijection of $\mathbb{R}_{\geq 0}$ by

$$\phi(x) = \log(1+x) \quad \text{with} \quad \phi^{-1}(x) = e^x - 1$$

and let

$$h_\Lambda(x) = \left(\phi \circ E_\Lambda \circ \phi^{-1}\right)(x).$$

While it is true that for all $\Lambda \in (0, e)$ the orbit of any positive real number under the orbit of E_Λ converges to a real attracting fixed point, the maps E_Λ are not all contractions on the positive real line. The maps h_Λ have this added benefit.

Lemma 5.4.2. *Let $\Lambda \in (0, e)$. Then h_Λ is a strict contraction on the positive real line, i.e. $|h'_\Lambda(x)| < 1$ for all $x \in \mathbb{R}_{\geq 0}$.*

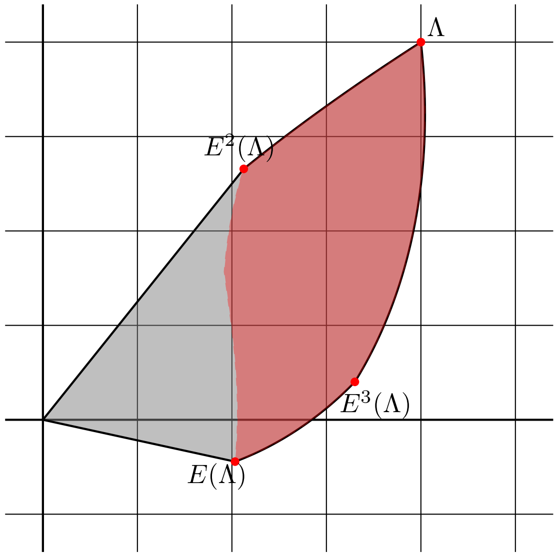

Figure 5.2: An example of a set $\widehat{V}_\Lambda$ and the subset formed by its image under $E = E_\Lambda$ for the parameter $\Lambda = 1 + i$. In this example the two radial arcs I_1 and I_2 end at the points $E_\Lambda(\Lambda)$ and $E^2_\Lambda(\Lambda)$.

Proof. We calculate that

$$h'_\Lambda(x) = -\frac{e^x}{1 + \frac{1}{\Lambda}e^{e^x - 1}}.$$

Therefore the inequality $|h'_\Lambda(x)| < 1$ is equivalent to the inequality $e^x - 1 < \frac{1}{\Lambda}e^{e^x-1}$. Note that for all real numbers y we have $y \leq e^{y-1}$ and thus, letting $y = e^x - 1$, we find $y \leq e^{y-1} < \frac{1}{\Lambda}e^y$, which is what was required. □

Lemma 5.4.3. *For $\Lambda \in \mathcal{U}_\infty \cap \mathbb{R}$ there is an bounded convex set K_Λ containing 0 that is strictly invariant under the map E_Λ.*

Proof. As discussed previously, for $\Lambda \in (-1/e, 1/e)$ we can take K_Λ to be the open unit disk. We may therefore assume that $\Lambda > 0$. The map ϕ, as defined above, can be extended to an holomorphic map on the half plane $\{z : \mathrm{Re}(z) > -1\}$. It follows that we can extend h_Λ to a map on a neighborhood of the positive real line.

Let $I = \phi([0, \Lambda])$. Because $[0, \Lambda]$ is forward invariant under E_Λ, the interval I is forward invariant under h_Λ. It follows from the fact that h_Λ is a strict contraction on I that for small enough ϵ the tubular neighborhood $I_\epsilon = \{z \in \mathbb{C} :$

$d(I,z) < \epsilon\}$ gets mapped strictly inside itself by h_Λ. That is, $h_\Lambda(I_\epsilon)$ is relatively compact in I_ϵ. It follows that $E_\Lambda(\phi^{-1}(I_\epsilon))$ is relatively compact in $\phi^{-1}(I_\epsilon)$.

All that remains to be shown is that $\phi^{-1}(I_\epsilon)$ is convex for ϵ small enough. Write $I = [a,b]$ and for a ball of radius ϵ centered at z write $B_\epsilon(z)$; then we see that

$$\exp(I_\epsilon) = \exp\left(\bigcup_{x\in[a,b]} B_\epsilon(x)\right) = \bigcup_{x\in[a,b]} \exp\left(B_\epsilon(x)\right) = \bigcup_{x\in[a,b]} e^x \exp(B_\epsilon(0))$$
$$= [e^a, e^b] \cdot \exp(B_\epsilon(0)).$$

If $\epsilon < 1$ then $\exp(B_\epsilon(0))$ is convex (see e.g. equation (5.2)), and thus $\phi^{-1}(I_\epsilon)$ is convex as a translation of an entrywise product of a positive real interval with a convex set. □

5.4.3. Conclusion of Theorem 5.1.1

Let K_1 be a closed set contained in $\mathcal{U}_\infty$. For $\Lambda \in K_1$ let K_Λ denote a bounded convex set containing 0 that is strictly invariant under the map E_Λ. Such sets exist by Lemmas 5.4.1 and 5.4.3. It follows from Lemma 5.2.20 that there exists a neighborhood U_Λ of Λ and a d_Λ such that for $d \geq d_\Lambda$ we have that $U_\Lambda \subseteq d\cdot\mathcal{U}_d$. The set K_1 is compact and can thus be covered by finitely such U_Λ, say $U_{\Lambda_1}, \ldots, U_{\Lambda_n}$. By taking $d_0 = \max_{i=1,\ldots,n} d_{\Lambda_i}$ we see that $K_1 \subseteq d \cdot \mathcal{U}_\infty$ for $d \geq d_0$.

Let K_2 be an open set containing $\overline{\mathcal{U}_\infty}$. We will show that for large enough d the sets $d \cdot \mathcal{U}_d$ are contained in K_2. In other words: we will show that the complement $(K_2)^c$ is contained in $d \cdot \overline{\mathcal{Z}_d}$ for d large enough. For this we use that zeros are dense outside the cardioid for any finite degree, which was proved in Chapter 3, building upon results from [BGGv20]:

Lemma 5.4.4 (See also Theorem 4.1.2)**.** *Let $d \geq 2$, the zero locus $\overline{\mathcal{Z}_d}$ contains the complement of the cardioid, i.e. $(\mathcal{C}_d)^c \subseteq \overline{\mathcal{Z}_d}$.*

Because the real interval $(-1/e, e)$ is contained in $\mathcal{U}_\infty$ there is a real number $\delta > 1$ such that $\delta \cdot (-1/e, e)$ is contained in K_2. It follows that we can cover the complement $(K_2)^c$ by the union of the complement of $(\delta \cdot \mathcal{C}_\infty)$ and a compact set K that does not intersect $\mathbb{R} \cup \mathcal{V}_\infty$.

The sets $d \cdot \mathcal{C}_d$ converge to $\mathcal{C}_\infty$ and thus by Lemma 5.4.4 it follows that $(\delta \cdot \mathcal{C}_\infty)^c \subseteq d \cdot \overline{\mathcal{Z}_d}$ for large enough d. It remains to be shown that we can choose d large enough such that the sets $d \cdot \overline{\mathcal{Z}_d}$ also contain K. By compactness of K it is sufficient to show that every $\Lambda \in K$ has a neighborhood X_Λ that is contained in $d \cdot \overline{\mathcal{Z}_d}$ for sufficiently large degree d.

Let $\Lambda_0 \in K$. Because Λ_0 is neither real nor a member of $\mathcal{V}_\infty$, Lemma 5.3.1 implies that $V_{\Lambda_0} = \mathbb{C}$. Let $Z_0 \in \mathbb{C}$ be a value such that $\Lambda_0 e^{-Z_0} = 100$. Choose

$g_\Lambda \in G_\Lambda$ such that $g_{\Lambda_0}(0) = 2Z_0$. It follows from Lemma 5.2.19 that there is a sequence of rooted graphs $(G_d, v_d) \in \mathcal{G}_d^{\lfloor \frac{d}{2} \rfloor}$ such that the maps $\Lambda \mapsto d \cdot R_{G_d,v_d}(\Lambda/d)$ converge uniformly on compact subsets to $\Lambda \mapsto \Lambda \cdot e^{-\frac{1}{2} g_\Lambda(0)}$. Therefore there is a d_0 and a neighborhood X_{Λ_0} such that for $d \geq d_0$ and $\Lambda \in X_{\Lambda_0}$ we have $|d \cdot R_{G_d,v_d}(\Lambda/d)| \geq 99$. The cardioid $\mathcal{C}_{\lfloor \frac{d}{2} \rfloor}$ is contained in a disk of radius

$$\frac{(\lfloor \frac{d}{2} \rfloor)^{\lfloor \frac{d}{2} \rfloor}}{(\lfloor \frac{d}{2} \rfloor - 1)^{\lfloor \frac{d}{2} \rfloor + 1}},$$

which for $d \geq 4$ is less than $99/d$. This means that the values $R_{G_d,v_d}(\Lambda/d)$ lie in the complement of $\mathcal{C}_{\lfloor \frac{d}{2} \rfloor}$ and thus in $\overline{\mathcal{Z}_{\lfloor \frac{d}{2} \rfloor}}$. It finally follows from Lemma 5.2.12 that for $d \geq d_0$ the sets $d \cdot \overline{\mathcal{Z}_d}$ contain X_{Λ_0}.

□

5.5. Near the real boundary of $\mathcal{U}_\infty$

5.5.1. Around e

For $\theta \in (0, \pi)$ we define $\gamma(\theta)$ as the unique positive real number that solves

$$\gamma(\theta)^2 - \sin(\gamma(\theta))^2 = \theta^2. \tag{5.3}$$

To keep the notation readable we will drop the dependency on θ and simply write $\gamma = \gamma(\theta)$. We define

$$\hat{\Lambda}(\theta) = \frac{\gamma + \theta}{\sin \gamma} e^{(\gamma - \theta)\cot \gamma} e^{\theta i}, \hat{c}(\theta) = \frac{\gamma - \theta}{\gamma + \theta} e^{2\theta \cot \gamma} \text{ and } \hat{Z}(\theta) = \frac{\gamma + \theta}{\sin \gamma} e^{-2\theta \cot \gamma} e^{-\gamma i}.$$

We can continuously extend these functions to be defined for $\theta = 0$ by setting $\gamma(0) = 0$, $\hat{\Lambda}(0) = e$, $\hat{c}(0) = 1$ and $\hat{Z}(0) = 1$. We define the composition $H_{\Lambda,c} = E_\Lambda \circ E_{\Lambda,c}$.

Lemma 5.5.1. *Let $\theta \in [0, \pi)$ then $\hat{Z}(\theta)$ is a fixed point of $H_{\hat{\Lambda}(\theta),\hat{c}(\theta)}$ with multiplier* 1.

Proof. We first check that $\hat{Z}(\theta)$ is a fixed point, and afterwards compute its multiplier.

As $H_{\Lambda,c}(Z) = E_\Lambda(E_{\Lambda,c}(Z)) = E_\Lambda(E_\Lambda(cZ))$ we calculate first

$$\hat{c}(\theta)\hat{Z}(\theta) = \left(\frac{\gamma - \theta}{\gamma + \theta} e^{2\theta \cot \gamma}\right) \cdot \left(\frac{\gamma + \theta}{\sin \gamma} e^{-2\theta \cot \gamma} e^{-\gamma i}\right) = \frac{\gamma - \theta}{\sin \gamma} e^{-\gamma i}.$$

Therefore its image

$$E_{\hat{\Lambda},\hat{c}}(\hat{Z}) = \hat{\Lambda} e^{-\hat{c}\hat{Z}} = \left(\frac{\gamma+\theta}{\sin\gamma} e^{(\gamma-\theta)\cot\gamma} e^{\theta i}\right)\left(e^{-(\gamma-\theta)\cot(\gamma)} e^{(\gamma-\theta)i}\right) = \frac{\gamma+\theta}{\sin\gamma} e^{\gamma i}$$

and

$$\begin{aligned} H_{\hat{\Lambda},\hat{c}}(\hat{Z}) = \hat{\Lambda} e^{-E_{\hat{\Lambda},\hat{c}}(\hat{Z})} &= \left(\frac{\gamma+\theta}{\sin\gamma} e^{(\gamma-\theta)\cot\gamma} e^{\theta i}\right)\left(e^{-(\gamma+\theta)\cot\gamma} e^{-(\gamma+\theta)i}\right) \\ &= \frac{\gamma+\theta}{\sin\gamma} e^{-2\theta\cot\gamma} e^{-\gamma i} = \hat{Z}. \end{aligned}$$

We see that $\hat{Z}$ is indeed a fixed point. The derivative of $H_{\Lambda,c}(Z)$ is

$$H'_{\Lambda,c}(Z) = c \cdot E_{\Lambda,c}(Z) \cdot H_{\Lambda,c}(Z).$$

and thus

$$H'_{\hat{\Lambda},\hat{c}}(\hat{Z}) = \hat{c} \cdot E_{\hat{\Lambda},\hat{c}}(\hat{Z}) \cdot H_{\hat{\Lambda},\hat{c}}(\hat{Z}) = E_{\hat{\Lambda},\hat{c}}(\hat{Z}) \cdot \left(\hat{c}\hat{Z}\right) = \frac{\gamma^2 - \theta^2}{\sin^2\gamma} = 1,$$

which completes the proof. □

Lemma 5.5.2. *If $\theta \in (0,\pi)$ then $\hat{c}(\theta) \in (0,1)$.*

Proof. Clearly for $\theta \in (0,\pi)$ the quantity γ is strictly larger than θ, and hence $\hat{c}(\theta)$ is strictly positive. Because $\gamma \in (0,\pi)$ the inequality $\cot(\gamma) < 1/\gamma$ is valid. Therefore we can write

$$\hat{c}(\theta) = \frac{\gamma-\theta}{\gamma+\theta} e^{2\theta\cot\gamma} < \frac{\gamma-\theta}{\gamma+\theta} e^{2\theta/\gamma} = \frac{1-x}{1+x} \cdot e^{2x},$$

where $x = \theta/\gamma$. The right-hand side of this equation describes a strictly decreasing function starting at 1 for $x = 0$, as one can check by calculating its derivative. It therefore follows that $\hat{c}(\theta) < 1$. □

From now on we assume that θ is sufficiently small such that both θ and γ lie in $(0,\pi/2)$, i.e. such that the complex numbers $e^{i\theta}$ and $e^{-i\gamma}$ lie in the first and fourth quadrant of the plane respectively. We define the set $T_\theta \subsetneq \mathbb{C}$ as the closed convex set bounded by the following two line segments

$$I_1 = \{t \cdot (\theta+\gamma)i : t \in [0,1]\}, \quad I_2 = \left\{t \cdot \frac{\pi/2-\theta}{\sin\gamma} e^{-i\gamma} : t \in [0,1]\right\}$$

and two infinite rays parallel to the real line starting at the endpoints of these segments

$$I_3 = \{(\theta+\gamma)i + t : t \in [0,\infty)\}, \quad I_4 = \left\{\frac{\pi/2-\theta}{\sin\gamma} e^{-i\gamma} + t : t \in [0,\infty)\right\}.$$

An explicit example of the set T_θ is illustrated in Figure 5.3.

Lemma 5.5.3. *For θ sufficiently small the region T_θ contains $\hat{Z}(\theta)$ and is forward invariant for $E_{\hat{\Lambda}(\theta)}$.*

Proof. As θ approaches 0 the modulus of $\hat{Z}(\theta)$ goes to 1, while the length of the line segment I_2 goes to infinity. It follows that for θ sufficiently small $\hat{Z}(\theta) \in I_2$, and hence $\hat{Z}(\theta) \in T_\theta$.

To show that T_θ is forward invariant for $E_{\hat{\Lambda}(\theta)}$ it is sufficient to show that its boundary gets mapped into T_θ, because $E_{\hat{\Lambda}(\theta)}$ is holomorphic. The image of I_1 is a circular arc of the form $\{|\hat{\Lambda}(\theta)|e^{\phi i} : \phi \in [-\gamma, \theta]\}$. One of the endpoints of this arc is Λ and the other has the same argument as the interval I_2. Therefore $E_{\hat{\Lambda}(\theta)}(I_1)$ lies in T_θ if the imaginary part of $\hat{\Lambda}(\theta)$ is less than or equal to $(\theta + \gamma)$ and the modulus $|\hat{\Lambda}(\theta)|$ is at most the length of I_2. The two inequalities involved are

$$\frac{\sin\theta}{\sin\gamma} e^{(\gamma-\theta)\cot\gamma} \leq 1 \quad \text{and} \quad (\gamma+\theta)e^{(\gamma-\theta)\cot\gamma} \leq \pi/2 - \theta. \tag{5.4}$$

As θ approaches 0 both left-hand sides approach 0 and thus the inequalities are satisfied for θ sufficiently small.

The image of I_2 is a part of a logarithmic spiral of the form

$$\{\hat{\Lambda} e^{t(\theta-\pi/2)\cot\gamma} e^{t(\pi/2-\theta)i} \mid t \in [0,1]\}.$$

This curve starts at $\hat{\Lambda}$ and rotates $\pi/2 - \theta > 0$ in anti-clockwise direction, since we assumed that $\theta \in (0, \pi/2)$. In particular this curve is in the upper half plane as $\arg\hat{\Lambda} + (\pi/2 - \theta) = \pi/2 < \pi$; moreover the endpoint is a positive multiple of i. In order to show that I_2 is in T_θ, it is enough to show that the maximal imaginary part for points on the curve is at most $(\theta + \gamma)$. By taking derivatives with respect to t, the maximal imaginary part occurs when

$$t^* = \frac{\gamma - \theta}{\pi/2 - \theta}.$$

The imaginary part at t^* is

$$\begin{aligned} |\hat{\Lambda}|e^{t^*(\theta-\pi/2)\cot\gamma}\sin(\theta + t^*(\pi/2-\theta)) &= \frac{\gamma+\theta}{\sin\gamma} e^{(\gamma-\theta)\cot\gamma} e^{(\theta-\gamma)\cot\gamma}\sin(\theta+\gamma-\theta) \\ &= \frac{\gamma+\theta}{\sin\gamma}\sin\gamma = \gamma+\theta. \end{aligned}$$

Now we are left with the images of I_3 and I_4. Both of them are straight lines connecting $0 \in T_\theta$ with one of the endpoints of the image of I_1 or I_2. As T_θ is convex and it contains the images of I_1 and I_2, the images of I_3 and I_4 are also contained in T_θ. □

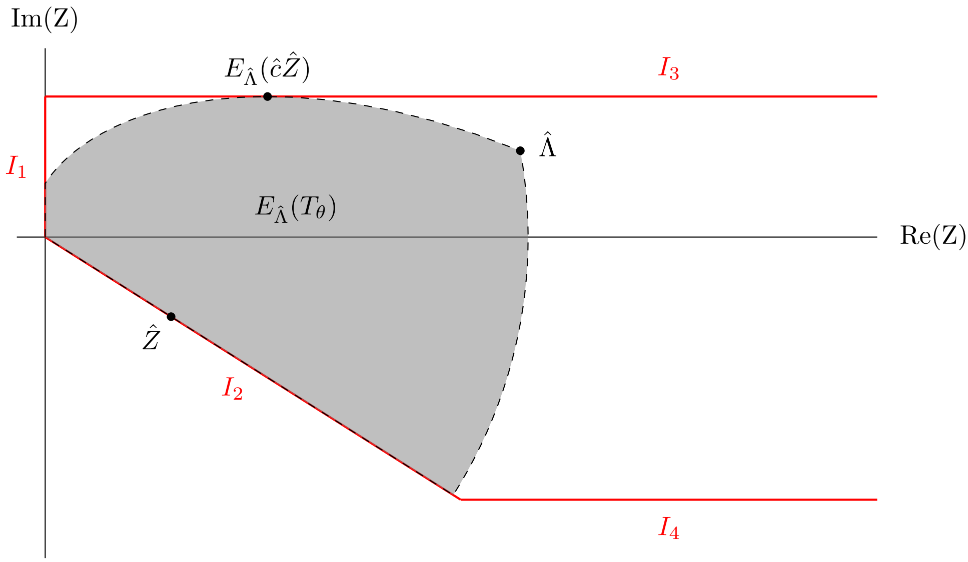

Figure 5.3: The region T_θ and its image under the map $E_{\hat{\Lambda}}$ for $\theta = 0.18$.

Corollary 5.5.4. *There exists a* $\theta_{max} > 0$ *such that the curves*

$$\Gamma = \{\hat{\Lambda}(\theta) : \theta \in [0, \theta_{max}]\} \quad \text{and} \quad \overline{\Gamma} = \{\overline{\hat{\Lambda}(\theta)} : \theta \in [0, \theta_{max}]\}$$

are contained in $\partial\mathcal{U}_\infty$.

Proof. Without loss of generality it is sufficient to prove that $\Gamma \subseteq \partial\mathcal{U}_\infty$ because the dynamics for Λ and $\overline{\Lambda}$ are conjugated. If $\theta = 0$, then $\hat{\Lambda}(0) = e$ and thus $\hat{\Lambda}(0) \in \partial\mathcal{U}_\infty$. So for the rest we may assume that $\theta > 0$, and therefore $\hat{\Lambda}(\theta) \notin \mathbb{R}$.

By the previous lemma we know that for any sufficiently small θ, the region T_θ is a convex forward $E_{\hat{\Lambda}}$-invariant set, hence $\hat{\Lambda} \in \mathcal{V}_\infty$.

On the other hand, we know that T_θ contains $\hat{Z}$, a neutral fixed point of $H_{\hat{\Lambda},\hat{c}}$, therefore by Lemma 5.3.9 $\hat{\Lambda} \notin \mathcal{U}_\infty$. In particular, $\hat{\Lambda} \in \partial\mathcal{U}_\infty$. $\square$

Using computer calculations it can be seen that the inequalities laid out in Lemma 5.5.3 needed for T_θ to be forward invariant hold up until at least $\theta = 0.18$. This implies that $\theta_{\max}$ is at least 0.18. In Figure 5.1 the curve Γ is drawn up until this angle.

5.5.2. Final remarks

In Section 4.5 the question was discussed whether the closure of the zeros of all spherically regular trees equals the closure of the zeros of all graphs, for the same degree bound. This question is not resolved in this chapter, but let us observe that for any $\Lambda_0 \in \Gamma$ and any $\varepsilon > 0$ the ball $B_{\varepsilon/d}(\Lambda_0/d)$ contains zeros of spherically regular trees for d sufficiently large.

To see this let us fix a $\Lambda_0 \in \Gamma$, and fix the corresponding c for which $H_{\Lambda_0,c}$ has a neutral fixed point Z of multiplier 1. Note that $H_{\Lambda,c}$ cannot have a persistently neutral fixed point in the neighborhood of Λ_0. It follows that in a punctured neighborhood of Λ_0, the corresponding fixed points are given by a multi-valued holomorphic function, where the number of points equals the multiplicity of the parabolic fixed point. The corresponding multipliers $\phi_i(\Lambda)$ are therefore also given by a multi-valued holomorphic function, and all converge to 1 as $\Lambda \to \Lambda_0$. The function $\Phi(\Lambda) = \prod_i(\phi_i(\lambda) - 1)$ therefore extends holomorphically to a full neighborhood of Λ_0, and has an isolated zero at $\Lambda = \Lambda_0$.

For fixed d define

$$h_{\lambda,c} := f_{d,\lambda} \circ f_{\lfloor cd \rfloor,\lambda},$$

and recall that $d \cdot h_{\Lambda/d,c}(\bullet/d)$ converges locally uniformly to $H_{\Lambda,c}$. It follows from the implicit function theorem that for large d the function $d \cdot h_{\Lambda/d,c}(\bullet/d)$ has fixed points near the fixed points of $H_{\Lambda,c}$, for Λ in a small annulus around Λ_0, whose multipliers are close to the multipliers of $H_{\Lambda,c}$. Hence for large d there exist holomorphic functions $\Phi_d = \prod_i(\phi_{i,d} - 1)$, in terms of the corresponding multipliers $\phi_{i,d}$, converging to the above function Φ. As above, the functions Φ_d can be extended to a full neighborhood of Λ_0.

It follows that for large d the functions Φ_d must also vanish at some point Λ_d in the neighborhood of Λ_0, meaning $h_{\Lambda/d,c}$ has a neutral fixed point with multiplier 1 for some $\lambda_d = \Lambda_d/d$ close to the parameter Λ_0/d. By Lemma 4.4.1 it follows that λ_d lies in the closure of the zeros of spherically regular trees.

Summary

Partition functions: zeros, unstable dynamics and complexity

The central objects of this thesis are the partition functions of two models: the hard-core model and the ferromagnetic Ising model. Let $G = (V, E)$ denote a finite simple graph. The hard-core model partition function, also known as the independence polynomial, is the generating function of the sizes of the independent sets in G, i.e.

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V: \\ I \text{ is independent}}} \lambda^{|I|}.$$

The partition function of the Ising model is given by

$$Z_G(\lambda, b) = \sum_{\sigma: V \to \{+,-\}} \lambda^{|n_+(\sigma)|} b^{\delta(\sigma)},$$

where $n_+(\sigma)$ is the set of vertices that get assigned the spin + under σ and $\delta(\sigma)$ is the number of edges that get different spins. We view b as a fixed real parameter in the interval $(0, 1)$ and thus think of both these functions as polynomials in the variable λ, from now on both denoted by Z_G.

For $\Delta \in \mathbb{Z}_{\geq 3}$ let $\mathcal{G}_\Delta$ denote the set of graphs with degree at most Δ. Furthermore let $\mathcal{Z}_\Delta$ denote the set of complex zeros of Z_G, where G runs over the whole class $\mathcal{G}_\Delta$. It was recently shown by Patel and Regts [PR17], building upon work of Barvinok [Bar16], that, if λ lies in an open zero-free neighborhood of 0, there exists a fully polynomial time algorithm (FPTAS) for approximating $Z_G(\lambda)$. This can be thought of as saying that zero-freeness implies that approximation is easy. The main result from chapters 2 and 3 is that, for respectively the ferromagnetic Ising model and the hard-core model, an inverse statement also holds.

Theorem (2.1.1 and 3.1.1). *The closure of the zeros $\overline{\mathcal{Z}_\Delta}$ is contained in the closure of parameters λ for which approximating $Z_G(\lambda)$ for $G \in \mathcal{G}_\Delta$ is #P-hard.*

Motivated by this theorem we would like to have a good understanding of the closure of the zeros $\overline{\mathcal{Z}_\Delta}$. For the ferromagnetic Ising model this set can be described explicitly [LY52b, PR20]. In fact, in this model, the closure of the zeros is equal to the closure of the zeros of the Cayley trees (rooted trees where the leaves all have the same distance to the root and the down-degree of all non-leaves is $\Delta - 1$). For the ferromagnetic Ising model the Cayley trees are thus in a certain sense extremal within the set of bounded degree graphs.

For the hard-core model it was shown in [PR19] that zeros of Cayley trees accumulate on the boundary of an explicitly given neighborhood of 0, denoted by Λ_Δ, that is otherwise zero-free for Cayley trees. Furthermore, using results from [BGGv20], it follows from Chapter 3 that zeros are dense outside Λ_Δ. The main result of Chapter 4 is that, contrary to the Ising model, Cayley trees are not extremal for the hard-core model.

Theorem (4.1.3)**.** *For $\Delta \in \{3, \ldots, 9\}$ there exist $\lambda \in \Lambda_\Delta$ with $G \in \mathcal{G}_\Delta$ such that $Z_G(\lambda) = 0$.*

The maximal zero free region, denoted by $\mathcal{U}_\Delta := \overline{\mathcal{Z}_\Delta}^c$, is thus in general strictly smaller than Λ_Δ. It seems very difficult to give a more explicit description or even make an accurate computer image of $\mathcal{U}_\Delta$ for any given Δ. In Chapter 5 it is analyzed what happens when Δ gets very large. One of the main results of Chapter 5 is that the rescaled domains $\Delta \cdot \mathcal{U}_\Delta$ converge to a limit set $\mathcal{U}_\infty$.

Theorem (5.1.1)**.** *The sets $\Delta \cdot \mathcal{U}_\Delta$ converge to a simply connected domain $\mathcal{U}_\infty$ in terms of the Hausdorff distance.*

In the course of Chapter 5 the set $\mathcal{U}_\infty$ is shown to possess certain topological properties of which a selection is collected in Theorem 5.1.2. Furthermore, the techniques described in Chapter 5 lead to an algorithm that produces a provably accurate computer image of $\mathcal{U}_\infty$ (see Figure 5.1).

Samenvatting

Partitiefuncties: nulpunten, instabiele dynamica en complexiteit

De twee centrale objecten in deze thesis zijn de partitiefuncties van het Ising-model en het harde-kern-model. Zij $G = (V, E)$ een eindige simpele graaf. De partitiefunctie van het harde-kern-model, ook wel het onafhankelijkheidspolynoom genoemd, is de genererende functie van de onafhankelijke verzamelingen van G:

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V: \\ I \text{ is onafhankelijk}}} \lambda^{|I|}.$$

De partitiefunctie van het Ising-model is

$$Z_G(\lambda, b) = \sum_{\sigma: V \to \{+,-\}} \lambda^{|n_+(\sigma)|} b^{\delta(\sigma)},$$

waarbij $n_+(\sigma)$ staat voor het aantal knopen dat + toegewezen krijgt door σ en $\delta(\sigma)$ voor het aantal lijnen waarvan de eindpunten twee verschillende spins hebben onder σ. In het vervolg nemen we b als een vaste reële parameter in het interval $(0, 1)$ en dus beschouwen we beide functies als polynomen in de variabele λ. We noteren deze polynomen met Z_G.

Laat $\Delta \in \mathbb{Z}_{\geq 3}$ en noteer met $\mathcal{G}_\Delta$ de verzameling van grafen waarvan de maximale graad van boven begrensd wordt door Δ. De vereniging van alle complexe nulpunten van Z_G, waarbij G door de verameling $\mathcal{G}_\Delta$ loopt, noteren we met $\mathcal{Z}_\Delta$. Patel en Regts [PR17] laten zien, voortbouwend op resultaten van Barvinok [Bar16], dat, als λ in een nulpuntsvrije omgeving rondom 0 ligt, er een algoritme bestaat om $Z_G(\lambda)$ te benaderen dat in polynomiale tijd uitgevoerd kan worden. Deze uitspraak zegt zoiets als dat de afwezigheid van nulpunten impliceert dat benaderen makkelijk is. In hoofdstukken 2 en 3 van deze thesis wordt aangetoond dat een inverse uitspraak ook waar is, zowel voor het ferromagnetische Ising-model als voor het onafhankelijkheidspolynoom.

Stelling (2.1.1 and 3.1.1)**.** *De afsluiting van de nulpunten* $\overline{\mathcal{Z}_\Delta}$ *is bevat in de afsluiting van de parameters* λ *waarvoor het benaderen van* $Z_G(\lambda)$ *voor* $G \in \mathcal{G}_\Delta$ *een* $\#\mathsf{P}$*-moeilijk probleem is.*

Deze stelling nemen we als motivatie om een beter begrip te krijgen van de afsluiting van de nulpunten $\overline{\mathcal{Z}_\Delta}$. Voor het ferromagnetische Ising-model is een expliciete beschrijving van deze verzameling bekend [LY52b, PR20]. In dit model is de afsluiting van de nulpunten gelijk aan de afsluiting van de nulpunten van Cayley-bomen. Dit zijn gewortelde bomen waarvan alle bladeren dezelfde afstand hebben tot de wortel, de wortel graad $\Delta - 1$ heeft, en de overige knopen graad Δ hebben. In een zekere zin zijn Cayley-bomen dus extremaal in de verzameling van begrensde graad grafen. Van het onafhankelijkheidspolynoom is het bekend dat de nulpunten van Cayley-bomen accumuleren op een bepaald expliciet gegeven gebied Λ_Δ [PR19]. Dit gebied is verder nulpuntsvrij voor Cayley-bomen. Gebruik makend van resultaten in [BGGv20] laten we in hoofdstuk 3 zien dat nulpunten dicht liggen buiten Λ_Δ. Het centrale resultaat uit hoofdstuk 4 is dat, in tegenstelling tot het Ising-model, Cayley-bomen in het algemeen niet extremaal zijn binnen de verzameling van begrensde graad grafen.

Stelling (4.1.3)**.** *Voor* $\Delta \in \{3, \ldots, 9\}$ *bestaan er* $\lambda \in \Lambda_\Delta$ *en* $G \in \mathcal{G}_\Delta$ *waarvoor* $Z_G(\lambda) = 0$.

De maximale nulpuntsvrije regio $\mathcal{U}_\Delta := \overline{\mathcal{Z}_\Delta}^c$ is dus in het algemeen kleiner dan Λ_Δ. Het lijkt moeilijk om een expliciete beschrijving te geven van $\mathcal{U}_\Delta$ voor een vaste Δ. Vooralsnog is er zelfs geen manier bekend om een accuraat computerplaatje van $\mathcal{U}_\Delta$ te maken. In hoofdstuk 5 bekijken we wat er gebeurt als Δ naar oneindig gaat. In dat hoofdstuk wordt onder andere bewezen dat de herschaalde verzamelingen $\Delta \cdot \mathcal{U}_\Delta$ convergeren.

Stelling (5.1.1)**.** *De verzamelingen* $\Delta \cdot \mathcal{U}_\Delta$ *convergeren naar een enkelvoudig samenhangend gebied* $\mathcal{U}_\infty$ *in de Hausdorffmetriek.*

In de loop van hoofdstuk 5 bewijzen we dat $\mathcal{U}_\infty$ enkele topologische eigenschappen heeft, waarvan de belangrijkste in stelling 5.1.2 staan. De technieken die in dat hoofdstuk beschreven worden maken het mogelijk om een bewijsbaar correct computerplaatje van $\mathcal{U}_\infty$ te maken (zie figuur 5.1).

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