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Conditional maximal distributions of processes related to higher-order heat-type equations

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Abstract

The conditional Feynman–Kac functional is used to derive the Laplace transforms of conditional maximum distributions of processes related to third- and fourth-order equations. These distributions are then obtained explicitly and are expressed in terms of stable laws and the fundamental solutions of these higher-order equations. Interestingly, it is shown that in the third-order case, a genuine non-negative real-valued probability distribution is obtained.

Keywords: Brownian motion; Maximal distribution; Feynman–Kac functional; Higher-order heat-type equations; Signed measures; Laplace transforms; Airy functions; Stable laws

1. Introduction

Our aim in this paper is to study the conditional maximal distributions of processes related to third- and fourth-order equations. This is carried out by deriving their Laplace transforms by means of the conditional Feynman–Kac (F–K) functional. Inversion of these Laplace transforms then enables us to present the distributions of the maxima in terms of stable laws and fundamental solutions of the equations.

The conditional F–K functional, which plays an important role in our analysis, was first introduced by Gelfand and Yaglom in 1956; the English version appeared in 1960. The F–K functional considered by Gelfand and Yaglom is evaluated there for the conditional Wiener measure.

An interesting series of applications of the conditional F–K functional can be found in Ladokhin (1962). In particular, he obtained the following distributions for the conditional Brownian motion:

$$\Pr\{\text{meas}(s<t\colon B(s)>0)<\beta \mid B(t)=0\}=\frac{\beta}{t} \quad \text{for } 0<\beta<t \tag{1.1}$$

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and

$$\Pr\{\text{meas}(s<t\text{: } B(s)>0)<\beta\,|\,B(t)>0\}$$
$$=\frac{2}{\pi}\arcsin\sqrt{\frac{\beta}{t}}-\frac{2}{\pi t}\sqrt{\beta(t-\beta)}\quad \text{for } 0<\beta<t, \tag{1.2}$$

where $B(t)$ denotes the standard Brownian motion.

In this paper, we introduce and apply the conditional F–K functional for processes $\{X(t),\ t>0\}$ related to heat-type equations of the form

$$\frac{\partial u}{\partial t}=c_n\frac{\partial^n u}{\partial x^n}\quad x\in\Re,\ \ t>0. \tag{1.3}$$

Processes related to equations like (1.3) have been examined by Krylov (1960), Daletsky (1969), Daletsky and Fomin (1965, 1991), and Hochberg (1978) for the special case $n=4,\ c_4=-1$.

The F–K functional for processes related to (1.3) was introduced by Krylov (1960) for $n=2k,\ c_{2k}=(-1)^{k+1}$, and more recently, for other cases, by Hochberg and Orsingher (1994). The conditional F–K functional for processes of higher order has been considered in a recent paper by Nikitin and Orsingher (1998).

Some applications of the unconditional generalized F–K functional are scattered throughout the literature. We concentrate here on the results referring to the distribution of the maximum. For our purposes, it is useful to record here the Laplace transforms of the maximal distributions (obtained by means of the unconditional F–K functional) for the cases $n=2$, $c_2=\frac{1}{2}$ (Brownian motion), $n=3$ (examined by Orsingher (1991) for $c_3=-1$) and $n=4$ (analyzed by Hochberg (1978) for $c_4=-1$):

$$\int_0^\infty e^{-ut}\frac{\partial}{\partial\lambda}P\left\{\max_{0\leqslant s\leqslant t}X(s)\leqslant\lambda\right\}dt=\sqrt{2}u^{-1/2}e^{-\sqrt{2u}\lambda},\quad n=2,\ c_2=\frac{1}{2}, \tag{1.4}$$

$$\int_0^\infty e^{-ut}\frac{\partial}{\partial\lambda}P\left\{\max_{0\leqslant s\leqslant t}X(s)\leqslant\lambda\right\}dt=\frac{2}{\sqrt{3}}u^{-2/3}e^{-(1/2)\sqrt[3]{u}\lambda}\sin\frac{\sqrt{3}\sqrt[3]{u}\lambda}{2},$$
$$n=3,\ c_3=-1, \tag{1.5}$$

$$\int_0^\infty e^{-ut}\frac{\partial}{\partial\lambda}P\left\{\max_{0\leqslant s\leqslant t}X(s)\leqslant\lambda\right\}dt=\sqrt{2}u^{-3/4}e^{-(1/\sqrt{2})\sqrt[4]{u}\lambda}\sin\frac{\sqrt[4]{u}\lambda}{\sqrt{2}},$$
$$n=4,\ c_4=-1. \tag{1.6}$$

It should be emphasized that by inverting the Laplace transform (1.4), the classical distribution of the maximum of Brownian motion is obtained.

While the unconditional F–K functional has been extensively applied and generalized in many directions, something similar has been done for its conditional version only recently (see Nikitin and Orsingher, 1998). Applying this generalized F–K functional, we obtain the Laplace transform for the conditional distribution of the maximum of processes related to Eq. (1.3) for $n=3$ and 4. We report here these Laplace transforms, together with their analog in the case $n=2$ (Brownian bridge):

$$\int_0^{\infty} e^{-ut} p(0,t) \frac{\partial}{\partial \lambda} P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t)=0 \right\} dt$$
$$= 2e^{-2\sqrt{2u}\lambda}, \quad n=2, \; c_2 = \frac{1}{2}, \tag{1.7}$$

$$\int_0^{\infty} e^{-ut} p(0,t) \frac{\partial}{\partial \lambda} P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t)=0 \right\} dt$$
$$= \frac{2}{\sqrt{3}\sqrt[3]{u}} e^{-(3/2)\sqrt[3]{u}\lambda} \sin \frac{\sqrt{3}\sqrt[3]{u}\lambda}{2}, \quad n=3, \; c_3 = \pm 1, \tag{1.8}$$

$$\int_0^{\infty} e^{-ut} p(0,t) \frac{\partial}{\partial \lambda} P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t)=0 \right\} dt$$
$$= \frac{1}{\sqrt{u}} e^{-\sqrt{2}\sqrt[4]{u}\lambda} [1 - \cos(\sqrt{2}\sqrt[4]{u}\lambda)], \quad n=4, \; c_4 = -1. \tag{1.9}$$

Formula (1.7) easily leads to the well-known distribution of the maximum of the Brownian bridge. It is important to underline the analogy of the conditional and unconditional Laplace transforms of the maximal distributions for the higher-order processes, in particular for $n=3$. In this last case, we note the striking result that, despite the fact that the governing equations differ in the two cases $c_3 = 1$ and -1, *the Laplace transforms of the maximal distributions coincide*. The conditioning assumption that $X(t)=0$ plays a fundamental role in explaining this, by compensating the different form of the density of the governing measure.

In our last section, we extract the explicit distributions of the maxima from the Laplace transforms above. Surprisingly, it turns out that in the third-order case, one gets a *genuine probability distribution* that is expressed in terms of the fundamental solution and of the stable law of index $\frac{1}{3}$.

The same conclusion has not yet been obtained for fourth-order processes, although some qualitative features of the distribution lead us to believe that the maximum is a true random variable, as in the third-order case.

2. The conditional Feynman–Kac functional

Now let $\{x(t),\; t>0\}$ be a set of functions x: $t \in [0,\infty) \to x(t)$. On this set, we introduce the measure induced by the fundamental solution of Eq. (1.3).

In the particular case $n=2$, the fundamental solution of (1.3) is given by

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad x \in \Re, \; t > 0 \tag{2.1}$$

and the related measure is the well-known Wiener measure.

For $n=3,\ c_3=1$, the fundamental solution reads

$$p(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\alpha x+\mathrm{i}\alpha^3 t}\,\mathrm{d}\alpha$$
$$=\frac{1}{\sqrt[3]{3t}}Ai\left(-\frac{x}{\sqrt[3]{3t}}\right), \tag{2.2}$$

where

$$Ai(z)=\frac{1}{\pi}\int_0^{\infty}\cos\left(\alpha z+\frac{\alpha^3}{3}\right)\mathrm{d}\alpha.$$

For $n=4,\ c_4=-1$ the fundamental solution

$$p(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\alpha x-\alpha^4 t}\,\mathrm{d}\alpha \tag{2.3}$$

cannot be expressed in terms of known functions, as it can in the previous case.

Functions (2.2) and (2.3) are sign-varying and therefore induce, on the set of sample paths x, signed measures P. These quasi-measures (with unbounded variation) have been analyzed by many authors (e.g., Daletsky and Fomin, 1991). Usually they are constructed for cylinder sets of the form

$$C=\{x\colon a_j\leqslant x(t_j)\leqslant b_j,\ 1\leqslant j\leqslant n\}$$

via the formula

$$P(C)=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\prod_{j=1}^{n}P(x_j-x_{j-1},t_j-t_{j-1})\,\mathrm{d}x_j$$

and then extended to the field of sets generated by $x(t_j)$. With respect to these measures, it is possible to define mean values

$$Ef=\int f\,\mathrm{d}P. \tag{2.4}$$

In our analysis, the conditional F–K functional plays a fundamental role. In parallel to the definition of Ladokhin (1962), we write it as follows:

$$E\{\mathrm{e}^{-\int_0^t V(X(s))\,\mathrm{d}s}\,|\,X(t)=x\}=\lim_{n\to\infty}\int\cdots\int_{\Re^n}\mathrm{e}^{-\sum_{j=1}^{n}V(X(t_{j-1}))(t_j-t_{j-1})}$$
$$\times P\left\{\bigcap_{j=1}^{n}X(t_{j-1})\in\mathrm{d}x_{j-1}\,|\,X(t)=x\right\}, \tag{2.5}$$

where V is a piecewise-continuous function and P is the signed measure related to Eq. (1.3).

For the functional (2.5), the following result holds:

Theorem 2.1. *If V is a piecewise continuous function and*

$$E\left\{\mathrm{e}^{-\int_0^t V(X(s))\,\mathrm{d}s}\,|\,X(t)=x\right\}=p(x,t)^{-1}\psi(x,t), \tag{2.6}$$

then ψ is the solution of the Cauchy problem

$$\frac{\partial\psi}{\partial t} = c_n \frac{\partial^n \psi}{\partial x^n} - V\psi, \quad x \in \Re, \ t > 0,$$
$$\psi(x,0) = \delta(x), \tag{2.7}$$

where δ is the Dirac delta function and p is the fundamental solution of (1.3).

While it is straightforward that ψ satisfies the initial condition $\psi(x,0) = \delta(x)$, the proof that it satisfies Eq. (2.7) can be done, as in Hochberg and Orsingher (1994), by means of semigroups of operators. For $n = 2$, $c_2 = \frac{1}{2}$. Theorem 2.1 coincides with the result obtained by Gelfand and Yaglom (1960) and reported in Ladokhin (1962).

It is easy to see that for the conditional measures appearing in (2.6), the following is true:

$$\begin{aligned} E\left\{ e^{-\int_0^t V(X(s))\,ds} \,|\, X(t) = x \right\} &= p(x,t)^{-1} E\left\{ e^{-\int_0^t V(X(s))\,ds} \,|\, X(0) = x \right\} \\ &= p(x,t)^{-1}\psi(x,t). \end{aligned} \tag{2.8}$$

In the application of Theorem 2.1, the most difficult problem is to find the general solution ψ to the equation in (2.7); this is carried out by resorting to the Laplace transform

$$\varphi(x,u) = \int_0^\infty e^{-ut}\psi(x,t)\,dt, \quad u > 0. \tag{2.9}$$

It is easily shown that φ solves the equation

$$-\psi(x,0) + u\varphi(x,u) = c_n \frac{\partial^n \varphi}{\partial x^n} - V\varphi. \tag{2.10}$$

By integrating (2.10) in $(-\varepsilon, +\varepsilon)$ and then letting $\varepsilon \to 0$, we derive the important boundary condition

$$c_n \left.\frac{\partial^{n-1}\varphi}{\partial x^{n-1}}\right|^{x=0^+} - c_n \left.\frac{\partial^{n-1}\varphi}{\partial x^{n-1}}\right|^{x=0^-} = -1. \tag{2.11}$$

3. Laplace transforms of the conditional maximal distributions for third- and fourth-order processes

We now illustrate the method of derivation of the Laplace transforms for conditional maximal distributions. We start by considering the potential function

$$V(x) = \begin{cases} \beta & \text{for } x > \lambda > 0, \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for which functional (2.5) becomes

$$\begin{aligned} p(x,t)^{-1}\psi_\beta(x,t) &= E\left\{ e^{-\int_0^t V(X(s))\,ds} \,|\, X(t) = x \right\} \\ &= E\left\{ \exp\left\{ -\beta \int_0^t 1_{[s\,:\,X(s)>\lambda]}\,ds \right\} 1_{[\max_{0\leqslant s\leqslant t} X(s) > \lambda]} \right. \\ &\quad \left. + 1_{[\max_{0\leqslant s\leqslant t} X(s)\leqslant\lambda]} \,|\, X(t) = x \right\}. \end{aligned} \tag{3.2}$$

Since, with non-zero measure, the set (λ,∞) is visited by processes X, we have, letting $\beta\to\infty$,

$$\lim_{\beta\to\infty} p(x,t)^{-1}\psi_\beta(x,t) = P\left\{\max_{0\leqslant s\leqslant t} X(s)\leqslant\lambda \mid X(t)=x\right\}. \tag{3.3}$$

Thus, the technique used for deriving the conditional distribution of the maximum is based upon solving the Cauchy problem (2.7) for V as in (3.1) and then performing limit (3.3). We illustrate this technique for the special cases where Eq. (1.3) is considered for $n=3$, $c_3=\pm1$ and $n=4$, $c_4=-1$.

A delicate point is to ascertain that the solution satisfies the matching conditions at the points $x=\lambda$ and 0. For $x=\lambda$, we take the continuity conditions

$$\left.\frac{\partial^k\varphi}{\partial x^k}\right|^{\lambda^+} = \left.\frac{\partial^k\varphi}{\partial x^k}\right|^{\lambda^-}, \quad k=0,\dots,n-1,$$

while for $x=0$ we take

$$\left.\frac{\partial^k\varphi}{\partial x^k}\right|^{0^+} = \left.\frac{\partial^k\varphi}{\partial x^k}\right|^{0^-}, \quad k=0,\dots,n-2$$

together with condition (2.11). Since we are interested only in bounded solutions, we must restrict ourselves to the cases where solutions of Eq. (2.10) vanish for $|x|\to\infty$.

We start by considering processes related to third-order equations.

Theorem 3.1. *Let* $p(x,t)$ (*represented by* (2.2)) *be the fundamental solution of*

$$\frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial x^3} \quad \textit{with } p(x,0)=\delta(x),\ t>0,\ x\in\Re. \tag{3.4}$$

Then, for any $\lambda>0$ *and* $u>0$, *the following equality holds*:

$$\int_0^\infty e^{-ut} p(0,t)\frac{\partial}{\partial\lambda} P\left\{\max_{0\leqslant s\leqslant t} X(s)\leqslant\lambda \mid X(t)=0\right\} dt$$

$$= \frac{2}{\sqrt{3}\sqrt[3]{u}} e^{-(3/2)\sqrt[3]{u}\lambda} \sin\frac{\sqrt{3}\sqrt[3]{u}\lambda}{2}. \tag{3.5}$$

Proof. With the choice $n=3$, $c_3=1$, Eq. (2.10) for the Laplace transform of the F–K functional becomes

$$-\psi(x,0)+u\varphi(x,u) = \frac{\partial^3\varphi}{\partial x^3} - V\varphi. \tag{3.6}$$

Restricting ourselves to bounded solutions of (3.6) and assuming V to be as in (3.1), we readily obtain

$$\varphi(x,u) = \begin{cases} Ae^{x\varphi_1\sqrt[3]{u+\beta}} + Be^{x\varphi_2\sqrt[3]{u+\beta}} & \text{for } x>\lambda,\\ Ce^{x\varphi_0\sqrt[3]{u}} + De^{x\varphi_1\sqrt[3]{u}} + Ee^{x\varphi_2\sqrt[3]{u}} & \text{for } 0<x<\lambda,\\ Fe^{x\varphi_0\sqrt[3]{u}} & \text{for } x<0, \end{cases} \tag{3.7}$$

where $\varphi_k = e^{i2k\pi/3}$, $k=0,1,2$.

The six constants appearing in (3.7) are obtained by imposing the conditions:

$$\left.\frac{\partial^k \varphi}{\partial x^k}\right|^{x=\lambda^+} = \left.\frac{\partial^k \varphi}{\partial x^k}\right|^{x=\lambda^-}, \quad k=0,1,2,$$
$$\left.\frac{\partial^k \varphi}{\partial x^k}\right|^{x=0^+} = \left.\frac{\partial^k \varphi}{\partial x^k}\right|^{x=0^-}, \quad k=0,1, \tag{3.8}$$
$$\left.\frac{\partial^2 \varphi}{\partial x^2}\right|^{x=0^+} - \left.\frac{\partial^2 \varphi}{\partial x^2}\right|^{x=0^-} = -1.$$

Since we are interested only in $\varphi(x,0)$, our calculations can be drastically reduced to the evaluation of F. After many calculations involving Vandermonde determinants, we obtain

$$F = \frac{1}{\sqrt[3]{u^2}} \frac{\sqrt[3]{u} - \sqrt[3]{u+\beta}}{(\varphi_2-\varphi_1)(\sqrt[3]{u}\varphi_0 - \sqrt[3]{u+\beta}\varphi_1)(\sqrt[3]{u}\varphi_0 - \sqrt[3]{u+\beta}\varphi_2)}$$
$$\times \left[\frac{\varphi_2 e^{\lambda\sqrt[3]{u}(\varphi_2-\varphi_0)}(\sqrt[3]{u}\varphi_2 - \sqrt[3]{u+\beta}\varphi_1)}{(\varphi_2-\varphi_0)} - \frac{\varphi_1 e^{\lambda\sqrt[3]{u}(\varphi_1-\varphi_0)}(\sqrt[3]{u}\varphi_1 - \sqrt[3]{u+\beta}\varphi_2)}{(\varphi_1-\varphi_0)} \right]$$
$$+ \frac{1}{\sqrt[3]{u^2}} \frac{1}{(\varphi_2-\varphi_0)(\varphi_1-\varphi_0)}. \tag{3.9}$$

On passing to the limit as $\beta \to \infty$, we extract from (3.9) the following expression:

$$\lim_{\beta\to\infty} F = \frac{1}{\sqrt[3]{u^2}} \left\{ \frac{e^{\lambda\sqrt[3]{u}(\varphi_2-\varphi_0)}}{(\varphi_2-\varphi_0)(\varphi_2-\varphi_1)} - \frac{e^{\lambda\sqrt[3]{u}(\varphi_1-\varphi_0)}}{(\varphi_1-\varphi_0)(\varphi_2-\varphi_1)} \right.$$
$$\left. + \frac{1}{(\varphi_2-\varphi_0)(\varphi_1-\varphi_0)} \right\}$$
$$= \frac{1}{3\sqrt[3]{u^2}} \left\{ 1 - e^{-(3/2)\sqrt[3]{u}\lambda} \left[\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{u}\lambda\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{u}\lambda\right) \right] \right\}. \tag{3.10}$$

Differentiating now with respect to λ, we obtain formula (3.5). □

Remark 3.1. For processes related to the third-order equation

$$\frac{\partial p}{\partial t} = -\frac{\partial^3 p}{\partial x^3} \quad \text{with } p(x,0)=\delta(x), \ t>0, \ x\in\Re, \tag{3.11}$$

we again have

$$\int_0^\infty e^{-ut} p(0,t) \frac{\partial}{\partial\lambda} P\left\{ \max_{0\leqslant s\leqslant t} X(s) \leqslant \lambda \,|\, X(t)=0 \right\} dt$$
$$= \frac{2}{\sqrt{3}\sqrt[3]{u}} e^{-(3/2)\sqrt[3]{u}\lambda} \sin\frac{\sqrt{3}\sqrt[3]{u}\lambda}{2}, \tag{3.12}$$

which is identical to (3.5). This rather striking result is due to the conditioning event $\{X(t)=0\}$; indeed, this makes the sample paths of the processes connected with (3.4) and (3.11) behave in the same way, despite the fact that the fundamental solution of (3.4) is given by (2.2) and that of (3.11) is given by

$$p(x,t) = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right).$$

We point out that in this case, the derivation of result (3.12) is somewhat more complicated than that of (3.5) in Theorem 3.1 because the general, bounded solution of (2.10) for $n=3$, $c_3=-1$, is given by

$$\varphi(x,u) = \begin{cases} A e^{x\varphi_1 \sqrt[3]{u+\beta}} & \text{for } x > \lambda, \\ B e^{x\varphi_0 \sqrt[3]{u}} + C e^{x\varphi_1 \sqrt[3]{u}} + D e^{x\varphi_2 \sqrt[3]{u}} & \text{for } 0 < x < \lambda, \\ E e^{x\varphi_0 \sqrt[3]{u}} + F e^{x\varphi_2 \sqrt[3]{u}} & \text{for } x < 0, \end{cases} \tag{3.13}$$

where $\varphi_k = e^{i(2k+1)\pi/3}$, $k=0,1,2$. Thus $\varphi(0,u) = E+F$, and this implies that we need to perform twice as many calculations as in the proof of Theorem 3.1.

Remark 3.2. Note that in the unconditional case, the distributions of the maximum and minimum differ (because of the asymmetry of density (2.2); see Orsingher (1991)), whereas it is easy to check that in the conditional case,

$$P\left\{\max_{0\leqslant s\leqslant t} X(s) \leqslant \lambda \,|\, X(t)=0\right\} = P\left\{\min_{0\leqslant s\leqslant t} X(s) > -\lambda \,|\, X(t)=0\right\}, \quad \lambda > 0.$$

We now pass to the case of fourth-order equations.

Theorem 3.2. *Let $p(x,t)$ (represented by (2.3)) be the fundamental solution of*

$$\frac{\partial p}{\partial t} = -\frac{\partial^4 p}{\partial x^4} \quad \text{with } p(x,0)=\delta(x),\ t>0,\ x \in \Re. \tag{3.14}$$

Then, for any $\lambda > 0$ and $u > 0$, the following equality holds:

$$\int_0^\infty e^{-ut} p(0,t) \frac{\partial}{\partial \lambda} P\left\{\max_{0\leqslant s\leqslant t} X(s) \leqslant \lambda \,|\, X(t)=0\right\} dt$$
$$= \frac{1}{\sqrt{u}} e^{-\sqrt{2}\sqrt[4]{u}\lambda}(1-\cos\sqrt{2}\sqrt[4]{u}\lambda). \tag{3.15}$$

Proof. For the choice $n=4$, $c_4=-1$, Eq. (2.10) becomes

$$-\psi(x,0) + u\varphi(x,u) = -\frac{\partial^4 \varphi}{\partial x^4} - V\varphi. \tag{3.16}$$

Eq. (3.16), for V defined as in (3.1), becomes

$$\frac{\partial^4 \varphi}{\partial x^4} = -(u+\beta)\varphi \quad \text{for } x > \lambda,$$

$$\frac{\partial^4 \varphi}{\partial x^4} = -u\varphi \quad \text{for } 0 < x < \lambda \quad \text{and} \quad x < 0$$

and possesses the following bounded, general integral:

$$\varphi(x,u)=\begin{cases} A\mathrm{e}^{x\varphi_2\sqrt[4]{u+\beta}}+B\mathrm{e}^{x\varphi_3\sqrt[4]{u+\beta}} & \text{for } x>\lambda,\\ C\mathrm{e}^{x\varphi_1\sqrt[4]{u}}+D\mathrm{e}^{x\varphi_2\sqrt[4]{u}}+E\mathrm{e}^{x\varphi_3\sqrt[4]{u}}+F\mathrm{e}^{x\varphi_4\sqrt[4]{u}} & \text{for } 0<x<\lambda,\\ G\mathrm{e}^{x\varphi_1\sqrt[4]{u}}+H\mathrm{e}^{x\varphi_4\sqrt[4]{u}} & \text{for } x<0, \end{cases} \tag{3.17}$$

where $\varphi_k=\mathrm{e}^{\mathrm{i}(2k-1)\pi/4}$, $k=1,2,3,4$ are the roots of -1.

The eight conditions at $x=0$ and λ read

$$\begin{aligned} \left.\frac{\partial^k\varphi}{\partial x^k}\right|^{x=\lambda^+} &= \left.\frac{\partial^k\varphi}{\partial x^k}\right|^{x=\lambda^-}, \quad k=0,1,2,3,\\ \left.\frac{\partial^k\varphi}{\partial x^k}\right|^{x=0^+} &= \left.\frac{\partial^k\varphi}{\partial x^k}\right|^{x=0^-}, \quad k=0,1,2,\\ \left.\frac{\partial^3\varphi}{\partial x^3}\right|^{x=0^+} &- \left.\frac{\partial^3\varphi}{\partial x^3}\right|^{x=0^-} =1. \end{aligned} \tag{3.18}$$

After lengthy calculations, we obtain

$$\begin{aligned} \varphi(0,u) &= G+H\\ &= \frac{1}{4\sqrt[4]{u^3}}\frac{\sqrt[4]{u}-\sqrt[4]{u+\beta}}{(\sqrt[4]{u}\varphi_1-\sqrt[4]{u+\beta}\varphi_2)(\sqrt[4]{u}\varphi_1-\sqrt[4]{u+\beta}\varphi_3)}\\ &\quad\times[(1-\mathrm{i})\mathrm{e}^{\lambda\sqrt[4]{u}(\varphi_2-\varphi_1)}(\sqrt[4]{u}\varphi_2-\sqrt[4]{u+\beta}\varphi_3)\\ &\quad-\mathrm{e}^{\lambda\sqrt[4]{u}(\varphi_3-\varphi_1)}(\sqrt[4]{u}\varphi_3-\sqrt[4]{u+\beta}\varphi_2)]\\ &\quad+\frac{1}{4\sqrt[4]{u^3}}\frac{\sqrt[4]{u}-\sqrt[4]{u+\beta}}{(\sqrt[4]{u}\varphi_4-\sqrt[4]{u+\beta}\varphi_2)(\sqrt[4]{u}\varphi_4-\sqrt[4]{u+\beta}\varphi_3)}\\ &\quad\times[(1+\mathrm{i})\mathrm{e}^{\lambda\sqrt[4]{u}(\varphi_3-\varphi_4)}(\sqrt[4]{u}\varphi_3-\sqrt[4]{u+\beta}\varphi_2)\\ &\quad-\mathrm{e}^{\lambda\sqrt[4]{u}(\varphi_2-\varphi_4)}(\sqrt[4]{u}\varphi_2-\sqrt[4]{u+\beta}\varphi_3)]+\frac{\sqrt{2}}{4\sqrt[4]{u^3}}. \end{aligned} \tag{3.19}$$

Expression (3.19) considerably simplifies on passing to the limit as $\beta\to\infty$:

$$\begin{aligned} \lim_{\beta\to\infty}\varphi(0,u) &= \int_0^\infty \mathrm{e}^{-ut}p(0,t)P\left\{\max_{0\leqslant s\leqslant t}X(s)<\lambda\,|\,X(t)=0\right\}\mathrm{d}t\\ &= \frac{\sqrt{2}}{4\sqrt[4]{u^3}}\{1+\mathrm{e}^{-\sqrt{2}\sqrt[4]{u}\lambda}[\cos(\sqrt{2}\sqrt[4]{u}\lambda)-\sin(\sqrt{2}\sqrt[4]{u}\lambda)-2]\}. \end{aligned} \tag{3.20}$$

Upon differentiating with respect to λ, we straightforwardly obtain (3.15). □

Remark 3.3. The technique used to derive the results of Theorems 3.1 and 3.2 can easily be adapted to derive the maximal distribution of the Brownian bridge. In this case, we obtain that

$$\int_0^\infty e^{-ut} p(0,t) P\left\{ \max_{0 \leqslant s \leqslant t} X(s) < \lambda \,|\, X(t) = 0 \right\} dt$$

$$= \frac{1 - e^{-2\sqrt{2u}\lambda}}{\sqrt{2u}}$$

$$= 2\int_0^\lambda e^{-2\sqrt{2u}y}\, dy$$

$$= \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-ut}(1 - e^{-2\lambda^2/t})\, dt. \tag{3.21}$$

Hence,

$$P\left\{ \max_{0 \leqslant s \leqslant t} X(s) < \lambda \,|\, X(t) = 0 \right\} = 1 - e^{-2\lambda^2/t}, \quad \lambda > 0, \tag{3.22}$$

since $p(0,t) = 1/\sqrt{2\pi t}$.

4. Conditional distributions of the maximum

We now derive the distributions of the conditional maxima by inverting the Laplace transforms (3.10) and (3.20) obtained in the previous section. These distributions are expressed in terms of the fundamental solutions of Eq. (1.3) and of the stable laws on $\mathfrak{R}^+$.

We start our analysis by observing that in both cases examined above, we have

$$\lim_{\lambda \to 0^+} \int_0^\infty e^{-ut} p_k(0,t) P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} dt = 0, \quad k = 3, 4. \tag{4.1}$$

This implies that the maximum immediately starts increasing, as is the case for Brownian motion.

From (3.10), we also have that

$$\lim_{\lambda \to \infty} \int_0^\infty e^{-ut} p_3(0,t) P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} dt = \frac{1}{3\sqrt[3]{u^2}} \tag{4.2}$$

and, considering that $p_3(0,t) = \Gamma(\frac{1}{3})/2\pi\sqrt{3}\sqrt[3]{t}$, we conclude that

$$P\left\{ \max_{0 \leqslant s \leqslant t} X(s) < \infty \,|\, X(t) = 0 \right\} = 1. \tag{4.3}$$

A result similar to (4.3) is valid also for the fourth-order case, as the reader can check for himself.

The distribution laws that we extract from the Laplace transforms obtained in the previous sections are expressed in terms of the fundamental solutions (2.2) and (2.3) and also of the stable laws of degrees $\frac{1}{3}$ and $\frac{1}{4}$, respectively. For the sake of completeness, we recall that the stable law $S_\alpha(\sigma, \beta, \mu)$ has characteristic function

$$E e^{iuS_\alpha(\sigma,\beta,\mu)} = \begin{cases} \exp\{-\sigma^\alpha |u|^\alpha [1 - i\beta(\operatorname{sign} u)\operatorname{tg}\frac{\pi\alpha}{2}] + i\mu u\}, & \alpha \neq 1, \\ \exp\{-\sigma |u| [1 + i\beta\frac{2}{\pi}(\operatorname{sign} u)\lg|u|] + i\mu u\}, & \alpha = 1 \end{cases} \tag{4.4}$$

for $\sigma \geqslant 0,\ 0 < \alpha \leqslant 2,\ -1 \leqslant \beta \leqslant 1,\ \mu \in \Re,\ u \in \Re$. The subclass of stable laws $S_\alpha(\sigma, 1, 0)$ has support $(0, \infty)$ and possesses Laplace transform

$$Ee^{-uS_\alpha(\sigma,1,0)} = \begin{cases} \exp\left\{ -\dfrac{\sigma^\alpha}{\cos \pi\alpha/2} u^\alpha \right\}, & \alpha \neq 1, \\ \exp\left\{ \sigma \dfrac{2u}{\pi} \lg u \right\}, & \alpha = 1, \end{cases} \tag{4.5}$$

where $u \geqslant 0,\ \sigma \geqslant 0,\ 0 < \alpha \leqslant 2$. See, e.g., Samorodnitsky and Taqqu (1994) for details.

In our analysis, the following Laplace transforms emerge:

$$L_{1/3}(u, \lambda) = e^{-\sqrt[3]{u}\lambda} \tag{4.6a}$$

and

$$L_{1/4}(u, \lambda) = e^{-\sqrt{2}\sqrt[4]{u}\lambda}. \tag{4.6b}$$

It is clear from (4.5) that the Laplace transform $L_{1/3}(u, \lambda)$ refers to the stable law with parameters $\alpha = \frac{1}{3}$, $\sigma = (\sqrt{3}\lambda/2)^3$, whose density will be denoted by $\bar{p}_{1/3}(\lambda, t)$. Analogously, the Laplace transform $L_{1/4}(u, \lambda)$ refers to the stable law with parameters $\alpha = \frac{1}{4}$, $\sigma = (\sqrt{2}\lambda \cos \pi/8)^4$, and its density will be denoted by $\bar{p}_{1/4}(\lambda, t)$.

In what follows, we also need the following Laplace transform formula:

$$\int_0^\infty e^{-ut} t^\gamma \, dt = \frac{\Gamma(\gamma+1)}{u^{\gamma+1}}, \quad \gamma > -1, \tag{4.7}$$

which can be straightforwardly checked.

Theorem 4.1. *For processes related to third-order equations, we have, for* $0 \leqslant \lambda < \infty$,

$$P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} = 1 - \frac{2\pi\sqrt{3}\sqrt[3]{t}}{\Gamma(1/3)} \int_0^t \bar{p}_{1/3}(\lambda, s) p_3(\lambda, t-s) \, ds. \tag{4.8}$$

Proof. The Laplace transform (3.10) can be written as follows:

$$\int_0^\infty e^{-ut} p_3(0,t) P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} dt$$

$$= \frac{1}{3\sqrt[3]{u^2}} - e^{-\lambda\sqrt[3]{u}} \frac{e^{-(\lambda/2)\sqrt[3]{u}}}{3\sqrt[3]{u^2}} \left\{ \cos\frac{\sqrt{3}}{2}\lambda\sqrt[3]{u} + \sqrt{3}\sin\frac{\sqrt{3}}{2}\lambda\sqrt[3]{u} \right\}. \tag{4.9}$$

From formula (4.7), we have

$$\frac{1}{3}\int_0^\infty e^{-ut} \frac{dt}{\sqrt[3]{t}\,\Gamma(2/3)} = \frac{1}{3\sqrt[3]{u^2}}. \tag{4.10}$$

The term $e^{-\lambda\sqrt[3]{u}}$ coincides with (4.6a) and refers to a stable law on $(0, \infty)$ of index $\frac{1}{3}$, as noted above.

The connection between (4.9) and $p_3(\lambda, t)$ is given by the following result, which is proved in Orsingher (1991):

$$\int_0^\infty e^{-ut} p_3(x,t) \, dt$$

$$= \frac{1}{\pi} \int_0^\infty e^{-ut} \, dt \int_0^\infty \cos(\alpha x - \alpha^3 t) \, d\alpha$$

$$= \frac{e^{-(1/2)\sqrt[3]{ux}}}{3\sqrt[3]{u^2}} \left\{ \cos\frac{\sqrt{3}}{2}\sqrt[3]{ux} + \sqrt{3}\sin\frac{\sqrt{3}}{2}\sqrt[3]{ux} \right\}, \quad u > 0,\ x > 0. \tag{4.11}$$

This enables us to write down (4.9) as follows:

$$\int_0^\infty e^{-ut} p_3(0,t) P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} dt$$

$$= \int_0^\infty e^{-ut} \frac{dt}{3\sqrt[3]{t}\,\Gamma(2/3)} - \int_0^\infty e^{-ut} (\bar{p}_{1/3} * p_3)(\lambda, t)\, dt, \tag{4.12}$$

where

$$(\bar{p}_{1/3} * p_3)(\lambda, t) = \int_0^t \bar{p}_{1/3}(\lambda, s) p_3(\lambda, t-s)\, ds. \tag{4.13}$$

Considering that

$$p_3(0,t) = \Gamma(\tfrac{1}{3})/2\pi\sqrt{3}\sqrt[3]{t},$$

we readily obtain result (4.8). □

Remark 4.1. From the well-known fact that p_3 decreases exponentially as $\lambda \to \infty$, we conclude from (4.8) that result (4.3) holds. Proving that

$$\lim_{\lambda \to 0^+} P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} = 0$$

(as emerged from (4.1)) requires a bit more care. As $\lambda \to 0$, formula (4.5) shows that

$$\bar{p}_{1/3}(\lambda, s) \to \delta(s)$$

and thus

$$\int_0^t \bar{p}_{1/3}(\lambda, s) p_3(\lambda, t-s)\, ds \to \int_0^t \delta(s) p_3(0, t-s)\, ds = p_3(0,t) = \frac{\Gamma(1/3)}{2\pi\sqrt{3}\sqrt[3]{t}}.$$

We point out that the structure of formula (4.8) differs substantially from the corresponding formula for the Brownian bridge, in that it involves the values of $p_3(\lambda, s)$ *for all* $s \in [0, t]$.

Remark 4.2. We now show that (4.8) is a *true probability distribution*, despite the fact that the governing measure is signed. This has been shown to be the case also for other functionals related to these processes such as sojourn times, whether conditional or not (cf. Orsingher, 1991; Nikitin and Orsingher, 1998).

Consider first that

$$\frac{\partial}{\partial \lambda} P\left\{ \max_{0 \leqslant s \leqslant t} X(s) \leqslant \lambda \,|\, X(t) = 0 \right\} = -\frac{2\pi\sqrt{3}\sqrt[3]{t}}{\Gamma(1/3)} \left\{ \int_0^t \frac{\partial}{\partial \lambda} \bar{p}_{1/3}(\lambda, s) p_3(\lambda, t-s)\, ds \right.$$

$$\left. + \int_0^t \bar{p}_{1/3}(\lambda, s) \frac{\partial}{\partial \lambda} p_3(\lambda, t-s)\, ds \right\}. \tag{4.14}$$

Since $\bar{p}_{1/3}$ is a probability density and the law p_3 given by (2.2) is a non-negative, exponentially decreasing function for $\lambda > 0$ (as the reader can ascertain by consulting Abramowitz and Stegun (1970), p. 446, where the Airy functions are depicted), we conclude that the second term on the right-hand side of (4.14) is positive for $\lambda > 0$.

To prove that the first term is positive as well, we write $\bar{p}_{1/3}$ in the following manner (see Samorodnitsky and Taqqu, 1994, p. 5):

$$\bar{p}_{1/3}(\lambda,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left\{-\mathrm{i}ut-\lambda\frac{\sqrt{3}}{2}\sqrt[3]{|u|}\left(1-\mathrm{i}(\operatorname{sign}u)\operatorname{tg}\frac{\pi}{6}\right)\right\}\,\mathrm{d}u$$

$$=\frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-(\sqrt{3}/2)\sqrt[3]{u}\lambda}\cos\left(ut+\frac{3}{2}\lambda\sqrt[3]{u}\right)\,\mathrm{d}u. \tag{4.15}$$

The idea underlying the proof is to express $(\partial/\partial\lambda)\bar{p}_{1/3}(\lambda,t)$ in terms of p_3. In light of formula (4.11), this can be done as follows:

$$\frac{\partial}{\partial\lambda}\bar{p}_{1/3}(\lambda,t)=-\frac{\sqrt{3}}{2\pi}\int_0^{\infty}\sqrt[3]{u}\mathrm{e}^{-(\sqrt{3}/2)\sqrt[3]{u}\lambda}$$
$$\times\left\{\cos\left(ut+\frac{3}{2}\lambda\sqrt[3]{u}\right)+\sqrt{3}\sin\left(ut+\frac{3}{2}\lambda\sqrt[3]{u}\right)\right\}\,\mathrm{d}u$$
$$=-\frac{3\sqrt{3}}{2\pi}\int_0^{\infty}u\mathrm{e}^{ut/\sqrt{3}}\frac{1}{3\sqrt[3]{u^2}}\mathrm{e}^{-ut/\sqrt{3}-(\sqrt{3}/2)\sqrt[3]{u}\lambda}$$
$$\times\left\{\cos\left(ut+\frac{3}{2}\lambda\sqrt[3]{u}\right)+\sqrt{3}\sin\left(ut+\frac{3}{2}\lambda\sqrt[3]{u}\right)\right\}\,\mathrm{d}u$$
$$=-\frac{3\sqrt{3}}{2\pi}\int_0^{\infty}u\mathrm{e}^{ut/\sqrt{3}}\left\{\int_0^{\infty}\mathrm{e}^{-us}p_3\left(\frac{2\sqrt[3]{u^2}t}{\sqrt{3}}+\sqrt{3}\lambda,s\right)\,\mathrm{d}s\right\}\,\mathrm{d}u. \tag{4.16}$$

We now resort to the case of fourth-order processes. Our next theorem shows that in this case, the conditional distribution of the maximum has a considerably more complicated structure.

Theorem 4.2. *For processes related to fourth-order equations, we have, for* $0\leqslant\lambda<\infty$,

$$P\left\{\max_{0\leqslant s\leqslant t}X(s)\leqslant\lambda\,|\,X(t)=0\right\}$$
$$=1+\frac{4\pi\sqrt[4]{t}}{\Gamma(1/4)}p_4(2\lambda,t)+\frac{2\Gamma(3/4)}{\Gamma(1/4)}\int_0^t\sqrt[4]{\frac{t}{(t-s)^3}}\frac{\partial}{\partial\lambda}p_4(2\lambda,s)\,\mathrm{d}s$$
$$-2\int_0^t\sqrt[4]{\frac{t}{s}}\bar{p}_{1/4}(\lambda,t-s)\,\mathrm{d}s. \tag{4.17}$$

Proof. The Laplace transform (3.20) can be written in the following way:

$$\int_0^{\infty}\mathrm{e}^{-ut}p_4(0,t)P\left\{\max_{0\leqslant s\leqslant t}X(s)\leqslant\lambda\,|\,X(t)=0\right\}\,\mathrm{d}t$$
$$=\frac{1}{2\sqrt{2}\sqrt[4]{u^3}}+\frac{1}{2\sqrt{2}\sqrt[4]{u^3}}\mathrm{e}^{-\lambda\sqrt{2}\sqrt[4]{u}}\{\cos(\lambda\sqrt{2}\sqrt[4]{u})+\sin(\lambda\sqrt{2}\sqrt[4]{u})\}$$
$$-\frac{1}{\sqrt{2}\sqrt[4]{u^3}}\mathrm{e}^{-\lambda\sqrt{2}\sqrt[4]{u}}\sin(\lambda\sqrt{2}\sqrt[4]{u})-\frac{1}{\sqrt{2}\sqrt[4]{u^3}}\mathrm{e}^{-\lambda\sqrt{2}\sqrt[4]{u}}. \tag{4.18}$$

Bearing in mind formula (4.7) and the form of the Laplace transform of stable laws of index $\frac{1}{4}$, we can easily invert the first and fourth terms of (4.18). Furthermore, we have

$$\int_0^\infty e^{-ut} p_4(x,t)\,dt = \frac{1}{2\pi}\int_0^\infty e^{-ut}\left\{\int_{-\infty}^{+\infty} e^{-i\alpha x-\alpha^4 t}\,d\alpha\right\}dt$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x}{u+\alpha^4}\,d\alpha$$
$$= \frac{\sqrt{2}}{4\sqrt[4]{u^3}} e^{-x\sqrt[4]{u}/\sqrt{2}}\left[\cos\frac{x\sqrt[4]{u}}{\sqrt{2}} + \sin\frac{x\sqrt[4]{u}}{\sqrt{2}}\right], \quad u>0,\ x>0 \tag{4.19}$$

(cf. Gradshtein and Ryzhik, 1994, formula 3.727, p. 447).

In view of the obvious identity

$$\frac{2}{\sqrt{2}\sqrt[4]{u^3}}\int_0^\lambda e^{-x\sqrt{2}\sqrt[4]{u}}\sin(x\sqrt{2}\sqrt[4]{u})\,dx$$
$$= \frac{1}{2u} - \frac{1}{2u} e^{-\lambda\sqrt{2}\sqrt[4]{u}}\{\sin(\lambda\sqrt{2}\sqrt[4]{u}) + \cos(\lambda\sqrt{2}\sqrt[4]{u})\} \tag{4.20}$$

and using Eq. (4.19), we can write

$$\frac{1}{\sqrt{2}\sqrt[4]{u^3}} e^{-\lambda\sqrt{2}\sqrt[4]{u}}\sin(\lambda\sqrt{2}\sqrt[4]{u})$$
$$= \frac{\partial}{\partial\lambda}\int_0^\lambda \frac{1}{\sqrt{2}\sqrt[4]{u^3}} e^{-x\sqrt{2}\sqrt[4]{u}}\sin(x\sqrt{2}\sqrt[4]{u})\,dx$$
$$= -\frac{1}{4u}\frac{\partial}{\partial\lambda} e^{-\lambda\sqrt{2}\sqrt[4]{u}}[\sin(\lambda\sqrt{2}\sqrt[4]{u}) + \cos(\lambda\sqrt{2}\sqrt[4]{u})]$$
$$= -\frac{1}{\sqrt{2}\sqrt[4]{u}}\int_0^\infty e^{-ut}\frac{\partial}{\partial\lambda} p_4(2\lambda,t)\,dt$$
$$= -\int_0^\infty e^{-ut}\frac{dt}{\sqrt{2}\sqrt[4]{t^3}\Gamma(\frac{1}{4})} \times \int_0^\infty e^{-ut}\frac{\partial}{\partial\lambda} p_4(2\lambda,t)\,dt$$
$$= -\int_0^\infty e^{-ut}\left\{\int_0^t \frac{1}{\sqrt{2}\sqrt[4]{(t-s)^3}\Gamma(1/4)}\frac{\partial}{\partial\lambda} p_4(2\lambda,s)\,ds\right\}dt. \tag{4.21}$$

Finally, taking into account that

$$p_4(0,t) = \frac{\Gamma(1/4)}{4\pi\sqrt[4]{t}}$$

and using the identity

$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) = \pi\sqrt{2},$$

we conclude the proof of the theorem. □

Remark 4.3. Considering that

$$\bar{p}_{1/4}(\lambda, t-s) \to \delta(t-s)$$

and

$$\lim_{\lambda \to 0^+} \frac{\partial}{\partial \lambda} p_4(2\lambda, t-s) = 0,$$

we easily conclude that the expression in formula (4.17) tends to zero as $\lambda \to 0^+$. To check that the expression in formula (4.17) tends to one as $\lambda \to +\infty$ is straightforward.

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