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# Difference Families in $Z_{2d+1} \oplus Z_{2d+1}$ and Infinite Translation Designs in $Z \oplus Z$

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**Abstract.** We analyse 3-subset difference families of $\mathbf{Z}_{2d+1} \oplus \mathbf{Z}_{2d+1}$ arising as reductions (mod $2d + 1$) of particular families of 3-subsets of $\mathbf{Z} \oplus \mathbf{Z}$. The latter structures, namely *perfect d-families*, can be viewed as 2-dimensional analogues of difference triangle sets having the least scope. Indeed, every perfect $d$-family is a set of base blocks which, under the natural action of the translation group $\mathbf{Z} \oplus \mathbf{Z}$, cover all edges $\{(x, y), (x', y')\}$ such that $|x - x'|, |y - y'| \leq d$. In particular, such a family realises a translation invariant $(G, K_3)$-design, where $V(G) = \mathbf{Z} \oplus \mathbf{Z}$ and the edges satisfy the above constraint. For that reason, we regard perfect families as part of the hereby defined *translation designs*, which comprise and slightly generalise many structures already existing in the literature. The geometric context allows some suggestive additional definitions. The main result of the paper is the construction of two infinite classes of $d$-families. Furthermore, we provide two sporadic examples and show that a $d$-family may exist only if $d \equiv 0, 3, 8, 11 \pmod{12}$.

**Key words.** Difference family, (infinite) translation design, scope, perfect family, pure family.

## 1. Introduction and Basics

The decomposition of a graph into copies isomorphic to a given sub-graph is a well-known research topic which has been considered – and is still being – by several authors. In many cases, the marriage between combinatorics and algebra has brought about a definitely better understanding of graph decompositions and, on the other hand, has provided a new, compelling way of looking at some algebraic structures. The large use of *difference families* over the years is an emblematic example of the above interplay.

**Definition 1.1.** *If $S$ is a subset of an abelian group $(H, +)$, let $\Delta S$ stand for $\{s - s' : s, s' \in S, s \neq s'\}$. Every family $\mathcal{F} = \{S_i \subset H : i \in I, |S_i| \text{ constant}\}$ is called a difference family if $\bigcup_{i \in I} \Delta S_i$ (shortly, $\Delta\mathcal{F}$) is equal to $H \setminus \{0\}$ with no repeated element.*

Notice that in the above definition $H$ and $I$ need not be finite. As valuable references on difference families with $H$ finite, we cite [1, 4–6]. The following notion generalises – to the possibly infinite size – one of the contexts where difference families are extensively used.

**Definition 1.2.** *Let $H$, $Q$, $G$ be respectively an abelian group, a graph whose vertex set is equal to $H$, and a sub-graph of $Q$. If $k \in H$ and $L$ is any subgraph of $Q$, let $k + L$ denote the graph $L'$ such that $V(L') = k + V(L)$ and that $\{h, h'\} \in E(L)$ if and only if $\{k + h, k + h'\} \in E(L')$. Finally, let $\simeq$ denote the graph isomorphism relation. A $(Q, H, G)$-translation design is a family $\{A_j : j \in J\}$ of sub-graphs of $Q$ satisfying*

$$I)\ \ A_j \simeq G \text{ for all } j \text{ and } \bigcup_{j \in J} E(A_j) = E(Q) \text{ with no repetition.}$$

$$II)\ \ \text{For any } h \in H \text{ and } j \in J,\ \ h + A_j = A_{j'} \text{ for some } j'.$$

The above definition implies that $h + \varepsilon \in E(Q)$ for all $\varepsilon \in E(Q)$, $h \in H$. In particular, $Q$ must be regular. We remark that $V(Q)$ and $J$ may have infinite size. The next claim provides the – as elementary as basic – recipe for recovering a particular kind of translation design from a suitable difference family.

**Property 1.3.** *Having denoted the complete graph of order $v$ by $K_v$, let $H$ be an abelian group and $\{A_j : j \in J\}$ be a $(K_{|H|}, H, K_v)$-translation design for some $v$. If $\mathcal{F}$ is a difference family of $H$ such that $S \in \mathcal{F}$ implies $S = V(A_j)$ for some $j$, then the families $\{A_j : j \in J\}$ and $\{h + S : h \in H, S \in \mathcal{F}\}$ coincide. Conversely, if $\mathcal{F} = \{S_i \subseteq H : i \in I, |S_i| = v\, \forall i\}$ is a difference family of an abelian group $H$, then the family $\{h + S_i : h \in H, i \in I\}$ is a $(K_{|H|}, H, K_v)$-translation design.*

In keeping with the standard terminology, we call *base blocks* the elements of a difference family. The well-known concept of *cyclic BIBD* (balanced incomplete block design) is a particular case of Definition 1.2, obtained by setting $H = \mathbf{Z}_u$ (the ring of integers (mod $u$)), $Q = K_u$ and $G = K_v$ for some suitable $u, v$. Pioneering works – as well as more recent findings – related to cyclic $BIBD$'s are collected in [4, 5]. Further, we mention [2] as a most recent paper. The weaker notion of *Optical Orthogonal Code* (see e.g. [2, 9, 22]) is obtained by dropping the hypothesis $Q = K_u$ in the above setting. Other classical combinatorial structures, such as for example *cyclic cycle systems* (also known as *cyclic $(K_v, C_k)$-designs*; see e.g. [3, 7, 8, 11, 13–21] for cyclic cycle systems or simply cycle systems), are easily definable in terms of $(Q, H, G)$-translation designs.

In the present paper we shall focus on the base blocks of some particular $(Q, \mathbf{Z} \oplus \mathbf{Z}, K_v)$-translation designs:

**Definition 1.4.** *A set $A$ made up of 3-subsets of $\mathbf{Z} \oplus \mathbf{Z}$ is termed a perfect $d$-family if $\Delta A = [-d, d] \times [-d, d] \setminus \{(0, 0)\}$. If, in addition, every 3-subset generates some nonproportional differences (thus, it generates precisely 3 mutually nonproportional differences), $A$ is termed a pure $d$-family.*

For example, the sets

$$\{(0,0),(1,0),(2,2)\},\{(0,0),(2,0),(3,3)\},\{(0,0),(3,0),(2,3)\},$$

$$\{(0,0),(1,1),(3,2)\},\{(0,0),(0,-1),(2,-2)\},\{(0,0),(0,-2),(3,-3)\},$$

$$\{(0,0),(0,-3),(3,-2)\},\{(0,0),(1,-1),(2,-3)\}$$

form a pure 3-family. Instead, an example of 3-subset that generates proportional differences is $\{(0,0),(2,3),(6,9)\}$ – here $d$ is at least 9. Every perfect $d$-family yields a difference family in $\mathbf{Z}_{2d+1} \oplus \mathbf{Z}_{2d+1}$. The required base blocks are in fact obtained by reducing (mod $2d+1$) the elements of every 3-subset. In that sense, perfect $d$-families can be viewed as 2-dimensional analogues of *difference triangle sets* [10] made up of $n$ 3-subsets and having the least *scope* [2] subject to $n$ fixed. In details, a difference triangle set is a family $\mathcal{F}$ of $n$ subsets of $\mathbf{Z}$ of the same size $k$, such that $\Delta\mathcal{F}$ is not a multiset. The scope of $\mathcal{F}$ is the maximum of $\Delta\mathcal{F}$. If the scope is the smallest admissible, subject to $n$ and $k$ fixed (thus, if it is equal to $k(k-1)n/2$), then reducing (mod $k(k-1)n+1$) yields a so-termed *perfect* difference family [2] which generates a cyclic $(k(k-1)n+1, k, 1)$-design. In the present 2-dimensional context, we define the *scope* of a family $\mathcal{F}$ of $n$ 3-subsets as the largest component occurring in a pair, over all pairs of $\Delta\mathcal{F}$. A little calculation can show that this number is not smaller than $\lceil(\sqrt{6n+1}-1)/2\rceil$. As a consequence, a family $\mathcal{F}$ is a perfect family if and only if its scope is equal to $(\sqrt{6n+1}-1)/2$ and $\Delta\mathcal{F}$ is not a multiset.

Alternatively, a perfect $d$-family can be regarded as a set of base blocks for a $(Q_d, \mathbf{Z} \oplus \mathbf{Z}, K_3)$-translation design such that

$$E(Q_d) = \{\{(x,y),(x',y')\} : |x-x'|, |y-y'| \le d, (x,y) \ne (x',y')\}.$$

In geometrical terms, the $((2d+1)^2-1)/6$ subsets of a perfect $d$-family can be represented as affine triangles, none of which is degenerate if and only if the family is pure, and whose edges correspond to all possible nonzero vectors (up to the sign) with integral entries ranging in $[-d, d]$. Figure 1 deals with the previously defined 3-family.

The necessary condition $3|2d^2+2d$, immediately arising from the above discussion, flows into the stronger

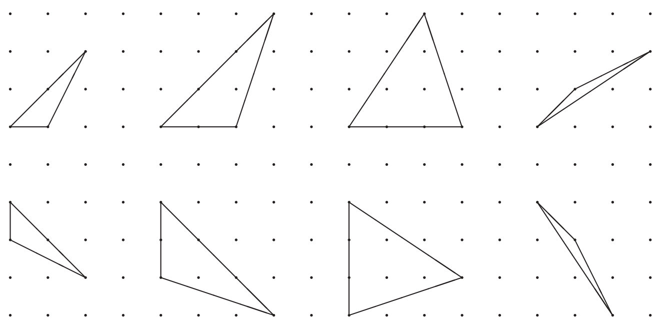

**Fig. 1.** Geometric interpretation of a pure 3-family

**Proposition 1.5.** *If a perfect $d$-family exists, then $d \equiv 0, 3, 8, 11$ (mod 12).*

*Proof.* For each subset $S$ of the family let us select three pairs $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, among the six of $\Delta S$, such that $x_i \geq 0$ for all $i$ and, if $x_i = 0$ for all $i$, such that $y_i > 0$ for all $i$. Assuming that $x_2 \leq x_1 \geq x_3$ and, if $x_i = 0$ for all $i$, that $y_2 < y_1 > y_3$, a geometrical argument yields with few difficulties $(x_1, y_1) = (x_2, y_2) + (x_3, y_3)$. As a consequence, the sum of these three pairs is a vector whose entries are even, and the same property still holds, clearly, when reversing the sign of all negative entries $y_i$ and summing the selected pairs over all the subsets of the family. However, by summing up all the above pairs in a different fashion, one realises that the resulting entries are even if and only if $d(d+1)/2$ is even. Indeed, the contribute of $(x, y)$ is double and hence negligible when $xy \neq 0$; instead, the remaining $2d$ pairs $\{(x, 0), (0, x) \colon 1 \leq x \leq d\}$ must be such that $\sum_{1\ lex \leq d} x\ (= d(d+1)/2)$ is even. The constraint on $d$, just obtained, can be rephrased as $d \equiv 0, 3 \pmod 4$. Now we observe that the former constraint $3 | 2d^2 + 2d$ is equivalent to $d \equiv 0, 2 \pmod 3$. By applying the Chinese Remainder Theorem to the two congruences, we reach the end of the proof. □

In the next section we exhibit a perfect, not pure, $d$-family for every $d \in \{12 \cdot 2^n : n \geq 0\} \cup \{20 \cdot 4^n : n \geq 0\} \cup \{8\}$. Thus, we settle the existence question for infinitely many instances of the forms $d \equiv 0$ and $d \equiv 8 \pmod{12}$. Our constructions avail of Skolem sequences. In the last section we draw some brief conclusions, by also providing a (so far) sporadic example of pure 8-family.

## 2. Two Infinite Classes of Perfect $d$-families

In this section we establish the following results.

**Theorem 2.1.** *For every $n \geq 0$ there exists a perfect, not pure, $12 \cdot 2^n$-family.*

**Theorem 2.2.** *For every $n \geq 0$ there exists a perfect, not pure, $20 \cdot 4^n$-family. Furthermore, there exists a perfect, not pure, 8-family.*

Before proving the above claims, we introduce two classical notions.

**Definition 2.3.** *Let $t$ be a positive integer. A Skolem sequence of order $t$ is a set of couples $\{\{a_i, b_i\} \colon 1 \leq i \leq t\}$ such that $\bigcup_{1 \leq i \leq t} \{a_i, b_i\} = \{1, 2, \ldots, 2t\}$ and $b_i - a_i = i$ for all $i$. A hooked Skolem sequence of order $t$ is a similar set of couples such that $\bigcup_{1 \leq i \leq t} \{a_i, b_i\} = \{1, 2, \ldots, 2t-1, 2t+1\}$ and $b_i - a_i = i$ for all $i$.*

For example, the couples $\{1, 2\}$, $\{4, 6\}$, $\{5, 8\}$, $\{3, 7\}$ and the couples $\{1, 2\}$, $\{3, 5\}$ realise a Skolem sequence and a hooked Skolem sequence of order 4 and 2 respectively. The following well-known result – a proof of which can be found e.g. in [4] – is crucial for our purposes.

**Proposition 2.4.** *A Skolem sequence of order $t$ exists if and only if $t \equiv 0, 1$ (mod 4). A hooked Skolem sequence of order $t$ exists if and only if $t \equiv 2, 3$ (mod 4).*

We are now ready for the proof of the two claimed results.

*Proof of Theorem 2.1.* Let us denote $2^n$ by $M$. Assuming that every 3-block of the family contains $(0, 0)$, we describe the family under construction by means of couples $\{(x_1, y_1), (x_2, y_2)\}$, instead of terns $\{(0, 0), (x_1, y_1), (x_2, y_2)\}$. In order to better explain the idea lying behind the next definitions, we let $\phi(x, y)$ stand for $(y, -x)$. The symbol $\phi$ should be interpreted as the 90-degree clockwise rotation of a vector. To begin with, we introduce the family

$$\mathcal{A} = \bigcup_{\substack{1 \leq i \leq 12M-1 \\ i \neq 8M}} \{\{(12M, i), (i, 12M)\}, \{\phi(12M, i), \phi(i, 12M)\}\}.$$

Clearly,

$$\Delta\mathcal{A} = \bigcup_{\substack{1 \leq i \leq 12M-1 \\ i \neq 8M}} \{(\pm 12M, \pm i), (\pm i, \pm 12M), (\pm(12M - i), \pm(12M - i))\},$$

where the sign choices are assumed to be independent and, therefore, give rise to 12 differences for each $i$. Let us now consider a Skolem sequence $\{\{a_i, b_i\} \colon 1 \leq i \leq 4M\}$. We define

$$\mathcal{B} = \bigcup_{1 \leq i \leq 4M} \{\{(4M + a_i, 0), (4M + b_i, 0)\}, \{\phi(4M + a_i, 0), \phi(4M + b_i, 0)\}\},$$

thus obtaining

$$\Delta\mathcal{B} = \bigcup_{1 \leq i \leq 12M} \{(\pm i, 0), (0, \pm i)\}.$$

Then we define

$$\mathcal{C} = \bigcup_{\substack{6M+1 \leq j \leq 12M-1 \\ 12M - j \leq i \leq 6M-1}} \{\{(\pm i, j), (\pm j, 12M - 1 - i)\}, \{\phi(\pm i, j), \phi(\pm j, 12M - 1 - i)\}\},$$

where the sign choices for each couple are assumed to coincide (thus, only 4 couples are generated for any fixed $j, i$). It follows that $\Delta\mathcal{C}$ contains the differences

$$\bigcup_{\substack{6M+1 \leq j \leq 12M-1 \\ 12M - j \leq i \leq 6M-1}} \{(\pm(j - i), \pm(i + j + 1 - 12M)), (\pm(i + j + 1 - 12M), \pm(j - i))\}$$

(for any choice of the $\pm$'s) and the differences related to the edges passing through $(0, 0)$, which the reader can recover from the definition of $\mathcal{C}$. No further difference is produced by $\mathcal{C}$. Let us remove from the above family the couples $\{(\pm 4M, 8M), (\pm 8M, 8M - 1)\}$ and $\{\phi(\pm 4M, 8M), \phi(\pm 8M, 8M - 1)\}$. The resulting family, which we denote by $\tilde{\mathcal{C}}$, is the one actually employed. The lost differences shall be generated in the end. Now we introduce the families

$$\mathcal{D}_m = \bigcup_{1 \le i \le j \le 3\frac{M}{m}-1} \{\{(\pm m(2i-1), m(2j+1)), (\pm(12M - m(2j+1)), m(2i-1))\},$$

$$\{\phi(\pm m(2i-1), m(2j+1)), \phi(\pm(12M - m(2j+1)), m(2i-1))\}\},$$

where $m$ ranges over all powers of 2 of the interval $[1, M]$, and the sign choices for each couple are assumed to coincide. We have that $\Delta\mathcal{D}_m$ contains

$$\bigcup_{1 \le i \le j \le 3\frac{M}{m}-1} \{(\pm(12M - 2m(i+j)), \pm 2m(j-i+1)),$$

$$(\pm 2m(j-i+1), \pm(12M - 2m(i+j)))\}$$

(for any choice of the $\pm$'s) and, as above, the remaining differences of $\Delta\mathcal{D}_m$ correspond to all edges passing through $(0, 0)$.

The next couples will fill the gaps left by the above families. Before defining them, let us summarise how the already existing couples work. Notice that if any difference $(x, y)$ with $x, y \ge 0$ has been generated, than the same is true more generally for $(\pm x, \pm y)$ and $(\pm y, \pm x)$. In the following discussion we assume, therefore, that the differences $(x, y)$ are such that $0 \le x \le y \le 12M$. It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ generate all the admissible differences $(x, y)$ satisfying either $x = 0$, or $y = 12M$, or $x = y$, except $(4M, 4M)$, $(8M, 12M)$ and $(12M, 12M)$. Concerning $\mathcal{C}$, the related differences are characterised either by $x \ne y \ne 12M$ and $x + y \ge 12M$, or by $x \ne 0$, $x + y$ odd and $x + y \le 12M - 1$, as it can be realised with few difficulties. However, from these differences we must remove $(1, 4M)$, $(4M, 8M)$, $(8M - 1, 8M)$, as prescribed. Finally, the differences arising from each $\mathcal{D}_m$ are precisely those satisfying $x + y \le 12M - 2$ and either $x, y \equiv m \pmod{2m}$ or $(x, y \equiv 0 \pmod{2m}, x + y \not\equiv 0 \pmod{4m})$. In Fig. 2 we have outlined the case $M = 4$. The $x$ axis is vertical and the $y$ entry increases from right to left. The triangle boundary is covered by $\Delta\mathcal{A}$ and $\Delta\mathcal{B}$ with the exception of the three diamonds, whereas the interior lines pass through all points covered by $\Delta\mathcal{C}$. Each vertical line, together with the corresponding oblique line at the same position from the left, refers to some fixed $j$ and is parametrised by $i$. In particular, the two bolder lines refer to $j = 44$. Dots, circles and squares are covered by $\mathcal{D}_m$ with $m = 1, 2, 4$ respectively. In particular, the larger dots and the bolder circles refer to $m = 1, i = 2$ and to $m = 2, i = 2$ respectively. Notice that, besides the six differences removed along the way, the only difference to be still generated is $(2M, 6M)$. Indeed, $\Delta\mathcal{D}_M$ does not cover the unique pair of the form $(x = 2M + 4\alpha M, y = 2M + 4\beta M)$ subject to $x + y \le 12M - 2$. It easily follows that the family

$$\begin{aligned}\mathcal{Z} = \{&\{(\pm 1, 4M), (\mp(8M-1), -8M)\}, \{\phi(\pm 1, 4M), \phi(\mp(8M-1), -8M)\},\\ &\{(2M, 6M), (6M, 2M)\}, \{\phi(2M, 6M), \phi(6M, 2M)\},\\ &\{(8M, 4M), (-4M, -8M)\}, \{\phi(8M, 4M), \phi(-4M, -8M)\}\}\end{aligned}$$

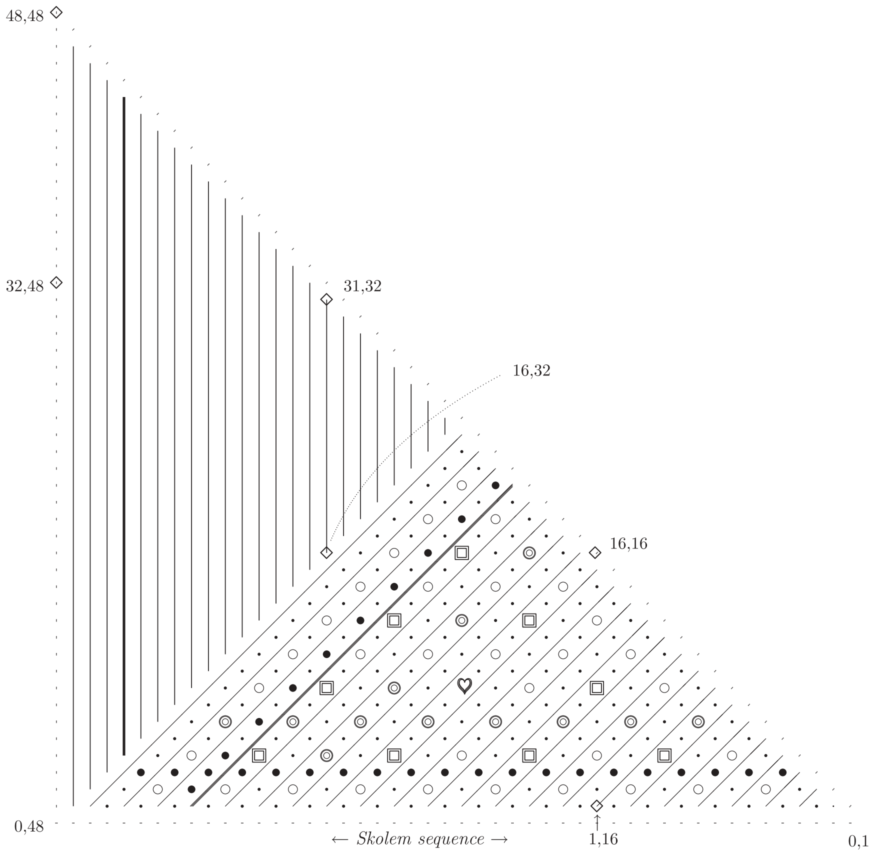


**Fig. 2.** The case $M = 4$

is what we need to conclude the construction (the diamonds and the heart in Fig. 2 represent the 6+1 last differences). The entire family is, therefore, $\mathcal{A} \cup \mathcal{B} \cup \tilde{\mathcal{C}} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \ldots \cup \mathcal{D}_{2^t} \cup \ldots \cup \mathcal{D}_M \cup \mathcal{Z}$. □

As the proof of the next theorem has much in common with the above proof, we allow ourselves to provide a more succinct explanation of the techniques employed.

*Proof of Theorem 2.2.* Let us denote $4^n$ by $M$. We start with establishing the former part of the claim. The required sub-families are defined as follows.

$$\mathcal{A} = \bigcup_{\substack{1 \le i \le 20M-1 \\ i \ne 2}} \{\{(20M, i), (i, 20M)\}, \{\phi(20M, i), \phi(i, 20M)\}\}.$$

We have that

$$\Delta\mathcal{A} = \bigcup_{\substack{1 \le i \le 20M-1 \\ i \ne 2}} \{(\pm 20M, \pm i), (\pm i, \pm 20M), (\pm(20M - i), \pm(20M - i))\}.$$

Because $(20M-2) \equiv 6 \pmod{12}$, it follows that $(20M-2)/3 \equiv 2 \pmod 4$. Let us denote this last fraction by $\mu$, and choose a hooked Skolem sequence $\{\{a_i, b_i\} : 1 \leq i \leq \mu\}$. We define

$$\mathcal{B} = \bigcup_{1 \leq i \leq \mu} \{\{(\mu + a_i, 0), (\mu + b_i, 0)\}, \{\phi(\mu + a_i, 0), \phi(\mu + b_i, 0)\}\},$$

thus obtaining

$$\Delta\mathcal{B} = \bigcup_{\substack{1 \leq i \leq 20M-1 \\ i \neq 20M-2}} \{(\pm i, 0), (0, \pm i)\}.$$

The next family is

$$\mathcal{C} = \bigcup_{\substack{10M+1 \leq j \leq 20M-1 \\ 20M-j \leq i \leq 10M-1}} \{\{(\pm i, j), (\pm j, 20M-1-i)\}, \{\phi(\pm i, j), \phi(\pm j, 20M-1-i)\}\},$$

where the sign choices must coincide. Therefore, $\Delta\mathcal{C}$ contains the differences

$$\bigcup_{\substack{10M+1 \leq j \leq 20M-1 \\ 20M-j \leq i \leq 10M-1}} \{(\pm(j-i), \pm(i+j+1-20M)), (\pm(i+j+1-20M), \pm(j-i))\}$$

(for any choice of the $\pm$'s) and the differences related to the edges passing through $(0, 0)$. In the present proof, some prescribed couples shall be removed from the family $\mathcal{D}_1$ (still to define) rather than from $\mathcal{C}$. We thus proceed with defining

$$\mathcal{D}_m = \bigcup_{1 \leq i \leq j \leq 5\frac{M}{m}-1} \{\{(\pm m(2i-1), m(2j+1)), (\pm(20M - m(2j+1)), m(2i-1))\},$$
$$\{\phi(\pm m(2i-1), m(2j+1)), \phi(\pm(20M - m(2j+1)), m(2i-1))\}\},$$

where $m$ ranges in $\{2^q : 0 \leq q \leq n\}$ and the sign choices for each couple are assumed to coincide. It follows that $\Delta\mathcal{D}_m$ is made up of the set

$$\bigcup_{1 \leq i \leq j \leq 5\frac{M}{m}-1} \{(\pm(20M - 2m(i+j)), \pm 2m(j-i+1)),$$
$$(\pm 2m(j-i+1), \pm(20M - 2m(i+j)))\}$$

(for any choice of the $\pm$'s) and of the further differences related to all edges passing through $(0, 0)$. We turn $\mathcal{D}_1$ into $\tilde{\mathcal{D}}_1$ by removing the couples $\{(\pm 1, 3), (\pm(20M - 3), 1)\}$, $\{\phi(\pm 1, 3), \phi(\pm(20M-3), 1)$. Then we add the family

$$\mathcal{E} = \{\{(\pm 2M, 6M), (\pm 14M, 2M)\}, \{\phi(\pm 2M, 6M), \phi(\pm 14M, 2M)\},$$

$$\{(\pm 2M, 10M), (\pm 10M, 6M)\}, \{\phi(\pm 2M, 10M), \phi(\pm 10M, 6M)\}\}$$

and the final family

$$\mathcal{Z} = \{\{(0, 20M-2), (20M, 20M)\}, \{\phi(0, 20M-2), \phi(20M, 20M)\},$$

$$\{(20M, -2)(20M-2, 2-20M)\}, \{\phi(20M, -2), \phi(20M-2, 2-20M)\},$$

$$\{(-2, 20M-4), (1, 20M-3)\}, \{\phi(-2, 20M-4), \phi(1, 20M-3)\},$$

$$\{(1, 3-20M), (1, 3)\}, \{\phi(1, 3-20M), \phi(1, 3)\}\}.$$

Reasoning as in the above proof, it could be checked that all these sub-families behave together in the expected way. Consequently, $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \tilde{\mathcal{D}}_1 \cup \mathcal{D}_2 \cup \ldots \cup \mathcal{D}_{2^t} \cup \ldots \cup \mathcal{D}_M \cup \mathcal{E} \cup \mathcal{Z}$ is the required $20M$-family (notice that the $\heartsuit$ gap of the previous theorem is replaced by a 6-difference gap, which $\mathcal{E}$ completely fills).

The latter claim of the theorem can be regarded as an extension of the former, by setting $M = 2/5$ and making the due adaptations. In more details, we define $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{Z}$ as in the general case. We do not need any further couple (with little effort, the reader may indeed realise that the current 6-difference gap corresponds to $\mathcal{D}_1 \setminus \tilde{\mathcal{D}}_1$, where $\tilde{\mathcal{D}}_1$ is empty). $\square$

## 3. A Pure Family and Some Remarks

Although we regard Skolem sequence as extremely noble objects, in any context might they be employed, we must point out that the families $\mathcal{B}$ of the previous section – arising from Skolem sequences possibly hooked – are the very ones which prejudice pureness. In order to construct a pure family one conceivably needs to give up Skolem sequences and other similar structures which, on the contrary, reveal so useful in the one-dimensional case. Unfortunately, besides the pure 3-family exhibited in the Introduction, we have so far succeeded in constructing no pure $d$-family but the following, with $d = 8$ (as usual, couples replace terns):

$$\mathcal{F} = \mathcal{F}_0 \bigcup \phi(\mathcal{F}_0), \text{ with}$$

$$\mathcal{F}_0 = \{\{(1,0), (6,8)\}, \{(2,0), (-3,8)\}, \{(3,0), (3,8)\},$$
$$\{(4,0), (-4,4)\}, \{(5,0), (7,8)\}, \{(6,0), (-1,8)\}, \{(7,0), (8,8)\},$$
$$\{(-2,8), (3,3)\}, \{(-4,8), (2,2)\}, \{(-6,8), (1,1)\},$$
$$\{(\pm 7,1), (\pm 6,7)\}, \{(\pm 7,2), (\pm 5,7)\}, \{(\pm 7,3), (\pm 4,7)\},$$
$$\{(\pm 6,2), (\pm 5,6)\}, \{(\pm 6,3), (\pm 4,6)\}, \{(\pm 5,3), (\pm 4,5)\}, \{(\pm 1,3), (\pm 5,1)\}\},$$

where the $\pm$'s of each couple must coincide. By plotting the absolute values of the involved differences as already done in the above section, the interior of the resulting 48-point triangle is easily seen to be filled up in almost the same way as in the construction of Theorem 2.2. Instead, the boundary elements are grouped in a rather peculiar way, which in our opinion seems difficult to generalise (in particular, thereis no interaction between the boundary and the interior). Also the pure

3-family defined at the beginning of this paper seems a hard starting point for constructing an infinite class, mainly because the amount of information it provides is quite scanty.

Needless to say, it is highly desirable to settle the existence question for all admissible cases (mod 12), by exhibiting both pure and nonpure families. Some tricky constructions might be required that possibly avoid the operator $\phi$. Another challenging problem, of a different nature, might be that of finding an infinite family $\mathcal{F}_n$ of perfect (or pure) $d_n$-families with increasing $d_n$, such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all $n$, so as to yield an infinite limit construction and, consequently, a difference family in $\mathbf{Z} \oplus \mathbf{Z}$. Finally, we emphasise that these questions are not endemic of 3-subsets. For, it is clear that the above concepts can be extended to cycle systems, polygon systems and many other combinatorial entities.

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