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A REPRESENTATION THEORY FOR POLYNOMIAL COFRACTIONALITY IN VECTOR AUTOREGRESSIVE MODELS

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We extend the representation theory of the autoregressive model in the fractional lag operator of Johansen (2008, *Econometric Theory* 24, 651–676). A recursive algorithm for the characterization of cofractional relations and the corresponding adjustment coefficients is given, and it is shown under which condition the solution of the model is fractional of order d and displays cofractional relations of order $d-b$ and polynomial cofractional relations of order $d-2b, \ldots, d-cb \geq 0$ for integer c; the cofractional relations and the corresponding moving average representation are characterized in terms of the autoregressive coefficients by the same algorithm. For $c=1$ and $c=2$ we find the results of Johansen (2008).

1. INTRODUCTION

Since the papers by Granger and Joyeux (1980) and Engle and Granger (1987) introduced the concepts of fractional processes and fractional cointegration (cofractionality), the attention of many scholars has been devoted both to testing and estimating the cointegrating relationships for fractional processes. Examples are Cheung and Lai (1993), Baillie and Bollerslev (1994), Dueker and Startz (1998), Kim and Phillips (2001), Marinucci and Robinson (2001), Breitung and Hassler (2002), Davidson (2002), Velasco (2003), Dittmann (2004), Dolado and Marmol (2004), Nielsen (2008), Hualde (2006, 2007), among others. For surveys, see Baillie (1996) and Henry and Zaffaroni (2003). Johansen and Nielsen (2010) discuss likelihood inference for an autoregressive (AR) model that generates fractionality and cofractionality.

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Based on the idea in Granger (1986) and with the aim of modeling both the cofractional and polynomial cofractional relations and the adjustment toward them, Johansen (2008) proposes the following error correction mechanism:

$$\Delta^d X_t = \Delta^{d-2b}(\alpha\beta' L_b X_t - \Gamma\Delta^b L_b X_t) + \sum_{i=1}^{k} \Psi_i \Delta^d L_b^i X_t + \epsilon_t, \tag{1.1}$$

where X_t is $p \times 1$, ϵ_t is independent and identically distributed with mean 0 and positive definite variance $\Omega > 0$ (denoted by $i.i.d.(0, \Omega)$), $\Delta := 1 - L,\ 0 < 2b \leq d \in \mathbb{R}$, and

$$L_b := 1 - (1 - L)^b$$

is the fractional lag operator. A necessary and sufficient condition on the parameters of the model ensures that the solution of (1.1) is fractional of order d (denoted by $X_t \in \mathcal{F}(d)$) and such that $\beta' X_t$ is of order $d - b$. Hence, the model allows for cofractionality, as there exist linear combinations of X_t that are fractional of a lower order than the process itself. The model also allows for polynomial cofractionality as $\beta' X_t \in \mathcal{F}(d - b)$ and $\Delta^b X_t \in \mathcal{F}(d - b)$ can be combined in such a way that the order of fractionality is reduced to $d - 2b$, but no further decrement is possible. The condition, representation, and interpretation of (1.1) are the fractional counterpart of the well-known concepts in the $I(2)$ model for polynomial cointegration (see Johansen, 1996). This is immediately seen by letting $d = 2$ and $b = 1$ in (1.1) and recognizing the standard model for variables that are integrated of order two.

It is very unlikely to observe economic series that can be described by processes that are integrated of order higher than two and in this sense the interest in these processes is indeed of a more abstract nature (for a study of these processes, see Johansen, 1988; Stock and Watson, 1993; Gregoir and Laroque, 1993; la Cour, 1998; Gregoir, 1999; Bauer and Wagner, 2007; Franchi, 2007). In the fractional case things are different, as there is no evident reason why a process of order $d - 2b$ could be a plausible description of the data whereas one of order $d - 3b$, say, would not.

The aim of this paper is to analyze a richer cofractional structure through an extension of (1.1) that we call the $VAR_{d,b,c}(k)$ model for polynomial cofractionality. A recursive algorithm for the characterization of the restrictions on the AR coefficients that define cofractional relations and the corresponding adjustments is given, and it is shown under which condition the solution of the model is fractional of order d and displays cofractional relations of order $d - b$ and polynomial cofractional relations of order $d - 2b, \ldots, d - cb \geq 0$ for integer c. The cofractional relations and the corresponding moving average representation are characterized in terms of the AR coefficients by the same algorithm. For $c = 1$ and $c = 2$ we find the results of Johansen (2008). In Franchi and Paruolo (2008) an extension of the same idea is used to characterize common cyclical features in stationary vector autoregressive (VAR) models.

The following notation is used throughout: $a := b$ and $b =: a$ indicate that a is defined by b, $\delta_{i,j}$ is Kronecker's delta ($\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise), and $\sum_{h=a}^{b} \cdot := 0$ for $b < a$. For any $p \times r$ matrix γ of full rank $r \leq p$, $sp(\gamma)$ is the space spanned by the columns of γ; with $\gamma_\perp$ of dimension $p \times p - r$ we indicate a basis of the orthogonal complement of $sp(\gamma)$, so that $\gamma' \gamma_\perp = 0$ and $\gamma_\perp' \gamma = 0$. Furthermore we define $\bar{\gamma} := \gamma(\gamma'\gamma)^{-1}$, denote with $P_\gamma := \bar{\gamma}\gamma' = \gamma\bar{\gamma}'$ the orthogonal projector matrix onto $sp(\gamma)$, and let $M_\gamma := I - P_\gamma = P_{\gamma_\perp}$ be the orthogonal projector matrix onto $sp(\gamma_\perp)$.

2. DEFINITIONS

The definition of order of fractionality is taken from Johansen (2008).

DEFINITION 2.1. *If $\sum_{i=0}^{\infty} ||C_i||^2 < \infty$ and $C(z) = \sum_{i=0}^{\infty} C_i z^i$, $|z| < 1$ can be extended to a continuous function on the boundary $|z| = 1$, we call the linear process $X_t = C(L)\epsilon_t$ fractional of order zero, $\mathcal{F}(0)$, if the spectrum at 1 is different from 0, i.e., if $f_X(1) = (1/2\pi)C(1)\Omega C(1)' \neq 0$. For such processes we denote by $\mathcal{F}(0)_+$ the class of asymptotically stationary processes of the form*

$$X_t^+ = \begin{cases} \sum_{i=0}^{t-1} C_i \epsilon_{t-i} & t = 1, 2, \ldots \\ 0 & t = 0, -1, \ldots \end{cases}.$$

If $\Delta_+^d X_t - \mu_t \in \mathcal{F}(0)_+$ for some deterministic function μ_t that depends on initial values we say that X_t is fractional of order d and write $X_t \in \mathcal{F}(d)$.

The notions of cofraction matrix polynomial and polynomially cofractional relations are introduced in the following definition.

DEFINITION 2.2. *Let $X_t \in \mathcal{F}(d)$ and $\gamma_n(u) := \sum_{i=0}^{n} \gamma_{n,i}(1-u)^i$ be such that $\gamma_n'(L_b)X_t \in \mathcal{F}(d_n)$, where $d_n := d - (n+1)b$ and $0 \leq d_n < d$. If for any $\varphi \neq 0$ and any polynomial $\phi(u) := \gamma_n(u) + (1-u)^{n+1}\psi(u)$ one has*

$$\varphi'\phi'(L_b)X_t \in \mathcal{F}(d_n),$$

we say that $\gamma_n(u)$ is a cofraction matrix polynomial and $\gamma_n'(L_b)X_t$ are polynomially cofractional relations.

Hence, a cofraction matrix polynomial is such that the order of fractionality of the linear combination $\gamma_{n,0}'X_t + \cdots + \gamma_{n,n}'\Delta^{nb}X_t$ cannot be reduced either by taking linear combinations or by including additional powers of $\Delta^b X_t$. We remark that n in $\gamma_n(u)$ is the degree of the cofraction matrix polynomial and because $n = 0$ is allowed for in Definition 2.2, cofractionality is included in the previous definition as a special case. Note that $\varphi = 0$ is excluded from this definition because $\varphi'\phi'(L_b)X_t = 0$ is not fractional of any order.

The model we propose in (3.1) in the next section is such that $\mathbb{R}^p$ is partitioned into $c+1$ mutually orthogonal subspaces, i.e., $I = \sum_{i=0}^{c} P_{\beta_i}$ where β_i has dimension $p \times r_i$ and full column rank and in each $sp(\beta_i)$ the properties of X_t are different. That is, $\gamma' X_t \in \mathcal{F}(d)$ for any $\gamma \in sp(\beta_c)$, and for $i = 0, \ldots, c-1$, we combine $\beta_i' X_t$ and powers of $\Delta^b X_t$ and define polynomially cofractional relations of degree $c-i-1$ in L_b and order of fractionality $d-(c-i)b$. The coefficients, the order, and the degree of the cofraction matrix polynomials are all expressed in terms of the parameters of the $VAR_{d,b,c}(k)$ model.

3. THE $VAR_{d,b,c}(k)$ MODEL

Consider the $VAR_{d,b,c}(k)$ model

$$\Pi_c(L)X_t := \Pi(L_b)\Delta^{d-cb}X_t = \epsilon_t, \tag{3.1}$$

where X_t is $p \times 1$, $\Pi(u)$ is a matrix polynomial of finite degree k in

$$u := 1-(1-z)^b \in \mathbb{C} \tag{3.2}$$

such that $\det \Pi(1) = 0$, $c > 0$ is the order of the pole of $\Pi(u)^{-1}$ at $u = 1$, b, d satisfy $0 < cb \leq d \in \mathbb{R}$, and ϵ_t is $i.i.d.(0, \Omega)$.

For $c = 1$ and $c = 2$ the error correction formulation of model (3.1) is found in Johansen (2008). When the pole of $\Pi(u)^{-1}$ at $u = 1$ is of order one, the characteristic function of (3.1) is $\Pi_1(z) := (1-z)^{d-b}\Pi(u)$, and the matrix polynomial $\Pi(u)$ is reparametrized as

$$\Pi(u) = (1-u)I + \Pi(1)u - (1-u)\sum_{i=1}^{k-1} \Psi_{1i} u^i, \tag{3.3}$$

where $\Pi(1) = -\alpha\beta'$ has reduced rank. When the pole of $\Pi(u)^{-1}$ at $u = 1$ is of order two, the characteristic function of (3.1) is $\Pi_2(z) := (1-z)^{d-2b}\Pi(u)$, and $\Pi(u)$ is reparametrized as

$$\Pi(u) = (1-u)^2 I + \Pi(1)u + \Gamma(1-u)u - (1-u)^2 \sum_{i=1}^{k-2} \Psi_{2i} u^i, \tag{3.4}$$

where $\Pi(1) = -\alpha\beta'$ and $\alpha_\perp' \Gamma \beta_\perp = \xi\eta'$ have reduced rank. Note that the possibility of reparameterizing $\Pi(u)$ as in (3.3) or (3.4) has nothing to do with the order of the pole. When the $I(1)$ condition $\det \alpha_\perp' \dot{\Pi}(1)\beta_\perp \neq 0$ holds, the first one provides the relevant error correction representation, and thus it is chosen in Johansen (2008). When the pole has order two the relevant error correction formulation is provided by the second one.

Hence, the representation analysis of (3.1) for $c = 1$ and $c = 2$ is found in Johansen (2008), where it is shown that $\beta' X_t \in \mathcal{F}(d-b)$ are the only cofractional relations when $c = 1$. For $c = 2$, the model displays the polynomial cofractional relations $\beta' X_t + \bar{\alpha}' \dot{\Pi} \Delta^b X_t \in \mathcal{F}(d-2b)$ and the cofractional relations $\eta' \bar{\beta}_\perp' X_t \in \mathcal{F}(d-b)$. Here we characterize the polynomial cofractional relations of (3.1) for general c.

The characteristic function of (3.1) is

$$\Pi_c(z) := (1-z)^{d-cb}\Pi(u), \qquad |z| \leq 1, \tag{3.5}$$

and we assume a unit root in $\Pi(u)$. Then $\operatorname{rank}\Pi(1) \leq p-1$, $\det\Pi(u) =: (1-u)^m g(u)$, where $m > 0$ and $g(u)$ is a scalar polynomial such that $g(1) \neq 0$; moreover, $\operatorname{adj}\Pi(u) =: (1-u)^a G(u)$ where $0 \leq a < m$ and the matrix polynomial $G(u)$ satisfies $G(1) \neq 0$. The reason is that when $\operatorname{rank}\Pi(1) < p-1$, $\operatorname{adj}\Pi(1) = 0$, and thus each entry of $\operatorname{adj}\Pi(u)$ contains the factor $1-u$. When $\operatorname{rank}\Pi(1) = p-1$, $\operatorname{adj}\Pi(1) \neq 0$, and thus $a = 0$ (see Franchi, 2007). The inverse of $\Pi(u)$ is then equal to

$$\Pi(u)^{-1} = \frac{G(u)}{(1-u)^{m-a}g(u)}, \qquad u \neq \{u : \det\Pi(u) = 0\}, \tag{3.6}$$

and because $G(1) \neq 0$ and $g(1) \neq 0$, this shows that the order of the pole at the unit root is equal to $m-a$, i.e., $c = m-a > 0$.

It then follows (see eqns. (3.2) and (3.6)) that the inverse of the characteristic function in (3.5) can be written as

$$\Pi_c(z)^{-1} = \frac{1}{(1-z)^d}\frac{G(u)}{g(u)}, \qquad |z| \leq 1, \tag{3.7}$$

and this shows that the pole at $z = 1$ has order d. Under the additional requirement that the roots of $g(u) = 0$ are outside $\mathbb{C}_b := \{u \in \mathbb{C} : |z| \leq 1\}$ (for the proof that a necessary and sufficient condition for $\Pi(L_b)X_t = \epsilon_t$ to be in $\mathcal{F}(0)$ is that the roots of $\det\Pi(u) = 0$ are outside $\mathbb{C}_b$, see Johansen, 2008), this determines the order of fractionality of the solution of the model, as is shown in Theorem 3.1.

THEOREM 3.1. *Let the roots of* $\det\Pi(u) = 0$ *be either at* $u = 1$ *or* $u \notin \mathbb{C}_b$. *Then the solution of (3.1) is fractional of order* d, $X_t \in \mathcal{F}(d)$.

Proof. Because the roots of $g(u)$ are outside $\mathbb{C}_b$ there is no z in the closed unit disk for which $1-(1-z)^b$ is equal to a root. Thus the function $C(z) := \frac{G(1-(1-z)^b)}{g(1-(1-z)^b)}$ is regular on the unit disk and continuous for $|z| \leq 1$, and by Lemma 10 in Johansen (2008) its expansion $C(z) = \sum_{n=0}^{\infty} C_n z^n$, $|z| < 1$ can be used to define the stationary process $Y_t = C(L)\epsilon_t$. The spectrum of Y_t at zero is $f_Y(1) = (1/2\pi)C(1)\Omega C(1)'$, and because $C(1) = \frac{G(1)}{g(1)} \neq 0$ we have $f_Y(1) \neq 0$. Then $\Delta_+^d X_t = Y_t^+ \in \mathcal{F}(0)_+$ implies $X_t \in \mathcal{F}(d)$, and the proof is complete. ■

Under the assumption that the roots of $\det\Pi(u) = 0$ are either at $z = 1$ or outside $\mathbb{C}_b$, Theorem 3.1 shows that the order of fractionality of the process X_t in (3.1) is given by the order of the pole of the inverse of its characteristic function at the unit root, exactly as it is in the univariate case (for the application of the same idea to the integer case, see Franchi, 2007).

By the same reasoning, the order of fractionality of $\gamma_n'(L_b)X_t$ is given by the order of the pole of $\gamma_n'(u)\Pi_c(z)^{-1}$ at $z = 1$. In the next lemma we state necessary and sufficient conditions for $\gamma_n(u)$ to be a cofraction matrix polynomial.

LEMMA 3.1. *Let $G(u)$ be as in (3.7), $\gamma_n(u) := \sum_{i=0}^{n} \gamma_{n,i}(1-u)^i$, and $\phi(u) := \gamma_n(u) + (1-u)^{n+1}\psi(u)$, where $\psi(u)$ is a matrix polynomial. Then a necessary and sufficient condition for $\gamma_n(u)$ to be a cofraction matrix polynomial is that*

$$\gamma_n'(u)G(u) = (1-u)^{n+1}\mu'(u),$$

where $\mu'(1) + \psi'(1)G(1)$ has full row rank.

Proof. (Suff.) If $\gamma_n'(u)G(u) = (1-u)^{n+1}\mu'(u)$ where $\mu'(1) + \psi'(1)G(1)$ has full row rank, then $\varphi'\phi'(u)G(u) = (1-u)^{n+1}\varphi'\nu'(u)$ where $\nu'(1) = \mu'(1) + \psi'(1)G(1)$. Hence for any $\varphi \neq 0$, $\varphi'\phi'(u)\Pi_c^{-1}(u)$ has a pole of order $d_n := d - (n+1)b$ at $z = 1$ so that $\varphi'\phi'(L_b)X_t \in \mathcal{F}(d_n)$. (*Nec.*) If $\gamma_n(u)$ is a cofraction matrix polynomial then $\varphi'\phi'(u)\Pi_c^{-1}(u)$ has a pole of order $d_n := d - (n+1)b$ at $z = 1$ for any $\varphi \neq 0$. This implies (see eqn. (3.7)) $\phi'(u)G(u) = (1-u)^{n+1}\nu'(u)$ where $\nu'(1)$ has full row rank, and substituting $\phi(u) := \gamma_n(u) + (1-u)^{n+1}\psi(u)$ one finds $\gamma_n'(u)G(u) = (1-u)^{n+1}\mu'(u)$ where $\mu'(u) := \nu'(u) - \psi'(u)G(u)$. Hence $\mu'(1) + \psi'(1)G(1) = \nu'(1)$ has full row rank. This completes the proof. ■

To characterize the cofractional relations it is thus necessary to understand the structure of $G(u)$ at $u = 1$. This is achieved in the next section.

4. THE REDUCED RANK STRUCTURE OF $\Pi(u)$

It is well known (see Johansen, 1996) under which restrictions on its coefficients the matrix polynomial $\Pi(u) = \sum_{i=0}^{k} \Pi_i(1-u)^i$ has an inverse with a pole of order one or two at $u = 1$. The order of the pole of $\Pi(u)^{-1}$ is one if and only if $\Pi(1) = -\alpha_0\beta_0'$ has rank $r_0 < p$ and $M_{\alpha_0}\Pi_1 M_{\beta_0} = -\alpha_1\beta_1'$ has rank $r_1 = p - r_0$, where $M_\gamma := I - P_\gamma = P_{\gamma_\perp}$ and the square matrices $(\alpha_0 : \alpha_1)$, $(\beta_0 : \beta_1)$ are nonsingular with orthogonal blocks. This is called the $I(1)$ condition. The pole has order two if and only if $M_{\alpha_0}\Pi_1 M_{\beta_0} = -\alpha_1\beta_1'$ has rank $r_1 < p - r_0$ and $M_{(\alpha_0:\alpha_1)}\theta_2 M_{(\beta_0:\beta_1)} = -\alpha_2\beta_2'$ has rank $r_2 = p - r_0 - r_1$, where $\theta_2 := \Pi_2 + \Pi_1\bar{\beta}_0\bar{\alpha}_0'\Pi_1$ and the square matrices $(\alpha_0 : \alpha_1 : \alpha_2)$, $(\beta_0 : \beta_1 : \beta_2)$ are nonsingular with orthogonal blocks. This is called the $I(2)$ condition.

The characterization of the reduced rank matrices is important because it allows the cofraction matrices and the corresponding adjustment coefficients to be defined. When the order of the pole is greater than two, the additional reduced rank restriction $M_{(\alpha_0:\alpha_1)}\theta_2 M_{(\beta_0:\beta_1)} = -\alpha_2\beta_2'$ of rank $r_2 < p - r_0 - r_1$ holds, but the next matrix that would either define an additional orthogonal subspace if $c > 3$ or stop the iteration if $c = 3$ is not known. The intuition is that it is of the form $M_{(\alpha_0:\alpha_1:\alpha_2)}\theta_3 M_{(\beta_0:\beta_1:\beta_2)} = -\alpha_3\beta_3'$ for some θ_3 that needs to be found out.

We next present necessary and sufficient conditions on $\Pi(u)$ at $u = 1$ for its inverse to have a pole of any given order $c > 0$ at $u = 1$. These conditions are of reduced rank type and are stated in term of matrices α_i, β_i, and $\theta_{i,j}$ that are functions of the AR coefficients Π_i; see (4.1) and (4.2) in Section 4.1.

The following notation is employed: whenever we have a $p \times p$ reduced rank matrix π of rank r, say, and we write $\pi = -\alpha\beta'$, this is understood to be a rank decomposition, i.e., α and β are full rank matrices of dimension $p \times r$, bases of the column and row spaces, respectively. The matrices α and β are not unique, but the results do not depend on the particular choice made, as shown in Section 4.2.

4.1. Rank Decompositions

Consider $\Pi(u) = \sum_{i=0}^{k} \Pi_i (1-u)^i$. We take α_i (respectively, β_i) to indicate bases of column - (respectively, row) spaces of certain matrices derived from $\Pi(u)$, where $(\alpha_0 : \alpha_1 : \cdots : \alpha_c)$ and $(\beta_0 : \beta_1 : \cdots : \beta_c)$ are square nonsingular matrices with orthogonal blocks. The number of blocks is given by the order of the pole of $\Pi(u)^{-1}$ at $u = 1$ and vice versa. To simplify notation, we further define $a_i := (\alpha_0 : \cdots : \alpha_{i-1})$ so that one has the partition $I = P_{a_i} + M_{a_i}$. For $i = 0$, we let $M_{a_0} := I$ and employ similar notation for $b_i := (\beta_0 : \cdots : \beta_{i-1})$. The matrices α_i and β_i are defined by the rank decompositions in (4.2), where $\theta_{i,j}$ is defined for $i = 0, \ldots, c$ and $j \geq 1$ as follows: $\theta_{0,j} := \Pi_{j-1}$, and for $l \geq 1$,

$$\theta_{l,j} := \theta_{l-1,j+1} + \theta_{l-1,1} \sum_{h=0}^{l-2} \bar{\beta}_h \bar{\alpha}_h' \theta_{h+1,j} \tag{4.1}$$

with $\sum_{h=a}^{b} \cdot := 0$ for $b < a$; α_i and β_i are defined from the rank decompositions

$$M_{a_i} \theta_{i,1} M_{b_i} = -\alpha_i \beta_i', \tag{4.2}$$

where $r_i := \operatorname{rank} M_{a_i} \theta_{i,1} M_{b_i}$ satisfies $r_i < p - \sum_{h=0}^{i-1} r_h$ for $i < c$ and $r_c = p - \sum_{h=0}^{c-1} r_h$. Next we prove the equivalence between the reduced rank restrictions in (4.2) and the value of c in (3.6).

THEOREM 4.1 (Rank decompositions). *Let α_i, β_i, and $\theta_{i,j}$ be defined in (4.1) and (4.2). Then the following conditions are equivalent:*

(i) $\Pi(u)^{-1}$ *has a pole of order c at $u = 1$;*

(ii) *for $0 \leq i \leq n \leq c$, one has*

$$-\alpha_i \beta_i' G_{n-i} + M_{a_i} \sum_{h=1}^{n-i} \theta_{i+1,h} G_{n-i-h} = \delta_{n,c} g_0 M_{a_i} \tag{4.3}$$

and

$$-G_{n-i} \alpha_i \beta_i' + \sum_{h=1}^{n-i} G_{n-i-h} \theta_{i+1,h} M_{b_i} = \delta_{n,c} g_0 M_{b_i}, \tag{4.4}$$

where $\delta_{i,j}$ is Kronecker's delta ($\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise);

(iii) *for $i = 0, \ldots, c$, one has*

$$M_{a_i} \theta_{i,1} M_{b_i} = -\alpha_i \beta_i',$$

where α_i and β_i are $p \times r_i$ matrices of full column rank r_i, with $r_i < p - \sum_{h=0}^{i-1} r_h$ for $i < c$ and $r_c = p - \sum_{h=0}^{c-1} r_h$. Moreover, $(\alpha_0 : \alpha_1 : \cdots : \alpha_c)$ and $(\beta_0 : \beta_1 : \cdots : \beta_c)$ are square nonsingular matrices with orthogonal blocks and

$$G_0 = -g_0 \bar{\beta}_c \bar{\alpha}_c'. \tag{4.5}$$

Proof. See the Appendix.

The conditions (4.2) are reduced rank conditions for $i = 0, \ldots, c-1$, whereas the terminal condition for $i = c$ is a full rank condition. To see that this is the case, let $a_{i\perp}$ and $b_{i\perp}$ be bases of $sp(M_{a_i})$ and $sp(M_{b_i})$, respectively. Pre- and postmultiply (4.2) by $a_{i\perp}'$ and $b_{i\perp}$, respectively, to find

$$a_{i\perp}' \theta_{i,1} b_{i\perp} = -\xi_i \eta_i',$$

where $\xi_i := a_{i\perp}' \alpha_i$ and $\eta_i := b_{i\perp}' \beta_i$ are $p - \sum_{h=0}^{i-1} r_h \times r_i$ matrices of full column rank r_i. Hence one has $r_i < p - \sum_{h=0}^{i-1} r_h$ for $i < c$ if and only if $a_{i\perp}' \theta_{i,1} b_{i\perp}$ is singular and $r_c = p - \sum_{h=0}^{c-1} r_h$ if and only if $a_{c\perp}' \theta_{c,1} b_{c\perp}$ is nonsingular. For $c = 1$ and $c = 2$ one finds the $I(1)$ and $I(2)$ conditions.

Each iteration defines the orthogonal subspaces $sp(\xi_i)$ and $sp(\xi_{i\perp})$; a basis of the first one is used for constructing α_i and part of the second subspace for α_{i+1} in the next iteration. Smaller and smaller dimensional subspaces are met at any iteration until the full rank matrix $a_{c\perp}' \theta_{c,1} b_{c\perp}$ is found and no additional subspace can be defined.

As a consequence of Theorem 4.1 one has the result in Corollary 4.1. This will be used in Theorem 4.2 to characterize the cofractional relations.

COROLLARY 4.1. *Let α_i, β_i, and $\theta_{i,j}$ be defined in (4.1) and (4.2); then for $0 \leq i \leq s+i \leq c$, one has*

$$\beta_i' G_s = \bar{\alpha}_i' \sum_{h=1}^{s} \theta_{i+1,h} G_{s-h} - \delta_{i+s,c} g_0 \bar{\alpha}_i'. \tag{4.6}$$

Proof. Let $s := n - i$ in (4.3) and premultiply by $\bar{\alpha}_i'$. Because $\bar{\alpha}_i' M_{a_i} = \bar{\alpha}_i' - \bar{\alpha}_i' P_{a_i} = \bar{\alpha}_i'$ one has (4.6). ■

Next we discuss whether the conclusions reached in Theorem 4.1 depend on the choice of α_i and β_i.

4.2. Uniqueness

We want to show that (4.2) is invariant with respect to choices of α_i and β_i. Construct a new basis $\lambda_i = \alpha_i \omega$ and $\mu_i = \beta_i \omega^{-1\prime}$ for some ω full rank so that $M_{a_i} \theta_{i,1} M_{b_i} = -\lambda_i \mu_i'$. Then

$$\bar{\lambda}_i = \bar{\alpha}_i \omega^{-1\prime} \quad \text{and} \quad \bar{\mu}_i = \bar{\beta}_i \omega$$

imply

$$\bar{\mu}_i \bar{\lambda}_i' = \bar{\beta}_i \bar{\alpha}_i',$$

and this shows that $\theta_{i,j}$ in (4.1) is invariant under changes of basis of $sp(\alpha_i)$ and $sp(\beta_i)$. Now consider $sp(M_{a_i}) = sp(a_{i\perp})$ and choose a new basis $\psi_i = a_{i\perp}\omega$ for some nonsingular ω. Then $P_{\psi_i} = P_{a_{i\perp}}$ implies $P_{\psi_i}\theta_{i,1}M_{b_i} = M_{a_i}\theta_{i,1}M_{b_i}$, and this shows that (4.2) is invariant under changes of basis of $sp(M_{a_i})$. Because the same holds for $sp(M_{b_i})$, the conclusions reached in Theorem 4.1 do not depend on the particular choices of α_i and β_i.

4.3. Cofraction Matrix Polynomials

Combining (3.7) with $G(u) = G_0 + (1-z)^b R(u)$ and (4.5) one has

$$\Pi_c(z)^{-1} = -\frac{g_0 \bar{\beta}_c \bar{\alpha}_c'}{(1-z)^d g(u)} + \frac{1}{(1-z)^{d-b}} \frac{R(u)}{g(u)}.$$

Hence $\gamma'\Pi_c(z)^{-1}$ has a pole of order d at 1 for $\gamma \in sp(\beta_c)$, whereas the pole has at most order $d-b$ for $\gamma \in sp(\beta_0 : \cdots : \beta_{c-1})$. It then follows that a cofraction matrix polynomial $\gamma_n(u) := \sum_{h=0}^{n} \gamma_{n,h}(1-u)^h$ must satisfy $\gamma_{n,0} = \beta_i$ for some $i \neq c$. The remaining coefficients are defined in Theorem 4.2 using the results of Corollary 4.1.

THEOREM 4.2 (Cofraction matrix polynomials). *Let α_i, β_i, and $\theta_{i,j}$ be defined in (4.1) and (4.2). Then for $i = 0, \ldots, c-1$,*

$$\gamma_{c-i-1}'(u) := \beta_i' - \bar{\alpha}_i' \sum_{h=1}^{c-i-1} \theta_{i+1,h}(1-u)^h \tag{4.7}$$

is a cofraction matrix polynomial, and $\gamma_{c-i-1}'(L_b)X_t \in \mathcal{F}(d-(c-i)b)$ are the corresponding polynomially cofractional relations.

Proof. See the Appendix.

Because no reduction in the order of fractionality of $\gamma_{c-i-1}'(L_b)X_t$ can be achieved either by taking linear combinations or by adding higher order terms to $\gamma_{c-i-1}(u)$, $\gamma_{c-i-1}'(u)$ in (4.7) is a cofraction matrix polynomial. Note that the highest reduction in the order of fractionality of X_t is achieved by the transformation $\gamma_{c-1}'(u)$, which is such that $\gamma_{c-1}'(L_b)X_t \in \mathcal{F}(d-cb)$. Next we show that it is impossible to achieve higher reductions.

THEOREM 4.3 (Minimality). *There is no matrix polynomial $\varphi(u)$ such that $\varphi(1) \neq 0$ and $\varphi'(L_b)X_t \in \mathcal{F}(d_\varphi)$ where $d_\varphi < d-cb$.*

Proof. By contradiction. Assume there exists $\varphi(u)$ such that $\varphi'(L_b)X_t \in \mathcal{F}(d_\varphi)$ where $d_\varphi < d-cb$. This implies that there exist $\ell > c$ and a matrix polynomial

$\nu(u)$ such that $\varphi'(u)\Pi_c(z)^{-1} = \frac{\nu'(u)}{(1-z)^{d-\ell b} g(u)}$ where $\nu(1) \neq 0$. Substituting (3.7) in the last equation one finds

$$\varphi'(u)G(u) = (1-u)^{\ell}\nu'(u). \tag{4.8}$$

Postmultiplying both sides of (4.8) by $\Pi(u)$, substituting $G(u)\Pi(u) = (1-u)^c g(u) I$, and rearranging terms one has

$$\nu'(u)\Pi(u) = (1-u)^{c-\ell} g(u)\varphi'(u).$$

Because $\ell > c$ and $g(1)\varphi'(1) \neq 0$, the right-hand side has a pole at 1, which is a contradiction because $\nu(u)$ and $\Pi(u)$ are polynomials. Hence $\ell \leq c$. This completes the proof. ■

Hence $\gamma_{c-1}(u)$ in (4.7) is minimal in the sense that it achieves the highest reduction in the order of fractionality by including the minimum number of terms. One could conjecture that a similar property is shared by every $\gamma_{c-i-1}(u)$ in (4.7) with respect to $\varphi(u)$ such that $\varphi(1) \neq 0$ belongs to $sp(\beta_i : \ldots : \beta_c)$. A proof of this statement would require extending (4.3) and (4.4) for $c+1 \leq n \leq c + \deg g(u)$, so that one would be able to fully characterize additional G coefficients, as we do here only for G_0; see (4.5). As Theorem 4.1 is sufficient to prove what we need, namely, that $\gamma_{c-i-1}(u)$ is a cofraction matrix polynomial, we do not cover this case here.

5. THE REPRESENTATION THEOREM

The results in Theorems 3.1 and 4.2 are collected in Theorem 5.1.

THEOREM 5.1. *Let the roots of* $\det \Pi(u) = 0$ *be either at* $u = 1$ *or* $u \notin \mathbb{C}_b$ *and let* α_i, β_i, *and* $\theta_{i,j}$ *be as in (4.1) and (4.2). Then for* $d \geq cb > 0$, *the solution of (3.1) is the* $\mathcal{F}(d)$ *process*

$$X_t = C_c \Delta_+^{-d}\epsilon_t + C_{c-1}\Delta_+^{-d+b}\epsilon_t + \cdots + C_1 \Delta_+^{-d+(c-1)b}\epsilon_t + \Delta_+^{-d+cb} Y_t + \mu_t, \tag{5.1}$$

where Y_t *is stationary and* μ_t *depends on initial values. The* C *matrices are*

$$C_{c-n} = \sum_{k=0}^{n} G_k c_{n-k}, \tag{5.2}$$

where $G(u)$ *and* $g(u)$ *are defined in (3.6) and* $c_n := \frac{1}{n!}\left(\frac{d^n}{du^n} g(u)^{-1}\right)\Big|_{u=1}$ *is a scalar. For* $i = 0, \ldots, c-1$, *one has the polynomial cofractional relations*

$$\gamma'_{c-i-1}(L_b)X_t := \beta_i' X_t - \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k}\Delta^{kb} X_t \tag{5.3}$$

of order $d-(c-i)b$. *For* $i = c$ *no cofractionality is present, i.e.,* $\beta_c' X_t \in \mathcal{F}(d)$.

Proof. See the Appendix.

From the moving average representation in (5.1) we see that X_t is composed of $\mathcal{F}(d)$ down to $\mathcal{F}(d-cb)$ processes that are generated by cumulating ϵ_t and Y_t. Each of the components is loaded into X_t through the corresponding C coefficient, which is a linear combination of the G matrices with scalar coefficients. Combining (5.2) and (4.5), one finds that $C_c = -\bar{\beta}_c\bar{\alpha}_c'$. Hence one has

$$\beta_c' X_t \in \mathcal{F}(d).$$

The other C coefficients are more complicated and not very interesting in themselves; what is important is to understand which linear combinations of the process and its fractional differences have lower order of fractionality. These are the polynomial cofractional relations described in (5.3), and they have the following characteristics:

$$\gamma_{c-1}'(L_b)X_t := \beta_0' X_t - \bar{\alpha}_0' \sum_{k=1}^{c-1} \theta_{1,k}\Delta^{kb} X_t \in \mathcal{F}(d-cb),$$

$$\gamma_{c-2}'(L_b)X_t := \beta_1' X_t - \bar{\alpha}_1' \sum_{k=1}^{c-2} \theta_{2,k}\Delta^{kb} X_t \in \mathcal{F}(d-(c-1)b),$$

$$\vdots$$

$$\gamma_1'(L_b)X_t := \beta_{c-2}' X_t - \bar{\alpha}_{c-2}'\theta_{c-1,1}\Delta^{b} X_t \in \mathcal{F}(d-2b).$$

Hence, when the coefficient of X_t is taken to be β_0, one can transform the process in such a way that the order goes from $\mathcal{F}(d)$ to $\mathcal{F}(d-cb)$, when it is taken to be β_1 from $\mathcal{F}(d)$ to $\mathcal{F}(d-(c-1)b)$, and so on, up to β_{c-2} for which the transformed process is in $\mathcal{F}(d-2b)$. When one starts with β_{c-1} only cofractionality is present, and one has

$$\gamma_0'(L_b)X_t := \beta_{c-1}' X_t \in \mathcal{F}(d-b).$$

We remark that no reduction in the order of fractionality of the polynomial cofractional relations can be achieved either by taking linear combinations or by adding higher order terms. Moreover, because $sp(\beta_0 : \beta_1 : \cdots : \beta_{c-1} : \beta_c) = \mathbb{R}^p$ the characterization of the properties of X_t is complete. Note that for $c = 1$ the result in Theorem 5.1 specializes into $sp(\beta_0 : \beta_1) = \mathbb{R}^p$,

$$X_t = C_1\Delta_+^{-d}\epsilon_t + \Delta_+^{-d+b}Y_t + \mu_t \in \mathcal{F}(d),$$

$$\beta_0' X_t \in \mathcal{F}(d-b) \quad \text{and} \quad \beta_1' X_t \in \mathcal{F}(d).$$

For $c = 2$ one has $sp(\beta_0 : \beta_1 : \beta_2) = \mathbb{R}^p$,

$$X_t = C_2\Delta_+^{-d}\epsilon_t + C_1\Delta_+^{-d+b}\epsilon_t + \Delta_+^{-d+2b}Y_t + \mu_t \in \mathcal{F}(d),$$

$$\beta_0' X_t + \bar{\alpha}_0'\Pi_1\Delta^b X_t \in \mathcal{F}(d-2b),$$

$$\beta_1' X_t \in \mathcal{F}(d-b), \quad \text{and} \quad \beta_2' X_t \in \mathcal{F}(d).$$

These are found, respectively, in Theorems 8 and 9 in Johansen (2008).

6. CONCLUSION

We have extended the study of the representation theory of an AR model that generates fractional, cofractional, and polynomial cofractional processes. The moving average representation reveals that the solution is composed of different fractional processes that can be canceled by specific linear combinations. The model is a parametric characterization of the fractional counterpart of the well-known phenomenon of cointegration and allows for modeling both the stable relations and the adjustment toward them.

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APPENDIX

For the proof of Theorem 4.1 we make repeated use of the following result.

LEMMA 6.1. *Let $G \neq 0$ and assume $\Phi G = G\Phi = 0$. This implies*

(i) $\Phi = -\alpha\beta'$ *with* α *and* β *of dimension* $p \times r$ *and full column rank* $r < p$, $sp(G) \subseteq sp(\beta_\perp)$ *and* $sp(G') \subseteq sp(\alpha_\perp)$*;*

(ii) if moreover $sp(G) \subseteq sp(b)$ *and* $sp(G') \subseteq sp(a)$*, then* (i) *implies* $sp(G) \subseteq \mathcal{B} := sp(b) \cap sp(\beta_\perp)$ *and* $sp(G') \subseteq \mathcal{A} := sp(a) \cap sp(\alpha_\perp)$ *where, because* $G \neq 0$, $\dim \mathcal{A} = \dim \mathcal{B} > 0$.

Proof.

(i) Because $\det \Phi \neq 0$ implies $G = 0$, one has $\Phi = -\alpha\beta'$ with α and β of dimension $p \times r$ and full column rank $r < p$, so that $sp(G) \subseteq sp(\beta_\perp)$ and $sp(G') \subseteq sp(\alpha_\perp)$.

(ii) From $sp(G) \subseteq sp(b)$ and (i) one has $sp(G) \subseteq \mathcal{B} := sp(b) \cap sp(\beta_\perp)$, and because $G \neq 0$ it must be that $\mathcal{B} \neq \{0\}$ and thus $\dim \mathcal{B} > 0$. Similarly one shows that $sp(G') \subseteq \mathcal{A} := sp(a) \cap sp(\alpha_\perp)$ and $\dim \mathcal{A} > 0$. ■

Proof of Theorem 4.1. $(i) \Rightarrow (ii)$ Assume (3.6). This implies

$$\Pi(u)G(u) = G(u)\Pi(u) = (1-u)^c g(u) I. \tag{A.1}$$

First we show (4.3). Consider the product $\Pi(u)G(u) = \sum_{n=0}^{N} A_n (1-u)^n$ where $N := c + \deg g(u)$ and $A_n := \sum_{h=0}^{n} \Pi_h G_{n-h}$. Note that (A.1) implies $A_n = \delta_{n,c} g_0 I$ for $n = 0, \ldots, c$, where $\delta_{i,j}$ is Kronecker's delta ($\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise). Substituting $\Pi_0 = -\alpha_0 \beta_0'$ and $\theta_{1,h} = \Pi_h$ one finds $A_n = -\alpha_0\beta_0' G_n + \sum_{h=1}^{n} \theta_{1,h} G_{n-h}$. Next we show by induction that for $0 \leq i \leq n \leq c$ one has

$$M_{a_i} A_n = -\alpha_i \beta_i' G_{n-i} + M_{a_i} \sum_{h=1}^{n-i} \theta_{i+1,h} G_{n-i-h}. \tag{A.2}$$

For $i=0$ one has $M_{a_0}:=I$ so that (A.2) holds by definition. Next we assume (A.2) for any $0\le i\le \ell$ and show that it holds for $i=\ell+1\le n\le c$. Let $s:=n-i\ge 0$ and recall that $A_{i+s}=\delta_{i+s,c}g_0 I$ for $i+s=0,\dots,c$. This implies $\bar\alpha_i' A_{i+s}=\delta_{i+s,c}g_0\bar\alpha_i'$ for $i+s=0,\dots,c$. Premultiplying (A.2) by $\bar\alpha_i'$ one has $\bar\alpha_i' A_{i+s}=-\beta_i' G_s+\bar\alpha_i'\sum_{h=1}^{s}\theta_{i+1,h}G_{s-h}$, because $\bar\alpha_i' M_{a_i}=\bar\alpha_i'-\bar\alpha_i' P_{a_i}=\bar\alpha_i'$. Hence

$$\beta_i' G_s=\bar\alpha_i'\sum_{h=1}^{s}\theta_{i+1,h}G_{s-h}-\delta_{i+s,c}g_0\bar\alpha_i'. \tag{A.3}$$

Next consider (A.2) for $i=\ell$. Premultiply by $M_{a_{\ell+1}}$ and change index in the summation on the right-hand side to find

$$M_{a_{\ell+1}}A_n=M_{a_{\ell+1}}\theta_{\ell+1,1}G_{n-\ell-1}+M_{a_{\ell+1}}\sum_{h=1}^{n-\ell-1}\theta_{\ell+1,h+1}G_{n-\ell-1-h}=:a+b\quad\text{(say)}.$$

Using the projection identity $I=M_{b_{\ell+1}}+P_{b_{\ell+1}}$, one finds

$$a=M_{a_{\ell+1}}\theta_{\ell+1,1}M_{b_{\ell+1}}G_{n-\ell-1}+M_{a_{\ell+1}}\theta_{\ell+1,1}\sum_{k=0}^{\ell}\bar\beta_k\beta_k' G_{n-\ell-1}.$$

Substituting in the first term from (4.2) and in the second term from (A.3), and interchanging the order of summation in the second term one has

$$a=-\alpha_{\ell+1}\beta_{\ell+1}'G_{n-\ell-1}+M_{\alpha_{\ell+1}}\sum_{h=1}^{n-\ell-1}(\theta_{\ell+1,1}\sum_{k=0}^{\ell}\bar\beta_k\bar\alpha_k'\theta_{k+1,h})G_{n-\ell-1-h}$$

because $\delta_{n-\ell-1+k,c}=0$ for $0\le k\le\ell$. Summing $a+b$ and using (4.1) one finds

$$M_{a_{\ell+1}}A_n=-\alpha_{\ell+1}\beta_{\ell+1}'G_{n-\ell-1}+M_{a_{\ell+1}}\sum_{h=1}^{n-\ell-1}\theta_{\ell+2,h}G_{n-\ell-1-h}.$$

This shows that (A.2) holds for $i=\ell+1$ and completes the proof by induction. Substituting (A.2) in $M_{a_i}A_n=\delta_{n,c}g_0M_{a_i}$ one has (4.3).

Next we show (4.4). Because $g(u)$ is a scalar polynomial, $\Pi(u)$ and $G(u)$ commute; see (A.1). Hence one has $\Pi'(u)G'(u)=(1-u)^c g(u)I$, and by transposing the corresponding (4.3) one finds

$$-G_{n-i}\beta_{*,i}\alpha_{*,i}'+\sum_{h=1}^{n-i}G_{n-i-h}\theta_{i+1,h}^{*}M_{a_{*,i}}=\delta_{n,c}g_0M_{a_{*,i}}, \tag{A.4}$$

where $\alpha_{*,i}$, $\beta_{*,i}$, and $\theta_{i,j}^{*}$ are defined by the rank decomposition of $\Pi'(u)$. We observe that the rank decompositions of $\Pi(u)$ and $\Pi'(u)$ share the same c and the same ranks r_i, with the interchange of the column and row spaces due to transposition, i.e., $\beta_{*,i}=\alpha_i$ and $\alpha_{*,i}=\beta_i$ for $i=0,\dots,c$. Substituting in (A.4) one finds (4.4), and this completes the proof.

$(ii)\Rightarrow(iii)$ Assume (4.3) and (4.4). First fix $n=i=0$ in (4.3) and (4.4) to find $\alpha_0\beta_0'G_0=G_0\alpha_0\beta_0'=0$. Hence Lemma 6.1 applies because $G_0\neq 0$, and one has $G_0=$

$M_{b_1}G_0M_{a_1}$ and $r_0 < p$; see (ii). Next let $n = i+1$ in (4.3) and (4.4). Premultiply the first by $M_{a_{i+1}}$ and postmultiply the second by $M_{b_{i+1}}$ to find

$$M_{a_{i+1}}\theta_{i+1,1}G_0 = \delta_{i+1,c}g_0M_{a_{i+1}} \tag{A.5}$$

and

$$G_0\theta_{i+1,1}M_{b_{i+1}} = \delta_{i+1,c}g_0M_{b_{i+1}}. \tag{A.6}$$

We proceed by induction. For $i = 0$, substituting $G_0 = M_{b_1}G_0M_{a_1}$ in (A.5) and (A.6) one has $M_{a_1}\theta_{1,1}M_{b_1}G_0 = G_0M_{a_1}\theta_{1,1}M_{b_1} = 0$. Hence Lemma A.1 applies, and one has (4.2) for $i = 1$ and $G_0 = M_{b_2}G_0M_{a_2}$. Next we assume (4.2) for $i = \ell < c-1$ and show that the same holds for $i = \ell+1 \le c-1$. Substitute $G_0 = M_{b_{\ell+1}}G_0M_{\ell+1}$ derived from the induction assumption in (A.5) and (A.6) for $i = \ell$; one has $M_{a_{\ell+1}}\theta_{\ell+1,1}M_{b_{\ell+1}}G_0 = G_0M_{a_{\ell+1}}\theta_{\ell+1,1}M_{b_{\ell+1}} = 0$. Hence Lemma A.1 applies, and one has (4.2) for $i = \ell+1$ and $G_0 = M_{b_{\ell+2}}G_0M_{a_{\ell+2}}$. This completes the proof by induction. Next let $n = i = c$ in (A.5) to find $-\alpha_c\beta_c'G_0 = g_0M_{a_c}$. Pre- and postmultiplication by $\bar{\alpha}_c'$ and α_c, respectively, gives $\beta_c'G_0\alpha_c = -g_0I$, and one finds $G_0 = -g_0\bar{\beta}_c\bar{\alpha}_c'$. This shows (4.5). Finally (A.5) for $i = c-1$ implies $P_{\alpha_c}\theta_{c,1}P_{\beta_c} = -\alpha_c\beta_c'$, and one has (4.2) for $i = c$. This completes the proof.

$(iii) \Rightarrow (i)$ Assume that (4.2) holds for $0 \le i \le c$. Hence $(\alpha_0 : \alpha_1 : \cdots : \alpha_c)$ and $(\beta_0 : \beta_1 : \cdots : \beta_c)$ are square nonsingular matrices with orthogonal blocks and $G_0 = -g_0\bar{\beta}_c\bar{\alpha}_c'$. For $i = 0$, (4.2) implies $\Pi_0 = -\alpha_0\beta_0'$ of rank $r_0 < p$. Hence $\det\Pi(u) = (1-u)^m g(u)$ for some $m > 0$ and $g(1) \ne 0$ and $\operatorname{adj}\Pi(u) = (1-u)^a G(u)$ for some $0 \le a < m$ and $G(1) \ne 0$. This implies

$$\Pi(u)G(u) = (1-u)^w g(u)I, \tag{A.7}$$

where $w := m-a > 0$, $G(1) \ne 0$, and $g(1) \ne 0$. We want to show that $w = c$ so that $\Pi(u)^{-1} = \frac{G(u)}{(1-u)^c g(u)}$ has a pole of order c at 1. By applying the same arguments of the $(i) \Rightarrow (ii)$ part of the proof one has

$$-\alpha_i\beta_i'G_{n-i} + M_{a_i}\sum_{h=1}^{n-i}\theta_{i+1,h}G_{n-i-h} = \delta_{n,w}g_0M_{a_i}. \tag{A.8}$$

Next we show that $w < c$ and $w > c$ both lead to a contradiction, so that it must be $w = c$. Suppose $w < c$; (A.8) for $i = n = w$ gives $-\alpha_w\beta_w'G_0 = g_0M_{a_w}$, which is a contradiction because $\beta_w'\beta_c = 0$. Next suppose $w > c$; (A.8) implies $\beta_i'G_0 = 0$ for $i = 0,\ldots,c$, which is a contradiction because $G_0 \ne 0$ and $(\beta_0 : \cdots : \beta_c)$ is nonsingular. Hence $w = c$ so that $\Pi(u)^{-1} = \frac{G(u)}{(1-u)^c g(u)}$ has a pole of order c at 1 because $G(1) \ne 0$ and $g(1) \ne 0$. This completes the proof. ■

Proof of Theorem 4.2. We want to show that for $i = 0,\ldots,c-1$, $\gamma_{c-i-1}(u)$ in (4.7) satisfies the conditions in Lemma 3.1.

First note that

$$(1-u)^k G(u) = \sum_{h=k}^{c-i-1} G_{h-k}(1-u)^h + (1-u)^{c-i}R_k(u), \tag{A.9}$$

where

$$R_k(1) = G_{c-i-k}. \tag{A.10}$$

Next recall that for $i = 0, \dots, c-1$, one has $\beta_i' G_0 = 0$, see (4.5), and

$$\beta_i' G_h = \bar{\alpha}_i' \sum_{k=1}^{h} \theta_{i+1,k} G_{h-k}, \qquad h = 1, \dots, c-i-1, \tag{A.11}$$

see (4.6); hence

$$\beta_i' G(u) = \sum_{h=1}^{c-i-1} \beta_i' G_h (1-u)^h + (1-u)^{c-i} \beta_i' R_0(u) =: a + b \quad \text{(say)}. \tag{A.12}$$

Substituting (A.11) in a and rearranging terms one finds

$$\sum_{h=1}^{c-i-1} \beta_i' G_h (1-u)^h = \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} \sum_{h=k}^{c-i-1} G_{h-k} (1-u)^h. \tag{A.13}$$

From (A.9) one has $\sum_{h=k}^{c-i-1} G_{h-k}(1-u)^h = (1-u)^k G(u) - (1-u)^{c-i} R_k(u)$. Hence (A.13) becomes

$$\sum_{h=1}^{c-i-1} \beta_i' G_h (1-u)^h = \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} (1-u)^k G(u) - (1-u)^{c-i} \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} R_k(u),$$

and (A.12) is rewritten as

$$\beta_i' G(u) = \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} (1-u)^k G(u) + (1-u)^{c-i} \mu_i'(u),$$

where

$$\mu_i'(u) := \beta_i' R_0(u) - \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} R_k(u).$$

By collecting the terms that multiply $G(u)$ on the left-hand side we write

$$\gamma_{c-i-1}'(u) G(u) = (1-u)^{c-i} \mu_i'(u), \tag{A.14}$$

where $\gamma_{c-i-1}'(u) := \beta_i' - \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} (1-u)^k$.

Next we show that $\mu_i'(1) + \psi' G_0$ has full row rank for any $\psi \neq 0$. Using (A.10) one finds

$$\mu_i'(1) = \beta_i' G_{c-i} - \bar{\alpha}_i' \sum_{k=1}^{c-i-1} \theta_{i+1,k} G_{c-i-k}.$$

Substituting $\beta_i' G_{c-i} = \bar{\alpha}_i' \sum_{h=1}^{c-i} \theta_{i+1,h} G_{c-i-h} - g_0 \bar{\alpha}_i'$ derived from (4.6) for $s = c-i$, one has

$$\mu_i'(1) = \bar{\alpha}_i' \theta_{i+1,c-i} G_0 - g_0 \bar{\alpha}_i'. \tag{A.15}$$

Because $G_0 \alpha_i = 0$ for $0 \leq i \leq c-1$ (see eqn. (4.5)), one has $(\mu_i'(1) + \psi' G_0)\alpha_i = -g_0 I$. This shows that $\mu_i'(1) + \psi' G_0$ has full row rank for any $\psi \neq 0$ so that Lemma 3.1 applies.

Hence, for $i = 0, \ldots, c-1$, $\gamma_{c-i-1}(u)$ in (4.7) is a cofraction matrix polynomial. This completes the proof. ∎

Proof of Theorem 5.1. The inverse of the characteristic function of (3.1) is

$$\Pi_c(z)^{-1} = \frac{1}{(1-z)^d} \frac{G(u)}{g(u)}.$$

Because $g(1) \neq 0$, we can expand $\frac{G(u)}{g(u)}$ in $0 < |1-u| < \delta$, for some $\delta > 0$, as

$$\frac{G(u)}{g(u)} = \sum_{n=0}^{\infty} B_n (1-u)^n,$$

where

$$B_n := \sum_{k=0}^{n} G_k c_{n-k} \text{ and } c_n := \frac{1}{n!} \left(\frac{d^n}{du^n} g(u)^{-1} \right) \Big|_{u=1}.$$

Substituting $u = 1-(1-z)^b$ we then have

$$\Pi_c(z)^{-1} = \frac{C_c}{(1-z)^d} + \frac{C_{c-1}}{(1-z)^{d-b}} + \cdots + \frac{C_1}{(1-z)^{d-(c-1)b}} + (1-z)^{cb-d} F(z),$$

where $C_{c-n} := B_n$ for $n = 0, \ldots, c$. This shows (5.2).

Because the roots of $g(u)$ are outside $\mathbb{C}_b$ there is no z in the closed unit disk for which $1-(1-z)^b$ is equal to a root. Thus $F(z)$ is regular on the unit disk and continuous for $|z| \leq 1$, and by Lemma 10 in Johansen (2008) its expansion $F(z) = \sum_{n=0}^{\infty} F_n z^n$, $|z| < 1$ can be used to define the stationary process $Y_t = F(L)\epsilon_t$. The application of the operator $\Pi_c(L)_+^{-1}$ to the equation $\Pi_c(L)X_t = \epsilon_t$ gives the result in (5.1), and $X_t \in \mathcal{F}(d)$ follows from $C_c = \frac{G(1)}{g(1)} \neq 0$.

By Theorem 4.2, the function

$$h_i(u) := (1-z)^{d-(c-i)b} \gamma'_{c-i-1}(u) \Pi_c(z)^{-1} = \frac{\mu'_i(u)}{g(u)}$$

has poles at the roots of $g(u)$ and no singularity at $u = 1$, and it is such that $h_i(1) \neq 0$. Because the roots of $g(u)$ are outside $\mathbb{C}_b$ there is no z in the closed unit disk for which $1-(1-z)^b$ is equal to a root. Thus the compound function

$$f_i(z) := h_i(1-(1-z)^b)$$

is regular on the unit disk, continuous for $|z| \leq 1$, and such that $f_i(1) = h_i(1) \neq 0$. Hence the coefficients of the expansion $f_i(z) = \sum_{n=0}^{\infty} f_n z^n$, $|z| < 1$ can be used to define the $\mathcal{F}(0)$ process $f_i(L)\epsilon_t$ (see Johansen, 2008, Lem. 1.). Then we write

$$\Delta_+^{d-(c-i)b} \gamma'_{c-i-1}(L_b) X_t = f_i(L)_+ \epsilon_t \in \mathcal{F}(0)_+$$

so that

$$\gamma'_{c-i-1}(L_b) X_t \in \mathcal{F}(d-(c-i)b)$$

immediately follows. This completes the proof. ∎