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Reformulation of Some Power Indices in Weighted Voting Games

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Abstract In TU-games in characteristic function form, some power indices can be expressed by means of different formulas, which potentially allow remarkable advantages in calculation. By exploiting the powerful instrument of average essentialities of coalitions, we propose new representations for the Banzhaf value, the normalized Banzhaf index, the coalition value and the Myerson value. When considering weighted voting games, some of these formulas only depend on the number of feasible coalitions exceeding the winning quota, and this may imply a considerable reduction of computational cost in algorithms.

JEL Classifications C71
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1. Introduction

Power indices constitute a fundamental instrument for the evaluation of individual and coalitional payoff of players in weighted voting games. As a matter of fact, it is well-known that several power indices of cooperative TU-games in characteristic function form admit a number of different representations, often based on analytic or combinatorial properties. An obvious example is the Shapley value, which in addition to its traditional formulation due to Shapley (1953), can also be expressed by integrating partial derivatives of the multilinear extension of the game as in Owen (1995), or by exploiting the peculiar instrument of average essentialities of coalitions as defined by Gambarelli (1990). The main idea on which his work was based concerned the drawing up of a new formula for the Shapley value, which turned out to be really advantageous in algorithms, especially in the reduction of calcula-

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tion time. Moreover, it was very useful to deduce relationships between that value and the barycenter of the imputation set of the related game.

Essentially, the present work aims at finding alternative formulations for some of the most common values by means of average essentialities, expressed either in classical or in adapted form. Basically, the objectives to meet are two: to reduce computational cost of possible calculation algorithms and to open some doors in the fascinating topic of geometric properties of values in imputation simplices, first introduced by Shapley (1953), and subsequently analyzed by Gambarelli (1990).

As far as the first one is concerned, the search for a better way to express a power index makes sense particularly: sparing operations for calculation or storage to be used is very important in algorithms, especially when dealing with a big number of players.

On the other hand, the position of a value in the imputation set of the related game deserves a special attention: in detail, necessary and sufficient conditions in order that the barycenter of the imputation set can coincide with a power index can easily be provided. Besides, one can straightaway construct simplices or even imputation sets whose barycenter is a specific value, and that may lead directly to the construction of games endowed with interesting properties.

In section 2, notation and definitions which will be used in the following are introduced. In section 3, the Banzhaf value and the normalized Banzhaf index are investigated. The new formula for the Banzhaf value enunciated in Proposition 3.1 immediately implies a particular representation for the normalized Banzhaf index in weighted voting games, only depending on the number of coalitions of all possible sizes, overtaking the majority quota. Remark 3.3 shows how a calculation algorithm based on this formula might be less expensive, in terms of both data-storing and operations needed to be performed.

Section 4 examines the coalitional value, originally introduced by Owen (1982), which suits situations in which political parties make agreements in order to form preliminary coalitions. At a price of some modifications in definitions, a new representation for this value can be written down. A numerical example at the end of this section expounds the possible advantages of this formula as far as data-storing is concerned.

An adequate power index for voting models appears to be the Myerson value, i.e. the Shapley value for graph-restricted games. It seems natural to regard the vertices of a graph as the players of a cooperative game and its edges as the feasible alliances. The subject of relationship between Myerson value and properties of graphs has been widely studied, for instance by Algaba et al. (2001) and Fernandez et al. (2002). Section 5 is devoted to the

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development of some further considerations about the Myerson value. Besides exposing it analogously to the Shapley value by means of appropriate redefinitions of the average essentialities, one can use some basic notions of graph theory to achieve a new representation. Proposition 5.9 shows that fixing some hypotheses on the considered weighted voting game, the payoff for each player only depends on the number of winning feasible coalitions she takes part in, which is strictly related to the number of links she has as a vertex of the graph. In order to clarify the previous formula, a numerical example about a 5-person weighted majority game follows. The strong impression one has is that notwithstanding the remarkable difficulties which the formulation of the Myerson value generally involves, average essentialities might provide new interesting results.

2. Preliminary definitions

Let us consider a TU-game in characteristic function form $v: 2^N \to R$, whose set of players is $N = \{1,...,n\}$. The notation we will use is the same as in Gambarelli (1990, 445), i.e. let us call:

$S(s,0)$ the set of the $\binom{n}{s}$ s-player coalitions;

$S(s,+i)$ the set of the $\binom{n-1}{s-1}$ s-player coalitions including the i-th element;

$S(s,-i)$ the set of the $\binom{n-1}{s}$ s-player coalitions excluding the i-th element.

All those sets are lexicographically ordered, and calling $S_j(s,k)$ the j-th element of the set $S(s,k)$, with $k = 0$ or $k = \pm i$, $i \in N$, we give the following definitions:

Definition 2.1 For every coalition $S \subseteq N$, the *essentiality* $e(S)$ is the number:

$$e(S) := v(S) - \sum_{i \in S} v(\{i\})$$

Particularly $e_j(s,k) := e\big(S_j(s,k)\big)$, for all $s \in N$, $k = -n, -n+1, ..., -1, 0, 1, ..., n-1, n$; $e(S)$ can even be thought of as the excess of any coalition S with respect to the payoff vector whose j-th coordinate is the characteristic value of the j -th player, as stated by Owen (1995, 319).

Definition 2.2 For $s=1,...,n$, the *average essentiality of the* s *-player coalitions* is:

$$a(s,0)=\frac{1}{\binom{n}{s}}\sum_{j=1}^{\binom{n}{s}} e_j(s,0)$$

Definition 2.3 For $s=1,...,n$, and $i\in S$, the *average essentiality of the* s*-player coalitions excluding the* i *-th* is:

$$a(s,-i)=\frac{1}{\binom{n-1}{s}}\sum_{j=1}^{\binom{n-1}{s}} e_j(s,-i)$$

Remark 2.4 The previous definition is compatible with the convention

$$a(n,-i)=0, \quad \forall i\in N$$

Definition 2.5 For $s=1,...,n$ and $i\in N$, the *average essentiality of the* s*-player coalitions including the* i *-th* is:

$$a(s,+i)=\frac{1}{\binom{n-1}{s-1}}\sum_{j=1}^{\binom{n-1}{s-1}} e_j(s,+i)$$

Definition 2.6 Given the nonnegative integers $q,w_1,\ldots,w_n$, such that

$$0<q\le\sum_{j=1}^{n} w_j$$

we denote with $v\equiv[q;w_1,\ldots,w_n]$ the simple *weighted voting game* on N defined by

$$v(S)=1 \text{ if } w(S)\ge q, v(S)=0 \text{ if } w(S)<q, \text{for all } S\subseteq N$$

where $w(S)=\sum_{i\in S} w_i$

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w_i is the number of votes of the i-th player, and q is the quota needed for a coalition to win. In order to avoid triviality for such a game, it is better to assume that no player can reach the majority quota on her own, that is $w_i < q$ for all $i \in N$.

3. A new formula for the Banzhaf value

It is sensible to recall the classical definition of the Banzhaf value as stated by Banzhaf (1965):

$$\tilde{\beta}(v) = \left(\tilde{\beta}_1(v), \ldots, \tilde{\beta}_n(v)\right), \quad \text{with} \quad \tilde{\beta}_i(v) = \frac{v(N) \cdot \theta_i}{\sum_{j=1}^{n} \theta_j}$$

where $\theta_i = \sum_{i \in S} \left(v(S) - v(S \setminus \{i\})\right)$ is the number of swings for the i-th player.

Proposition 3.1 The Banzhaf value of any n-person TU-game v in characteristic function form can be expressed by the following formula:

$$\tilde{\beta}_i(v) = \frac{v(N)\left(\sum_{s=2}^{n}\left[\binom{n}{s} a(s,0) - 2\binom{n-1}{s} a(s,-i)\right] + 2^{n-1} v(\{i\})\right)}{\sum_{j=1}^{n}\left(\sum_{s=2}^{n}\left[\binom{n}{s} a(s,0) - 2\binom{n-1}{s} a(s,-j)\right] + 2^{n-1} v(\{j\})\right)}$$

for all $i \in N$

Proof We can express θ_i as follows:

$$\theta_i = \sum_{s=2}^{n} \sum_{j=1}^{\binom{n-1}{s-1}} \left(v\left(S_j(s,+i)\right) - v\left(S_j(s-1,-i)\right)\right) + v(\{i\})$$

Since the following equality holds for all $i, s \in N$:

$$\sum_{j=1}^{\binom{n}{s}} v\left(S_j(s,0)\right) = \sum_{j=1}^{\binom{n-1}{s}} v\left(S_j(s,-i)\right) + \sum_{j=1}^{\binom{n-1}{s-1}} v\left(S_j(s,+i)\right)$$

a simple relation on the average essentialities follows:

$$\binom{n}{s}a(s,0)=\binom{n-1}{s}a(s,-i)+\binom{n-1}{s-1}a(s,+i) \tag{1}$$

Hence, the number of swings can be written in terms of average essentialities:

$$\theta_i=\sum_{s=2}^{n}\left[\begin{array}{l}\binom{n-1}{s-1}a(s,+i)+\sum\limits_{k\in S_j(s,+i)}v(\{k\})-\binom{n-1}{s-1}a(s-1,-i)\\ -\sum\limits_{k\in S_j(s-1,-i)}v(\{k\})\end{array}\right]+v(\{i\})$$

By exploiting (1) and the straightforward claim that $a(1,\text{-}i)=0$, for all $i\in N$, we have that:

$$\begin{aligned}\theta_i&=\sum_{s=2}^{n}\left[\begin{array}{l}\binom{n}{s}a(s,0)-\binom{n-1}{s}a(s,-i)-\binom{n-1}{s-1}a(s-1,-i)\\ +\binom{n-1}{s-1}v(\{i\})\end{array}\right]+v(\{i\})\\ &=\sum_{s=2}^{n}\left[\binom{n}{s}a(s,0)-2\binom{n-1}{s}a(s,-i)\right]+2^{n-1}v(\{i\}).\end{aligned}$$

Consequently, the i-th coordinate of the Banzhaf value will be given by the expression:

$$\tilde{\beta}_i(v)=\frac{v(N)\left(\sum\limits_{s=2}^{n}\left[\binom{n}{s}a(s,0)-2\binom{n-1}{s}a(s,-i)\right]+2^{n-1}v(\{i\})\right)}{\sum\limits_{j=1}^{n}\left(\sum\limits_{s=2}^{n}\left[\binom{n}{s}a(s,0)-2\binom{n-1}{s}a(s,-j)\right]+2^{n-1}v(\{j\})\right)} \qquad \square$$

Next corollary provides a representation of the normalized Banzhaf index for a voting game which only depends on the number of winning coalitions of every cardinality, except 1 and n. In the following, we will denote with the symbol $\#A$ the cardinality of the set A.

Corollary 3.2 The normalized Banzhaf index $\beta(v)$ of a weighted voting game $v\equiv[q;w_1,\ldots,w_n]$ can be expressed like that:

$$\beta_i(v) = \frac{1 + \sum_{s=2}^{n-1}\left(K(s,0) - 2K(s,-i)\right)}{n + n\sum_{s=2}^{n-1} K(s,0) - 2\sum_{j=1}^{n}\left(\sum_{s=2}^{n-1} K(s,-j)\right)}, \quad \forall i \in N \tag{2}$$

where

$K(s,-i) : \#\{T \in S(s,-i) : w(T) \geq q\}$
$K(s,0) := \#\{T \in S(s,0) : w(T) \geq q\}$
$\forall s \in N, \quad \forall i = 1,\ldots,n$

Proof If v is a simple weighted voting game, obviously we have that

$$a(s,0) = \frac{\sum_{j=1}^{\binom{n}{s}} v(S_j)}{\binom{n}{s}} = \frac{\#\{T \in S(s,0) : w(T) \geq q\}}{\binom{n}{s}}$$

$$a(s,-i) = \frac{\sum_{j=1}^{\binom{n-1}{s}} v(S_j)}{\binom{n-1}{s}} = \frac{\#\{T \in S(s,\text{-}i) : w(T) \geq q\}}{\binom{n-1}{s}}$$

Then, if we call

$$K(s,-i) := \#\{T \in S(s,\text{-}i) : w(T) \geq q\}$$

$\forall s \in N, \quad \forall i = 1,\ldots,n$, the representation of the normalized Banzhaf index becomes

$$\beta_i(v) = \frac{1 + \sum_{s=2}^{n-1}\left(K(s,0) - 2K(s,-i)\right)}{n + n\sum_{s=2}^{n-1} K(s,0) - 2\sum_{j=1}^{n}\left(\sum_{s=2}^{n-1} K(s,-j)\right)}, \quad \forall i \in N \qquad \square$$

Remark 3.3 Is the formula obtained in Corollary 3.2 useful to reduce the computational cost of algorithms? What follows is a succinct comparison between two algorithm projects, respectively based on the usage of the original formulation and of the new one. In every computation algorithm for the normalized Banzhaf index of a simple game a verification process, attainable through an 'if-then-else' cycle, is associated to each coalition S : if the sum of seats (or votes) of the coalition players exceeds the quota q , the value is 1, and it is 0 otherwise. The precise number of verifications to make can be evaluated in both methods. Dealing with the traditional definition of the Banzhaf-Coleman index, in order to calculate the number θ_i of swings for the i -th player, we have to check all the coalitions of $S(s,+i)$ and of $S(s,-i)$, with $s=2,\ldots,n-1$, i.e. a total number of

$$\sum_{s=2}^{n-1}\left(\binom{n-1}{s-1}+\binom{n-1}{s}\right)=\sum_{s=2}^{n-1}\binom{n}{s}=\sum_{s=0}^{n}\binom{n}{s}-\binom{n}{0}-\binom{n}{1}-\binom{n}{n}=2^n-n-2$$

sets for each person of the game. Since the number of players is n , $n2^n-n^2-2n$ verifications are needed.

As far as the aspects of data-storing are concerned, all the coalitions' characteristic values one needs to memorize should be 2^n-n-2 , since the empty set and the one-player coalitions have value 0 and the complete player set N has obviously value 1. Let us now analyze how an algorithm based on the new formula should work: we need to calculate all the $K(s,0)$, with $s=2,\ldots,n-1$, and all the $K(s,-i)$ as we defined them in Corollary 3.2. To this aim, we need to check 2^n-n-2 coalitions for the $K(s,0)$; on the other hand, each $K(s,-i)$ requires the verification of:

$$\sum_{s=2}^{n-1}\binom{n-1}{s}=2^{n-1}-n \text{ sets.}$$

So totally, we have

$$2^n-n-2+n\left(2^{n-1}-n\right)=2^n\left(1+\frac{n}{2}\right)-n^2-n-2$$

coalitions. The memory needed to store all those values is definitely less than the one required by the previous method. In fact, the $K(s,0)$ are $n-2$, the $K(s,-i)$ are $n-2$ too, so in total $n-2+n(n-2)=n^2-n-2$ entries are needful.

For instance, an algorithm for a 6-person weighted voting game requires $6\cdot 2^6-6^2-12=336$ verifications with the traditional formula, but only

$2^6 \cdot \left(1+6/2\right)-36-6-2=212$ verifications with the new one. The values to be stored are $2^6-6-2=56$ with the original formula, whereas they are only $6^2-6-2=28$ with the new one.

So it is easy to realize that such a representation, only depending on the number of coalitions exceeding the quota q, becomes considerably suitable for games with a large number of players.

4. A new formula for the coalition value

Realistic models for voting games are sometimes constructed investigating likely political phenomena such as preliminary alliances amongst parties (or firms, if our aim is an application to financial games). In other words, when some players agree to form an a priori union, an apter power index able to evaluate both the marginal contribution of a player inside her coalition and power of the coalition herself is needed.

The algebraic instrument which suits this model is a partition of the player set N into so-called a priori unions, i.e. pairwise disjoint subsets of N whose union is N itself. A value taking a priori unions into account is the Owen-Banzhaf coalition value, initially proposed by Owen (1982) and in recent years newly discussed and axiomatized by Laruelle and Valenciano (2003) and by Albizuri et al. (2003). Actually, Albizuri et al. (2003) take this value as a starting point and subsequently define the configuration value, a sort of generalization to a more complex case, in which the a priori coalitions are not necessarily disjoint.

First of all, it is convenient to recall the definition of the coalition value, as it was formalized by Owen's axiomatization (1995, 303-307). Given a TU-game $v:2^N \to R$ and a subset $M:=\{1,...,m\}$ strictly contained in N, we can fix a partition $B=\{B_1,\dots,B_m\}$ of N (which Albizuri et al. (2003) call a *coalition structure*), where:

$$\bigcup_{k=1}^{m} B_k = N,\ B_i \cap B_j = \varnothing \ \ \forall \ \ i,j \in M,\ i \neq j$$

Consider the quotient TU-game $u = v/B : 2^M \to R$, such that:

$$u(H) = v\left(\bigcup_{h\in H} B_h\right),\ \forall \ \ H \subseteq M$$

one can easily notice that if $v \equiv \left[q;w_1,\dots,w_n\right]$ is a weighted majority game, u

is a weighted majority game too: $u \equiv \left[q; w(B_1), \ldots, w(B_m)\right]$.

The coalition value $\varphi(B,v) = \left(\varphi_1(B,v), \ldots, \varphi_n(B,v)\right)$ is expressed by Albizuri et al. (2003) as follows:

$$\varphi_i(B,v) = \sum_{\substack{C \subseteq B \\ B_p \notin C}} \frac{(m-|C|-1)!|C|!}{m!} \cdot \sum_{\substack{S \subseteq B_p \\ i \in S}} \frac{(|S|-1)!\left(|B_p|-|S|\right)!}{|B_p|!} \left(v\left(A_C \cup S\right) - v\left(A_C \cup (S \setminus \{i\})\right)\right)$$

$\forall \; i \in N$, where

$$A_C = \bigcup_{B_q \in C} B_q$$

A plain relation between the coalitional value and the Shapley value holds, so the justification for this formula is somewhat intuitive: given a coalition structure B, we consider all her substructures C not containing the a priori set which includes the i-th player. Then we regard the union of all coalitions of C as a unique set A_C subsequently, we consider the set system formed by all the unions between a A_C and all coalitions containing i, and we calculate I's Shapley value referred to this coalition structure.

This adapted Shapley value is represented by the internal sum; on the other hand, the external sum extends this calculation to all substructures C.

In fact, the coefficients in the external sum are the classical Shapley value factors: m, $|C|$, and $m-|C|$ are the cardinalities of $B, B \setminus (B \setminus C)$ and $B \setminus C$ respectively. Hence, to obtain i-th player's coalitional value, a sort of double omputation of a Shapley value needs to be performed. In the case of weighted voting games on an a priori coalition structure, the value can be reformulated with the help of some slight changes in the definitions of average essentialities.

Theorem 4.1 If $v \equiv [q; w_1, \ldots, w_n]$ is a weighted voting game defined on a player set $N = \{1,\ldots,n\}$, and $B = \{B_1, \ldots, B_m\}$ is a coalition structure, the coalition value of the quotient game v/B can be expressed as follows:

$$\varphi_i(B,v) = \sum_{\substack{C \subseteq B \\ B_p \notin C}} \frac{1}{(m-|C|)\binom{m}{|C|}} \left(\sum_{s=1}^{|B_p|} \frac{K_C(s,+i) - K_C(s-1,-i)}{|B_p|} \right) \tag{3}$$

for all $i \in N$, where

$$K_C(s,+i) := \#\left\{T \in S(s,+i) : T \subseteq B_p, w\left(A_C \cup T\right) \geq q\right\}$$
$$K_C(s-1,-i) := \#\left\{T \in S(s,+i) : T \subseteq B_p, w\left(A_C \cup (T \setminus \{i\})\right) \geq q\right\}$$

$$K_C(s-1,-i) := \#\left\{T \in S(s,+i) : T \subseteq B_p, w\left(A_C \cup (T \setminus \{i\})\right) \geq q\right\}$$

for every $C \subseteq B$ and for every $B_p \in B \setminus C$

Proof First, we assume that no single player can obtain the majority by herself, because in that case all the other players would be dummies, i.e. their value would be 0. Let us fix a subset C of B. We can define the average essentiality of the coalitions of the kind $A_C \cup T$:

$$a_C(s,+i) := \frac{1}{\binom{|B_p|-1}{|T|-1}} \sum_{T \subseteq B_p} v\left(A_C \cup T\right), \ \forall \ T \in S(s,+i)$$

Analogously, we define the average essentiality of the coalitions $A_C \cup (T \setminus \{i\})$ as follows:

$$a_C(s-1,-i) := \frac{1}{\binom{|B_p|-1}{|T|-1}} \sum_{T \subseteq B_p} v\left(A_C \cup (T \setminus \{i\})\right), \ \forall \ T \in S(s,+i)$$

In both definitions B_p is the only element of B containing the i-th player.

Consequently, by defining:

$$K_C(s,+i) := \#\left\{T \in S(s,+i) : T \subseteq B_p, w\left(A_C \cup T\right) \geq q\right\}$$

$$K_C(s-1,-i) := \#\left\{T \in S(s,+i) : T \subseteq B_p, w\left(A_C \cup (T \setminus \{i\})\right) \geq q\right\}$$

for all $i, s \in N$,the formula (3) easily follows. □

Example 4.2 Because of the complexity of the coalition value formula, it is rather difficult to arrange an analysis like the one in Remark 3.3; in fact the number of coalitions whose characteristic value should be verified strongly depends on the a priori structure B. To avoid weightening the treatment, we can just set up a numerical example in which the number of K_C to be stored is smaller than that of all the sets $A_C \cup T$ needed to calculate the coalitional value.

Consider the 7-person weighted voting game

$$v \equiv [15;1,2,3,4,5,6,7]$$

and the a priori union

$$B = \{\{1,3\},\{2,4,6\},\{5,7\}\}$$

By the traditional calculation it is simple to see that the coalitional value of this game is:

$$\varphi(B,v) = \left(0,\frac{1}{18},\frac{1}{3},\frac{5}{36},\frac{1}{6},\frac{5}{36},\frac{1}{6}\right)$$

Let us compare the two formulas. We can notice that as far as the elements 1,3,5,7 are concerned, the calculation of their coalition value requires the knowledge of the values of 12 coalitions.

For instance, let us examine player 1; the four substructures C_i to be considered are:

$$C_1 = \varnothing, \quad C_2 = \{5,7\}, \quad C_3 = \{2,4,6\}, \quad C_4 = \{\{5,7\},\{2,4,6\}\}$$

Consequently, excluding N, $\varnothing$ and all one-player coalitions as usual, the sets whose value is necessary to compute $\varphi_i(B,v)$ are 12, in detail:

$$\begin{array}{llllll} \{1,3\}, & \{5,7,1,3\}, & \{5,7,3\}, & \{5,7,1\}, & \{5,7\}, & \{2,4,6,1\} \\ \{2,4,6\}, & \{2,4,6,1,3\}, & \{2,4,6,3\}, & \{2,4,6,5,7,1\} & & \\ \{2,4,6,5,7\}, & \{2,4,5,6,7,3\} & & & & \end{array}$$

With the new formula, since 1 belongs to a 2-player a priori coalition, one needs to store:

$$\begin{array}{llll} K_{C_1}(2,+1), & K_{C_2}(2,+1), & K_{C_2}(1,+1), & K_{C_2}(1,-1) \\ K_{C_2}(0,-1), & K_{C_3}(2,+1), & K_{C_3}(1,+1), & K_{C_3}(1,-1) \\ K_{C_3}(0,-1), & K_{C_4}(1,+1), & K_{C_4}(1,-1), & K_{C_4}(0,-1) \end{array}$$

exactly 12 values, so no memory is spared in these cases.

But if we consider the elements belonging to a 3-player a priori union, that is 2, 4 or 6, the situation changes. In fact, to compute $\varphi_2(B,v)$ by means of the traditional formula we need to know the values of the following coalitions:

$$\begin{array}{l}\{2,4,6\},\quad \{2,4\},\quad \{2,6\},\quad \{1,3,2,4,6\}\quad \{1,3,4,6\}\\ \{1,3,2,6\},\quad \{1,3,6\},\quad \{1,3,2,4\},\quad \{1,3,4\},\quad \{1,3,2\}\\ \{1,3\},\quad \{5,7,2,4,6\},\quad \{5,7,4,6\},\quad \{5,7,2,6\},\quad \{5,7,6\}\\ \{5,7,2,4\},\quad \{5,7,4\},\quad \{5,7,2\},\quad \{5,7\},\quad \{1,3,5,7,4,6\}\\ \{1,3,5,7,2,4\},\quad \{1,3,5,7,4\},\quad \{1,3,5,7,2,6\}\\ \{1,3,5,7,6\},\quad \{1,3,5,7,2\},\quad \{1,3,5,7\}\end{array}$$

totally 26 subsets of N.

In this case, the four substructures to consider are:

$$C_1 = \varnothing,\quad C_2 = \{5,7\},\quad C_3 = \{1,3\},\quad C_4 = \{\{5,7\},\{1,3\}\}$$

Consequently, the formula (3) requires the memorization of

$$\begin{array}{llll}K_{C_1}(3,+1), & K_{C_1}(2,-1), & K_{C_1}(2,+1), & K_{C_2}(3,+1)\\ K_{C_2}(2,-1), & K_{C_2}(2,+1), & K_{C_2}(1,-1), & K_{C_2}(1,+1)\\ K_{C_2}(0,-1), & K_{C_3}(3,+1), & K_{C_3}(2,-1), & K_{C_3}(2,+1)\\ K_{C_3}(1,-1), & K_{C_3}(1,+1), & K_{C_3}(0,-1), & K_{C_4}(2,+1)\\ K_{C_4}(2,-1), & K_{C_4}(1,+1), & K_{C_4}(1,-1), & K_{C_4}(0,-1)\end{array}$$

which are 20 in total.

To sum up, with the original formula the values to be stored are $12 \cdot 4 + 3 \cdot 26 = 126$, whereas with the new one they are only $12 \cdot 4 + 3 \cdot 20 = 108$. Hence, this method appears more advantageous as the number of players belonging to large a priori coalitions increases.

5. Some notes about the Myerson value

In this section some preliminary classical definitions are proposed, in order to characterize the value of Myerson (1977, 225-229), for n-person TU-games. The notation used is the same that was exploited by Algaba et al. (2001) and Fernandez et al. (2002).

Definition 5.1 Let $N = \{1,2,...n\}$ be a finite set of players and F a set system of coalitions. F is called *union stable* if for all $S_1, S_2 \in F$ with $S_1 \cap S_2 \neq \varnothing$ it is satisfied that $S_1 \cup S_2 \in F$.

Definition 5.2 Given the union stable set system $F \subseteq 2^N$, the elements belonging to F are called *feasible coalitions*.

Definition 5.3 If $F \subseteq 2^N$, for every $S \subseteq N$, a set $T \subseteq S$ is called a *component of* S *in* F if $T \in F$ and there exists no $T' \in F$ such that $T \subset T' \subseteq S$, i.e. if T is a maximal feasible subset of S in F.

Definition 5.4 Given an n-person game $v : 2^N \to R$ in characteristic function form and the union stable set system $F \subseteq 2^N$, the *F-restricted game* (or *graph-restricted game*) $v^F : 2^N \to R$ is defined by:

$$v^F(S) = \sum_{T \in C_F(S)} v(T), \quad \forall S \subseteq N$$

where $C_F(S)$ is the collection of all the components of S in F.

Let us recall the classical definition of the Shapley value in the most general case (see Shapley, 1953):

Definition 5.5 The Shapley value of the n-person TU-game $v : 2^N \to R$ is the vector $\Phi(v) = (\Phi_1(v), \Phi_2(v), \ldots, \Phi_n(v))$, where:

$$\Phi_i(v) = \sum_{\substack{i \in S \\ S \subseteq N}} \frac{(n-|S|)!(|S|-1)!}{n!} \left(v(S) - v(S \setminus \{i\})\right)$$

Definition 5.6 If v is an n-person TU-game and $F \subseteq 2^N$ is a union stable set system, the Myerson value of the game v is the Shapley value of the related F-restricted game v^F, i.e. the vector $\mu(v) = (\mu_1(v), \ldots, \mu_n(v))$ such that:

$$\mu_i(v) = \Phi_i\left(v^F\right)$$

Gambarelli's theorem (1990) yields the following representation for the Shapley value:

Theorem 5.7 The Shapley value of an n-person game in characteristic function form v can be expressed as follows:

$$\Phi_i(v) = v(\{i\}) + \sum_{s=2}^{n} \frac{a(s,0)}{s} - \sum_{s=2}^{n-1} \frac{a(s,-i)}{s}, \quad \forall i \in N$$

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Proof See Gambarelli (1990). □

Let us redefine the average essentialities of the coalitions as follows:

$$a_F(s,0) = \frac{1}{\binom{n}{s}} \sum_{j=1}^{\binom{n}{s}} \left(v^F\left(S_j(s,0)\right) - \sum_{k \in S_j(s,0)} v^F\left(\{k\}\right) \right)$$

$$a_F(s,-i) = \frac{1}{\binom{n-1}{s}} \sum_{j=1}^{\binom{n-1}{s}} \left(v^F\left(S_j(s,-i)\right) - \sum_{k \in S_j(s,-i)} v^F\left(\{k\}\right) \right)$$

$$a_F(s,+i) = \frac{1}{\binom{n-1}{s-1}} \sum_{j=1}^{\binom{n-1}{s-1}} \left(v^F\left(S_j(s,+i)\right) - \sum_{k \in S_j(s,+i)} v^F\left(\{k\}\right) \right)$$

Consequently, adapting the relation (1) between the average essentialities:

$$na_F(s,0) = (n-s)a_F(s,-i) + sa_F(s,+i), \quad \forall i \in N, \quad \forall s = 1,\ldots,n$$

we can also express the coordinates of the Myerson value:

$$\mu_i(v) = v^F(\{i\}) + \frac{v^F(N)}{n} - \sum_{j=1}^{n} \frac{v^F(\{j\})}{n} + \sum_{s=2}^{n-1} \frac{a_F(s,+i) - a_F(s,0)}{n-s}$$

By definition of graph-restricted game, it is obvious that in general if v is a non-trivial weighted voting game, v^F is a simple game in $(0,1)$ normalization too.

In fact, since the components of one-element subsets are still one-element subsets, we have that if $F \subseteq 2^N$ is a union stable set system, the coordinates of the Myerson value of such a game are:

$$\mu_i(v) = \frac{1}{n} + \sum_{s=2}^{n-1} \frac{a_F(s,+i) - a_F(s,0)}{n-s} \tag{4}$$

Substantially, given a non-trivial weighted voting game $v \equiv [q; w_1, \ldots, w_n]$, a graph

$G = (N,E)$ and a union stable set system F, the corresponding graph-restricted game is defined as follows:

$$v^F(S)=\#\left\{T\in C_F(S):\sum_{j\in T}w_j\ge q\right\}\text{ if }C_F(S)\ne\varnothing,\ v^F(S)=0$$

otherwise.

In (4), the average essentialities are given by

$$a_F(s,+i)=\frac{1}{\binom{n-1}{s-1}}\sum_{j=1}^{\binom{n-1}{s-1}}\#\left\{T\in C_F\left(S_j(s,+i)\right):\sum_{k\in T}w_k\ge q\right\}$$

$$a_F(s,0)=\frac{1}{\binom{n}{s}}\sum_{j=1}^{\binom{n}{s}}\#\left\{T\in C_F\left(S_j(s,0)\right):\sum_{k\in T}w_k\ge q\right\}$$

From now on we will call

$$T(s,\pm i):=S(s,\pm i)\cap F,\quad \forall i\in N\cup\{0\},\quad \forall s=1,\dots,n$$

Definition 5.8 If $G=(N,E)$ is a connected graph and $i\in N$, the *degree of* i (deg(i)) is the number of vertices of G which are incident with i.

Obviously, given a graph G and a graph-restricted game v^F, for every $i\in N$,

$$\deg(i)=\sum_{s=2}^{n-1}(s-1)\cdot\left|T(s,+i)\right|+n-1\text{ if }N\in F$$

$$\deg(i)=\sum_{s=2}^{n-1}(s-1)\cdot\left|T(s,+i)\right|\text{ if }N\notin F$$

where F is the union stable set system on which the game is defined.

Now we can decompose deg(i) in sums of contributions coming from all coalitions containing the i-th player, the winning and the losing ones, i.e. we can define, for all $s=2,...,n-1$:

$$\deg_{w,s}(i):=(s-1)\cdot\#\left\{S\in T(s,+i):\sum_{j\in S}w_j\ge q\right\}$$

$$\deg_{l,s}(i) := (s-1)\cdot \#\left\{S \in T(s,+i) : \sum_{j\in S} w_j < q\right\}$$

respectively the number of winning and losing coalitions of F containing the player i, so that

$$\deg(i) = n-1\sum_{s=2}^{n-1}\left(\deg_{w,s}(i) + \deg_{l,s}(i)\right)$$

if the grand coalition N belongs to F. We can connect the notion of degree of a vertex of a graph to the average essentialities and consequently to the Myerson value.

Proposition 5.9 If $v \equiv [q; w_1,\ldots,w_n]$ is a weighted voting game and v^F is the corresponding F-restricted game on a stable set system F associated to a graph $G=(N,E)$, if for every $S \in S(s,+i)\setminus F$, either $C_F(S) = \varnothing$ or no components of S in F are winning coalitions, then the Myerson value for the i-th player is:

$$\mu_i(v) = \frac{1}{n} - \sum_{s=2}^{n-1}\frac{a_F(s,0)}{n-s} + \sum_{s=2}^{n-1}\frac{\deg_{w,s}(i)}{s(s-1)\binom{n-1}{s}} \tag{5}$$

Proof Call $S'(s,\pm i) := S(s,\pm i)\setminus F$ and $S'_k(s,\pm i)$ its elements, ordered lexicographically, for every $i \in N \cup \{0\}$, and for all $s = 1,\ldots,n$, we have the following decomposition for $a_F(s,+i)$:

$$a_F(s,+i) = \frac{\#\left\{T \in S(s,+i)\cap F : \sum_{k\in T} w_k \geq q\right\}}{\binom{n-1}{s-1}} + \frac{\sum_{j=1}^{|S'(s,+i)|}\#\left\{T \in C_F\left(S'_j(s,+i)\right) : \sum_{k\in T} w_k \geq q\right\}}{\binom{n-1}{s-1}}$$

Observe that if for every $S \in S'(s,+i)$ it is satisfied that $C_F(S) = \varnothing$ or if no components of S in F reach the winning quota q, the second summand vanishes, and by definition of $\deg_{w,s}(i)$:

$$a_F(s,+i) = \frac{1}{\binom{n-1}{s-1}} \left[\frac{\deg_{w,s}(i)}{s-1} \right]$$

Consequently, the following representation for the Myerson value holds:

$$\mu_i(v) = \frac{1}{n} + \sum_{s=2}^{n-1} \frac{\dfrac{\deg_{w,s}(i)}{\binom{n-1}{s-1}(s-1)} - a_F(s,0)}{n-s}$$
$$= \frac{1}{n} - \sum_{s=2}^{n-1} \frac{a_F(s,0)}{n-s} + \sum_{s=2}^{n-1} \frac{\deg_{w,s}(i)}{s(s-1)\binom{n-1}{s}}, \quad \forall\, i \in N$$

□

Example 5.10 Consider the 5-person weighted voting game v with $w_k = k$ for $k = 1,2,3,4,5$ and majority quota $q = 8$, and the relative graph-restricted game on the union stable set system

$$F = \{\{1,2\},\ \{2,4\},\ \{1,2,4\},\ \{1,2,3,5\},\ \{2,3,4,5\},\ N\}$$

which is defined by:
$v^F(S) = 1$ if $S \in F$ and $\sum_{j\in S} w_j \geq 8$; $v^F(S) = 0$ if $S \in F$ and $\sum_{j \in S} w_j < 8$; $v^F(S) = 0$ otherwise.

Obviously, all the 2-player coalitions have characteristic value 0, just like $\{1,2,4\}$, the only 3-player coalition belonging to F. The 3-player coalitions whose components belong to F, consequently, have characteristic value 0 too, and besides the remaining 3-player coalitions have empty collection of components.

As far as the 4-player coalitions are concerned, the two of them appearing in F have characteristic value 1, the remaining ones have the following collection of components:

$$C_F(\{1,2,3,4\}) = \{1,2,4\} \Rightarrow v^F(\{1,2,3,4\}) = v(\{1,2,4\}) = 0$$
$$C_F(\{1,2,4,5\}) = \{1,2,4\} \Rightarrow v^F(\{1,2,4,5\}) = v(\{1,2,4\}) = 0$$
$$C_F(\{1,3,4,5\}) = \varnothing \Rightarrow v^F(\{1,3,4,5\}) = 0$$

hence the hypotheses of Proposition 5.9 are satisfied. We can perform the calculation of the Myerson value either by means of the traditional formula or by applying (5). It is straightforward to verify that

$$\mu(v) = \left(\frac{1}{20}, \frac{3}{10}, \frac{3}{10}, \frac{1}{20}, \frac{3}{10}\right)$$

On the other hand, in formula (5) we have that

$$a_F(2,0) = \frac{1}{\binom{5}{2}} = \frac{1}{10}, \quad a_F(3,0) = 0, \quad a_F(4,0) = \frac{2}{\binom{5}{4}} = \frac{2}{5}$$

so the first summand for all the coordinates of $\mu(v)$ is $-1/5$; in order to calculate $\deg_{w,s}(i)$ for all $i \in N$ and $s = 2,3,4$, it is sufficient to see how many winning coalitions of size s including the i-th player are contained in F. One can easily notice that

$$\begin{aligned}
&\deg_{w,2}(1) = \deg_{w,2}(2) = \deg_{w,2}(4) = 0 \\
&\deg_{w,2}(3) = \deg_{w,2}(5) = 1, \quad \deg_{w,3}(i) = 0 \quad \forall i \in N \\
&\deg_{w,4}(1) = \deg_{w,4}(4) = 3 \cdot 1 = 3 \\
&\deg_{w,4}(2) = \deg_{w,4}(3) = \deg_{w,4}(5) = 3 \cdot 2 = 6
\end{aligned}$$

So the coordinates of the Myerson value are:

$$\mu_1(v) = -\frac{1}{5} + \frac{3}{4 \cdot 3 \cdot \binom{4}{4}} = \frac{1}{20}; \quad \mu_2(v) = -\frac{1}{5} + \frac{6}{4 \cdot 3 \cdot \binom{4}{4}} = \frac{3}{10}$$

$$\mu_3(v) = -\frac{1}{5} + \frac{6}{4 \cdot 3 \cdot \binom{4}{4}} = \frac{3}{10}; \quad \mu_4(v) = -\frac{1}{5} + \frac{3}{4 \cdot 3 \cdot \binom{4}{4}} = \frac{1}{20}$$

$$\mu_5(v) = -\frac{1}{5} + \frac{6}{4 \cdot 3 \cdot \binom{4}{4}} = \frac{3}{10}$$

Remark 5.11 The hypotheses for the application of (5) are undoubtedly strict, but just like the representation (2), interesting profits from its usage might be drawn. In fact, to compute a Myerson value the needful data are only the value of the grand coalition (which is not necessarily 1), the $n-2$ average essentialities $a_F(s,0)$, and all the $n(n-2)$ $\deg_{w,s}(i)$, so only $n-2+n(n-2)+1 = n^2-n-1$ values, instead of all the 2^n-n-1 characteristic values of any TU-game in which all one-player coalitions have null value.

6. Further developments

The average essentialities of coalitions provide a useful instrument for the expression of power indices, and sometimes bring computational advantages, as we have shown in Remarks 3.3 and 5.11 and in Example 4.2. Anyway, we only sketched some outlines of algorithms based on these formulas. It would be interesting to check their actual efficiency by writing down the complete algorithms precisely and implementing them.

Other possible developments concern the links between the Myerson value and the degrees of the single players: specific properties of graphs may help in order to simplify the expression of that value also in cases different from the one analyzed in Proposition 5.9.

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