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# Minimax Multi-District Apportionments

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**Abstract** The problem of seat apportionment in electoral systems turns out to be quite complex, since no apportionment method exists which succeeds in verifying all the principal fairness criteria. Gambarelli (1999) introduced an apportionment technique which is custom made for each case, respects Hare minimum, Hare maximum and Monotonicity and satisfies other criteria in order of preference. In this paper a generalization of that method is proposed, in order to extend it to the multi-district election case, where criteria should be respected at a global as well as at a local level. An existence theorem and a generating algorithm are supplied.

*Keywords* apportionment, Hare quota, power index, Banzhaf, minimax

## 1. Introduction

Apportionments are a typical problem of the world of politics, as there is a need to assign seats to parties in proportion to the number of votes, or constituencies in proportion to the population. The problem consists in transforming an ordered set of nonnegative integers, the 'votes', into a set of integers, the 'seats', respecting some specific fairness conditions. Several methods have been constructed, but paradoxes and contradictions are likely to occur in many cases [e.g., Brams (1976)]. Starting from some results by Balinski, Demange and Young (1982, 1989), Gambarelli (1999) proposed an apportionment technique respecting the principal criteria of electoral systems. The approach was related to one-district elections and involves the determination


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of an order of preference for the satisfaction of criteria and introduces the concept of 'minimax solution'.

In this paper we propose a generalization of the above method to multi-district systems, where further problems arise. In such situations, the total apportionment depends not only on the global number of votes, but also on the votes obtained by parties in every district.

The basic apportionment criteria are presented in the following Section. Section 3 synthetically features the most known apportionment techniques. In Section 4 we recall the minimax method for one-district apportionments. The generalization of this method to the multi-district case is presented in Section 5. The ordering of the new criteria is discussed in Section 6. A theorem on the existence of the solutions is presented in Section 7. Section 8 shows further criteria to refine the solution. Section 9 supplies a comparison with the results of classical methods. An overview on the Banzhaf index and an algorithm generating solutions are reported in the Appendices.

## 2. Criteria

Apportionment can be defined as the process of allotting indivisible objects (seats) amongst players (parties) entitled to various shares. The related literature is quite vast: see for instance, Hodge and Klima (2005) for an overview.

Let $v = (v_1, \ldots, v_n)$ be the vector of valid votes won by the $n$ parties ($n \geq 2$) of an electoral system, where $s^T$ is the total number of seats to be assigned. We call:

$v^T = \sum_{i=1}^n v_i$ the total number of votes,

$h_i = \dfrac{s^T}{v^T} v_i$ the *Hare quota* of the $i$-th party,

$S^0$ the set of $n$-dimensional integer allotments $s = (s_1, \ldots, s_n)$ such that $\sum_{i=1}^n s_i = s^T$.

The problem consists in detecting, amongst all possible seat allotments, the aptest one to represent the Hare quota vector $\widetilde{h} = (h_1, \ldots, h_n)$. Obviously, if $\widetilde{h}$ is an integer vector, it turns out to be the best solution. Otherwise, a rounding allotment procedure is necessary, which should satisfy some fairness criteria. The most well known of these criteria are

(a) *Hare maximum* No party can obtain more seats than the ones it wins by rounding up its Hare quota.

(b) *Hare minimum* No party can obtain fewer seats than the ones it wins by rounding down its Hare quota.

(c) *Monotonicity* For any pair of parties, the one entitled to fewer votes cannot win more seats than the other one.

(d) *Superadditivity* A party formed by the union of two parties must at least gain a number of seats equal to the sum of the seats won by the single parties.

(e) *Symmetry* The apportionment must not depend on the order in which parties are considered. In particular, two parties having the same amount of votes must achieve the same number of seats.

It is well-known that no apportionment method exists which conjointly verifies all the above conditions, for all possible vectors of votes. For instance, consider a system in which there are only two parties gaining exactly the same amount of votes, and where an odd number of seats must be allotted. In such a case, criterion e) cannot be fulfilled, then an exogenous criterion must be applied. Analogously, it can be proved that c) and e) cannot be respected in general, if a) and d) hold and vice versa.

Moreover, some paradoxes may occur: 'Alabama', 'Population', 'New States' and so on [e.g. Brams (1976)].

Hence, the problem we are going to face is the search for a suitable compromise solution.

## 3. Classical apportionment methods

Hereafter some traditional apportionment techniques will be recalled and applied to a simple apportionment problem.

*Example 1* Let 19, 15, 6 be the valid votes obtained by the parties A, B, C, and 4 seats be shared among these parties.

The *Method of Largest Remainders* (also known as *Hamilton's Method*) assigns the initial seats according to the Hare minimum quotas and the remaining ones to the parties having the largest fractional parts of their quotients among the remainders.

In this case, party A and party B initially win one seat each. Subsequently, party A with 0.9 and party C with 0.6 obtain the last two seats. Hence Hamilton's Method provides the seat allotment $(2, 1, 1)$.

The *Method of the Greatest Divisors* (also known as *Method of d'Hondt* or *Jefferson's Method*) allots seats to the parties having the highest quotients after dividing their respective shares by 1, then by 2, then by 3, and so on. In our case, only the division by 2 is needed, because the quotients it generates are 9.5 for A, 7.5 for B, 3 for C. Consequently, the highest among all quotients are

19, 15, 9.5 and 7.5, so A and B gain two seats each, and no seat is assigned to C.

The *Method of the Greatest Divisors with quota* (also known as *Balinski-Young Method*) is an apportionment technique similar to that of d'Hondt, except for the impossibility for each party to exceed its Hare maximum quota: when a party reaches its Hare maximum quota, it does not participate in the seat allotment any longer. In this example, no party can exceed that quota, so the apportionments generated by the Method of d'Hondt and by Balinski-Young Method coincide.

For further apportionment techniques see for instance Nurmi (1982), Holubiec and Mercik (1994) and Hodge and Klima (2005).

## 4. The method of mimimax

The minimax method is inspired by the nucleolus (see Schmeidler, 1969).

### 4.1 *Preliminary definitions*

Let $s$ be a seat vector of $S^0$ and $v$ be a vote vector in $\mathbb{R}^n_+$. Consider the simplex:

$$\overline{X} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \ : \ \sum_{k=1}^{n} x_k = 1 \right\}.$$

Given a transform $t : \mathbb{R}^n_+ \longrightarrow \overline{X}$, we call

$$t(s) = \overline{s} = (\overline{s}_1, \ldots, \overline{s}_n), \ \ t(v) = \overline{v} = (\overline{v}_1, \ldots, \overline{v}_n).$$

For all $s \in S^0$, $\in \mathbb{R}^n_+$, $i, j = 1, \ldots, n$, $i \neq j$, we call

$e_j(v, s) = \overline{s}_j - \overline{v}_j$ the *bonus* of the $j$-th component;

$c_{ij}(v, s) = e_i(v, s) - e_j(v, s)$ the *complaint* of the $j$-th party against the $i$-th party;

$c(v, s)$ the *complaint vector*, i.e. the vector whose components are the non-negative complaints listed in non-increasing order.

The previous definitions allow us to establish a relation on $S^0$. For all $s'$, $s'' \in S^0$ we say that:

$s'$ is *indifferent to* $s''$ with respect to $v$ ($s' \approx s''$) if and only if $c(v, s') = c(v, s'')$.

$s'$ is *preferable to* $s''$ with respect to $v$ ($s' \succ s''$) if and only if $k \in \mathbb{Z}_+$ exists such that:

1. $c_k(v, s') < c_k(v, s'')$;
2. $c_h(v, s') = c_h(v, s'')$ for all $h < k$.

It is easy to prove that $\approx$ is an equivalence relation and that $\succ$ is a total order in the set $S^0$. Consequently, this relation determines a preference for the apportionment vectors of $S^0$.

Observe that, if a transform $t^* : \mathbb{R}^n_+ \longrightarrow \overline{X}$ exists such that $t^*(s) = t^*(v)$, then all bonuses and consequently all parties' complaints vanish.

We call *t-minimax criterion* (or *t-criterion*) the criterion which consists in keeping only the seat allotments not preferred, with respect to the distribution of votes, by other apportionments, and discarding all the others.

Gambarelli's method (1999) consists in the following procedure.

An order of importance of criteria to be applied, is preliminarily fixed: $C_1, C_2, \ldots, C_k$.

Then we call:

$S^1$ the subset of $S^0$ obtained after applying criterion $C_1$;

$S^2$ the subset of $S^1$ obtained after applying criterion $C_2$;
and so on until $S^k$.

We call $C_1C_2 \ldots C_k$-*solution* the set $S^k$ of allotments which respect the criteria $C_1, C_2, \ldots, C_k$, applied in sequence.

The first criterion to be applied in this method is called the *F-criterion*, and consists in discarding all seat apportionments violating at least one of the basic criteria: Hare maximum, Hare minimum and Monotonicity. Gambarelli (1999: 446) proved that the $F$-criterion applied as the first criterion $C_1$ in the sequence of criteria determining the solution, generates a non-empty set of seat allotments.

### 4.2 *The N-criterion*

Let $(x_1, \ldots, x_n)$ be a nonnegative integer vector such that $x_1 + x_2 + \ldots + x_n = x^T$. Consider the normalization map $N : \mathbb{R}^n_+ \longrightarrow \overline{X}$ such that

$$N(x_1, \ldots, x_n) = \left( \frac{x_1}{x^T}, \ldots, \frac{x_n}{x^T} \right) = (\overline{x}_1, \ldots, \overline{x}_n).$$

The *Normalization criterion* (or the *N-criterion*) is the $t$-minimax criterion associated to the transform $t = N$. Notice that the apportionments generated by the application of the $N$-criterion coincide with those provided by Hamilton's Method. Anyway, the next criterion to be applied will furthermore restrict the solution set achieved as yet.

### 4.3 *The $\beta$-criterion*

This criterion is based on the Banzhaf normalized power index (1965). Some notes on this index (here simply called *$\beta$-power index* are supplied in Appendix A. We consider the $\beta$-power index particularly suitable for electoral systems, because of its proportionality properties in the allotment of seats.

In order to enunciate the second minimax criterion, we will consider the transform $\overline{\beta} : \mathbb{R}^n_+ \longrightarrow \overline{X}$, associating to every vote distribution $v = (v_1, \ldots, v_n)$ and to every seat allotment $s = (s_1, \ldots, s_n)$ the Banzhaf normalized power indices $\overline{\beta}(v)$ and $\overline{\beta}(v)$ of the related voting games with simple majority quota. The *$\beta$-criterion* is the $t$-minimax criterion associated to the transform $t = \overline{\beta}$.

### 4.4 *Minimax solutions*

The two previous minimax criteria allow us to make use of two different sequences of criteria: by choosing $C_1 = F$, $C_2 = N$, $C_3 = \beta$, we will have the $FN\beta$-solution; by choosing $C_1 = F$, $C_2 = \beta$, $C_3 = N$, the method will generate $F\beta N$-solution. Gambarelli (1999: 455) proved that each $FN\beta$-solution and each $F\beta N$-solution consist of a non-empty sets of seat allotments. The next example shows an application of the minimax method.

*Example 2* Let 8 seats be assigned to the 4 parties $A, B, C, D$ entitled to the votes (50, 30, 15, 5).

If we apply the $F$-criterion, then all seat apportionments are discarded except $s_1 = (4, 2, 1, 1)$, $s_2 = (4, 3, 1, 0)$ and $s_3 = (4, 2, 2, 0)$.

In fact, Hare quotas are: 4 for party $A$, 2.4 for party $B$, 1.2 for party $C$, and 0.4 for party $D$, so all the remaining seat distributions would violate either Hare maximum or Hare minimum. Subsequently, $N$-criterion is applied to the apportionment set:

$$S^1 = \{(4, 2, 1, 1),\ (4, 3, 1, 0),\ (4, 2, 2, 0)\}$$

the normalized vector of votes is $\overline{v} = (0.5, 0.3, 0.15, 0.05)$;

the normalized vectors of the seat distributions are respectively:
$\overline{s}_1 = (0.5, 0.25, 0.125, 0.125)$,
$\overline{s}_2 = (0.5, 0.375, 0.125, 0)$,
$\overline{s}_3 = (0.5, 0.25, 0.25, 0)$;

the bonuses are:
$e(v, s_1) = (0, -0.05, -0.025, 0.075)$,
$e(v, s_2) = (0, 0.075, -0.025, -0.05)$,
$e(v, s_3) = (0, -0.05, 0.1, -0.05)$.

Consequently, the three complaint vectors are:
$c(v, s_1) = (0.125, 0.100, 0.075, 0.050, 0.025, 0.025)$,
$c(v, s_2) = (0.125, 0.100, 0.075, 0.050, 0.025, 0.025)$,
$c(v, s_3) = (0.150, 0.150, 0.100, 0.050, 0.050, 0.000)$.

According to the previously defined relation, $s_1 \approx s_2 > s_3$.

The application of the $N$-criterion causes the elimination of $s_3$. Then the set of 'surviving' allotments is

$S^2 = \{(4, 2, 1, 1),\ (4, 3, 1, 0)\}$.

The next step is the application of the $\beta$-criterion. The Banzhaf normalized power index of votes for simple majority can be obtained after some computations: $\overline{\beta}(v) = (0.7, 0.1, 0.1, 0.1)$.

The $\beta$-index of seats of each apportionment, for simple majority, has to be computed for $s_1 = (4, 2, 1, 1)$ and $s_2 = (4, 3, 1, 0)$:

$\overline{\beta}(s_1) = (0.7, 0.1, 0.1, 0.1)$,
$\overline{\beta}(s_2) = (0.6, 0.2, 0.2, 0)$.

The bonuses are respectively:
$\overline{\beta}(s_1) - \overline{\beta}(v) = (0, 0, 0, 0)$;
$\overline{\beta}(s_2) - \overline{\beta}(v) = (-0.1, 0.1, 0.1, -0.1)$.

So this last criterion yields the unique $FN\beta$-solution:

$S^3 = \{(4, 2, 1, 1)\}$.

In general, $S^3$ may be composed by more than one seat allotment.

## 5. Multi-district apportionments

Our aim is to extend the minimax method to the multi-district case.

### 5.1 *A leading example*

We will show our model using the following

*Example 3* An electoral system is composed of two districts (to which 6 and 5 seats must be assigned) and three parties $A$, $B$, $C$. The valid votes obtained are shown in Table 1.

The local Hare quotas are reported in Table 2. The last row of Table 2 shows the global Hare quotas, i.e. the Hare quotas of the totals of Table 1. Notice that the sum of local Hare quotas differs from the global Hare quotas. Table

Table 1 — The votes of Example 3

| Votes | Party A | Party B | Party C | Totals |
|---|---|---|---|---|
| District I | 50 | 60 | 10 | 120 |
| District II | 10 | 10 | 60 | 80 |
| Totals | 60 | 70 | 70 | 200 |

Table 2 — The local and global Hare quotas of Example 3

| Hare quotas | Party A | Party B | Party C | Totals |
|---|---|---|---|---|
| District I | 2.500 | 3.000 | 0.500 | 6 |
| District II | 0.625 | 0.625 | 3.750 | 5 |
| Totals | 3.125 | 3.625 | 4.250 | 11 |
| Global Hare quotas | 3.300 | 3.850 | 3.850 | 11 |

Table 3 — The normalized votes and $\beta$-indices of votes of Example 3

| | Party A | Party B | Party C | Totals |
|---|---|---|---|---|
| *Local normalized votes* | | | | |
| District I | 0.416 | 0.500 | 0.083 | 1 |
| District II | 0.125 | 0.125 | 0.750 | 1 |
| *Global normalized votes* | 0.300 | 0.350 | 0.350 | 1 |
| *Local $\beta$-indices of votes* | | | | |
| District I | 1/5 | 3/5 | 1/5 | 1 |
| District II | 0 | 0 | 1 | 1 |
| *Global $\beta$-indices of votes* | 1/3 | 1/3 | 1/3 | 1 |

3 shows the related normalized votes and $\beta$-indices of votes. In this case the last row shows these data at the global level, too.

### 5.2 *Data and variables*

We will utilize the following indices:

$d$ to denote the districts ($d = 1, \dots, n_d$) and

$p$ to denote the parties ($p = 1, \dots, n_p$).

A multi-district apportionment problem is based on the following data:

$V = [v_{dp}]$, the matrix of valid votes obtained by the $p$-th party in the $d$-th district;

$a = [a_d]$, the vector of the seats to be assigned to the $d$-th district (in our example (6, 5)).

Other computation parameters are:

$S = [s_{dp}]$, any integer matrix whose elements are seat distributions which respect the total seats to be allotted in the districts, i.e.

$$\sum_{p=1}^{n_p} s_{dp} = a_d \quad (d = 1, ..., n_d).$$

$b = [b_p]$, the total seats assigned to the $p$-th party in matrix $S$, i.e.

$$\sum_{d=1}^{n_d} s_{dp} = b_p \quad (p = 1, ..., n_p).$$

Let $S^0$ be the set of matrices $S$ respecting the above conditions.

We will generalize the definitions of section 4.1 as follows.
For all $v \in V$, $s \in S^0$, $p, q = 1, \dots, n_p$, $p \neq q$, we call

$e_{dp}(V, S) = \bar{s}_{dp} - \bar{v}_{dp}$ the *bonus* of the $p$-th party in the $d$-th district;

$c_{dpq}(V, S) = e_{dp}(V, S) - e_{dq}(V, S)$ the *complaint* of the $p$-th party against the $q$-th party in the $d$-th district;

$c(V, S) = (c_1(V, S), \dots, c_k(V, S))$ the $S$-*complaint vector*, i.e. the vector whose components are the non-negative complaints of the whole matrix $S$, listed in non-increasing order.

The above definitions allow us to establish on $S^0$ the same preference relationship introduced in section 4.1.

### 5.3 *Solutions*

We will use the same concept of solution introduced in Section 4.1, with the simple substitution of $S^k$ with $\mathbb{S}^k$ for all involved $k$.

For an easier understanding of the criteria used, we will present them together with the construction of the solution to example 3. Obviously, the order of criteria can be changed depending on the importance given to them. Here the following sequence is used:

*$F_G$-criterion* ($F$-criterion for the global apportionments);

*$N_G$-criterion* ($N$-criterion for the global apportionments);

*$\beta_G$-criterion* ($\beta$-criterion for the global apportionments);

*$N_L$-criterion* ($N$-criterion for the local apportionments).

*$\beta_L$-criterion* ($\beta$-criterion for the local apportionments).

The *$F_G$-criterion*, *$N_G$-criterion* and *$\beta_G$-criterion* are no other than the corresponding criteria presented in section 4.1, applied to global Hare quotas of the votes. In our example the application of the $F_G$-criterion leads to the only matrices where total seats per party are: (3, 4, 4), inasmuch this distribution is the only one which respects monotonicity, the Hare minimum and Hare maximum at global level. As the $F_G$-criterion supplies only one allocation of total seats, the $N_G$-criterion and the $\beta_G$-criterion maintain the set of the above matrices unchanged.

The $N_L$-criterion consists in keeping only the matrices which minimize the $S$-complaint vectors, according to what is indicated in the $t$-minimax criterion presented in section 4.1, using $t = N$. In our example, to help the search for such matrices, we can focus on the only ones that respect the Hare minimum and Hare maximum in all the districts, as they are preferable to all the others. These are shown in the upper part of Table 4. In the same tables the rounded normalized seats $\bar{s}_{dp}$, the bonuses $e_{dp}$ and the complaints $c_{dpq}$ are shown.

The maximum values of the $S$-complaint vectors of the four matrices are respectively 0.20, 0.25, 0.225, 0.425. The matrix which corresponds to the minimum of such values is the first. Then the $F_G N_G \beta_G N_L$-solution is unique and is the matrix shown in Table 5.

The $\beta_L$-criterion consists in keeping only those matrices which minimize the $S$-complaint vectors, according to what is indicated in the $t$-minimax cri-

Table 4 — The computations to obtain the $F_G N_G \beta_G N_L \beta_L$-solution of Example 3

| $S$ | A | B | C | A | B | C |
|---|---|---|---|---|---|---|
| District I | 3 | 3 | 0 | 2 | 4 | 1 |
| District II | 0 | 1 | 4 | 1 | 0 | 4 |
| $\bar{s}_1 p$ | 0.5 | 0.5 | 0 | $0.\overline{3}$ | $0.\overline{6}$ | 0 |
| $\bar{s}_2 p$ | 0 | 0.2 | 0.8 | 0.2 | 0 | 0.8 |
| $e_1 p$ | $0.08\overline{3}$ | 0 | $-0.08\overline{3}$ | $-0.08\overline{3}$ | $0.1\overline{6}$ | $-0.08\overline{3}$ |
| $e_2 p$ | −0.125 | 0.075 | 0.050 | 0.075 | −0.125 | 0.050 |
| $c_1 pq$ | 0.17 | 0.08 | 0.08 | 0.25 | 0.25 | 0 |
| $c_2 pq$ | 0.2 | 0.175 | 0.025 | 0.2 | 0.175 | 0.025 |
| | | | | | | |
| District I | 2 | 3 | 1 | 1 | 4 | 1 |
| District II | 1 | 1 | 3 | 2 | 0 | 3 |
| $\bar{s}_1 p$ | $0.\overline{3}$ | 0.5 | $0.1\overline{6}$ | $0.1\overline{6}$ | $0.1\overline{6}$ | $0.1\overline{6}$ |
| $\bar{s}_2 p$ | 0.2 | 0.2 | 0.6 | 0.4 | 0 | 0.6 |
| $e_1 p$ | $-0.08\overline{3}$ | 0 | $-0.08\overline{3}$ | −0.85 | $0.1\overline{6}$ | $-0.08\overline{3}$ |
| $e_2 p$ | 0.075 | 0.075 | 0.150 | 0.275 | −0.125 | −0.150 |
| $c_1 pq$ | 0.17 | 0.08 | 0.08 | 0.42 | 0.33 | 0.08 |
| $c_2 pq$ | 0.225 | 0.225 | 0 | 0.425 | 0.4 | 0.025 |

Table 5 — The $F_G N_G \beta_G N_L \beta_L$-solution of Example 3

| | | |
|---|---|---|
| 3 | 3 | 0 |
| 0 | 1 | 4 |

Table 6 — The $F_G N_G \beta_G \beta_L$-solution of Example 3

| | | |
|---|---|---|
| 2 | 3 | 1 |
| 1 | 1 | 3 |

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 1 | 2 |

Table 7 — The $F_G N_G \beta_G \beta_L N_L$-solution of Example 3

| | | |
|---|---|---|
| 2 | 3 | 1 |
| 1 | 1 | 3 |

terion presented in section 4.1, using $t = \beta$. In our example, due to uniqueness, the $F_G N_G \beta_G N_L \beta_L$-solution coincides with the $F_G N_G \beta_G N_L$-solution.

If we want to change the order of the two local criteria, we must apply the $\beta_L$-criterion to the matrices of the $F_G N_G \beta_G$-solution. Observe that, among such matrices, there are two, and only two, which perfectly respect the power indices of the votes, shown in Table 3. These matrices (shown in Table 6) lead to null $S$-complaint vectors and therefore are the $F_G N_G \beta_G \beta_L$-solution.

It is now easy to verify that the $F_G N_G \beta_G \beta_L N_L$-solution is the one shown in Table 7.

Observe that all the above solutions remain the same if the order of $N_G$-criterion and $\beta_G$-criterion is exchanged, as mentioned at the beginning of the presentation of these criteria. However, the two solutions obtained by inverting the order of local criteria are different. An example is now given in which these solutions coincide.

*Example 4* An electoral system is made up of two districts (to which 20 and 80 seats must be assigned) and two parties $A$, $B$. The valid votes obtained are shown in Table 8.

The local and global Hare quotas are reported in Table 9. Observe that all Hare quotas are integer numbers. Table 10 shows the related normalized votes and $\beta$-indices of votes.

It is easy to verify that the $F_G N_G$-solution is the set of matrices shown in Table 11, varying the integer $k$ from 0 to 20. Similarly for the $F_G \beta_G$-solution. Obviously, the $F_G N_G \beta_G$-solution and the $F_G \beta_G N_G$-solution coincide with the above solutions.

Now we will continue with the calculations of the $F_G N_G \beta_G N_L$-solution (see Table 12). After some algebra we obtain that 5 is the value of $k$ which minimizes $\max\{(4-k)/10,\ (9-k)/40\}$. Therefore the $F_G N_G \beta_G N_L$-solution is made up of the only matrix shown in Table 13 and coincides with the $F_G N_G \beta_G N_L \beta_L$-solution.

Going on to the calculation of the $F_G N_G \beta_G \beta_L N_L$-solution, it is easy to verify that such a solution is the set of matrices shown in Table 11 for which $20 - k > k$ and $30 + k < 50 - k$. These matrices correspond to the values of $k$ between 0 and 9. $k = 5$ is included in these. Therefore the $F_G N_G \beta_G \beta_L N_L$-solution coincides with the $F_G N_G \beta_G N_L \beta_L$-solution.

Table 8 — The votes of Example 4

| Votes | Party A | Party B | Totals |
|---|---|---|---|
| District I | 320 | 80 | 400 |
| District II | 4680 | 4920 | 9600 |
| Totals | 5000 | 5000 | 10000 |

Table 9 — The local and global
Hare quotas of Example 4

| Hare quotas | Party A | Party B | Totals |
|---|---|---|---|
| District I | 16 | 4 | 20 |
| District II | 39 | 41 | 80 |
| Totals | 55 | 45 | 100 |
| Global Hare Quotas | 50 | 50 | 100 |

Table 10 — The normalized votes and
$\beta$-indices of votes of Example 4

| | Party A | Party B | Totals |
|---|---|---|---|
| *Local normalized votes* | | | |
| District I | 0.8000 | 0.2000 | 1 |
| District II | 0.4875 | 0.5125 | 1 |
| *Global normalized votes* | 0.5 | 0.5 | 1 |
| *Local $\beta$-indices of votes* | | | |
| District I | 1 | 0 | 1 |
| District II | 0 | 1 | 1 |
| *Global $\beta$-indices of votes* | 0.5 | 0.5 | 1 |

Table 11 — The solutions of Example 4

| Solutions | Party A | Party B | Totals |
|---|---|---|---|
| District I | $20 - k$ | $k$ | 20 |
| District II | $30 + k$ | $50 - k$ | 80 |
| Totals | 50 | 50 | 100 |

Table 12 — The computations to obtain the $F_G N_G \beta_G N_L \beta_L$-solution of Example 4

| | Party A | Party B |
|---|---|---|
| $\bar{s}_{1p}$ | $(20 - k)/20$ | $k/20$ |
| $\bar{s}_{2p}$ | $(30 + k)/80$ | $(50 - k)/80$ |
| $e_{1p}$ | $(4 - k)/20$ | $(k - 4)/20$ |
| $e_{2p}$ | $(9 - k)/80$ | $(k - 9)/80$ |

| | |
|---|---|
| $c_{1pq}$ | $(4 - k)/10$ |
| $c_{2pq}$ | $(9 - k)/40$ |

## 6. On the ordering of criteria

In the examples in the last Section we gave greater importance to global level criteria than to those at a local level; however, there is no change in the technique if the order is permuted. However, it seems reasonable to apply the $F_G$-criterion first, as this guarantees respect to the will of the entire electorate. Complaints are often heard about the misrepresentations of parliamentary majorities, due to local roundings. Such dissatisfaction seems reasonable inasmuch as a Parliament represents the entire population. Subsequently, the choice of order of the criteria depends on the national situation which it is applied to. In particular, the choice of priority between adhering to normalized votes or to power indices in the first case gives preference to the proportional aspect; in the second case to the majority aspect, which is essential for democracy.

## 7. On the existence of solutions

*Theorem 1* For every multi-district apportionment problem, all solutions having the $F_G$-criterion as the first criterion, are not empty.

Table 13 — The $F_G N_G \beta_G N_L \beta_L$-solution and the $F_G N_G \beta_G \beta_L N_L$-solution of Example 4

| | |
|---|---|
| 15 | 5 |
| 35 | 45 |

*Proof* In our hypotheses at least one distribution of seats $b = (b_1, \ldots, b_n)$ which is able to verify the $F$-criterion exists (see the end of section 4.1). It is known that, given any two integer vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ having equal sums of the components, at least one integer matrix $(n_d \times n_p)$ exists, which has such vectors as totals of row and column. Each one of the other criteria $C_{k+1}$ generates a nonempty subset of the $S^k$-solution, inasmuch it chooses, from these matrices, only the optimal ones according to that criterion; however, in the case of equal optimality, it keeps them all. □

An algorithm for the automatic computation of the solution is shown in Appendix B.

## 8. Further criteria

In cases of non-uniqueness, further criteria can be added and applied in order to restrict the solution set, using the same techniques. For instance, after the five criteria have been applied, it is again possible to choose whether to give preference to the $\beta_L$-criterion or the $N_L$-criterion. Taking into consideration the corresponding complaints, it is possible to keep only those matrices for which the maximum of such a vector corresponds to a minimum number of votes. Resorting to this method a further restriction of the solution set can be obtained. The uniqueness of the final matrix, however, cannot be guaranteed; e.g. in the case where the global Hare quotas are of the type shown in the example in Section 2. In such cases it is therefore necessary to apply other methods, based for example on the candidates' ages, draws and so on.

## 9. A comparison with other methods

Tables 14 and 15 show the allocations assigned by the principal classical methods of rounding (presented in Section 3) in the cases of examples 3 and 4, indicating some criteria which are violated.

Regarding Table 14, we add that all apportionments respect, at a local level, symmetry, monotonicity and Hare minimum; at global level Hare minimum and power index. Note that the new solutions respect all the criteria at global level; in particular the solutions in the last column respect all the criteria, contrary to classical methods.

Regarding Table 15, we add that all apportionments respect: at a local level, symmetry, monotonicity and power index; at global level, monotonicity. Note that the new solutions respect all the criteria at global level, contrary to classical methods.

## 10. Conclusions

The concept of solution proposed here avoids most of the distortions which arise when using classical methods and, when unavoidable, minimizes their negative effects. The procedures to obtain the solutions are simply applicable to automatic computation. The majority of classical techniques were developed before computers existed, or at least before they came into common use. We think it is now time to get up-to-date with electoral regulations, too.

## Appendix

### A *Some notes on the normalized Banzhaf power index*

In the Theory of Cooperative Games, a power index is a function which assigns shares of power to the players as a quantitative measure of their influence in voting situations. For instance, suppose that a system is composed of three parties without particular propensity for special alliances, and that a simple majority is required. If the allotment of seats is (40, 30, 30), any reasonable power index will assign an equal power allotment of (1/3, 1/3, 1/3). If the seat allotment of the three parties is (60, 30, 10), then any reasonable index would give a power share of (1, 0, 0), since the first party attains the majority by itself. Some complications occur if the seat allotment is (50, 30, 20). If $A, B, C$ are the three parties, we can remark that $A$ is crucial for the three coalitions $\{A, B, C\}$, $\{A, B\}$ and $\{A, C\}$, i.e. such coalitions attain the majority with party $A$ and lose it without $A$. On the other hand, party $B$ is only crucial for the coalition $\{A, B\}$ and party $C$ is only crucial for the coalition $\{A, C\}$. In general, the power indices are based on the crucialities of the parties. In particular, the Banzhaf index (1965) assigns to each party the number of coalitions for which it is crucial. In our example, the assigned powers are (3, 1, 1).

The *Banzhaf normalized power index* assigns to each party a quota of the unity proportional to the number of coalitions for which it is crucial. In our example, the assigned powers are (3/5, 1/5, 1/5).

In addition to John F. Banzhaf, several authors independently introduced various indices having the same normalization: James S. Coleman (1971), Lionel S. Penrose (1946) and, according to a particular interpretation, Luther

Table 14 — The allocations assigned by various methods in the case of example 3

| Method | Hamilton | | | Hondt-Jefferson | | | Balinski-Young | | | $F_G N_G \beta_G N_L \beta_L$ and $F_G \beta_G N_G N_L \beta_L$ | | | $F_G N_G \beta_G \beta_L N_L$ and $F_G \beta_G N_G \beta_L N_L$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | A | B | C | A | B | C | A | B | C | A | B | C | A | B | C |
| District I | 3 | 3 | 0 | 3 | 3 | 0 | 3 | 3 | 0 | 3 | 3 | 0 | 2 | 3 | 1 |
| District II | 1 | 0 | 4 | 0 | 0 | 5 | 0 | 1 | 4 | 0 | 1 | 4 | 1 | 1 | 3 |
| Totals | 4 | 3 | 4 | 3 | 3 | 5 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 |
| *Local Breaks* | | | | | | | | | | | | | | | |
| Hare max. | | | | | | X | | | | | | | | | |
| Power indices | | | X | | | X | | | X | | | X | | | |
| *Global Breaks* | | | | | | | | | | | | | | | |
| Symmetry | | | X | | | | | | | | | | | | |
| Monotonicity | | | X | | | | | | | | | | | | |
| Hare max. | | | | | | X | | | | | | | | | |

Table 15 — The allocations assigned by various methods in the case of Example 4

| Method | Hamilton<br>Hondt-Jefferson<br>Balinski-Young | | $F_G N_G \beta_G N_L \beta_L$<br>$F_G \beta_G N_G N_L \beta_L$<br>$F_G \beta_G N_G \beta_L N_L$ | |
|---|---|---|---|---|
| | A | B | A | B |
| District I | 16 | 4 | 15 | 5 |
| District II | 39 | 41 | 35 | 45 |
| Totals | 55 | 45 | 50 | 50 |
| | | | | |
| Local Breaks | | | | X |
| Hare min. | | | | X |
| Hare max. | | | | |
| | | | | |
| Global Breaks | | | | |
| Symmetry | | X | | |
| Hare min. | | X | | |
| Hare max. | | X | | |
| Power indices | | X | | |

Martin in the XVIII century (see Riker, (1986) and Felsenthal and Machover (2005)). That is the reason why this index should be mentioned as 'Banzhaf-Coleman-Martin-Penrose Normalized power index'.

A combinatorial interpretation is shown in Palestini (2005). For the automatic computation in general cases, we suggest the algorithm by Bilbao et al. (2000). The algorithm by Gambarelli (1996) takes into account previous computations, when the seats vary recursively. Then (with reference to the Appendix B) it is more suitable for the application of the $\beta_L$-criterion, if computed before the $N_L$-criterion.

Overviews of further power indices can be found in Gambarelli (1983), Holubiec and Mercik (1994), Gambarelli and Owen (2004).

## B *An algorithm generator of the solutions*

We show an algorithm for the automatic generation of the solutions having as first criteria $F_G N_G \beta_G$ or $F_G \beta_G N_G$. Notice that this procedure can be easily structured for parallel processing, so that the time of computation can be considerably reduced.

*Input*

$V$ the valid votes.

$a$ the seats to be assigned to the districts.
'Global option' of the ordering of criteria at the global level
($F_G N_G \beta_G$ or $F_G \beta_G N_G$).
'Local option' of the ordering of criteria at the local level
($N_L \beta_L$ or $\beta_L N_L$).

*Output*
$S_1, S_2, \ldots, S_n$ the set of survived matrices.

*Working area*
$\overline{N^v}$ the matrix of normalized votes.
$\overline{\beta^v}$ the matrix of the Banzhaf normalized power indices of votes.
$\overline{N^s}$ the matrix of normalized seats.
$\overline{\beta^s}$ the matrix of the Banzhaf normalized power indices of the seats.
$R_1, R_2, \ldots, R_n$ the set of matrices survived to the first local criterion.
$B$ the set of vectors $b$ generated by criteria $F_G N_G \beta_G$ or $F_G \beta_G N_G$.
$c_{CUR}$ the vector $c(V, S)$ at the current step.
$c_{MIN}$ the minimum vector $c_{CUR}$ of the past steps.
$f_d$, $f_p$ pointers to set $S$.
$S$ the matrix in construction:

| | | | | | |
|---|---|---|---|---|---|
| $s_{11}$ | $s_{12}$ | $s_{13}$ | $\cdots$ | $s_{1n_p}$ | $a_1$ |
| $s_{21}$ | $s_{22}$ | $s_{23}$ | $\cdots$ | $s_{2n_p}$ | $a_2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_{n_d 1}$ | $s_{n_d 2}$ | $s_{n_d 3}$ | $\cdots$ | $s_{n_d n_p}$ | $a_{n_d}$ |
| $b_1$ | $b_2$ | $b_3$ | $\cdots$ | $b_{n_p}$ | $\sum b_d = \sum a_p$ |

*Procedure*
Read the input data.
Compute $\overline{V}$.
Compute $\overline{\beta}$ using Bilbao et al. (2000).
Compute $B$ according to the global option.
Set maximum values to $c_{MIN}$.

For every $b$ of $B$:
Set $(n_d, n_p)$ as first pointers.
Move $n_d$ to $f_d$ and $n_p$ to $f_p$.
For all $S$ of the current $b$:
Set $S$ (move $a$ and $b$ to the arrays of the totals and move zeroes to all $s_{dp}$).

Call the subroutine 'Construction of the next $S$'.
Update $f_d$, $f_p$.
Call the subroutine 'Generation of solution' using $R_k$ as output.
Return.
Return

Set maximum values to $c_{MIN}$.
Move $n$ to $m$.
Varying $t$ from 1 to $m$:
Move $R_t$ to $S$.
Call subroutine 'Generation of solution' using $S_n$ as output.
Return
End

*Subroutine 'generation of solution'*
If the local option is $N_L$,
compute $\overline{N^s}$
else
compute $\overline{\beta^s}$ using Gambarelli (1996) (case $\beta_L N_L$)
or Bilbao et al. (2000) (case $N_L \beta_L$).
During the above computation, construct $c_{CUR}$ and compare it with $c_{MIN}$.
Just if $c_{CUR} > c_{MIN}$ exit.
When the construction of the normalized matrix is ended:
If $c_{CUR} = c_{MIN}$
move $n + 1$ to $n$
else
move 1 to $n$
move $c_{CUR}$ to $c_{MIN}$.
Move $S$ to output.
Exit

*Subroutine 'construction of the next S'*
If $\min\{a_{n_d}, b_{n_p}\} = a_{n_d}$, then
move $a_{n_d}$ to $s_{n_d n_p}$,
move 0 to all the other elements of the last row
and to $a_{n_d}$,
move $(b_{n_p} - a_{n_d})$ to $b_{n_p}$,
and iterate the procedure on the submatrix
obtained by eliminating the last row,
i.e. decreasing $n_d$ by 1.
If $\min\{a_{n_d}, b_{n_p}\} = b_{n_p}$, then
move $b_{n_p}$ to $s_{n_d n_p}$,

move 0 to all other elements of the last column
and to $b_{n_p}$,
move $(a_{n_d} - b_{n_p})$ to $b_{n_p}$,
and iterate the procedure on the submatrix
obtained by eliminating the last column,
i.e. decreasing $n_p$ by 1.

At the end of the procedure we obtain $a_1 = b_1$;
this number will be moved to $s_{11}$.
Exit.

*Example* In example 3 the construction sequence of the first $S$ is:

| | | | |
|---|---|---|---|
| | | | 6 |
| | | | 5 |
| 3 | 4 | 4 | 11 |

| | | | |
|---|---|---|---|
| 0 | | | 6 |
| 4 | | | 1 |
| 3 | 4 | 0 | 7 |

| | | | |
|---|---|---|---|
| 0 | | | 6 |
| 0 | 1 | 4 | 1 |
| 3 | 3 | 0 | 6 |

| | | | |
|---|---|---|---|
| 3 | 0 | | 3 |
| 0 | 1 | 4 | 0 |
| 3 | 0 | 0 | 3 |

| | | | |
|---|---|---|---|
| 3 | 3 | 0 | 0 |
| 0 | 1 | 4 | 0 |
| 0 | 30 | 0 | 0 |

| | | |
|---|---|---|
| 2 | 4 | 0 |
| 1 | 0 | 4 |

| | | |
|---|---|---|
| 3 | 2 | 1 |
| 0 | 2 | 3 |

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 0 | 3 | 2 |

. . .

| | | |
|---|---|---|
| 0 | 2 | 4 |
| 3 | 2 | 0 |

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