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A convergent scheme for a non local Hamilton Jacobi equation modelling dislocation dynamics

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Abstract We study dislocation dynamics with a level set point of view. The model we present here looks at the zero level set of the solution of a non local Hamilton Jacobi equation, as a dislocation in a plane of a crystal. The front has a normal speed, depending on the solution itself. We prove existence and uniqueness for short time in the set of continuous viscosity solutions. We also present a first order finite difference scheme for the corresponding level set formulation of the model. The scheme is based on monotone numerical Hamiltonian, proposed by Osher and Sethian. The non local character of the problem makes

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it not monotone. We obtain an explicit convergence rate of the approximate solution to the viscosity solution. We finally provide numerical simulations.

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1 Introduction

The object of this article is to prove the convergence for a first order finite difference scheme that approximates the solution of a non local Hamilton Jacobi equation, describing a model for dislocation dynamics.

The model looks at a dislocation (see [4] for a physical presentation of the model for dislocation dynamics) in a 2D plane as the zero level set of a continuous function u, solving the following equation:

$$\begin{cases} u_t(x,y,t) = \big(c^0 \star [u](x,y,t)\big)\, |\nabla u(x,y,t)| & \mathbb{R}^2 \times (0,\overline{T}), \\ u(x,y,0) = u^0(x,y) & \mathbb{R}^2, \end{cases} \tag{1}$$

where $[u]$ is the characteristic function of the set $\{u \geq 0\}$, defined by

$$[u] = \begin{cases} 1 & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases} \tag{2}$$

Here ∇ indicates the gradient with respect to the spatial variables, the kernel $c^0(x,y)$ depends only on the space variables and $\star$ denotes the convolution in space. Another level set model that allows for dislocation loops and intersections of dislocation lines (operation of FrankRead sources, cross-slip and climb) was proposed in [13].

The equation

$$\begin{cases} \rho_t(x,y,t) = \big(c^0 \star \rho(x,y,t)\big)\, |\nabla \rho(x,y,t)| & \mathbb{R}^2 \times (0,\overline{T}), \\ \rho(x,y,0) = \rho^0(x,y) & \mathbb{R}^2, \end{cases}$$

has been studied in the case of *discontinuous viscosity solutions* ρ in [1,4], where short time existence and uniqueness results are proved under certain conditions. This equation is related to (1) since one expects that $\rho = [u]$ for $\rho^0 = [u^0]$.

The paper is organized as follows. In Sect. 2 we present the approximation scheme for the problem (1). In Sect. 3 we present our main results: existence of continuous solution of the problem (1) and convergence of the corresponding approximation solution. These results are respectively proved in Sects. 6 and 7. In Sect. 4 and 5 we establish auxillary regularity results respectively for the continuous and discrete problem. In Sect. 8 we propose some numerical simulations.

2 Setting of the problem

We build a first order finite difference scheme that uses a monotone numerical Hamiltonian for the norm of the spatial gradient, the discrete convolution for the non local speed and forward Euler scheme for the time derivative.

Given a mesh size $\Delta x, \Delta y, \Delta t$ and a lattice $Q_{\mathrm{d}}^{\overline{T}} = Q_{\mathrm{d}} \times \{0, \ldots, (\Delta t)N_{\overline{T}}\}$ where $Q_{\mathrm{d}} = \left\{(i\Delta x, j\Delta y) : (i,j) \in \mathbb{Z}^2\right\}$ and $N_{\overline{T}}$ is the integer part of $\overline{T}/\Delta t$, we will denote with (x_i, y_j, t_n) the node $(i\Delta x, j\Delta y, n\Delta t)$ and with $v^n = (v^n_{i,j})_{i,j}$ the values of the numerical approximation of the exact solution $u(x_i, y_j, t_n)$. We set $\Delta X = (\Delta x, \Delta y)$ so that its Euclidean norm $|\Delta X|$ is the space mesh size. We shall assume throughout that $|\Delta X| \leq 1$ and $\Delta t \leq 1$.

The main difficulty with (1) is clearly due to the dependence of the velocity on the solution itself. This requires the availability of the solution we are intending to approximate. We solve this problem by fixing the solution on each time step $[t_n, t_{n+1}]$ and we apply a monotone scheme S:

$$\begin{cases} v^{n+1}_{i,j} = S([v^n], v^n, i, j) & n = 0, \ldots, N_{\overline{T}} - 1, \\ v^0_{i,j} = u^0(x_i, y_j). \end{cases} \tag{3}$$

The scheme S is an explicit marching scheme:

$$S([v^n], v^n, i, j) = v^n_{i,j} + \Delta t H_{\mathrm{d}}([v^n], D^+_x v^n_{i,j}, D^+_y v^n_{i,j}, D^-_x v^n_{i,j}, D^-_y v^n_{i,j}, i, j), \tag{4}$$

where

$$[v^n]_{i,j} = \begin{cases} 1 & \text{if } v^n_{i,j} \geq 0, \\ 0 & \text{if } v^n_{i,j} < 0, \end{cases} \tag{5}$$

and the discrete numerical Hamiltonian reads:

$$H_{\mathrm{d}}([v^n], D^+_x v^n_{i,j}, D^+_y v^n_{i,j}, D^-_x v^n_{i,j}, D^-_y v^n_{i,j}, i, j) = \begin{cases} c_{i,j}([v^n])E^+ & \text{if } c_{i,j}([v^n]) \geq 0, \\ c_{i,j}([v^n])E^- & \text{if } c_{i,j}([v^n]) < 0. \end{cases} \tag{6}$$

Here E^+, E^- are numerical monotone Hamiltonians that approximate the Euclidean norm. For concreteness, we shall take those proposed by Osher and Sethian in [10]:

$$E^+ = \left\{\max(D^+_x v^n_{i,j}, 0)^2 + \max(D^+_y v^n_{i,j}, 0)^2 + \min(D^-_x v^n_{i,j}, 0)^2 + \min(D^-_y v^n_{i,j}, 0)^2\right\}^{\frac{1}{2}},$$
$$E^- = \left\{\min(D^+_x v^n_{i,j}, 0)^2 + \min(D^+_y v^n_{i,j}, 0)^2 + \max(D^-_x v^n_{i,j}, 0)^2 + \max(D^-_y v^n_{i,j}, 0)^2\right\}^{\frac{1}{2}}.$$

$D^+_x v^n_{i,j}$ and $D^-_x v^n_{i,j}$ are the standard forward and backward first difference, i.e. for a general function $f_{i,j}$:

$$D_x^+ f_{i,j} = \frac{f_{i+1,j} - f_{i,j}}{\Delta x},$$
$$D_x^- f_{i,j} = \frac{f_{i,j} - f_{i-1,j}}{\Delta x},$$
$$D_y^+ f_{i,j} = \frac{f_{i,j+1} - f_{i,j}}{\Delta y},$$
$$D_y^- f_{i,j} = \frac{f_{i,j} - f_{i,j-1}}{\Delta y}.$$

The non-local velocity in (6) is the discrete convolution

$$c_{i,j}([v^n]) = \sum_{l,m\in\mathbb{Z}} \overline{c}^{\,0}_{i-l,j-m}[v^n]_{l,m}\Delta x\Delta y, \tag{7}$$

with

$$\overline{c}^{\,0}_{i,j} = \frac{1}{|Q_{i,j}|}\int_{Q_{i,j}} c^0(x,y)\mathrm{d}x\mathrm{d}y \tag{8}$$

where $Q_{i,j}$ is the square cell

$$Q_{i,j} = [x_i - \Delta x/2, x_i + \Delta x/2] \times [y_j - \Delta y/2, y_j + \Delta y/2].$$

3 Main results

3.1 Notations

To state our result, we need some notations.

$\mathrm{BV}(\mathbb{R}^2)$ is the space of functions on $\mathbb{R}^2$ with bounded variation:

$$\mathrm{BV}(\mathbb{R}^2) = \{u : \mathbb{R}^2 \to \mathbb{R} : |u|_{\mathrm{BV}(\mathbb{R}^2)} < \infty\},$$

for $|u|_{\mathrm{BV}(\mathbb{R}^2)} = |u|_{L^1(\mathbb{R}^2)} + |Du|_{M(\mathbb{R}^2)}$ where $|\mu|_{M(\mathbb{R}^2)} = |\mu|(\mathbb{R}^2)$ designates the total variation of the measure μ. We also denote by $\mathrm{Lip}(\mathbb{R}^2)$, resp. $\mathrm{Lip}(\mathbb{R}^2 \times [0,\overline{T}))$, the set of the globally Lipschitz functions in space, respectively in space and time. The functions can be unbounded, but they will have at most linear growth.

Following [3], given a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, we define the quantities

$$|u|_{L^1_{\mathrm{unif}}(\mathbb{R}^2)} = \sup_{(x,y)\in\mathbb{R}^2} \int_{Q(x,y)} |u|, \quad |u|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |u|_{L^\infty(Q(x,y))},$$

where $Q(x,y)$ is the unit square centred at (x,y):

$$Q(x,y) = \left\{ (x',y') \in \mathbb{R}^2 : |x - x'| \leq \frac{1}{2}, |y - y'| \leq \frac{1}{2} \right\}.$$

We denote respectively by $L^1_{\text{unif}}(\mathbb{R}^2)$ and $L^\infty_{\text{int}}(\mathbb{R}^2)$ the space that consists of the functions for which these quantities are finite.

We will make use of the following inequality (see [3] for the proof):

Lemma 3.1 (Convolution inequality) *For every $f \in L^\infty_{int}(\mathbb{R}^2)$ and $g \in L^1_{unif}(\mathbb{R}^2)$, the convolution product $f \star g$ is bounded and satisfies*

$$|f \star g|_{L^\infty(\mathbb{R}^2)} \leq |f|_{L^\infty_{int}(\mathbb{R}^2)} |g|_{L^1_{unif}(\mathbb{R}^2)}.$$

For a function $u(x,y,t)$, we denote with L_u the Lipschitz constant of u with respect to (x,y) and with L'_u the Lipschitz constant of u with respect to t, i.e.

$$|u(x,y,t) - u(x',y',t')| \leq L_u |(x,y) - (x',y')| + L'_u |t - t'|$$

for all x, x', y, y', t, t'.

Functions in (x,y,t) on the lattice Q^T_{d} (with nodes $(x_i, y_j, t_n) = (i\Delta x, j\Delta y, n\Delta t)$) are written indifferently $f^n_{i,j}$ or f^n.

We denote by L_{f^n} the spatial Lipschitz constant of f^n:

$$\left| f^n_{i,j} - f^n_{l,m}| \leq L_{f^n} |(x_i, y_j) - (x_l, y_m) \right|$$

for all x_i, y_j, x_l, y_m. We will also use the discrete 1-norm defined by:

$$|f|_1 = \sum_{i,j} |f(x_i, y_j)| \Delta x \Delta y.$$

3.2 Existence and convergence

We state our existence and uniqueness result for the problem (1), whose natural framework is the theory of continuous viscosity solutions (see for instance [5, 6, 8]).

Theorem 1 (Short time existence and uniqueness) *Let $u^0 \in Lip(\mathbb{R}^2)$ satisfying*

$$\frac{\partial u^0}{\partial y}(x,y) > b > 0 \quad in\ \mathbb{R}^2\ a.e. \tag{9}$$

Let c^0 verify

$$c^0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV(\mathbb{R}^2). \tag{10}$$

Then there exists

$$T^* = \min\left\{ \frac{1}{\left|c^0\right|_{BV(\mathbb{R}^2)}} \ln\left(1 + \frac{b}{2L_{u^0}}\right), \quad \frac{b}{L_{u^0}} \frac{1}{16|c^0|_{L^\infty_{int}(\mathbb{R}^2)}} \right\}$$

for which there exists a unique viscosity solution $u(x,y,t) \in Lip(\mathbb{R}^2 \times [0,T^*))$ *of the problem* (1) *in* $\mathbb{R}^2 \times [0,T^*)$. *The solution verifies*

$$|\nabla u(x,y,t)| < 2L_{u^0} \quad on\ \mathbb{R}^2 \times [0,T^*)\ a.e. \tag{11}$$

$$\frac{\partial u}{\partial y}(x,y,t) > b/2 > 0 \quad on\ \mathbb{R}^2 \times [0,T^*)\ a.e. \tag{12}$$

Our second main result is a convergence result for the scheme (3) to the solution of the problem (1), with an estimate of the rate of convergence. It turns out that we can obtain the same rate as the one for fronts with a velocity that is independent of u.

Theorem 2 (Numerical error estimate) *Let us consider the viscosity solution* $u(x,y,t)$ *of* (1) *on* $\mathbb{R}^2 \times [0,T^*)$, *with initial condition* u^0 *satisfying the assumptions of Theorem 1. Let* $v^n_{i,j}$ *be the numerical solution of the scheme* (3). *Assume that the time step* Δt *satisfies*

$$\Delta t = \lambda_x \Delta x, \quad \Delta t = \lambda_y \Delta y \tag{13}$$

with λ_x, λ_y *positive constant such that*

$$0 < \lambda_x, \lambda_y \leq \frac{1}{2\sqrt{2}|c^0|_{L^1}}.$$

Then there exist a time $0 < T^*_d < T^*$ *and a positive constant C, depending only on* λ_x, λ_y, $|c^0|_{L^\infty_{int}}$, $|c^0|_{BV}$, $|\nabla u^0|_{L^\infty}$, b (*where b is defined in Theorem 1*) *such that*

$$\sup_{i,j\in\mathbb{Z}} |u(x_i,y_j,n\Delta t) - v^n_{i,j}| \leq C\,|\Delta t|^{\frac{1}{2}}, \quad n = 0, \ldots N_{T^*_d}. \tag{14}$$

Remark 3.1 We could actually use the CFL estimate

$$0 < \lambda_x, \quad \lambda_y \leq \frac{1}{2\sqrt{2}\sup_{i,j}|c_{i,j}([v^n])|} \tag{15}$$

that corresponds to a non uniform time step or the estimate

$$0 < \lambda_x, \quad \lambda_y \leq \frac{1}{2\sqrt{2}|\bar{c}^{\,0}|_1}$$

that is uniform in time but still depends on the mesh. This CFL conditions are weaker than the one given in the statement of the Theorem by virtue of the inequalities

$$\sup_{i,j} |c_{i,j}([v^n])| \leq |\overline{c}^{\,0}|_1 \leq |c^0|_{L^1}.$$

Our numerical schemes will freely use these refined conditions.

Remark 3.2 We can choose H_{d} as any monotone and consistent numerical Hamiltonian; for instance you can replace the Osher–Sethian with the one proposed by Rouy and Tourin [11].

Remark 3.3 F. Sabac has proved in [12] that the rate of convergence 1/2 in L^1 of monotone finite difference schemes for hyperbolic conservation laws is optimal. From this result the convergence rate 1/2 in L^∞ for Hamilton Jacobi equation seems sharp.

Remark 3.4 From the proof of Theorem 2, it can be easily seen that we do not only get an error estimate between the continuous and the numerical solutions, but also get an error estimate between the position of the continous and of the numerical level sets.

4 Preliminaries for the continuous problem

Lemma 4.1 (A priori regularity for the eikonal equation) *Consider the eikonal equation*

$$u_t = c(x,y,t)|\nabla u| \qquad in\ \mathbb{R}^2 \times (0,\overline{T}), \qquad u(\cdot,0) = u^0 \qquad in\ \mathbb{R}^2. \tag{16}$$

Suppose that the velocity c is bounded and continuous with respect to all the variables and is Lipschitz continuous in space with Lipschitz constant L_c. Suppose also that the initial data is Lipschitz continuous and satisfies

$$|\nabla u^0| \leq B^0, \quad \frac{\partial u^0}{\partial y} \geq b^0 \qquad a.e.\ in\ \mathbb{R}^2,$$

for some constants $B^0 > 0$ and $b^0 > 0$.

Then, the eikonal equation (16) *has a unique viscosity solution u with at most linear growth in space. Moreover, u is Lipschitz continuous and we have the estimates*

$$|\nabla u(\cdot,t)| \leq B(t), \quad \left|\frac{\partial u(\cdot,t)}{\partial t}\right| \leq C(t), \quad \frac{\partial u(\cdot,t)}{\partial y} \geq b(t) \quad a.e.\ in\ \mathbb{R}^2 \times (0,\overline{T}),$$

for the functions

$$B(t) = B^0 \mathrm{e}^{L_c t}, \quad C(t) = |c|_{L^\infty} B^0 \mathrm{e}^{L_c t}, \quad b(t) = b^0 - B^0 \left(\mathrm{e}^{L_c t} - 1\right).$$

Proof The solvability of the eikonal equation (16) in the set of the continuous functions with at most linear growth is classical (see e.g. Barles [5] for a proof). The Lipschitz bounds in space and time are also standard (see e.g. Crandall and Lions [7]; see also [4] for a complete proof that is more adapted to the present equation).

To establish the lower bound on $\partial u(\cdot,t)/\partial y$, we simply consider for $\lambda > 0$, the function

$$\overline{u}^{\lambda}(x,y,t) = u(x,y+\lambda,t) - \lambda b(t).$$

We check easily that this is a supersolution satisfying $\overline{u}^{\lambda}(x,y,0) \geq u(x,y,0)$. The comparison principle shows that we have

$$\overline{u}^{\lambda} \geq u$$

i.e.

$$u(x,y+\lambda,t) - u(x,y,t) \geq \lambda b(t) \quad \text{for every } \lambda > 0.$$

This is the integral form of the differential bound in the statement of the lemma. □

The next lemma concerns the estimate of the distance of the level sets of two functions u^1 and u^2, which are assumed to be increasing in one direction as a function of $|u^1 - u^2|_{L^\infty}$. We measure the distance of two level sets as the area of the difference of the two characteristic functions associated.

Lemma 4.2 (Estimate on the characteristic functions) *Let* $u^1 \in C(\mathbb{R}^2)$ *satisfy*

$$\frac{\partial u^1}{\partial y} \geq b$$

in the distributions' sense for some $b > 0$ *and* $u^2 \in L^\infty_{loc}(\mathbb{R}^2)$. *Then, we have the estimate*

$$\left|[u^2] - [u^1]\right|_{L^1_{\text{unif}}} \leq \frac{2}{b}\left|u^2 - u^1\right|_{L^\infty}. \tag{17}$$

Proof We fix $(\overline{x},\overline{y}) \in \mathbb{R}^2$ and we set $Q = Q(\overline{x},\overline{y})$. We set $M = |u^2 - u^1|_{L^\infty}$ and we redefine u^2 on a set of zero measure so that $|u^2 - u^1| \leq M$ everywhere. We fix $x \in [\overline{x} - 1/2, \overline{x} + 1/2]$ arbitrarily. Since $u^1(x,y) - by$ is nondecreasing in y and continuous, there is a unique y_x so that $u^1(x,y_x) = 0$. We claim that

$$\begin{aligned} A_x = \{y \in [\overline{y} - 1/2, \overline{y} + 1/2] \mid [u^2](x,y) \neq [u^1](x,y)\} \\ \subset [y_x - M/b, y_x + M/b]. \end{aligned}$$

Indeed, if $y_x + M/b < y \le \overline{y} + 1/2$, we have that

$$u^2(x,y) \ge u^1(x,y) - M \ge u^1(x,y_x) + b(y - y_x) - M > 0.$$

So, both $u^2(x,y)$ and $u^1(x,y)$ are positive; hence, $[u^2](x,y) = [u^1](x,y)$. Similarly, whenever $\overline{y} - 1/2 \le y < y_x - M/b$, both $u^2(x,y)$ and $u^1(x,y)$ are negative, hence $[u^2](x,y) = [u^1](x,y)$. The required inclusion follows.

For every $x \in [\overline{x} - 1/2, \overline{x} + 1/2]$, we deduce from the inclusion that

$$\int_{\overline{y}-1/2}^{\overline{y}+1/2} |[u^2](x,y) - [u^1](x,y)| \, \mathrm{d}y \le \mathrm{meas}(A_x) \le 2M/b.$$

Integrating with respect to x, we deduce that $|[u^2] - [u^1]|_{L^1\left(Q(\overline{x},\overline{y})\right)} \le 2M/b$. Taking the supremum over $(\overline{x},\overline{y})$ yields the bound of the lemma. □

Finally we will use the following result which is a simple adaptation of a result in [4]:

Proposition 4.3 (A stability result, [4]) *Let us consider for* $p = 1, 2$ *two different equations*

$$\begin{cases} u_t^p = c^p(x,y,t)\,|\nabla u^p| & \text{on } \mathbb{R}^2 \times (0,\overline{T}), \\ u^p(x,y,0) = u^0(x,y), \end{cases} \tag{18}$$

with initial condition u^0 *and velocity* c^p *satisfying the assumptions in Lemma 4.1.*

Then we have for every $t \in [0,\overline{T})$*:*

$$\begin{aligned} &|u^2(\cdot,\cdot,t) - u^1(\cdot,\cdot,t)|_{L^\infty(\mathbb{R}^2)} \\ &\quad \le \int_0^t |c^2(\cdot,s) - c^1(\cdot,s)|_{L^\infty(\mathbb{R}^2)} \max\Big(|\nabla u^1(\cdot,\cdot,s)|_{L^\infty(\mathbb{R}^2)}, \\ &\qquad\qquad |\nabla u^2(\cdot,\cdot,s)|_{L^\infty(\mathbb{R}^2)}\Big) \; \mathrm{d}s \end{aligned}$$

5 Preliminaries for the discrete problem

Let us consider the equation

$$u_t = c(x,y,t)|\nabla u|$$

and the monotone scheme

$$v_{i,j}^{n+1} = v_{i,j}^n + \Delta t \, H^{\mathrm{num}}(v^n; i,j) \tag{19}$$

with

$$H^{\mathrm{num}}(v^n;i,j) = \begin{cases} c^n_{i,j}E^+ & \text{if } c^n_{i,j} > 0, \\ c^n_{i,j}E^- & \text{if } c^n_{i,j} \le 0, \end{cases} \tag{20}$$

where E^+ and E^- have been defined in Sect. 2. We assume throughout that the CFL condition (15) holds with $c^n_{i,j}$ instead of $c_{i,j}([v^n])$. We establish in this section several estimates that are the discrete analogues of the results obtained previously.

Lemma 5.1 *If for some* $M^n > 0$ *we have*

$$\left|\frac{v^n_{i+1,j} - v^n_{i,j}}{\Delta x}\right|, \left|\frac{v^n_{i,j+1} - v^n_{i,j}}{\Delta y}\right| \le M^n, \quad \forall\, i,j \in \mathbb{Z}$$

and

$$M^{n+1} = M^n\left(1 + 2\Delta t \sup_{i,j\in\mathbb{Z}}\left(\left|\frac{c^n_{i+1,j} - c^n_{i,j}}{\Delta x}\right|, \left|\frac{c^n_{i,j+1} - c^n_{i,j}}{\Delta y}\right|\right)\right)$$

then

$$\frac{v^{n+1}_{i+1,j} - v^{n+1}_{i,j}}{\Delta x} \ge -M^{n+1}, \quad \forall\, i,j \in \mathbb{Z}. \tag{21}$$

Exchanging i and j and changing sign, we obtain the following corollary.

Lemma 5.2 (Discrete gradient estimate) *If for some* $M^0 > 0$ *we have*

$$\left|\frac{v^0_{i+1,j} - v^0_{i,j}}{\Delta x}\right|, \left|\frac{v^0_{i,j+1} - v^0_{i,j}}{\Delta y}\right| \le M^0, \quad \forall\, i,j \in \mathbb{Z}$$

and

$$M^{n+1} = M^n\left(1 + 2\Delta t \sup_{i,j\in\mathbb{Z}}\left(\left|\frac{c^n_{i+1,j} - c^n_{i,j}}{\Delta x}\right|, \left|\frac{c^n_{i,j+1} - c^n_{i,j}}{\Delta y}\right|\right)\right)$$

then

$$\left|\frac{v^n_{i+1,j} - v^n_{i,j}}{\Delta x}\right|, \left|\frac{v^n_{i,j+1} - v^n_{i,j}}{\Delta y}\right| \le M^n, \quad \forall\, i,j \in \mathbb{Z}\ \forall\, n \in \mathbb{N}$$

Proof of Lemma 5.1 We consider the function

$$w_{i,j}^n = v_{i+1,j}^n + M^n \Delta x.$$

By assumption, we have

$$w_{i,j}^n \geq v_{i,j}^n \quad \forall\, i,j \in \mathbb{Z}.$$

We will check that w is a discrete supersolution, i.e.

$$w_{i,j}^{n+1} - \left(w_{i,j}^n + \Delta t H^{\text{num}}(w^n; i,j)\right) \geq 0. \tag{22}$$

Since the scheme is monotone and $v_{i,j}^n$ is a solution, (22) will imply that

$$w_{i,j}^{n+1} \geq v_{i,j}^{n+1} \quad \forall\, i,j \in \mathbb{Z},$$

which is exactly (21).

Let us show that w^n is a discrete supersolution

$$\begin{aligned} & w_{i,j}^{n+1} - \left(w_{i,j}^n + \Delta t H^{\text{num}}(w^n; i,j)\right) \\ & \quad = v_{i+1,j}^{n+1} + M^{n+1}\Delta x - \left(v_{i+1,j}^n + M^n \Delta x + \Delta t H^{\text{num}}(w^n; i,j)\right) \\ & \quad = \left(M^{n+1} - M^n\right)\Delta x + \Delta t\left(H^{\text{num}}(v^n; i+1,j) - H^{\text{num}}(w^n; i,j)\right) \\ & \quad = \left(M^{n+1} - M^n\right)\Delta x + \Delta t\left(H^{\text{num}}(v^n; i+1,j) - H^{\text{num}}(v_{\cdot+1,\cdot}^n; i,j)\right). \end{aligned}$$

Assume that $c_{i,j}^n$ and $c_{i+1,j}^n$ have the same sign, e.g. that they are nonnegative. Then,

$$\begin{aligned} & w_{i,j}^{n+1} - \left(w_{i,j}^n + \Delta t H^{\text{num}}(w^n; i,j)\right) \\ & \quad = 2M^n \Delta t \Delta x \sup_{i,j\in\mathbb{Z}} \left(\left|\frac{c_{i+1,j}^n - c_{i,j}^n}{\Delta x}\right|, \left|\frac{c_{i,j+1}^n - c_{i,j}^n}{\Delta y}\right|\right) + (c_{i+1,j}^n - c_{i,j}^n)\Delta t E^+ \\ & \quad \geq 2M^n \Delta t \Delta x \left(\sup_{i,j\in\mathbb{Z}} \left(\left|\frac{c_{i+1,j}^n - c_{i,j}^n}{\Delta x}\right|, \left|\frac{c_{i,j+1}^n - c_{i,j}^n}{\Delta y}\right|\right) - \frac{c_{i+1,j}^n - c_{i,j}^n}{\Delta x}\right) \\ & \quad \geq 0. \end{aligned}$$

If $c_{i+1,j}^n$ and $c_{i,j}^n$ do not have the same sign, the conclusion prevails because of the estimate for $a, b \geq 0$:

$$\begin{aligned}|c^n_{i+1,j}a - c^n_{i,j}b| &\le \max(a,b)\max(|c^n_{i+1,j}|,|c^n_{i,j}|)\\ &\le \max(a,b)|c^n_{i+1,j} - c^n_{i,j}|\\ &\le \max(a,b)\left|\frac{c^n_{i+1,j}-c^n_{i,j}}{\Delta x}\right|\Delta x.\end{aligned}$$

This ends the proof of the lemma. □

In the same way we can prove:

Lemma 5.3 (Discrete gradient estimate from below) *If for some $b^0 > 0$, we have*

$$\frac{v^0_{i,j+1} - v^0_{i,j}}{\Delta y} \ge b^0, \quad \forall i,j \in \mathbb{Z}$$

and

$$b^{n+1} = b^n - 2\Delta t \sup_{i,j\in\mathbb{Z}}\left(\left|\frac{c^n_{i+1,j}-c^n_{i,j}}{\Delta x}\right|, \left|\frac{c^n_{i,j+1}-c^n_{i,j}}{\Delta y}\right|\right)M^n$$

then

$$\frac{v^n_{i,j+1} - v^n_{i,j}}{\Delta y} \ge b^n, \quad \forall\, i,j \in \mathbb{Z} \qquad \forall\, n \in \mathbb{N}. \tag{23}$$

Proof We consider the function

$$w^n_{i,j} = v^n_{i,j-1} + b^n\Delta y.$$

By assumption, we have

$$w^n_{i,j} \le v^n_{i,j} \quad \forall\, i,j \in \mathbb{Z}.$$

One can check easily that w is a discrete subsolution, i.e.

$$w^{n+1}_{i,j} - \left(w^n_{i,j} + \Delta t H^{\mathrm{num}}(w^n; i,j)\right) \le 0. \tag{24}$$

Since the scheme is monotone and $v^n_{i,j}$ is a solution, (24) implies that

$$w^{n+1}_{i,j} \le v^{n+1}_{i,j} \quad \forall\, i,j \in \mathbb{Z},$$

which is exactly (23). □

Proposition 5.4 (A numerical stability result) *Let us consider two numerical solutions v^n and w^n of the corresponding monotone scheme* (*with the same initial condition*)

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t H_1^{num}(v^n; i, j) \tag{25}$$

and

$$w_{i,j}^{n+1} = w_{i,j}^{n} + \Delta t H_2^{num}(w^n; i, j) \tag{26}$$

with H_1^{num}

$$H_1^{num}(v^n; i, j) = \begin{cases} c_{i,j}^{1,n} E^+ & \text{if } c_{i,j}^{1,n} \geq 0, \\ c_{i,j}^{1,n} E^- & \text{if } c_{i,j}^{1,n} < 0, \end{cases}$$

where E^+ *and* E^- *are defined in Sect. 2 and* H_2^{num} *defined similarly as* H_1^{num}.

Then there exist a positive constant C, depending on the discrete gradients estimate on v and w, such that

$$\sup_{Q_d^T} \left| v_{i,j}^{n+1} - w_{i,j}^{n+1} \right| \leq C\overline{T} \sup_{Q_d^T} \left| c_{i,j}^{1,n} - c_{i,j}^{2,n} \right|. \tag{27}$$

Proof We follow the proof of the Proposition 4.3 in [4]. We look at the numerical solution v^n in the scheme (26):

$$\begin{aligned} \left| \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} - H_2^{\text{num}}(v^n; i, j) \right| &= \left| H_1^{\text{num}}(v^n; i, j) - H_2^{\text{num}}(v^n; i, j) \right| \\ &\leq \max(E_1^+, E_1^-) \left| c_{i,j}^{1,n} - c_{i,j}^{2,n} \right|. \end{aligned}$$

We set $K = \sup_{i,j,n} |c_{i,j}^{1,n} - c_{i,j}^{2,n}| \max(E_1^+, E_1^-)$ (K can be bounded by a quantity that only depends on the estimates for $\|c\|_{L^\infty}$ and on the bounds of the discrete gradient of v) and we define $\hat{v}_{i,j}^n = v_{i,j}^n + nK\Delta t$. This discrete function verifies

$$\hat{v}_{i,j}^{n+1} \geq \hat{v}_{i,j}^{n} + \Delta t H_2^{\text{num}}(\hat{v}^n; i, j).$$

Since the scheme (26) is monotone, this implies

$$\hat{v}_{i,j}^{n} \geq w_{i,j}^{n},$$

i.e.

$$w_{i,j}^{n} - v_{i,j}^{n} \leq nK\Delta t.$$

Exchanging the role of v^n with w^n we finally obtain (27). □

We prove an estimate on the discrete characteristic functions.

Lemma 5.5 (Estimate on the discrete characteristic functions) *Assume that for some $b > 0$, we have*

$$\frac{v^1_{i,j+1} - v^1_{i,j}}{\Delta y} \geq b, \quad \forall\, i, j \in \mathbb{Z}.$$

Then, for all i, we get

$$\sum_{j \in \mathbb{Z}} \left| \left[v^2_{i,j} \right] - \left[v^1_{i,j} \right] \right| \Delta y \leq 2 \left(\frac{\sup_{i,j} |v^2_{i,j} - v^1_{i,j}|}{b} + \Delta y \right).$$

for every $v^2_{i,j}$.

Proof For each $l \in \mathbb{Z}$, let $m_l \in \mathbb{Z}$ such that

$$v^1_{l,m_l} < 0, \quad v^1_{l,m_l+1} \geq 0,$$

i.e.

$$\left[v^1_{l,m_l} \right] = 0, \quad \left[v^1_{l,m_l+1} \right] = 1.$$

Let

$$M := \sup_{(l,m) \in \mathbb{Z}^2} \left| v^2_{l,m} - v^1_{l,m} \right|$$

and fix $P \in \mathbb{N}$ such that $M < bP\Delta y \leq M + b\Delta y$. Then, for all $p \geq P$, we have

$$v^2_{l,m_l+p+1} \geq v^1_{l,m_l+p+1} - M \geq v^1_{l,m_l+1} + bP\Delta y - M \geq 0.$$

Similarly,

$$v^2_{l,m_l-p} \leq v^1_{l,m_l-p} + M \leq v^1_{l,m_l} - bP\Delta y + M < 0.$$

Therefore,

$$\left[v^2_{l,m_l+p+1} \right] = 1 = \left[v^1_{l,m_l+p+1} \right], \quad \left[v^2_{l,m_l-p} \right] = 0 = \left[v^1_{l,m_l-p} \right].$$

This implies:

$$\sum_{m \in \mathbb{Z}} \left| [v^2_{l,m}] - [v^1_{l,m}] \right| \Delta y \leq \sum_{m_l - P+1 \leq m \leq m_l + P} \Delta y \leq 2P\Delta y \leq 2 \left(\frac{M}{b} + \Delta y \right).$$

□

6 Proof of Theorem 1

Proof We define the set

$$E = \left\{ u \in L^\infty_{\text{loc}}(\mathbb{R}^2 \times [0, T^*)), \quad \left| \begin{array}{l} |\nabla u(x,y,t)| \le 2L_{u^0} \ a.e. \\ \dfrac{\partial u}{\partial y}(x,y,t) \ge \dfrac{b}{2} \ a.e. \\ \left|\dfrac{\partial u}{\partial t}(x,y,t)\right| \le 2L_{u^0}|c^0|_{L^\infty_{\text{int}}(\mathbb{R}^2)} \ a.e. \end{array} \right. \right\},$$

where T^* is to be defined later. We endow E with the topology of uniform convergence, e.g. with the distance

$$d(u^1, u^2) = \min\left(|u^2 - u^1|_{L^\infty(\mathbb{R}^2\times(0,T^*))}, \frac{b}{4} \right).$$

We note that E is a complete metric space.

We fix $\rho \in E$. Then, the convolution $c(x,y,t) := \left(c^0 \star [\rho(\cdot,\cdot,t)]\right)(x,y)$ satisfies the following properties.

Bounded in space and time:

$$\begin{aligned} &\sup_{\mathbb{R}^2\times[0,T^*)} \left(c^0 \star [\rho(\cdot,\cdot,t)]\right)(x,y) \\ &\quad \le \sup_{(0,T^*)} |[\rho(\cdot,\cdot,t)]|_{L^1_{\text{unif}}(\mathbb{R}^2)} |c^0|_{L^\infty_{\text{int}}(\mathbb{R}^2)} \\ &\quad \le |c^0|_{L^\infty_{\text{int}}(\mathbb{R}^2)}, \end{aligned}$$

because $|[\rho]| \le 1$.

Lipschitz in space: let $x, y, x', y' \in \mathbb{R}, t \in [0, T^*)$

$$\begin{aligned} &\left| \left(c^0 \star [\rho(\cdot,\cdot,t)]\right)(x,y) - \left(c^0 \star [\rho(\cdot,\cdot,t)]\right)(x',y') \right| \\ &\quad \le |c^0|_{BV(\mathbb{R}^2)}(|x - x'| + |y - y'|). \end{aligned}$$

Continuous in time. For all $\rho \in E$, for all $t_1, t_2 \in [0, T^*), x, y \in \mathbb{R}$

$$\begin{aligned} &\left| \left(c^0 \star [\rho(\cdot,\cdot,t_1)]\right)(x,t) - \left(c^0 \star [\rho(\cdot,\cdot,t_2)]\right)(x,y) \right| \\ &\quad \le |c^0|_{L^\infty_{\text{int}}(\mathbb{R}^2)} |[\rho(\cdot,\cdot,t_1)] - [\rho(\cdot,\cdot,t_2)]|_{L^1_{\text{unif}}(\mathbb{R}^2)} \\ &\quad \le |c^0|_{L^\infty_{\text{int}}(\mathbb{R}^2)} \frac{4}{b} |\rho(\cdot,\cdot,t_1) - \rho(\cdot,\cdot,t_2)|_{L^\infty(\mathbb{R}^2)} \\ &\quad \le \frac{8}{b} L_{u^0} |c^0|^2_{L^\infty_{\text{int}}(\mathbb{R}^2)} |t_1 - t_2|. \end{aligned}$$

We have used Lemma 4.2 for the second inequality, and the definition of the set E for the last inequality.

From Lemma 4.1, we can therefore define w as the unique viscosity solution of

$$\begin{cases} w_t = \left(c^0 \star [\rho]\right) |\nabla w| & \text{on } \mathbb{R}^2 \times (0, T^*), \\ w(\cdot,\cdot,0) = u^0(\cdot,\cdot). \end{cases}$$

We have that $w \in E$ as long as $b(t) \geq \frac{b}{2}$ and $B(t) \leq 2L_{u^0}$ on $[0, T^*)$ (where the expressions of $b(t)$ and $B(t)$ are given in Lemma 4.1 with $B^0 = L_{u^0}$, and $b^0 = b$). This holds true if

$$T^* \leq \min\left\{ \frac{1}{|c^0|_{\mathrm{BV}(\mathbb{R}^2)}} \ln\left(1 + \frac{b}{2L_{u^0}}\right), \frac{\ln 2}{|c^0|_{\mathrm{BV}(\mathbb{R}^2)}} \right\}.$$

Since $b \leq L_{u^0}$, the min reduces to the first term:

$$T^* \leq \min\left\{ \frac{1}{|c^0|_{\mathrm{BV}(\mathbb{R}^2)}} \ln\left(1 + \frac{b}{2L_{u^0}}\right) \right\}.$$

The operator Φ defined by $\Phi(\rho) = w$ therefore maps E into E. Furthermore, from the stability result (Proposition 4.3) and the estimate on the characteristic function (Lemma 4.2), we get for $w^i = \Phi(\rho^i)$

$$\begin{aligned} &|w^2 - w^1|_{L^\infty(\mathbb{R}^2 \times (0,T^*))} \\ &\quad \leq 2L_{u^0} T^* |c^0 \star [\rho^2] - c^0 \star [\rho^1]|_{L^\infty(\mathbb{R}^2 \times (0,T^*))} \\ &\quad \leq 2L_{u^0} T^* |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} \sup_{(0,T^*)} |[\rho^2](\cdot,\cdot,t) - [\rho^1](\cdot,\cdot,t)|_{L^1_{\mathrm{unif}}(\mathbb{R}^2)} \\ &\quad \leq 2L_{u^0} |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} T^* \min\left(\frac{4}{b} |\rho^2 - \rho^1|_{L^\infty(\mathbb{R}^2 \times (0,T^*))}, 1 \right) \\ &\quad \leq \frac{8L_{u^0}}{b} |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} T^* d(\rho^2, \rho^1). \end{aligned}$$

Choosing

$$T^* = \min\left\{ \frac{1}{|c^0|_{\mathrm{BV}(\mathbb{R}^2)}} \ln\left(1 + \frac{b}{2L_{u^0}}\right), \quad \frac{b}{L_{u^0}} \frac{1}{16|c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)}} \right\},$$

we conclude

$$d(w^2, w^1) \leq |w^2 - w^1|_{L^\infty(\mathbb{R}^2 \times (0,T^*))} \leq \frac{1}{2}\, d(\rho^2, \rho^1)$$

i.e. Φ is a contraction. By the fixed point theorem, we conclude that there exists a unique viscosity solution on $(0, T^*)$. This ends the proof of the theorem. □

7 Proof of Theorem 2

7.1 The abstract convergence theorem of Alvarez et al. [2]

In this section, we recall the abstract convergence result of Alvarez et al. [2] for the discretization of the non local eikonal equation

$$u_t = c[u]\,|\nabla u| \qquad \text{in} \;\; \mathbb{R}^2 \times (0, T^{**}), \qquad u(\cdot, 0) = u^0 \qquad \text{in} \;\; \mathbb{R}^2. \tag{28}$$

We suppose that the equation has a solution $u \in \mathrm{Lip}(\mathbb{R}^2 \times [0, T^{**}))$ and that $c[u] \in W^{1,\infty}(\mathbb{R}^2 \times [0, T^{**}))$.

Let $E^\Delta_T = \mathbb{R}^{\mathbb{Z}^2 \times \{0,\ldots,N_T\}}$ be the set of discrete functions defined on the mesh Q^T_{d} and $E^\Delta = E^\Delta_{T^{**}}$. We denote by $G^\Delta : E^\Delta \to E^\Delta$ the operator that associates to a given discrete velocity $c^\Delta \in E^\Delta$ the discrete solution v of the finite difference scheme

$$\begin{cases} v_I^{n+1} = v_I^n + \Delta t\, c^\Delta(X_I, t_n)\, E_{\mathrm{d}}^{\mathrm{sign}(c^\Delta(X_I,t_n))}(D^+ v^n, D^- v^n), \\ v_I^0 = u^0(X_I). \end{cases} \tag{29}$$

Here $E^\pm_{\mathrm{d}} = E^\pm_{\mathrm{d}}(p_x^+, p_y^+, p_x^-, p_y^-)$ is a suitable approximation of the Euclidean norm that is Lipschitz continuous, consistent with the Euclidean norm (i.e. $E^\pm_{\mathrm{d}}(p_x, p_y, p_x, p_y) = |(p_x, p_y)|$) and monotone

$$\frac{\partial E^+_{\mathrm{d}}}{\partial p_x^+} \geq 0, \quad \frac{\partial E^+_{\mathrm{d}}}{\partial p_y^+} \geq 0, \quad \frac{\partial E^+_{\mathrm{d}}}{\partial p_x^-} \leq 0, \quad \frac{\partial E^+_{\mathrm{d}}}{\partial p_y^-} \leq 0,$$
$$\frac{\partial E^-_{\mathrm{d}}}{\partial p_x^+} \leq 0, \quad \frac{\partial E^-_{\mathrm{d}}}{\partial p_y^+} \leq 0, \quad \frac{\partial E^-_{\mathrm{d}}}{\partial p_x^-} \geq 0, \quad \frac{\partial E^-_{\mathrm{d}}}{\partial p_y^-} \geq 0.$$

One readily checks that the Osher–Sethian Hamiltonian recalled in sect. 2 satisfies these properties.

For every mesh Δ and every $T \leq T^{**}$, we consider two subsets U^Δ and V^Δ of E^Δ and set $U^\Delta_T = U^\Delta \cap E^\Delta_T$ and $V^\Delta_T = V^\Delta \cap E^\Delta_T$. For all $T \leq T^{**}$, we assume that $G^\Delta(V^\Delta_T) \subset U^\Delta_T$ and that $(u)^\Delta_T \in U^\Delta_T$, where $(u)^\Delta_T$ is the restriction to Q^T_{d} of the continuous solution u of (28).

We also assume that the sets U^Δ and V^Δ are respectively equi-Lipschitz and equibounded in the sense that there is a constant K so that, for every mesh Δ,

$$|D^+ w| \leq K, \quad |c| \leq K, \qquad \text{for all } w \in U^\Delta \text{ and } c \in V^\Delta. \tag{30}$$

In addition, we suppose that the following uniform CFL condition is satisfied

$$\Delta t \leq \frac{L}{K}\,\Delta x, \quad \Delta t \leq \frac{L}{K}\,\Delta y \tag{31}$$

for

$$L^{-1} = 2\max\left(\left|\frac{\partial E_\mathrm{d}}{\partial p_x^+}\right|_{L^\infty} + \left|\frac{\partial E_\mathrm{d}}{\partial p_x^-}\right|_{L^\infty}, \left|\frac{\partial E_\mathrm{d}}{\partial p_y^+}\right|_{L^\infty} + \left|\frac{\partial E_\mathrm{d}}{\partial p_y^-}\right|_{L^\infty}\right)$$

and

$$K = \sup_{\Delta}\ \sup_{V^\Delta} |c^\Delta|_\infty.$$

For the Osher–Sethian Hamiltonian, one computes easily that $L = 1/2\sqrt{2}$. This guarantees that the scheme defined by (29) is monotone.

These assumptions imply the following stability of the operator G^Δ (see Proposition 5.4): there is a constant K so that, for every mesh Δ satisfying the CFL condition (31), for all $0 \le T \le T^{**}$ and all $c_1, c_2 \in V_T^\Delta$,

$$\sup_{Q_T^\Delta} |G^\Delta(c_2) - G^\Delta(c_1)| \le KT \sup_{Q_T^\Delta} |c_2 - c_1|. \tag{32}$$

Finally, we approximate the nonlocal velocity mapping $c : U \to V$ by a map $c^\Delta : U^\Delta \to V^\Delta$ so that $c^\Delta(U_T^\Delta) \subset V_T^\Delta$ for all $T \le T^{**}$. We make the following two assumptions.

Consistency for the discrete velocity c^Δ: There is a constant K such that, for every mesh Δ, for every $T \le T^{**}$, we have

$$\sup_{Q_\mathrm{d}^T} |c[u] - c^\Delta[u^\Delta]| \le K|\Delta X| \tag{33}$$

(where u is the solution of (28) and $u^\Delta = (u)_T^\Delta$ is the restriction of u to Q_T^Δ).

Stability property of the velocity c^Δ: There is a constant K so that, for all meshes Δ, for all $0 \le T \le T^{**}$ and all $w_1, w_2 \in U_T^\Delta$,

$$\sup_{Q_T^\Delta} |c^\Delta[w_2] - c^\Delta[w_1]| \le K\big(\sup_{Q_T^\Delta} |w_2 - w_1| + |\Delta X|\big). \tag{34}$$

We also suppose that c^Δ is stationary, i.e. that there is a mapping $\bar{c}^\Delta$ such that $c^\Delta[w](\cdot, t_n) = \bar{c}^\Delta\big[w(\cdot, t_n)\big]$. This implies that the explicit non-local scheme

$$\begin{cases} v_I^{n+1} = v_I^n + \Delta t\, c^\Delta[v](X_I, t_n)\, E_\mathrm{d}^{\mathrm{sign}(c^\Delta(X_I, t_n))}(D^+v^n, D^-v^n), \\ v_I^0 = u^0(X_I) \end{cases} \tag{35}$$

has a unique solution for all time, which we denote by v.

The convergence result of Alvarez et al. [2] is as follows.

Theorem 3 *Assume that $T \le T^{**} \wedge 1$. Then, under the previous assumptions, there exists a positive constant K' such that, for all $0 \le T \le T^{**} \wedge 1$,*

$$\sup_{Q_d^T} |u - v| \le \frac{K'\sqrt{T|\Delta X|}}{(1 - K'T)^+} \quad \textit{provided } |\Delta X| \le T/K'.$$

The constant K' only depends on the constants K, L, T^{**}, $|c[u]|_{W^{1,\infty}(\mathbb{R}^2 \times [0,T^{**}))}$ and on the Lipschitz constants of E^+, E^- and u^0.

7.2 Application of the abstract convergence result: proof of Theorem 2

In this part, we verify that the scheme presented in Sect. 2 satisfies the assumptions of the preceding subsection for $T^{**} = T^*/2$. Theorem 2 will then follow from Theorem 3 with $T_{\mathrm{d}}^* = \inf \left(T^{**} \wedge 1, \frac{1}{2K'} \right)$.

When

$$c[u] = c^0 \star [u]$$

the solvability of the non-local eikonal equation (28) is guaranteed by Theorem 1. Moreover, since $|[u]| \le 1$, we have the uniform estimate

$$|c[u]|_{W^{1,\infty}(\mathbb{R}^2 \times [0,T^*))} \le \left| c^0 \right|_{\mathrm{BV}(\mathbb{R}^2)}.$$

We set

$$U_T^\Delta = \left\{ w \in E^\Delta \mid |D_x^+ w|, |D_y^+ w| \le L_{u^0} \mathrm{e}^{2T|c^0|_{BV}}, \right.$$
$$\left. \frac{w_{i,j+1}^n - w_{i,j}^n}{\Delta y} \ge b - L_{u^0} (\mathrm{e}^{2T|c^0|_{\mathrm{BV}}} - 1) \right\}$$

and

$$V_T^\Delta = \left\{ c \in E^\Delta \mid |c| \le |c^0|_{L^1(\mathbb{R}^2)},\ |D_x^+ c|, |D_y^+ c| \le |c^0|_{\mathrm{BV}} \right\}.$$

By Lemmas 5.2 and 5.3, we see after a straightforward computation that $G^\Delta(V_T^\Delta) \subset U_T^\Delta$ for all $T \le T^*$. Moreover, by Lemma 4.1, $(u)_T^\Delta \in U_T^\Delta$ for all $T \le T^*$. By definition, the sets U^Δ and V^Δ are clearly equibounded in the sense of (30).

Now consider the non-local discrete velocity given by Eqs. (7) and (8)

$$c_{i,j}^\Delta[v] = \sum_{l,m \in \mathbb{Z}} \bar{c}_{i-l,j-m}^{\,0} [v]_{l,m} \Delta x \Delta y, \quad \bar{c}_{i,j}^{\,0} = \frac{1}{|Q_{i,j}|} \int_{Q_{i,j}} c^0(x,y) \mathrm{d}x \mathrm{d}y$$

It is clearly stationary.

In order to verify the remaining properties, we first note that c^Δ can be written as the continuous convolution

$$c^\Delta_{i,j}[v] = c^0 \star [v_\#](x_i, y_j), \tag{36}$$

where $v_\#$ is the piecewise constant lifting of v

$$v_\# = \sum_{i,j} v_{i,j}\chi_{Q_{i,j}}, \tag{37}$$

where $\chi_{Q_{i,j}}$ is the indicator function of $Q_{i,j}$. Indeed,

$$\begin{aligned}
c^\Delta_{i,j}[v] &= \sum_{l,m\in\mathbb{Z}} \bar{c}^{\,0}_{i-l,j-m}[v]_{l,m}\Delta x\Delta y \\
&= \sum_{l,m\in\mathbb{Z}} \left(\frac{1}{|Q_{i-l,j-m}|} \int_{Q_{i-l,j-m}} c^0(x,y)\mathrm{d}x\mathrm{d}y \right) [v]_{l,m}\Delta x\Delta y \\
&= \sum_{l,m\in\mathbb{Z}} \int_{Q_{l,m}} c^0(x_i - x, y_j - y)[v_\#](x,y)\mathrm{d}x\mathrm{d}y \\
&= (c^0 \star [v_\#])(x_i, y_j).
\end{aligned}$$

Since $|[v_\#]| \le 1$, we deduce that for all $u \in U^\Delta_T$ we have

$$|c^\Delta[v]| \le |c^0|_{L^1}, \quad |D^\pm c^\Delta[v]| \le |\nabla c^0 \star [v_\#]|_{L^\infty} \le |c^0|_{\mathrm{BV}}.$$

This implies in particular that $c^\Delta(U^\Delta_T) \subset V^\Delta_T$. Moreover, since

$$\Delta t = \lambda_x \Delta x, \quad \Delta t = \lambda_y \Delta y, \qquad \text{with } 0 < \lambda_x, \lambda_y \le \frac{1}{2\sqrt{2}|c^0|_{L^1}},$$

we see that the CFL condition (31) is satisfied (we recall that $L = 1/2\sqrt{2}$ for the Osher–Sethian Hamiltonian).

As concerns consistency (33), we deduce from Lemma 4.2 that

$$\begin{aligned}
\sup_{i,j} |c^\Delta_{i,j}[u^\Delta](\cdot, t_n) - c[u](x_i, y_j, t_n) &\le \sup |c^0 \star [u^\Delta_\#](\cdot, t_n) - c^0 \star [u](\cdot, t_n)| \\
&\le |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} |[u^\Delta_\#](\cdot, t_n) - [u](\cdot, t_n)|_{L^1_{\mathrm{unif}}(\mathbb{R}^2)} \\
&\le |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} \frac{4}{b} |u^\Delta_\#(\cdot, t_n) - u(\cdot, t_n)|_{L^\infty(\mathbb{R}^2)} \\
&\le |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} \frac{8L_{u^0}}{b} |\Delta X|.
\end{aligned}$$

Finally, to prove the stability (34), we note that

$$\begin{aligned}
&|c^{\Delta}_{i,j}[w^1] - c^{\Delta}_{i,j}[w^2]| \\
&\quad = |c^0 \star [w^1_{\#}](x_i, y_j) - c^0 \star [w^2_{\#}](x_i, y_j)| \\
&\quad \le |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} |[w^1_{\#}] - [w^2_{\#}]|_{L^1_{\mathrm{unif}}(\mathbb{R}^2)} \\
&\quad \le |c^0|_{L^\infty_{\mathrm{int}}(\mathbb{R}^2)} \sup_{x\in\mathbb{R}} \left(\int_{[x-\frac{1}{2},x+\frac{1}{2}]\times\mathbb{R}} |[w^1_{\#}] - [w^2_{\#}]| \mathrm{d}x' \mathrm{d}y' \right).
\end{aligned}$$

Setting $I^x_i = \left[x - \frac{1}{2}, x + \frac{1}{2}\right] \cap \left[x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}\right]$, we deduce from Lemma 5.5 that for all $x \in \mathbb{R}$

$$\begin{aligned}
\int_{[x-\frac{1}{2},x+\frac{1}{2}]\times\mathbb{R}} \left|\left[w^1_{\#}\right] - [w^2_{\#}]\right| \mathrm{d}x'\mathrm{d}y' &\le \sum_{i,j} \left|[w^1_{i,j}] - [w^2_{i,j}]\right| \, |I^x_i| \, \Delta y \\
&\le 2\left(\Delta y + \frac{2}{b} \sup_{i,j} \left|w^1_{i,j} - w^2_{i,j}\right| \right) \sum_i |I^x_i| \\
&\le 2\left(\Delta y + \frac{2}{b} \sup_{i,j} \left|w^1_{i,j} - w^2_{i,j}\right| \right)
\end{aligned}$$

for $T \le T^*/2$ which garantees that $\frac{w^n_{i,j+1} - w^n_{i,j}}{\Delta y} \ge b/2$ for every $w \in U^{\Delta}_T$. Combining the two estimates, we obtain (34).

8 Numerical tests and applications

Let us first recall that the numerical error estimate given in Theorem 2 shows that the scheme (3) gives accurate numerical solutions when the gradient in the direction y of the initial data is supposed bounded from below. In particular this implies that the dislocation line described by the set of discontinuity of $[u]$ is a graph in the y direction.

From a computational point of view, the evolution of a graph is not very convenient to implement. For this reason, we only consider here the case of initial datum which are assumed to be l_x-periodic in the x direction and l_y-periodic in the y direction. This will allow us to work in a periodic box

$$\left([0, l_x[\times[0, l_y[\right)_{\mathrm{per}}$$

and to compute numerically the convolution using Fast Fourier Transform. Our simulations will show that our scheme seems reasonable for these numerical tests.

From the physical point of view, a natural kernel $c^0 = c^0_\delta$ is proposed in Alvarez et al. [4] for the Peierls–Nabarro model. Its Fourier transform is given by:

$$\widehat{c^0_\delta}(\xi_x, \xi_y) = -\frac{1}{2}\left(\frac{\xi_x^2 + (\frac{1}{1-\nu})\xi_y^2}{\sqrt{\xi_x^2+\xi_y^2}}\right) \mathrm{e}^{-\delta\sqrt{\xi_x^2+\xi_y^2}}, \tag{38}$$

where ν is the Poisson ratio that takes values in $(-1, 0.5)$, and δ is a constant proportional to the size of the core of the dislocations. In our simulations we considered cases with $\delta = C|\Delta X|$ and $1 \lesssim C \leq 10$). Figure 1 represents the set $\{(x, c) \in \mathbb{R}^2 : \exists\, y \in \mathbb{R}, \text{ s.t. } c = c^0(x, y)\}$, where the kernel $c^0 = c^0_\delta$ is the discrete Fourier transform of expression (38) with $\delta = 0.5, \nu = 0.3$.

8.1 Computation of the discrete convolution

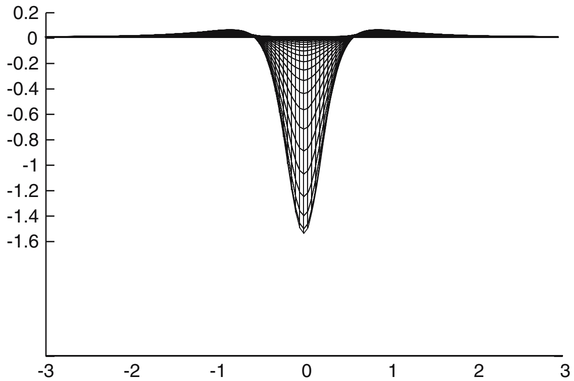
Our method to compute the convolution is simply to take the inverse Fourier transform of the product of the Fourier transform of c^0 with the Fourier transform of

$$w = [v^n].$$

This procedure is much less expensive from the computational point of view than the trivial direct discrete convolution. The main result of this subsection is Lemma 8.1 which shows explicitely how to compute the convolution in Fourier space.

We divide the interval $[0, l_x[$ in m_x intervals of length $\Delta x = l_x/m_x$ and do the same for $[0, l_y[$ in m_y intervals of length $\Delta y = l_y/m_y$. We introduce the notation M and L for the matrices $M = \begin{pmatrix} m_x & 0 \\ 0 & m_y \end{pmatrix}$, $L = \begin{pmatrix} l_x & 0 \\ 0 & l_y \end{pmatrix}$. We set $Q_M =$

Fig. 1 Anisotropic kernel in the Peierls–Nabarro model



$[0,l_x[\times[0,l_y[$ which is discretized in $Q_{\mathrm{d},M} = Q_\mathrm{d} \cap Q_M = \{(x_i,y_j),\ (i,j) \in \mathbb{Z}^2_M\}$ with $\mathbb{Z}^2_M = \{0,\ldots,m_x-1\} \times \{0,\ldots,m_y-1\}$.

From Eqs. (36) to (37), we see that the discrete velocity c^Δ_I is given by

$$c^\Delta_I = (c^0 \star w_\#)(X_I),$$

where $w_\#$ is the piecewise constant lifting of $w_I = [v^n_I]$ defined by

$$w_\# = \sum_{I\in\mathbb{Z}^2} w_I \chi_{Q_I}.$$

We recall that we assume that v^n_I satisfies the following periodic conditions

$$v^n_{i+m_x,j} = v^n_{i,j} = v^n_{i,j+m_y}. \tag{39}$$

As a consequence we see easily that w and c^Δ are also periodic, i.e. satisfy (39) and then are characterized by their values for $I \in \mathbb{Z}^2_M$.

On the one hand, for a general discrete function $f = (f_J)_{J\in\mathbb{Z}^2_M}$ we define its discrete Fourier transform by

$$(\widehat{f})_P = \sum_{J\in\mathbb{Z}^2_M} f_J\ \mathrm{e}^{-\mathrm{i}2\pi J^T\cdot M^{-1}\cdot P}, \quad P \in \mathbb{Z}^2_M.$$

and the inverse Fourier transform is given by

$$f_P = \frac{1}{m_x\, m_y} \sum_{J\in\mathbb{Z}^2_M} (\widehat{f})_J\ \mathrm{e}^{\mathrm{i}2\pi J^T\cdot M^{-1}\cdot P}, \quad P \in \mathbb{Z}^2_M. \tag{40}$$

On the other hand, we take the following definition for the Fourier transform of c^0 (using the notation $X = (x,y)$):

$$\widehat{c^0}(S) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{e}^{-iS\cdot X} c^0(X)\mathrm{d}X.$$

Then we have the following result

Lemma 8.1 (Computation of the discrete convolution by discrete Fourier transform) *We have*

$$(\widehat{c^\Delta})_P = \widehat{w}_P \cdot (\widehat{\widehat{c^0}})_P \quad \textit{for every} \quad P \in \mathbb{Z}^2_M \tag{41}$$

where

$$(\widehat{\widehat{c^0}})_P = \sum_{\{\xi=(\xi_x,\xi_y)=\pi L^{-1}\cdot(P+M\cdot K),\, K\in\mathbb{Z}^2\}} \widehat{c^0}(\xi)\cdot\frac{\sin(\xi_x\frac{\Delta x}{2})}{(\xi_x\frac{\Delta x}{2})}\cdot\frac{\sin(\xi_y\frac{\Delta y}{2})}{(\xi_y\frac{\Delta y}{2})}, \tag{42}$$

and $\widehat{c^0}$ *is the Fourier transform of the kernel* c^0.

From the numerical point of view, the main interest of this lemma is to compute the convolution as a simple multiplication in the Fourier space. This computation is very quick, using first the Fast Fourier Transform to compute $(\hat{v}_P)_P$, and then the inverse Fast Fourier Transform to compute $(c_J)_J$ in the real space.

In our model, we see from expression (38) that the coefficients $\widehat{c^0}(\xi)$ with $\xi = \pi L^{-1}\cdot(P+M\cdot K)$ decrease exponentially with K. In particular when $\delta/|\Delta X|$ is large enough, we see that $\widehat{c^0}(\xi)$ is quite well approximated by the first term in the serie (42). The choice of this first term depends on the values of P and provides the following approximation

$$\widehat{\widetilde{c}^0}_P \simeq \widehat{c^0}(\xi)\frac{\sin(\xi_x\frac{\Delta x}{2})}{(\xi_x\frac{\Delta x}{2})}\cdot\frac{\sin(\xi_y\frac{\Delta y}{2})}{(\xi_y\frac{\Delta y}{2})} \tag{43}$$

with $\xi = \pi L^{-1}\cdot(P+M\cdot K)$ and $K=(k_x,k_y)$ is defined by (for the subscripts $\alpha = x, y$)

$$\begin{cases} k_\alpha = 0 & \text{if } 0 \le p_\alpha < m_\alpha/2, \\ k_\alpha = -1 & \text{if } m_\alpha/2 \le p_\alpha \le m_\alpha - 1. \end{cases}$$

Proof of Lemma 8.1 Using definition (8) we get for $P\in\mathbb{Z}^2_M$:

$$\begin{aligned}(c^0\star w_\#)(X_P) &= \sum_{K\in\mathbb{Z}^2} w_K\int_{Q_K} c^0(X_P-X)\mathrm{d}X\\ &= \sum_{K\in\mathbb{Z}^2} w_K\bar{c}^{\,0}_{P-K}|Q_0|\\ &= \sum_{J\in\mathbb{Z}^2_M} w_J\left(\sum_{K\in\mathbb{Z}^2}\bar{c}^{\,0}_{P-J+M\cdot K}|Q_0|\right)\\ &= \sum_{J\in\mathbb{Z}^2_M} w_J\widetilde{c}^{\,0}_{P-J},\end{aligned}$$

where we define:

$$\widetilde{c}^{\,0}_P = |Q_0|\sum_{K\in\mathbb{Z}^2}\bar{c}^{\,0}_{P+M\cdot K}.$$

Then an easy but tedious computation shows that the coefficient $\left(\widehat{\widetilde{c}^0}_P\right)$ is related to $\widehat{c^0}$ by relation (42). This ends the proof of the Lemma. □

8.2 Numerical simulations

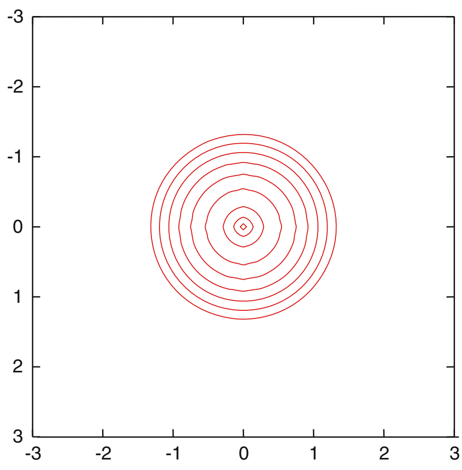
In all our simulations, we take $\Delta x = \Delta y$ and we plot the level sets each N iterations. We refer to the table for the values of the parameters used for Figs.2, 3, 4, 5, 6, 7 and 8

Fig.	ν	δ	$\Delta x = \Delta y$	Δt	N
2	0	0.5	0.04	0.05	160
3	0.33	0.5	0.04	0.05	160
4	0	0.5	0.04	0.05	100
5	0	0.02	0.04	5×10^{-3}	160
6	0	0.5	0.03	0.05	20
7	0	0.5	0.03	0.025	40
8	0	0.5	0.03	0.03	40

8.2.1 Isotropic/anisotropic collapse of a circle

We propose two tests regarding the shrinking of a circle: the isotropic and the anisotropic case. The problem (1) is approximated in $[-3,3]^2$ with $u^0(x,y) = (\max(2-x^2-y^2,0)^{\frac{1}{2}} - 0.5)$. In the isotropic case (Fig. 2 with $\nu = 0$) the circle shrinks in a self similar way. In the anisotropic case (Fig. 3 with $\nu = 0.33$) the circle shrinks changing shape like an ellipsoid elongated in the direction of the physical Burgers vector (see for instance [4] or the references quoted therein such as Hirth and Lothe [9] for more explanation).

Fig. 2 Isotropic shrinking of a circle



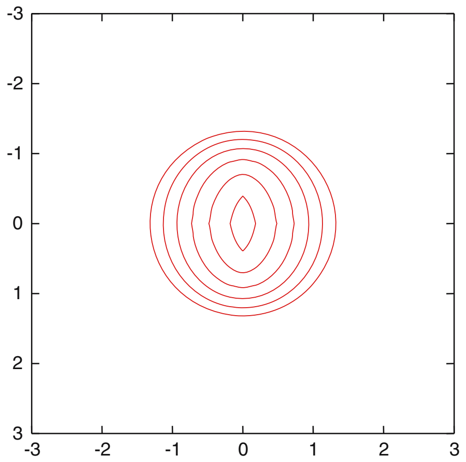


Fig. 3 Anisotropic shrinking of a circle

8.2.2 Convex/non-convex evolution depending on the core size δ

We consider the problem (1) in $[-3,3]^2$ with

$$u^0(\xi,\eta) = \begin{cases} 1.5 - (|\xi-\eta| + |\xi+\eta|) & |\xi-\eta| + |\xi+\eta| < 3, |\xi|, |\eta| < 1.5, \\ -1.5 & |\xi-\eta| + |\xi+\eta| \geq 3, |\xi|, |\eta| < 1.5, \\ -1.5 & |\xi|, |\eta| \geq 1.5, \end{cases}$$

where $\xi\,(x,y) = \frac{1}{2}\big(\sqrt{3}x + y\big), \eta(x,y) = \frac{1}{2}\big(\sqrt{3}y + x\big)$. Here, we are looking at the collapse of a non smooth front: the square.

In Fig. 4 we show a non-convex evolution of the square. This non-convex evolution is possible here, because we chose the size δ of the dislocation core

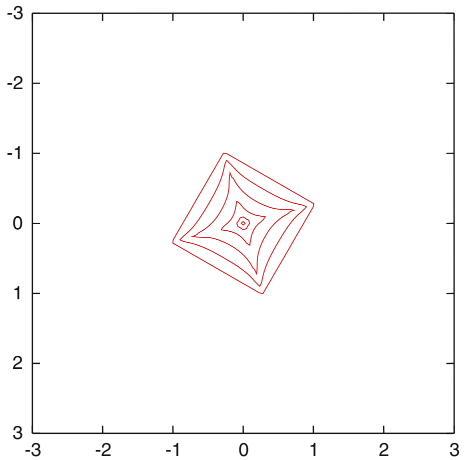


Fig. 4 Non convex shrinking of a square

large enough with respect to the size of the square. At these scales, the dislocation is so small, that its non-convex evolution has not really a physical meaning. In Fig. 5, we decrease δ by a factor 25, such that the new size δ of the core is now less than the size of the square. Here we recover a convex evolution. In fact the corners get smoothed and the square evolves approaching the shape of a circle. This is coherent with the fact that when δ goes to zero, it is physically expected that the limit evolution is decribed by mean curvature motion.

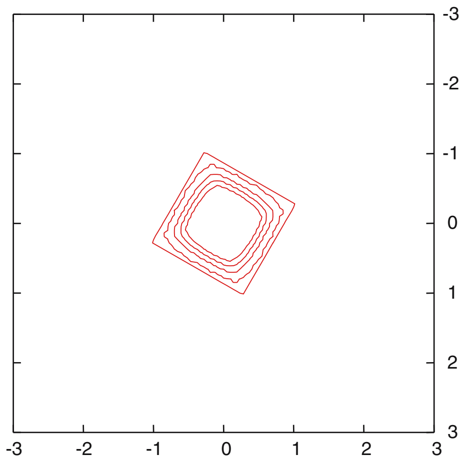
8.2.3 Non fattening evolution for relatively large δ

For mean curvature motion, it is known that fattening occurs if the starting shape is an eight curve with a double point. In that case, the normal vector at the double point is not well defined, and this leads to the phenomenon of fattening which corresponds to the development of a zone with non-empty interior where the associated viscosity solution has constant value, i.e. to a level set with non-empty interior. This is related to the non-unique evolution of the front. From a numerical point of view, it can be observed from the change of topology of the level sets close to the double point.

It turns out that for small enough δ, fattening also occurs for dislocation dynamics.

On the contrary, for relatively large enough δ, we do not observe fattening and the evolution of the front seems uniquely defined. On Figs. 6, 7, 8, we have considered different shapes of curves with a double point, more or less elongated in the x-direction or the y-direction. These simulations correspond respectively to the following initial data: $u^0(x,y) = y^4 - y^2 + x^2$, $u^0(x,y) = y^4 - 2y^2 + x^2$, $u^0(x,y) = y^4 - y^2 + 5x^2$. In each case, we have represented the three level sets corresponding to the values -0.01, 0, 0.01. We observe that these level sets have the same behaviour even at the double point which seems to indicate that there is no fattening.

Fig. 5 Shrinking of a square approaching a circle, with $\delta/\Delta x = 0.5$



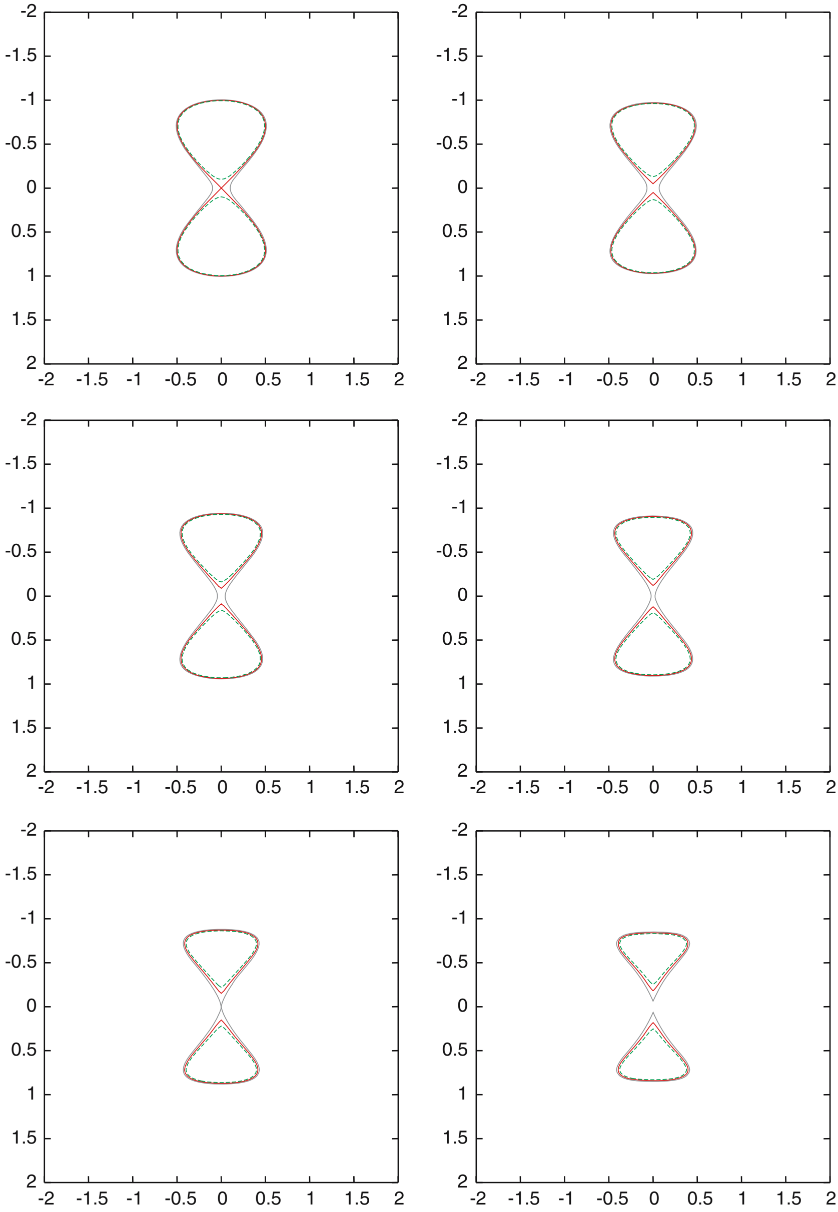


Fig. 6 Non fattening propagation, $\delta = 0.5$ (level sets $-0.01, 0, 0.01$)

8.3 A numerical difficulty

In our model, the dislocation line is represented by the zero level line of the solution u. Numerically the level lines are well defined when

$$|\nabla u| \geq C > 0.$$

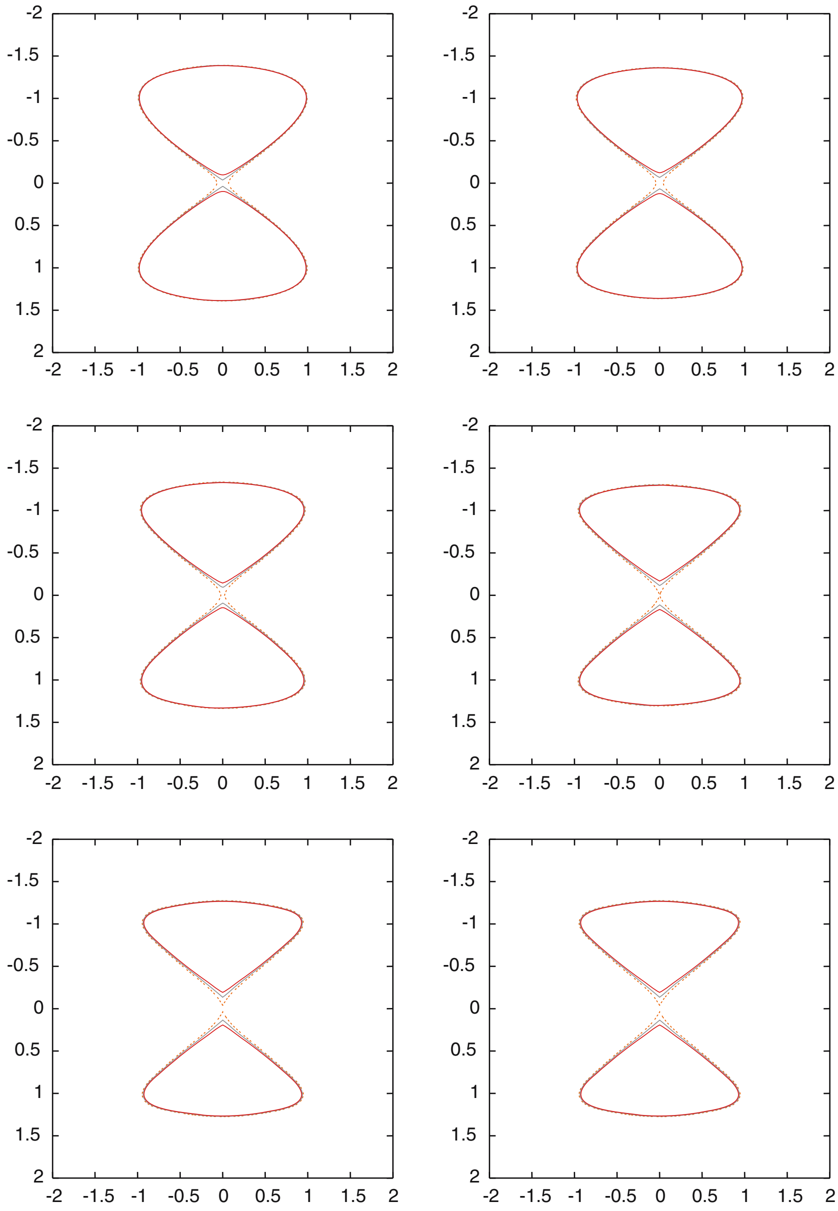


Fig. 7 Non fattening propagation, $\delta = 0.5$ (level sets $-0.01, 0, 0.01$)

We have seen in the proof of Theorem 2 that for $n\Delta t \le T_{\mathrm{d}}^*$:

$$\frac{v_{i,j+1}^n - v_{i,j}^n}{\Delta y} \ge b - L_{u^0}(\mathrm{e}^{-2T|c^0|_{\mathrm{BV}}} - 1).$$

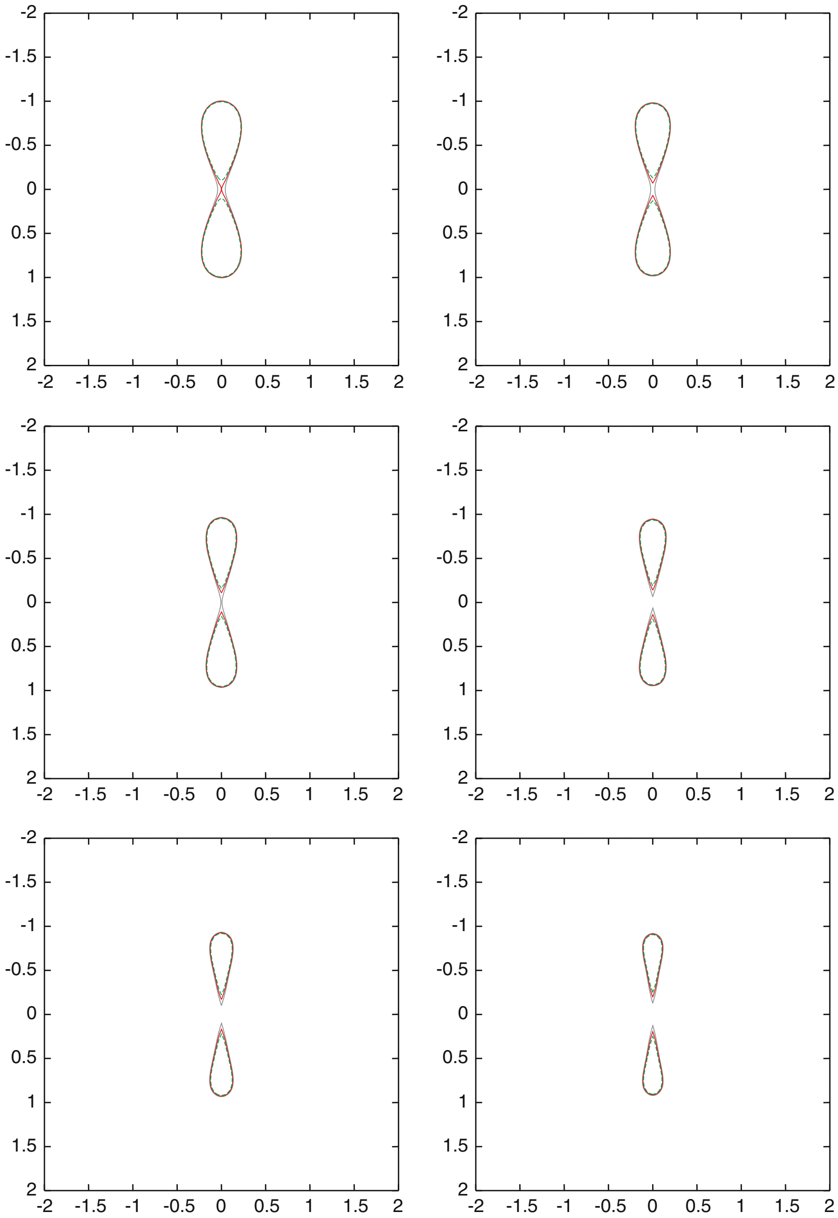


Fig. 8 Non fattening propagation, $\delta = 0.5$ (level sets $-0.01, 0, 0.01$)

For times larger than T_{d}^{*}, the gradient may vanish, and then we may loose any control on the level line, and the solution given by our scheme may be wrong. In these situations, we observe numerically some oscillations of the level line, which indicate that the numerical gradient is too small to ensure a good localization of the level line.

The numerical difficulty comes from the fact that the time T^*_{d} may be very small. We now show formally how this time scales with the size δ of the core.

We consider $c^0 = c^0_\delta$, whose Fourier transform is given by (38). We will now follow the dependence on the small parameter δ. First we remark that c^0_δ can be rewritten as

$$c^0_\delta(x,y) = \frac{1}{\delta^3} c^0_1\left(\frac{x}{\delta}, \frac{y}{\delta}\right),$$

where c^0_1 is a function undependent of δ. We deduce that

$$|c^0_\delta|_1 = \frac{|c^0_1|_1}{\delta}$$

and the Lipschitz constant of the velocity $c_\delta = c^0_\delta \star [u]$ is estimated by

$$L_{c_\delta} \simeq |c^0_\delta|_{\mathrm{BV}} \simeq \frac{|c^0_1|_{\mathrm{BV}}}{\delta^2}.$$

This scaling is crucial for the critical time $T^*_{\mathrm{d}} = T^*_{\mathrm{d},\delta}$, defined in Theorem 2, since

$$T^*_{\mathrm{d},\delta} \simeq \frac{1}{|c^0_\delta|_{\mathrm{BV}}} \simeq \delta^2 T^*_{\mathrm{d},1},$$

where $T^*_{\mathrm{d},1}$ is independent on δ. Similarly the CFL condition reads (see Theorem 2)

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{2\sqrt{2}|c^0_\delta(\cdot,\cdot)|_1} = \frac{\delta}{2\sqrt{2}|c^0_1(\cdot,\cdot)|_1}.$$

Ignoring all the constants independent on δ, we express the final time and CFL condition only with respect to δ:

$$\Delta t \simeq \delta \Delta x, \quad T^*_{\mathrm{d}} \simeq \delta^2.$$

In conclusion if we choose the parameter δ such that $\delta \simeq \Delta x$, then $T^*_{\mathrm{d}} \simeq \Delta t$. This implies that the gradient can get too small and in the worse case, one may have to reinitialize the gradient at each time iteration.

From our numerical simulations we have observed that if the ratio:

$$\frac{T/\Delta t}{\delta/\Delta x}$$

for final time T is small enough, (less than 15) then we are in the case where the gradient is far from zero and we are able to follow the evolution of the front.

Otherwise, for smaller δ or larger T, the numerical zero level set presents some oscillations, as it is the case in Fig. 5.

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