Journal of Combinatorial Theory, Series A 118 (2011) 558–585

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Sharp thresholds for hypergraph regressive Ramsey numbers

Lorenzo Carlucci [a,1], Gyesik Lee [b,2], Andreas Weiermann [c,3]

[a] Università di Roma "La Sapienza", Department of Computer Science, Via Salaria 113, 00198 Roma, Italy
[b] ROSAEC center, Seoul National University, 599 Gwanak-ro Gwanak-gu, 151-742 Seoul, Republic of Korea
[c] University of Ghent, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281, B9000 Gent, Belgium

ARTICLE INFO

Article history:
Received 5 November 2009
Available online 21 August 2010

Keywords:
Regressive Ramsey Theorem
Rapidly growing Ramsey functions
Independence results
Fast-growing hierarchies

ABSTRACT

The f-regressive Ramsey number $R_f^{\mathrm{reg}}(d,n)$ is the minimum N such that every coloring of the d-tuples of an N-element set mapping each $x_1,\dots,x_d$ to a color below $f(x_1)$ (when $f(x_1)$ is positive) contains a *min-homogeneous* set of size n, where a set is called min-homogeneous if every two d-tuples from this set that have the same smallest element get the same color. If f is the identity, then we are dealing with the standard regressive Ramsey numbers as defined by Kanamori and McAloon. The existence of such numbers for hypergraphs or arbitrary dimension is unprovable from the axioms of Peano Arithmetic. In this paper we classify the growth-rate of the regressive Ramsey numbers for hypergraphs in dependence of the growth-rate of the parameter function f. We give a sharp classification of the thresholds at which the f-regressive Ramsey numbers undergo a drastical change in growth-rate. The growth-rate has to be measured against a scale of fast-growing recursive functions indexed by finite towers of exponentiation in base ω (the first limit ordinal). The case of graphs has been treated by Lee, Kojman, Omri and Weiermann. We extend their results to hypergraphs of arbitrary dimension. From the point of view of Logic, our results classify the provability of the Regressive Ramsey Theorem for hypergraphs of

E-mail addresses: carlucci@di.uniroma1.it (L. Carlucci), gslee@ropas.snu.ac.kr (G. Lee), weierman@cage.ugent.be (A. Weiermann).

[1] The first author gratefully acknowledges partial support by grant no. 13393 from the John Templeton Foundation, and by a "Progetto Italia" fellowship from Telecom Italia.

[2] The second author was partially supported by the Engineering Research Center of Excellence Program of Korea Ministry of Education, Science and Technology (MEST) / Korea Science and Engineering Foundation (KOSEF), grant number R11-2008-007-01002-0.

[3] The third author gratefully acknowledges partial support by grants from DFG, NWO, FWO and from John Templeton Foundation grant no. 13396.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2010.08.004

fixed dimension in subsystems of Peano Arithmetic with restricted induction principles.

1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers including 0. A number $d \in \mathbb{N}$ is identified with the set $\{0, 1, \ldots, d-1\}$, which may also be sometimes denoted by $[d]$. The set of all *d-element subsets* of a set X is denoted by $[X]^d$. For a function $C : [X]^d \to \mathbb{N}$ we write $C(x_1, \ldots, x_d)$ for $C(\{x_1, \ldots, x_d\})$ under the assumption that $x_1 < \cdots < x_d$. Let $f : \mathbb{N} \to \mathbb{N}$ be a number-theoretic function. A function $C : [X]^d \to \mathbb{N}$ is called *f-regressive* if for all $s \in [X]^d$ such that $f(\min(s)) > 0$ we have $C(s) < f(\min(s))$. When f is the identity function we just say that C is *regressive*. A set H is *min-homogeneous* for C if for all $s, t \in [H]^d$ with $\min(s) = \min(t)$ we have $C(s) = C(t)$. We write

$$X \xrightarrow{\min} (m)^d_f$$

if for all f-regressive $C : [X]^d \to \mathbb{N}$ there exists $H \subseteq X$ such that $\operatorname{card}(H) = m$ and H is min-homogeneous for C. In case $d = 2$, we just write $X \xrightarrow{\min} (m)_f$. We denote by $(\mathrm{KM})^d_f$ the following statement

$$(\forall m)\ (\exists \ell)\ \bigl[\ell \xrightarrow{\min} (m)^d_f\bigr],$$

and abbreviate $(\forall d)[(\mathrm{KM})^d_f]$ as $(\mathrm{KM})_f$. Using a compactness argument and the Canonical Ramsey Theorem of Erdős and Rado, Kanamori and McAloon [6] proved that $(\mathrm{KM})_f$ is true for every choice of f. For f the identity function, the theorem has the notable property of being a Gödel sentence [4] for Peano Arithmetic [6] and is known as the Regressive Ramsey Theorem. It is equivalent to the famous Paris–Harrington Theorem (see [11,2,7]). The latter was the first example of a theorem from finite combinatorics that is undecidable in formal number theory. Not a few people consider the Regressive Ramsey Theorem to be more natural. The m-th regressive Ramsey number for d-hypergraphs and parameter function f is denoted by $R^{\mathrm{reg}}_f(d, m)$ and is defined as the smallest ℓ satisfying $\ell \xrightarrow{\min} (m)^d_f$. When f is the identity function we drop the subscript. Regressive Ramsey numbers for graphs have also been investigated by Kojman and Shelah [9]. They showed that $R^{\mathrm{reg}}(2, i)$ grows as the Ackermann function. More recently, Kojman, Lee, Omri and Weiermann computed the sharp thresholds on the parameter function f at which the f-regressive Ramsey numbers for graphs cease to be Ackermannian and become primitive recursive [8]. In this paper we extend the results of [8] to hypergraphs of arbitrary dimension. We classify the thresholds on f at which the f-regressive Ramsey number undergo an acceleration against the scale of fast-growing Hardy functions that naturally extends the Grzegorczyk hierarchy.

We introduce some terminology to describe the main result from [8]. Recall that the primitive recursive functions are the functions obtained from the successor function, projections and constant functions by closing under composition and recursion. The Ackermann function is the canonical example of a recursive function that eventually dominates every primitive recursive function. A function is said to be of *Ackermannian growth* if it eventually dominates every primitive recursive function. Let $B : \mathbb{N} \to \mathbb{N}^+$ be unbounded and non-decreasing. For an unbounded and non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ we define the inverse function $f^{-1} : \mathbb{N} \to \mathbb{N}$ as follows:

$$f^{-1}(n) := \begin{cases} m & \text{if } m = \min\{i\colon\ f(i) \geqslant n\} > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Note that for a strictly increasing f we have $f^{-1}(f(n)) = n$. Let $f_B(i) := i^{1/B^{-1}(i)}$. The main result of [8] says that the f_B-regressive Ramsey numbers for graphs are Ackermannian if and only if B is. For every f dominated by f_B, the f-regressive Ramsey number is primitive recursive if B is.

To state our main results, we need to introduce the so-called fast-growing hierarchy [13,14]. This hierarchy naturally extends the Grzegorczyk hierarchy of primitive recursive functions used in [8] to classify the threshold for Regressive Ramsey number for graphs. The hierarchy is indexed by notations for (constructive, countable) ordinals below the ordinal ε_0. The indexing by ordinal notations allows long iterations and diagonalization. We use the fact that any ordinal α below ε_0 can be written uniquely in (Cantor) normal form as $\sum_{i=k}^{0} c_i \cdot \omega^{\alpha_i}$, where $\alpha > \alpha_k > \cdots > \alpha_0$ and $c_i \geqslant 1$. We fix an assignment of "fundamental sequences" to ordinals below ε_0. A fundamental sequence for a limit ordinal λ is an infinite sequence $(\lambda_n)_{n\in\mathbb{N}}$ of smaller ordinals whose supremum is λ. We define the assignment $\cdot[\cdot] : \varepsilon_0 \times \mathbb{N} \to \varepsilon_0$ as follows by case distinction on the structure of the normal form of a limit ordinal α. We define $\alpha[x] := \gamma + \omega^{\lambda[x]}$, if $\alpha = \gamma + \omega^{\lambda}$ with λ limit. We define $\alpha[x] := \gamma + \omega^{\beta} \cdot x$, if $\alpha = \gamma + \omega^{\beta+1}$. We also set $\varepsilon_0[x] := \omega_{x+1}$, where $\omega_0(x) := x$, $\omega_{d+1}(x) := \omega^{\omega_d(x)}$ and $\omega_d := \omega_d(1)$. For technical reasons we extend the assignment to non-limit ordinals as follows: $(\beta+1)[x] := \beta$ and $0[x] := 0$. If f is a function and $d \geqslant 0$ we denote by f^d the d-th iteration of f, with $f^0(x) := x$. The fast-growing hierarchy is defined as follows, by induction on α,

$$F_0(x) := x + 1,$$
$$F_{\alpha+1}(x) := F_{\alpha}^{(x+1)}(x),$$
$$F_{\lambda}(x) := F_{\lambda[x]}(x), \quad \text{if } \lambda \text{ is a limit.}$$

The fast-growing hierarchy is well known in the study of formal systems of Arithmetic, where it can be used to classify the functions that have a proof of totality in the system. The correspondence is – roughly – as follows. A recursive function has a proof of totality in Peano arithmetic if and only if, for some $\alpha < \varepsilon_0$, it is primitive recursive *in* F_α (i.e., belongs to the class of functions obtained from the class of primitive recursive functions by adding F_α as an extra base function). For $d \geqslant 1$, a recursive function has a proof of totality in the subsystem of Peano arithmetic with induction restricted to d-quantifier induction (i.e., to predicates starting with d alternations of existential and universal quantifiers $\exists x_1 \forall x_2 \ldots$ followed by a quantifier-free predicate) if and only if it is primitive recursive in F_{ω_d}. Also, $F_{\omega_{d+1}}$ eventually dominates all functions that are primitive recursive in F_α for all $\alpha < \omega_{d+1}$, and F_{ε_0} eventually dominates all functions that are primitive recursive in F_{ω_d} for all $d \in \mathbb{N}$. Thus, each new level of exponentiation in the ordinal index corresponds to a *drastical* jump in growth-rate as well as in logical complexity, analogous to the jump between primitive recursive and Ackermannian growth rate.

Lee obtained in his PhD thesis [10] the following result. Let $d \geqslant 1$. For hypergraphs of dimension $d+1$, the $\log_d$-regressive Ramsey numbers are primitive recursive, but the $\log_{d \dot{-} 2}$-regressive Ramsey numbers grow as fast as F_{ω_d}. Here and in the rest of the paper $\log_d$ denotes the d-iterated binary logarithm. We say that a function f grows as fast as F_β, or that it has F_β-growth, if f eventually dominates every F_α for $\alpha < \beta$. This kind of drastical change in growth rate and proof complexity has been dubbed a "phase-transition" by Weiermann, who first observed it [15,16]. This turned out to be a pervasive phenomenon in formal arithmetic (see [17] for a survey), with tight connections to analytic combinatorics. Lee conjectured that $\log_{d-1}$-regressive Ramsey numbers, and $(\log_{d-1})^{1/\ell}$-regressive Ramsey numbers, for every ℓ, grow as fast as F_{ω_d}. Our results imply that Lee's conjecture is true and that we can also replace the constant ℓ with any function growing slower than the inverse of F_{ω_d}.

Theorem A *(Upper bounds). Let $d \geqslant 1$. Let $B : \mathbb{N} \to \mathbb{N}^+$ be unbounded and non-decreasing. Let $f : \mathbb{N} \to \mathbb{N}$ be such that for every i, $f(i) \leqslant (\log_{d-1}(i))^{1/B^{-1}(i)}$. If B is bounded by a function primitive recursive in F_α for some $\alpha < \omega_d$, then the same is true of $R_f^{\mathrm{reg}}(d+1,\cdot)$. If B is primitive recursive in F_α for some $\alpha < \omega_d$, then the same is true of $R_f^{\mathrm{reg}}(d+1,\cdot)$.*

Theorem B *(Lower bounds). Let $d \geqslant 1$. Let $B : \mathbb{N} \to \mathbb{N}^+$ be unbounded and non-decreasing. Let $f_B(i) := (\log_{d-1}(i))^{1/B^{-1}(i)}$. If B eventually dominates F_α for all $\alpha < \omega_d$ then $R_{f_B}^{\mathrm{reg}}(d+1,\cdot)$ eventually dominates F_α for all $\alpha < \omega_d$.*

In logical terms, this implies that proving the f-regressive Ramsey Theorem for hypergraphs of dimension $d+1$ *necessarily* requires $(d+1)$-quantifier induction if and only if f grows as f_B, with $B(i) = F_{\omega_d}(i)$.

Lemma 1.1. *Let $\beta \leqslant \varepsilon_0$. If the composition $f \circ g$ of two non-decreasing functions eventually dominates F_α for all $\alpha < \beta$, then either f or g eventually dominates all F_α for all $\alpha < \beta$.*

Proof. Suppose that f does not eventually dominate all F_α's for $\alpha < \beta$. Suppose g is eventually dominated by F_{α_1} for some $\alpha_1 < \beta$. Let p be F_{α_2}, for some $\alpha_2 < \beta$. Then $h(x) = p(g(n+1))$ is eventually dominated by some α such that $\varepsilon_0 > \alpha \geqslant \alpha_1 + \alpha_2$, by the properties of the fast-growing hierarchy. By hypothesis on $f \circ g$ there exists N such that, for all $n \geqslant N$, $f(g(n)) \geqslant h(n) = p(g(n+1))$. For all $i \geqslant g(N)$ there exists $n \geqslant N$ such that $g(n) \leqslant i \leqslant g(n+1)$. Then $f(i) \geqslant f(g(n)) \geqslant p(g(n+1)) \geqslant p(i)$. Since p was arbitrary, this proves that f eventually dominates all F_α's for $\alpha < \beta$, contra the assumption. $\square$

2. Upper bounds

In this section we show the upper bounds on f-regressive Ramsey numbers for $f(n) \leqslant (\log_{d-1}(n))^{1/F_\alpha^{-1}(n)}$ for $\alpha < \varepsilon_0$. Essentially, the bound for standard Ramsey functions [12] from Erdős and Rado's [3] is adapted to the case of regressive functions.

Definition 2.1. Let $C : [\ell]^d \to k$ be a coloring. Call a set H *s-homogeneous* for C if for any s-element set $U \subseteq H$ and for any $(d-s)$-element sets $V, W \subseteq H$ such that $\max U < \min\{\min V, \min W\}$, we have

$$C(U \cup V) = C(U \cup W),$$

$(d-1)$-homogeneous sets are called *end-homogeneous*.

Note that 0-homogeneous sets are homogeneous and 1-homogeneous sets are min-homogeneous. Let

$$X \to_s \langle m \rangle_k^d$$

denote that given any coloring $C : [X]^d \to k$, there is H s-homogeneous for C such that $\mathrm{card}(H) \geqslant m$. The following lemma shows a connection between s-homogeneity and homogeneity.

Lemma 2.2. *Let $s \leqslant d$ and assume*

(1) $\ell \to_s \langle p \rangle_k^d$,
(2) $p - d + s \to (m - d + s)_k^s$.

Then we have

$$\ell \to (m)_k^d.$$

Proof. Let $C : [\ell]^d \to k$ be given. Then assumption 1 implies that there is $H \subseteq \ell$ such that $|H| = p$ and H is s-homogeneous for C. Let $z_1 < \cdots < z_{d-s}$ be the last $d-s$ elements of H. Set $H_0 := H \setminus \{z_1, \ldots, z_{d-s}\}$. Then $\mathrm{card}(H_0) = p - d + s$. Define $D : [H_0]^s \to k$ by

$$D(x_1, \ldots, x_s) := C(x_1, \ldots, x_s, z_1, \ldots, z_{d-s}).$$

By assumption 2 there is Y_0 such that $Y_0 \subseteq H_0$, $\mathrm{card}(Y_0) = m - d + s$, and homogeneous for D. Hence $D \restriction [Y_0]^s = e$ for some $e < k$. Set $Y := Y_0 \cup \{z_1, \ldots, z_{d-s}\}$. Then $\mathrm{card}(Y) = m$ and Y is homogeneous for C. Indeed, we have for any sequence $x_1 < \cdots < x_d$ from Y,

$$C(x_1,\dots,x_d)=C(x_1,\dots,x_s,z_1,\dots,z_{d-s})=D(x_1,\dots,x_s)=e.$$

The proof is complete. □

Given d,s such that $s\leqslant d$ define $R^s_\mu(d,\cdot,\cdot):\mathbb{N}^2\to\mathbb{N}$ by

$$R^s_\mu(d,k,m):=\min\bigl\{\ell\colon\ \ell\to_s\langle m\rangle^d_k\bigr\}.$$

Then

- $R^0_\mu(1,k,m-d+1)=k\cdot(m-d)+1$,
- $R^d_\mu(d,k,m)=R^s_\mu(d,1,m)=m$,
- $R^s_\mu(d,k,d)=d$,
- $R^s_\mu(d,k,m)\leqslant R^{s-1}_\mu(d,k,m)$ for any $s>0$.

R^s_μ are called *Ramsey functions*. Then the standard Ramsey function for d-hypergraphs and two colors – which we denote by $R(d,k,m)$ – coincides with $R^0_\mu(d,k,m)$ and $R^{\mathrm{reg}}_{f_k}(d,m)=R^1_\mu(d,k,m)$ where f_k is the constant function with value k. Define a binary operation $*$ by putting, for positive natural numbers x and y,

$$x*y:=x^y.$$

Further, we put for $p\geqslant 3$,

$$x_1*x_2*\dots*x_p:=x_1*\bigl(x_2*\bigl(\dots*(x_{p-1}*x_p)\cdots\bigr)\bigr).$$

Erdős and Rado [3] gave an upper bound for $R(d,k,m)$: Given d,k,m such that $k\geqslant 2$ and $m\geqslant d\geqslant 2$, we have

$$R(d,k,m)\leqslant k*\bigl(k^{d-1}\bigr)*\bigl(k^{d-2}\bigr)*\dots*\bigl(k^2\bigr)*\bigl(k\cdot(m-d)+1\bigr).$$

The following theorem is provable in Primitive Recursive Arithmetic ($\mathrm{I}\Sigma_1$).

Theorem 2.3. *Let $2\leqslant d\leqslant m$, $0<s\leqslant d$, and $2\leqslant k$,*

$$R^s_\mu(d,k,m)\leqslant k*\bigl(k^{d-1}\bigr)*\bigl(k^{d-2}\bigr)*\dots*\bigl(k^{s+1}\bigr)*(m-d+s)*s.$$

In particular, for $s=1$, we have $R^{\mathrm{reg}}_{f_k}(2,m)\leqslant k^{m-1}$, where f_k is the constant function with value k.

Proof. The proof construction below generalizes Erdős and Rado [3]. We shall work with s-homogeneity instead of homogeneity.

Let X be a finite set. In the following construction we assume that $\operatorname{card}(X)$ is large enough. How large it should be will be determined after the construction has been defined. Throughout this proof the letter Y denotes subsets of X such that $\operatorname{card}(Y)=d-2$.

Let $C:[X]^d\to k$ be given and $x_1<\dots<x_{d-1}$ the first $d-1$ elements of X. Given $x\in X\setminus\{x_1,\dots,x_{d-1}\}$ put

$$C_{d-1}(x):=C(x_1,\dots,x_{d-1},x).$$

Then $\operatorname{Im}(C_{d-1})\subseteq k$, and there is $X_d\subseteq X\setminus\{x_1,\dots,x_{d-1}\}$ such that C_{d-1} is constant on X_d and

$$\operatorname{card}(X_d)\geqslant k^{-1}\cdot\bigl(\operatorname{card}(X)-d+1\bigr).$$

Let $x_d:=\min X_d$ and given $x\in X_d\setminus\{x_d\}$ put

$$C_d(x):=\prod\bigl\{C\bigl(Y\cup\{x_d,x\}\bigr)\colon\ Y\subseteq\{x_1,\dots,x_{d-1}\}\bigr\}.$$

Then $\mathrm{Im}(C_d) \subseteq k * \binom{d-1}{d-2}$, and there is $X_{d+1} \subseteq X_d \setminus \{x_d\}$ such that C_d is constant on X_{d+1} and

$$\mathrm{card}(X_{d+1}) \geqslant k^{-\binom{d-1}{d-2}} \cdot \bigl(\mathrm{card}(X_d) - 1\bigr).$$

Generally, let $p \geqslant d$, and suppose that $x_1, \dots, x_{p-1}$ and $X_d, X_{d+1}, \dots, X_p$ have been defined, and that $X_p \neq \varnothing$. Then let $x_p := \min X_p$ and for $x \in X_p \setminus \{x_p\}$ put

$$C_p(x) := \prod \bigl\{C\bigl(Y \cup \{x_p, x\}\bigr)\colon\ Y \subseteq \{x_1, \dots, x_{p-1}\}\bigr\}.$$

Then $\mathrm{Im}(C_p) \subseteq k * \binom{p-1}{d-2}$, and there is $X_{p+1} \subseteq X_p \setminus \{x_p\}$ such that C_p is constant on X_{p+1} and

$$\mathrm{card}(X_{p+1}) \geqslant k^{-\binom{p-1}{d-2}} \cdot \bigl(\mathrm{card}(X_p) - 1\bigr).$$

Now put

$$\ell := 1 + R^s_\mu(d-1, k, m-1).$$

Then $\ell \geqslant m \geqslant d$. If $\mathrm{card}(X)$ is sufficiently large, then $X_p \neq \varnothing$, for all p such that $d \leqslant p \leqslant \ell$, so that $x_1, \dots, x_\ell$ exist. Note also that $x_1 < \cdots < x_\ell$. For $1 \leqslant \rho_1 < \cdots < \rho_{d-1} < \ell$ put

$$D(\rho_1, \dots, \rho_{d-1}) := C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_\ell).$$

By definition of ℓ there is $Z \subseteq \{1, \dots, \ell - 1\}$ such that Z is s-homogeneous for D and $\mathrm{card}(Z) = m - 1$. Finally, we put

$$X' := \{x_\rho\colon\ \rho \in Z\} \cup \{x_\ell\}.$$

We claim that X' is min-homogeneous for C. Let

$$H := \{x_{\rho_1}, \dots, x_{\rho_d}\} \quad \text{and} \quad H' = \{x_{\eta_1}, \dots, x_{\eta_d}\}$$

be two subsets of X' such that $\rho_1 = \eta_1, \dots, \rho_s = \eta_s$ and

$$1 \leqslant \rho_1 < \cdots < \rho_d \leqslant \ell, \qquad 1 \leqslant \eta_1 < \cdots < \eta_d \leqslant \ell.$$

Since $x_{\rho_d}, x_\ell \in X_{\rho_d}$, we have $C_{\rho_{d-1}}(x_{\rho_d}) = C_{\rho_{d-1}}(x_\ell)$ and hence

$$C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_{\rho_d}) = C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_\ell).$$

Similarly, we show that

$$C(x_{\eta_1}, \dots, x_{\eta_{d-1}}, x_{\eta_d}) = C(x_{\eta_1}, \dots, x_{\eta_{d-1}}, x_\ell).$$

In addition, since $\{x_{\rho_1}, \dots, x_{\rho_{d-1}}\} \cup \{x_{\eta_1}, \dots, x_{\eta_{d-1}}\} \subseteq X'$, we have

$$D(\rho_1, \dots, \rho_{d-1}) = D(\eta_1, \dots, \eta_{d-1}),$$

i.e.,

$$C(x_{\rho_1}, \dots, x_{\rho_{d-1}}, x_\ell) = C(x_{\eta_1}, \dots, x_{\eta_{d-1}}, x_\ell).$$

This means that $C(H) = C(H')$ and proves that X' is max-homogeneous for C. This implies that X' is min-homogeneous for C.

We now return to the question of how large $\mathrm{card}(X)$ should be in order to ensure that the construction above can be carried through.

Set

$$t_d := k^{-1} \cdot \bigl(\mathrm{card}(X) - d + 1\bigr),$$
$$t_{p+1} := k^{-\binom{p-1}{d-2}} \cdot (t_p - 1) \quad (d \leqslant p < \ell).$$

Then we require that $t_\ell > 0$, where

$$\begin{aligned} t_\ell &= k^{-\binom{\ell-2}{d-2}} \cdot \left(k^{-\binom{\ell-3}{d-2}} \cdot \left(\cdots \left(k^{-\binom{d-1}{d-2}} \cdot (t_d - 1)\right) \cdots\right) - 1\right) \\ &= k^{-\binom{\ell-2}{d-2} - \cdots - \binom{d-1}{d-2}} \cdot t_d - k^{-\binom{\ell-2}{d-2} - \cdots - \binom{d-1}{d-2}} - \cdots - k^{-\binom{\ell-2}{d-2} - \binom{\ell-3}{d-2}} - k^{-\binom{\ell-2}{d-2}}. \end{aligned}$$

Since $k = k^{\binom{d-2}{d-2}}$, a sufficient condition on $\mathrm{card}(X)$ is then

$$\mathrm{card}(X) - d + 1 > k^{\binom{\ell-3}{d-2} + \cdots + \binom{d-2}{d-2}} + k^{\binom{\ell-4}{d-2} + \cdots + \binom{d-2}{d-2}} + \cdots + k^{\binom{d-2}{d-2}}.$$

A possible value is

$$\mathrm{card}(X) = d + \sum_{p=d-1}^{\ell-2} k^{\binom{p}{d-1}},$$

so that

$$\begin{aligned} R_\mu^s(d,k,m) &\leqslant d + \sum_{p=d-1}^{\ell-2} k^{\binom{p}{d-1}} \\ &\leqslant d + \sum_{p=d-1}^{\ell-2} k^{p^{d-1}} \\ &\leqslant d + \sum_{p=d-1}^{\ell-2} \left(k^{(p+1)^{d-1}} - k^{p^{d-1}}\right) \\ &= d + k^{(\ell-1)^{d-1}} - k^{(d-1)^{d-1}} \\ &\leqslant k^{(\ell-1)^{d-1}} \\ &= k^{R_\mu^s(d-1,k,m-1)^{d-1}}. \end{aligned}$$

Hence

$$R_\mu^s(d,k,m) * d \leqslant \left(k^d\right) * R_\mu^s(d-1,k,m-1) * (d-1).$$

After $(d-s)$ times iterated applications of the inequality we get

$$\begin{aligned} R_\mu^s(d,k,m) * d &\leqslant \left(k^d\right) * \left(k^{d-1}\right) * \cdots * \left(k^{s+1}\right) * R_\mu^s(s,k,m-d+s) * s \\ &= \left(k^d\right) * \left(k^{d-1}\right) * \cdots * \left(k^{s+1}\right) * (m-d+s) * s. \end{aligned}$$

This completes the proof. $\square$

Remark 2.4. Lemma 26.4 in [1] gives a slight sharper estimate for $s = d-1$:

$$R_\mu^{d-1}(d,k,m) \leqslant d + \sum_{i=d-1}^{m-2} k^{\binom{i}{d-1}}.$$

Corollary 2.5. *Let $2 \leqslant d \leqslant m$ and $2 \leqslant k$. Let f_k be the constant function with value k,*

$$R_{f_k}^{\mathrm{reg}}(d,m) \leqslant k * \left(k^{d-1}\right) * \left(k^{d-2}\right) * \cdots * \left(k^2\right) * (m-d+1).$$

Now we come back to f-regressiveness and prove the key upper bound of the present section. $2_d(x)$ is defined as follows: $2_0(x) := x$, $2_{d+1}(x) := 2^{2_d(x)}$, and $2_d := 2_d(1)$. We sometimes write 2_d^x instead of $2_d(x)$ for the sake of readability.

Lemma 2.6. *Given $d \geqslant 1$ and $\alpha \leqslant \varepsilon_0$ set $f_\alpha^d(i) := \lfloor \sqrt[F_\alpha^{-1}(i)]{\log_d(i)} \rfloor$. Then there exist $p, q \in \mathbb{N}$ depending (primitive-recursively) on d and α such that, for all m,*

$$R^{\mathrm{reg}}_{f_\alpha^{d-1}}(d+1, m) \leqslant 2_{d-1}^{F_\alpha(q)^{m+p}}.$$

Proof. Given d, α and m, let p be such that $d < p$, and for every x,

$$2_{d-1}^{x^{m+d}+1} + x \leqslant 2_{d-1}^{x^{m+p}}.$$

Let $q > p$ be so large that

$$(k) * (k^d) * \cdots * (k^2) * (m-d) < 2_{d-1}^{F_\alpha(q)^{m+d}+1}, \tag{2.1}$$

with $k := \lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1$. Now set

$$\ell := 2_{d-1}^{F_\alpha(q)^{m+d}+1} + F_\alpha(q) \leqslant 2_{d-1}^{F_\alpha(q)^{m+p}} =: N.$$

Let $C : [N]^{d+1} \to \mathbb{N}$ be any f_α^d-regressive function and

$$D : \left[F_\alpha(q), \ell\right]^{d+1} \to \mathbb{N}$$

be defined from C by restriction. Then for any $y \in [F_\alpha(q), \ell]$, we have

$$\begin{aligned} \sqrt[F_\alpha^{-1}(y)]{\log_{d-1}(y)} &\leqslant \sqrt[F_\alpha^{-1}(F_\alpha(q))]{\log_{d-1}\left(2_{d-1}\left(F_\alpha(q)^{m+p}\right)\right)} \\ &= \sqrt[q]{F_\alpha(q)^{m+p}}. \end{aligned}$$

Hence

$$\operatorname{Im}(D) \subseteq \left\lfloor F_\alpha(q)^{(m+p)/q} \right\rfloor + 1,$$

i.e., D is an $(\lfloor F_\alpha(q)^{(m+p)/q} \rfloor + 1)$-coloring.

By Corollary 2.5 and inequality 2.1 above, there is an $H \subseteq N$ min-homogeneous for D, hence for C, such that $\mathrm{card}(H) \geqslant m$. $\square$

Theorem 2.7. *Let $d \geqslant 1$, $\alpha < \omega_d$, $f_\alpha^d(i) := \lfloor \sqrt[F_\alpha^{-1}(i)]{\log_{d-1}(i)} \rfloor$.*

(1) $R^{\mathrm{reg}}_{\log^*}(\cdot,\cdot)$ *is primitive recursive.*
(2) $R^{\mathrm{reg}}_{\log_d}(d+1,\cdot)$ *is primitive recursive.*
(3) $R^{\mathrm{reg}}_{f_\alpha^d}(d+1,\cdot)$ *is primitive recursive in F_{ω_d}.*

Proof. (1) Let $m \geqslant d \geqslant 1$ be given. Choose x so large that $k = x + m$ satisfies

$$k * (k^{d-1}) * (k^{d-2}) * \cdots * (k^2) * (m-d+1) < 2_d^{x+m},$$

and $\ell := 2_d^{x+m}$ satisfies

$$\log^* \ell \leqslant k.$$

Thus, any $\log^*$-regressive coloring of $[\ell]^d$ is a k-coloring. We claim that $R^{\mathrm{reg}}_{\log^*}(d,m) \leqslant \ell$. Let $C : [\ell]^d \to \mathbb{N}$ be $\log^*$-regressive. By Theorem 2.3 we can find an $H \subseteq \ell$ min-homogeneous for C such that $\mathrm{card}(H) \geqslant m$.

(2) Let $d, m \geqslant 1$ be given. Let x be such that for $k := x + m$ and $\ell := 2_d^{x+m}$ we have

$$k * (k^d) * (k^{d-1}) * \cdots * (k^2) * (m-d) < 2_d^{x+m},$$

and

$$\lfloor \log_d(\ell) \rfloor \leqslant k.$$

Thus any $\log_d$-regressive coloring of $[\ell]^{d+1}$ is a k-coloring. We claim that $R^{\mathrm{reg}}_{\log_d}(d+1, m) \leqslant \ell$. Let $C : [\ell]^{d+1} \to \mathbb{N}$ be $\log_d$-regressive. By Theorem 2.3 we can find an $H \subseteq \ell$ min-homogeneous for C such that $\mathrm{card}(H) \geqslant m$.

(3) The assertion follows from Lemma 2.6. $\square$

It is also possible to work with variable iterations to obtain an upper bound for the Kanamori–McAloon principle with unbounded dimensions, as shown in Lee [10]. Let $|\cdot|_d$ be the d-times iterated binary length function.

Lemma 2.8. *Given $d \geqslant 2$ and $\alpha \leqslant \varepsilon_0$, let $g_\alpha(i) := |i|_{F_\alpha^{-1}(i)}$. Then, for some sufficiently large m,*

$$R^{\mathrm{reg}}_{g_\alpha}(d, m) \leqslant 2_{d+1}\bigl(F_\alpha(m)\bigr).$$

Proof. Given α, d, m define ℓ, N by

$$\ell := 2_d\bigl(F_\alpha(m)\bigr) + F_\alpha(m) \leqslant 2_{d+1}\bigl(F_\alpha(m)\bigr) =: N.$$

Let $C : [N]^d \to \mathbb{N}$ be any f_α-regressive function and

$$D : \bigl[F_\alpha(m), \ell\bigr]^d \to \mathbb{N}$$

be defined from C by restriction. Then for any $y \in [F_\alpha(m), \ell]$ we have

$$\begin{aligned} |y|_{F_\alpha^{-1}(y)} &\leqslant \bigl|2_{d+1}\bigl(F_\alpha(m)\bigr)\bigr|_{F_\alpha^{-1}(F_\alpha(m))} \\ &= \bigl|2_{d+1}\bigl(F_\alpha(m)\bigr)\bigr|_m \\ &< F_\alpha(m) \end{aligned}$$

if $m > d + 1$. Hence,

$$\mathrm{Im}(g) \subseteq F_\alpha(m).$$

In addition, we have for $k := F_\alpha(m)$

$$(k) * \bigl(k^{d-1}\bigr) * \cdots * \bigl(k^2\bigr) * (m - d + 1) < 2_d\bigl(F_\alpha(m)\bigr)$$

if m is large enough. By Theorem 2.3 we find H min-homogeneous for D, hence for C, such that $\mathrm{card}(H) \geqslant m$. $\square$

Theorem 2.9. *$R^{\mathrm{reg}}_{g_\alpha}(\cdot)$ is primitive recursive in F_{ε_0} for all $\alpha < \varepsilon_0$.*

Proof. The claim follows directly from Lemma 2.8. $\square$

3. Lower bounds

In this section we prove the lower bounds on the f-regressive Ramsey numbers for $f(n) = (\log_{d-1}(n))^{1/F^{-1}_{\omega_d}(n)}$, for all $d \geqslant 1$. The key arguments in Subsection 3.4 are a non-trivial adaptation of Kanamori and McAloon's [6], Section 3. Before being able to apply those arguments we need to develop – by bootstrapping – some relevant bounds for the parametrized Kanamori–McAloon principle. This is done in Subsection 3.3 by adapting the idea of the Stepping-up Lemma in [5]. We begin with the base case $d = 1$ which is helpful for a better understanding of the coming general cases. The following Subsection 3.1, covering the base case $d = 1$ of our main result, is already done in [10,8].

3.1. Ackermannian Ramsey functions

Throughout this subsection m denotes a *fixed* positive natural number. Set

$$h_\omega(i) := \left\lfloor \sqrt[F_\omega^{-1}(i)]{i} \right\rfloor \quad \text{and} \quad h_m(i) := \left\lfloor \sqrt[m]{i} \right\rfloor.$$

Define a sequence of strictly increasing functions $f_{m,n}$ as follows:

$$f_{m,n}(i) := \begin{cases} i+1 & \text{if } n = 0, \\ f_{m,n-1}^{(\lfloor \sqrt[m]{i} \rfloor)}(i) & \text{otherwise.} \end{cases}$$

Note that $f_{m,n}$ are strictly increasing.

Lemma 3.1. *$R_{h_m}^{\mathrm{reg}}(2, R(2, c, i+3)) \geqslant f_{m,c}(i)$ for all c and i.*

Proof. Let $k := R(2, c, i+3)$ and define a function $C_m : [R_{h_m}^{\mathrm{reg}}(2, k)]^2 \to \mathbb{N}$ as follows:

$$C_m(x, y) := \begin{cases} 0 & \text{if } f_{m,c}(x) \leqslant y, \\ \ell & \text{otherwise,} \end{cases}$$

where the number ℓ is defined by

$$f_{m,p}^{(\ell)}(x) \leqslant y < f_{m,p}^{(\ell+1)}(x)$$

where $p < c$ is the maximum such that $f_{m,p}(x) \leqslant y$. Note that C_m is h_m-regressive since $f_{m,p}^{(\lfloor \sqrt[m]{x} \rfloor)}(x) = f_{m,p+1}(x)$. Let H be a k-element subset of $R_{h_m}^{\mathrm{reg}}(2, k)$ which is min-homogeneous for C_m. Define a c-coloring $D_m : [H]^2 \to c$ by

$$D_m(x, y) := \begin{cases} 0 & \text{if } f_{m,c}(x) \leqslant y, \\ p & \text{otherwise,} \end{cases}$$

where p is as above. Then there is an $(i+3)$-element set $X \subseteq H$ homogeneous for D_m. Let $x < y < z$ be the last three elements of X. Then $i \leqslant x$. Hence, it suffices to show that $f_{m,c}(x) \leqslant y$ since $f_{m,c}$ is an increasing function.

Assume $f_{m,c}(x) > y$. Then $f_{m,c}(y) \geqslant f_{m,c}(x) > z$ by the min-homogeneity. Let $C_m(x, y) = C_m(x, z) = \ell$ and $D_m(x, y) = D_m(x, z) = D_m(y, z) = p$. Then

$$f_{m,p}^{(\ell)}(x) \leqslant y < z < f_{m,p}^{(\ell+1)}(x).$$

By applying $f_{m,p}$ we get the contradiction that $z < f_{m,p}^{(\ell+1)}(x) \leqslant f_{m,p}(y) \leqslant z$. $\square$

We are going to show that $R_{h_m}^{\mathrm{reg}}(2, \cdot)$ is not primitive recursive. This will be done by comparing the functions $f_{m,n}$ with the Ackermann function.

Lemma 3.2. *Let $i \geqslant 4^m$ and $\ell \geqslant 0$.*

(1) $(2i+2)^m < f_{m,\ell+2m^2}(i)$ *and* $f_{m,\ell+2m^2}((2i+2)^m) < f_{m,\ell+2m^2}^{(2)}(i)$.
(2) $F_n(i) < f_{m,n+2m^2}^{(2)}(i)$.

Proof. (1) By induction on k it is easy to show that $f_{m,k}(i) > (\lfloor \sqrt[m]{i} \rfloor)^k$ for any $i > 0$. Hence for $i \geqslant 4^m$,

$$f_{m,2m^2}(i) > \left(\left\lfloor \sqrt[m]{i} \right\rfloor \right)^{2m^2} \geqslant \left(\left\lfloor \sqrt[m]{i} \right\rfloor \right)^{m^2} \cdot 2^{m^2+m} \geqslant \left(\sqrt[m]{i+1} \right)^{m^2} \cdot 2^m = (2i+2)^m$$

since $2 \cdot \lfloor \sqrt[m]{i} \rfloor \geqslant \sqrt[m]{i+1}$. The second claim follows from the first one.

(2) By induction on n we show the claim. If $n=0$ it is obvious. Suppose the claim is true for n. Let $i \geqslant 4^m$ be given. Then by induction hypothesis we have $F_n(i) \leqslant f^{(2)}_{m,n+2m^2}(i)$. Hence

$$F_{n+1}(i) \leqslant F_n^{(i+1)}(i) \leqslant f^{(2i+2)}_{m,n+2m^2}(i) \leqslant f_{m,n+2m^2+1}\big((2i+2)^m\big) < f^{(2)}_{m,n+2m^2+1}(i).$$

The induction is now complete. $\square$

Corollary 3.3. *$F_n(i) \leqslant f_{m,n+2m^2+1}(i)$ for any $i \geqslant 4^m$.*

Theorem 3.4. *$R^{\mathrm{reg}}_{h_m}(2,\cdot)$ and $R^{\mathrm{reg}}_{h_\omega}(2,\cdot)$ are not primitive recursive.*

Proof. Lemma 3.1 and Corollary 3.3 imply that $R^{\mathrm{reg}}_{h_m}(2,\cdot)$ is not primitive recursive. For the second assertion we claim that

$$N(i) := R^{\mathrm{reg}}_{h_\omega}\big(2, R\big(2, i+2i^2+1, 4^i+3\big)\big) > F_\omega(i)$$

for all i. Assume to the contrary that $N(i) \leqslant A(i)$ for some i. Then for any $\ell \leqslant N(i)$ we have $A^{-1}(\ell) \leqslant i$, hence $\sqrt[i]{\ell} \leqslant \sqrt[A^{-1}(\ell)]{\ell}$. Hence

$$\begin{aligned} R^{\mathrm{reg}}_{h_\omega}\big(2, R\big(2, i+2i^2+1, 4^i+3\big)\big) &\geqslant R^{\mathrm{reg}}_{h_i}\big(2, R\big(2, i+2i^2+1, 4^i+3\big)\big) \\ &\geqslant f_{i,i+2i^2+1}\big(4^i\big) \\ &> F_\omega(i) \end{aligned}$$

by Lemma 3.1 and Corollary 3.3. Contradiction! $\square$

Now we are ready to begin with the general cases.

3.2. Fast-growing hierarchies

We introduce some variants of the fast-growing hierarchy and prove that they are still fast-growing, meaning they match-up with the original hierarchy.

Definition 3.5. Let $d > 0$, $c > 1$. Let ϵ be a real number such that $0 < \epsilon \leqslant 1$,

$$\begin{aligned} &B_{\epsilon,c,d,0}(x) := 2_d^{\lfloor \log_d(x) \rfloor^c}, \\ &B_{\epsilon,c,d,\alpha+1}(x) := B^{\lfloor \epsilon \cdot \sqrt[c]{\log_d(x)} \rfloor}_{\epsilon,c,d,\alpha}(x), \\ &B_{\epsilon,c,d,\lambda(x)} := B_{\epsilon,c,d,\lambda[\lfloor \epsilon \cdot \sqrt[c]{\log_d(x)} \rfloor]}(x). \end{aligned}$$

In the following we abbreviate $B_{\epsilon,c,d,\alpha}$ by B_α when ϵ, c, d are fixed.

Lemma 3.6. *Let c, d, ϵ be as above. For all $x > 0$,*

(1) $B_{i+1}(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}) \geqslant 2_d^{\lfloor \epsilon^{-1}\cdot(F_i(x)+1)\rfloor^c}$ *for all $i \in \omega$ and $x > 0$.*
(2) $B_\alpha(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}) \geqslant 2_d^{\lfloor \epsilon^{-1}\cdot(F_\alpha(x)+1)\rfloor^c}$ *for all $\alpha \geqslant \omega$ and $x > 0$.*

Proof. (1) We claim that $B_0^m(x) = 2_d^{\lfloor \log_d(x)\rfloor^{c^m}}$ for $m > 0$. Proof by induction on m. The base case holds trivially. For the induction step we calculate:

$$\begin{aligned}
B_0^{m+1}(x) &= B_0\big(B_0^m(x)\big)\\
&= 2_d^{\lfloor \log_d(B_0^m(x))\rfloor^c}\\
&= 2_d^{\lfloor \log_d(2_d^{\lfloor \log_d(x)\rfloor^{c^m}})\rfloor^c}\\
&= 2_d^{\lfloor \lfloor \log_d(x)\rfloor^{c^m}\rfloor^c}\\
&= 2_d^{\lfloor \log_d(x)\rfloor^{c^{m+1}}}.
\end{aligned}$$

We now claim that $B_{i+1}(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}) \geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_i(x)+1)\rfloor^c}$. Proof by induction on i. For $i=0$ we obtain

$$\begin{aligned}
B_1\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) &= B_0^{\lfloor \epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&= B_0^{\lfloor \epsilon\cdot\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor\rfloor}\big(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&\geqslant B_0^{x+1}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&= 2_d^{\lfloor \log_d(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c})\rfloor^{c^{x+1}}}\\
&= 2_d^{\lfloor \lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c\rfloor^{c^{x+1}}}\\
&= 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^{c^{x+2}}}\\
&\geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_0(x)+1)\rfloor^c}
\end{aligned}$$

since $x>0$ and $c>1$. For the induction step we compute

$$\begin{aligned}
B_{i+1}\big(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) &= B_i^{\lfloor \epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&\geqslant B_i^{x+1}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&\geqslant B_i^{x}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_{i-1}(x)+1)\rfloor^c}\big)\\
&\geqslant B_i^{x-1}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_{i-1}^2(x)+1)\rfloor^c}\big)\\
&\geqslant \cdots\\
&\geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_{i-1}^{x+1}(x)+1)\rfloor^c}\\
&= 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_i(x)+1)\rfloor^c}.
\end{aligned}$$

(2) We prove the claim by induction on $\alpha \geqslant \omega$. Let $\alpha=\omega$. We obtain

$$\begin{aligned}
B_\omega\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) &= B_{\omega[\epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c})}]}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&\geqslant B_{x+1}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\\
&\geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_x(x)+1)\rfloor^c}\\
&= 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_\omega(x)+1)\rfloor^c}.
\end{aligned}$$

For the successor case $\alpha+1$ we compute

$$\begin{aligned}
B_{\alpha+1}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) &= B_\alpha^{\lfloor \epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) \\
&= B_\alpha^{x+1}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) \\
&= B_\alpha^{x}\big(B_\alpha\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big)\big) \\
&\geqslant B_\alpha^{x}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_\alpha(x)+1)\rfloor^c}\big) \\
&\geqslant \cdots \\
&\geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_\alpha^{x+1}(x)+1)\rfloor^c} \\
&\geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_{\alpha+1}(x)+1)\rfloor^c}.
\end{aligned}$$

If λ is a limit we obtain

$$\begin{aligned}
B_\lambda\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) &= B_{\lambda[\lfloor \epsilon\cdot\sqrt[c]{\log_d(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c})}\rfloor]}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) \\
&\geqslant B_{d,\lambda[x+1]}\big(2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(x+1)\rfloor^c}\big) \\
&\geqslant 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_{\lambda[x+1]}(x)+1)\rfloor^c} \\
&= 2_d^{\lfloor 2\cdot\epsilon^{-1}\cdot(F_\lambda(x)+1)\rfloor^c}. \qquad \square
\end{aligned}$$

Theorem 3.7. *Let $d>0$, $c>1$. Let ϵ be a real number such that $0<\epsilon\leqslant 1$.*

(1) *$B_{\epsilon,c,d,\omega}$ eventually dominates all primitive recursive functions.*
(2) *$B_{\epsilon,c,d,\omega_d}$ eventually dominates F_α for all $\alpha<\omega_d$.*

Proof. Obvious by Lemma 3.6. $\square$

3.3. Bootstrapping

In this section we show how suitable iterations of the Regressive Ramsey Theorem for $(d+1)$-hypergraphs and parameter function $f(x)=\sqrt[c]{\log_{d-1}(x)}$ (for constant c) can be used to obtain min-homogeneous sets whose elements are "spread apart" with respect to the function $2_{d-1}(\log_{d-1}(x)^c)$ (i.e., $B_{\epsilon,c,d-1,0}$). This fact will be used next (Proposition 3.21) to show that one can similarly obtain from the same assumption even sparser sets (essentially sets whose elements are "spread apart" with respect to the function $F_{\omega_{d-1}^c}$).

For the sake of clarity we work out the proofs of the main results of the present section for the base cases $d=2$ and $d=4$ in detail in Section 3.3.1 before generalizing them in Section 3.3.2. We hope that this will improve the readability of the arguments.

Definition 3.8. We say that a set X is *f-sparse* if and only if for all $a,b\in X$ we have $f(a)\leqslant b$. We say that two elements a,b of a set X are *n-apart* if and only if there exist $e_1,\dots,e_n$ from X such that $a<e_1<\cdots<e_n<b$. We say that a set is *(f,n)-sparse* if and only if for all $a,b\in X$ such that a and b are n-apart we have $f(a)\leqslant b$.

Definition 3.9. Let X be a set of cardinality $>m\cdot k$. We define X/m as the set $\{x_0,x_m,x_{2m},\dots,x_{k\cdot m}\}$, where x_i is the $(i+1)$-th smallest element of X.

Thus, if a set X is (f,m)-sparse of cardinality $>k\cdot m$ we have that X/m is f-sparse and has cardinality $>k$.

3.3.1. $B_{\epsilon,2,1,0}$-sparse min-homogeneous sets – Base Cases

Given $P : [\ell]^d \to \mathbb{N}$ we call $X \subseteq \ell$ max-homogeneous for P if for all $U, V \in [X]^d$ with $\max(U) = \max(V)$ we have $P(U) = P(V)$.

Let $\mathrm{MIN}_k^d(m) := R_\mu(d, k, m)$, i.e., the least natural number ℓ such that for all partitions $P : [\ell]^d \to k$ there is a min-homogeneous $Y \subseteq \ell$ such that $\mathrm{card}(Y) \geqslant m$. Let $\mathrm{MAX}_k^d(m)$ be the least natural number ℓ such that for all partitions $P : [\ell]^d \to k$ there is a max-homogeneous $Y \subseteq \ell$ such that $\mathrm{card}(Y) \geqslant m$.

Let $k \geqslant 2$ and $m \geqslant 1$. Given an integer $a < k^m$ let $a = k^{m-1} \cdot a(m-1) + \cdots + k^0 \cdot a(0)$ be in the unique representation with $a(m-1), \ldots, a(0) \in \{0, \ldots, k-1\}$. Then $D^{(k,m)} : [k^m]^2 \to m$ is defined by

$$D^{(k,m)}(a, b) := \max\bigl\{j\colon\ a(j) \neq b(j)\bigr\}.$$

Lemma 3.10. *Let $k \geqslant 2$ and $m \geqslant 1$.*

(1) $\mathrm{MIN}^2_{k\cdot m}(m+2) > k^m$.
(2) $\mathrm{MAX}^2_{k\cdot m}(m+2) > k^m$.

Proof. Let us show the first item. Define $R_1 : [k^m]^2 \to k \cdot m$ as follows:

$$R_1(a, b) := k \cdot D(a, b) + b\bigl(D(a, b)\bigr),$$

where $D := D^{(k,m)}$. Assume $Y = \{a_0, \ldots, a_\ell\}$ with $a_0 < \cdots < a_\ell$ is min-homogeneous for R_1. We claim $\ell \leqslant m$. Let $c_i := D(a_i, a_{i+1})$, $i < \ell$. Since $m > c_0$ it is sufficient to show $c_{i+1} < c_i$ for every $i < \ell - 1$.

Fix $i < \ell - 1$. We have $D(a_i, a_{i+1}) = D(a_i, a_{i+2})$ since $R_1(a_i, a_{i+1}) = R_1(a_i, a_{i+2})$ by min-homogeneity. Hence for any $j > D(a_i, a_{i+1})$ we have $a_i(j) = a_{i+1}(j) = a_{i+2}(j)$ which means $c_i \geqslant c_{i+1}$. Moreover, $R_1(a_i, a_{i+1}) = R_1(a_i, a_{i+2})$ further yields $a_{i+1}(D(a_i, a_{i+1})) = a_{i+2}(D(a_i, a_{i+2}))$, hence $c_i = c_{i+1}$ cannot be true, since $a_{i+1}(D(a_{i+1}, a_{i+2})) \neq a_{i+2}(D(a_{i+1}, a_{i+2}))$.

For the proof of the second item define $R'_1 : [k^m]^2 \to k \cdot m$ as follows:

$$R'_1(a, b) := k \cdot D(a, b) + a\bigl(D(a, b)\bigr),$$

where $D := D^{(k,m)}$. Assume $Y = \{a_0, \ldots, a_\ell\}$ with $a_0 < \cdots < a_\ell$ is max-homogeneous for R'_1. We claim $\ell \leqslant m$. Let $c_i := D(a_i, a_{i+1})$, $i < \ell$. Since $m > c_{\ell-1}$ it is sufficient to show $c_{i+1} > c_i$ for every $i < \ell - 1$.

Fix $i < \ell - 1$. We have $D(a_i, a_{i+2}) = D(a_{i+1}, a_{i+2})$ since $R'_1(a_i, a_{i+2}) = R'_1(a_{i+1}, a_{i+2})$ by max-homogeneity. Hence for any $j > D(a_{i+1}, a_{i+2})$ we have $a_i(j) = a_{i+1}(j) = a_{i+2}(j)$ which means $c_i \leqslant c_{i+1}$. Moreover, $R'_1(a_i, a_{i+2}) = R'_1(a_{i+1}, a_{i+2})$ further yields $a_i(D(a_i, a_{i+2})) = a_{i+1}(D(a_{i+1}, a_{i+2}))$, hence $c_i = c_{i+1}$ cannot be true, since $a_i(D(a_i, a_{i+1})) \neq a_{i+1}(D(a_i, a_{i+1}))$. $\square$

Lemma 3.11. *Let $k, m \geqslant 2$.*

(1) $\mathrm{MIN}^3_{2k\cdot m}(2m+4) > 2^{k^m}$.
(2) $\mathrm{MAX}^3_{2k\cdot m}(2m+4) > 2^{k^m}$.

Proof. (1) Let $k, m \geqslant 2$ be positive integers and put $e := k^m$. Let R_1 and R'_1 be the partitions from Lemma 3.10. Define $R_2 : [2^e]^3 \to 2k \cdot m$ as follows:

$$R_2(u, v, w) := \begin{cases} R_1(D(u, v), D(v, w)) & \text{if } D(u, v) < D(v, w), \\ k \cdot m + R'_1(D(v, w), D(u, v)) & \text{if } D(u, v) > D(v, w), \end{cases}$$

where $D := D^{(2,e)}$. The case $D(u, v) = D(v, w)$ does not occur since we developed u, v, w with respect to base 2. Let $Y \subseteq 2^e$ be min-homogeneous for R_2. We claim $\mathrm{card}(Y) < 2m + 4$.

Assume $\mathrm{card}(Y) \geqslant 2m + 4$. Let $\{u_0, \ldots, u_{2m+3}\} \subseteq Y$ be min-homogeneous for R_2. We shall provide a contradiction. Let $d_i := D(u_i, u_{i+1})$ for $i < 2m + 3$.

Case 1: Assume there is some r such that $d_r < \cdots < d_{r+m+1}$. We claim that $Y' := \{d_r, \ldots, d_{r+m+1}\}$ is min-homogeneous for R_1 which would contradict Lemma 3.10.

Note that for all i, j with $r \leqslant i < j \leqslant r+m+2$ we have

$$D(u_i, u_j) = \max\bigl\{D(u_i, u_{i+1}), \ldots, D(u_{j-1}, u_j)\bigr\}.$$

We have therefore for $r \leqslant i < j \leqslant r+m+1$,

$$R_1(d_i, d_j) = R_1\bigl(D(u_i, u_{i+1}), D(u_{i+1}, u_{j+1})\bigr) = R_2(u_i, u_{i+1}, u_{j+1}).$$

By min-homogeneity of Y we obtain similarly

$$R_2(u_i, u_{i+1}, u_{j+1}) = R_2(u_i, u_{i+1}, u_{p+1}) = R_1(d_i, d_p)$$

for all i, j, p such that $r \leqslant i < j < p \leqslant r+m+1$.

Case 2: Assume there is some r such that $d_r > \cdots > d_{r+m+1}$. We claim that $Y' := \{d_{r+m+1}, \ldots, d_r\}$ is max-homogeneous for R_1' which would contradict Lemma 3.10.

Assume $r \leqslant i < j < p \leqslant r+m+1$, hence $u_i < u_j < u_p$ and $d_p < d_j < d_i$. Note that we also have $d_j = D(u_j, u_p)$ and $d_i = D(u_i, u_p)$. Hence

$$k \cdot m + R_1'(d_p, d_j) = k \cdot m + R_1'\bigl(D(u_p, u_{p+1}), D(u_j, u_p)\bigr) = R_2(u_j, u_p, u_{p+1}).$$

By min-homogeneity we obtain

$$\begin{aligned} k \cdot m + R_1'(d_p, d_i) &= k \cdot m + R_1'\bigl(D(u_p, u_{p+1}), D(u_i, u_p)\bigr) \\ &= R_2(u_i, u_p, u_{p+1}) \\ &= R_2(u_i, u_j, u_{j+1}) \\ &= k \cdot m + R_1'(d_j, d_i). \end{aligned}$$

Case 3: There is a local maximum of the form $d_i < d_{i+1} > d_{i+2}$. Note then that $D(u_i, u_{i+2}) = d_{i+1}$. Hence we obtain the following contradiction using the min-homogeneity: $k \cdot m > R_1(d_i, d_{i+1}) = R_2(u_i, u_{i+1}, u_{i+2}) = R_2(u_i, u_{i+2}, u_{i+3}) = k \cdot m + R_1'(d_{i+2}, d_{i+1}) \geqslant k \cdot m$.

Case 4: Cases 1 to 3 do not hold. Then there must be two local minima. But then inbetween we have a local maximum and we are back in Case 3.

(2) Similar to the first claim. Define R_2' just by interchanging R_1 and R_1' and argue as above interchanging the role of min-homogeneous and max-homogeneous sets. $\square$

Lemma 3.12. *Let $k, m \geqslant 2$.*

(1) $\mathrm{MIN}^4_{4k\cdot m}(2(2m+4)+2) > 2^{2^{k^m}}$.
(2) $\mathrm{MAX}^4_{4k\cdot m}(2(2m+4)+2) > 2^{2^{k^m}}$.

Proof. (1) Let $k, m \geqslant 2$ be positive integers and put $\ell := 2^{k^m}$. Let R_2 and R_2' be the partitions from Lemma 3.11. Let $D := D^{(2,\ell)}$. Then define $R_3 : [2^\ell]^4 \to 4k \cdot m$ as follows:

$$R_3(u, v, w, x) := \begin{cases} R_2(D(u,v), D(v,w), D(w,x)) & \text{if } D(u,v) < D(v,w) < D(w,x), \\ 2k \cdot m + R_2'(D(w,x), D(v,w), D(u,v)) & \text{if } D(u,v) > D(v,w) > D(w,x), \\ 0 & \text{if } D(u,v) < D(v,w) > D(w,x), \\ 2k \cdot m & \text{if } D(u,v) > D(v,w) < D(w,x). \end{cases}$$

The cases $D(u,v) = D(v,w)$ or $D(v,w) = D(w,x)$ don't occur since we developed u, v, w, x with respect to base 2.

Let $Y \subseteq 2^\ell$ be min-homogeneous for R_3. We claim $\mathrm{card}(Y) \leqslant 2(2m+4)+1$. Let $Y = \{u_0, \ldots, u_h\}$ be min-homogeneous for R_3, where $h := 2(2m+4)+1$. Put $d_i := D(u_i, u_{i+1})$ and $g := 2m+3$.

Case 1: Assume that there is some r such that $d_r < \cdots < d_{r+g}$. We claim that $Y' := \{d_r, \ldots, d_{r+g}\}$ is min-homogeneous for R_2 which would contradict Lemma 3.11.

Note again that for $r \leqslant i < j \leqslant r+g+1$ we have

$$D(u_i, u_j) = \max\bigl\{D(u_i, u_{i+1}), \ldots, D(u_{j-1}, u_j)\bigr\} = D(u_{j-1}, u_j).$$

Therefore for $r \leqslant i < p < q \leqslant r+g$,

$$\begin{aligned} R_2(d_i, d_p, d_q) &= R_2\bigl(D(u_i, u_{i+1}), D(u_{i+1}, u_{p+1}), D(u_{p+1}, u_{q+1})\bigr) \\ &= R_3(u_i, u_{i+1}, u_{p+1}, u_{q+1}). \end{aligned}$$

By the same pattern we obtain for $r \leqslant i < u < v \leqslant r+g$,

$$\begin{aligned} R_2(d_i, d_u, d_v) &= R_2\bigl(D(u_i, u_{i+1}), D(u_{i+1}, u_{u+1}), D(u_{u+1}, u_{v+1})\bigr) \\ &= R_3(u_i, u_{i+1}, u_{u+1}, u_{v+1}). \end{aligned}$$

By min-homogeneity of Y for R_3 we obtain then $R_2(d_i, d_p, d_q) = R_2(d_i, d_u, d_v)$. Thus Y' is min-homogeneous for R_2.

Case 2: Assume that there is some r such that $d_r > \cdots > d_{r+g}$. We claim that $Y' := \{d_{r+g}, \ldots, d_r\}$ is max-homogeneous for R_2' which would contradict Lemma 3.11.

Then for $r \leqslant i < p < q \leqslant r+g$,

$$\begin{aligned} 2k \cdot m + R_2'(d_q, d_p, d_i) &= 2k \cdot m + R_2'\bigl(D(u_{p+1}, u_{q+1}), D(u_{i+1}, u_{p+1}), D(u_i, u_{i+1})\bigr) \\ &= R_3(u_i, u_{i+1}, u_{p+1}, u_{q+1}). \end{aligned}$$

By the same pattern we obtain for $r \leqslant i < u < v \leqslant r+g$,

$$\begin{aligned} 2k \cdot m + R_2'(d_v, d_u, d_i) &= 2k \cdot m + R_2'\bigl(D(u_{u+1}, u_{v+1}), D(u_{i+1}, u_{u+1}), D(u_i, u_{i+1})\bigr) \\ &= R_3(u_i, u_{i+1}, u_{u+1}, u_{v+1}). \end{aligned}$$

By min-homogeneity of Y for R_3 we obtain then $R_2'(d_q, d_p, d_i) = R_2'(d_v, d_u, d_i)$. Thus Y' is max-homogeneous for R_2'.

Case 3: There is a local maximum of the form $d_i < d_{i+1} > d_{i+2}$. Then we obtain the following contradiction using the min-homogeneity

$$\begin{aligned} 0 &= R_3(u_i, u_{i+1}, u_{i+2}, u_{i+3}) \\ &= R_3(u_i, u_{i+2}, u_{i+3}, u_{i+4}) \\ &\geqslant 2k \cdot m \end{aligned}$$

since $D(u_i, u_{i+2}) = d_{i+1} > d_{i+2}$.

Case 4: Cases 1 to 3 do not hold. Then there must be two local minima. But then inbetween we have a local maximum and we are back in Case 3.

(2) Similar to the first claim. Define R_3' just by interchanging R_2 and R_2' and argue interchanging the role of min-homogeneous and max-homogeneous sets. □

We now show how one can obtain sparse min-homogeneous sets for certain functions of dimension 3 from the bounds from Lemma 3.11. It will be clear that the same can be done for functions of dimension 4 using the bounds from Lemma 3.12. In Section 3.3.2 we will lift the bounds and the sparseness results to the general case.

Lemma 3.13. *Let $f(i) := \lfloor\sqrt{\log(i)}\rfloor$. Let $\ell := 2^{(16\cdot 17+1)^2}$. Then there exists an f-regressive partition $P : [\mathbb{N}]^3 \to \mathbb{N}$ such that if Y is min-homogeneous for P and of cardinality not below $3\ell - 1$, then we have $2^{(\log(a))^2} \leqslant b$ for all $a, b \in \bar{Y}/4$, where*

$$\bar{Y} := Y \setminus \bigl(\{\textit{the first } \ell \textit{ elements of } Y\} \cup \{\textit{the last } \ell - 2 \textit{ elements of } Y\}\bigr).$$

Proof. Let $u_0 := 0$, $u_1 = \ell$ and $u_{i+1} := \mathrm{MIN}^3_{f(u_i)-1}(\ell+1) - 1$ for $i > 0$. Notice that $u_i < u_{i+1}$. This is because $u_i \geqslant 2^{(16\cdot 17+1)^2}$ implies by Lemma 3.11, letting $m = 8$,

$$\begin{aligned} u_{i+1} &= \mathrm{MIN}^3_{f(u_i)-1}(\ell+1) - 1 \\ &\geqslant \mathrm{MIN}^3_{f(u_i)-1}(20) - 1 \\ &\geqslant 2^{\lfloor \frac{f(u_i)-1}{16} \rfloor^8} \\ &> 2^{f(u_i)^4} \\ &= 2^{\log(u_i)^2} \\ &\geqslant u_i. \end{aligned}$$

Let $G_0 : [u_1]^3 \to 1$ be the constant function with the value 0 and for $i > 0$ choose $G_i : [u_{i+1}]^3 \to f(u_i) - 1$ such that every G_i-min-homogeneous set $Y \subseteq u_{i+1}$ satisfies $\mathrm{card}(Y) < \ell + 1$. Let $P : [\mathbb{N}]^3 \to \mathbb{N}$ be defined as follows:

$$P(x_0, x_1, x_2) := \begin{cases} G_i(x_0, x_1, x_2) + 1 & \text{if } u_i \leqslant x_0 < x_1 < x_2 < u_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then P is f-regressive by the choice of the G_i. Assume that $Y \subseteq \mathbb{N}$ is min-homogeneous for P and $\mathrm{card}(Y) \geqslant 3\ell - 1$ and $\bar{Y}$ is as described, i.e., $\mathrm{card}(\bar{Y}) \geqslant \ell + 1$. If $\bar{Y} \subset [u_i, u_{i+1}[$ then $\bar{Y}$ is G_i-min-homogeneous hence $\mathrm{card}(\bar{Y}) \leqslant \ell$ which is excluded. Hence each interval $[u_i, u_{i+1}[$ contains at most two elements from Y since we have omitted the last $\ell - 2$ elements from Y.

If a, b are in $\bar{Y}/4$. Then there are $e_1, e_2, e_3 \in \bar{Y}$ such that $a < e_1 < e_2 < e_3 < b$, and so there exists an $i \geqslant 1$ such that $a \leqslant u_i < u_{i+1} \leqslant b$. Hence $b \geqslant u_{i+1} \geqslant 2^{f(u_i)^4} \geqslant 2^{\log(a)^2}$ as above by Lemma 3.11. □

We just want to remark that $2^{(16\cdot 17+1)^2}$ is not the smallest number which satisfies Lemma 3.13.

3.3.2. $B_{\epsilon,c,d,0}$-sparse min-homogeneous sets – Generalization

We now show how the above results Lemma 3.12 and Lemma 3.13 can be generalized to arbitrary dimension. Let g_d be defined inductively as follows: $g_0(x) := x$, $g_{d+1}(x) := 2 \cdot g_d(x) + 2$. Thus

$$g_d(x) := \underbrace{2\big(\cdots\big(2(2}_{d} x + 2) + 2\big) \cdots\big) + 2,$$

i.e., d iterations of the function $x \mapsto 2x + 2$.

Lemma 3.14. *Let $d \geqslant 1$ and $k, m \geqslant 2$.*

(1) $\mathrm{MIN}^{d+1}_{2^{d-1}k\cdot m}(g_{d-2}(2m+4)) > 2_{d-1}(k^m)$.
(2) $\mathrm{MAX}^{d+1}_{2^{d-1}k\cdot m}(g_{d-2}(2m+4)) > 2_{d-1}(k^m)$.

Proof. (Sketch) By a simultaneous induction on $d \geqslant 1$. The base cases for $d \leqslant 2$ are proved in Lemma 3.10 and Lemma 3.11. Let now $d \geqslant 2$. The proof is essentially the same as the previous ones.

Let $R_d : [2_{d-1}(k^m)]^{d+1} \to 2^{d-1}k \cdot m$ (or $R'_d : [2_{d-1}(k^m)]^{d+1} \to 2^{d-1}k \cdot m$) be a partition such that every min-homogeneous set for R_d (or max-homogeneous set for R'_d) is of cardinality $< g_{d-2}(2m+4)$.

We define then $R_{d+1} : [2^{k^m}_d]^{d+2} \to 2^d k \cdot m$ as follows:

$$\begin{aligned} &R_{d+1}(x_1, \ldots, x_{d+2}) \\ &\quad := \begin{cases} R_d(d(x_1,x_2), \ldots, d(x_{d+1}, x_{d+2})) & \text{if } d(x_1,x_2) < \cdots < d(x_{d+1}, x_{d+2}), \\ 2^{d-1}k\cdot m + R'_d(d(x_{d+2}, x_{d+1}), \ldots, d(x_2, x_1)) & \text{if } d(x_1,x_2) > \cdots > d(x_{d+1}, x_{d+2}), \\ 0 & \text{if } d(x_1,x_2) < d(x_2,x_3) > d(x_3,x_4), \\ 2^{d-1}k\cdot m & \text{else.} \end{cases} \end{aligned}$$

And $R'_{d+1} : [2_d^{k^m}]^{d+2} \to 2^d k \cdot m$ is defined similarly by interchanging R_d and R'_d. Now we can argue analogously to Lemma 3.12. $\square$

We now state the key result of the present section, the Sparseness Lemma. Let $f(i) := \lfloor \sqrt[c]{\log_{d-1}(i)} \rfloor$. We show how an f-regressive function P of dimension $d+1$ can be defined such that all large min-homogeneous sets are $(2_{d-1}^{(\log_{d-1}(\cdot))^c}, 3)$-sparse.

Lemma 3.15 *(Sparseness Lemma). Given $c \geqslant 2$ and $d \geqslant 1$ let $f(i) := \lfloor \sqrt[c]{\log_{d-1}(i)} \rfloor$. And define $m := 2c^2$, $n := 2^{d-1} \cdot m$, and $\ell := 2_{d-1}((n \cdot (n+1)+1)^c)$. There exists an f-regressive partition $P_{c,d} : [\mathbb{N}]^{d+1} \to \mathbb{N}$ such that, if Y is*

- *min-homogeneous for $P_{c,d}$, and*
- $\operatorname{card}(Y) \geqslant 3\ell - 1$,

then we have $2_{d-1}^{(\log_{d-1}(a))^c} \leqslant b$ for all $a, b \in \bar{Y}/4$, where

$$\bar{Y} := Y \setminus \big(\{\textit{the first } \ell \textit{ elements of } Y\} \cup \{\textit{the last } \ell - 2 \textit{ elements of } Y\}\big).$$

Proof. Let $u_0 := 0, u_1 := \ell$ and $u_{i+1} := \mathrm{MIN}^{d+1}_{f(u_i)-1}(\ell+1) - 1$. Notice that $u_i < u_{i+1}$. This is because $u_i \geqslant \ell$ implies by Lemma 3.14,

$$\begin{aligned} u_{i+1} &= \mathrm{MIN}^{d+1}_{f(u_i)-1}(\ell+1) - 1 \\ &\geqslant \mathrm{MIN}^{d+1}_{f(u_i)-1}\big(g_{d-2}(2m+4)\big) - 1 \\ &\geqslant 2_{d-1}^{\lfloor \frac{f(u_i)-1}{2^{d-1} \cdot m} \rfloor^m} \\ &> 2_{d-1}^{f(u_i)^{m/2}} \\ &= 2^{\log(u_i)^c} \\ &\geqslant u_i. \end{aligned}$$

Note that $\ell > g_{d-2}(2m+4)$. Let $G_0 : [u_1]^{d+1} \to 1$ be the constant function with value 0 and for $i > 0$ choose $G_i : [u_{i+1}]^{d+1} \to f(u_i) - 1$ such that every G_i-min-homogeneous set $Y \subseteq u_{i+1}$ satisfies $\operatorname{card}(Y) \leqslant \ell$. Let $P : [\mathbb{N}]^{d+1} \to \mathbb{N}$ be defined as follows:

$$P_{c,d}(x_0, \ldots, x_d) := \begin{cases} G_i(x_0, \ldots, x_d) + 1 & \text{if } u_i \leqslant x_0 < \cdots < x_d < u_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_{c,d}$ is f-regressive by choice of the G_i's. Assume $Y \subseteq \mathbb{N}$ is min-homogeneous for $P_{c,d}$ and $\operatorname{card}(Y) \geqslant 3\ell - 1$. Let $\bar{Y}$ be as described, i.e., $\operatorname{card}(\bar{Y}) \geqslant \ell + 1$. If $\bar{Y} \subseteq [u_i, u_{i+1}[$ for some i then $\bar{Y}$ is min-homogeneous for G_i, hence $\operatorname{card}(\bar{Y}) \leqslant \ell$, which is impossible. Hence each interval $[u_i, u_{i+1}[$ contains at most two elements from $\bar{Y}$, since we have omitted the last $\ell - 2$ elements of Y.

Given $a, b \in \bar{Y}/4$ let $e_1, e_2, e_3 \in \bar{Y}$ such that $a < e_1 < e_2 < e_3 < b$. Then there exists an $i \geqslant 1$ such that $a \leqslant u_i < u_{i+1} \leqslant b$. Hence $b \geqslant u_{i+1} \geqslant 2^{f(u_i)^{m/2}} \geqslant 2^{\log(a)^c}$ as above by Lemma 3.14. $\square$

3.4. Capturing, glueing, compressing

Given $c \geqslant 2$ and $d \geqslant 1$ let $f_{c,d}(x) := \lfloor \sqrt[c]{\log_d(x)} \rfloor$. We first want to show that the regressive Ramsey function $R^{\mathrm{reg}}_{f_{c,d-1}}(d+1, \cdot)$ eventually dominates $B_{\epsilon,c,d,\omega^c_{d-1}}$ (for suitable choices of ϵ). Now let $f_{\omega_d,d-1}(x)$ be $\lfloor \sqrt[B^{-1}_{\omega_d}(\cdot)]{\log_{d-1}(x)} \rfloor$. We will conclude that the regressive Ramsey function $R^{\mathrm{reg}}_{f_{\omega_d,d-1}}(d+1, \cdot)$ eventually dominates B_{ω_d}. From the viewpoint of logic this implies that the Regressive Ramsey Theorem for $(d+1)$-hypergraphs with parameter function $f_{\omega_d,d-1}$ cannot be proved without induction on predicates with $(d+2)$ alternations of existential and universal quantifiers.

3.4.1. $B_{\omega_d^c}$-sparse min-homogeneous sets

We begin by recalling the definition of the "step-down" relation on ordinals from [7] and some of its properties with respect to the hierarchies defined in Section 3.2.

Definition 3.16. Let $\alpha < \beta \leqslant \varepsilon_0$ Then $\beta \to_n \alpha$ if for some sequence $\gamma_0, \dots, \gamma_k$ of ordinals we have $\gamma_0 = \beta, \gamma_{i+1} = \gamma_i[n]$ for $0 \leqslant i < k$ and $\gamma_k = \alpha$.

We first recall the following property of the $\to_n$ relation. It is stated and proved as Corollary 2.4 in [7].

Lemma 3.17. *Let $\beta < \alpha < \varepsilon_0$. Let $n > i$. If $\alpha \to_i \beta$ then $\alpha \to_n \beta$.*

Proposition 3.18. *Let $\alpha \leqslant \varepsilon_0$. For all $c \geqslant 2, d \geqslant 1$, let $f(x) = \lfloor \sqrt[c]{\log_d(x)} \rfloor$. Let $0 < \epsilon \leqslant 1$. Then we have the following*:

(1) *If $f(n) > f(m)$ then $B_{\epsilon,c,d,\alpha}(n) > B_{\epsilon,c,d,\alpha}(m)$.*
(2) *If $\alpha = \beta + 1$ then $B_{\epsilon,c,d,\alpha}(n) \geqslant B_{\epsilon,c,d,\beta}(n)$; if $\epsilon \cdot f(n) \geqslant 1$ then $B_{\epsilon,c,d,\alpha}(n) > B_{\epsilon,c,d,\beta}(n)$.*
(3) *If $\alpha \to_{\lfloor \epsilon \cdot f(n) \rfloor} \beta$ then $B_{\epsilon,c,d,\alpha}(n) \geqslant B_{\epsilon,c,d,\beta}(n)$.*

Proof. Straightforward from the proof of Proposition 2.5 in [7]. $\square$

We denote by $T_{\omega_d^c,n}$ the set $\{\alpha\colon\ \omega_d^c \to_n \alpha\}$. We recall the following bound from [7], Proposition 2.10.

Lemma 3.19. *Let $n \geqslant 2$ and $c, d \geqslant 1$. Then*

$$\operatorname{card}(T_{\omega_d^c,n}) \leqslant 2_{d-1}\big(n^{6c}\big).$$

Observe that, by straightforward adaptation of the proof of Lemma 3.19 (Proposition 2.10 in [7]), we accordingly have $\operatorname{card}(T_{\omega_d^c,f(n)}) \leqslant 2_{d-1}(f(n)^{6c})$ for f a non-decreasing function and all n such that $f(n) \geqslant 2$.

Definition 3.20. Let τ be a function of type k. We say that τ is *weakly monotonic on first arguments* on X (abbreviated *w.m.f.a.*) if for all $s, t \in [X]^k$ such that $\min(s) < \min(t)$ we have $\tau(s) \leqslant \tau(t)$.

In the rest of the present section, when ϵ, c, d are fixed and clear from the context, B_α stands for $B_{\epsilon,c,d,\alpha}$ for brevity.

Proposition 3.21 *(Capturing). Given $c, d \geqslant 2$ let $\epsilon = \sqrt[6c]{1/3}$. Put*

$$f(x) := \left\lfloor \sqrt[c]{\log_{d-1}(x)} \right\rfloor,$$

$$g(x) := \left\lfloor \sqrt[6c^2]{\log_{d-1}(x)} \right\rfloor,$$

$$h(x) := \left\lfloor \sqrt[6c]{\frac{1}{3}} \cdot \sqrt[6c^2]{\log_{d-1}(x)} \right\rfloor.$$

Then there are functions $\tau_1 : [\mathbb{N}]^2 \to \mathbb{N}$, $\tau_2 : [\mathbb{N}]^2 \to \mathbb{N}$, and $\tau_3 : [\mathbb{N}]^2 \to 2$, such that τ_1 is $2_{d-2}(\frac{1}{3}f)$-regressive, τ_2 is f-regressive, and the following holds: If $H \subseteq \mathbb{N}$ is of cardinality strictly larger than 2 and such that

(a) *H is min-homogeneous for τ_1,*
(b) *$\forall s, t \in [H]^2$ if $\min(s) < \min(t)$ then $\tau_1(s) \leqslant \tau_1(t)$ (i.e., τ_1 is w.m.f.a. on H),*
(c) *H is $2_{d-1}^{\lfloor \log_{d-1}(\cdot)^c \rfloor}$-sparse (i.e., $B_{\epsilon,c,d-1,0}$-sparse),*
(d) *$\min(H) \geqslant h^{-1}(2)$,*
(e) *H is min-homogeneous for τ_2, and*
(f) *H is homogeneous for τ_3,*

then for any $x < y$ in H we have $B_{\epsilon,c,d-1,\omega_{d-1}^c}(x) \leqslant y$, i.e., H is $B_{\epsilon,c,d-1,\omega_{d-1}^c}$-sparse.

Proof. Define a function τ_1 as follows:

$$\tau_1(x, y) := \begin{cases} 0 & \text{if } B_{\omega_{d-1}^c}(x) \leqslant y \text{ or } h(x) < 2, \\ \xi \dot{-} 1 & \text{otherwise, where } \xi = \min\{\alpha \in T_{\omega_{d-1}^c, h(x)}\colon\ y < B_\alpha(x)\}. \end{cases}$$

$\xi \dot{-} 1$ means 0 if $\xi = 0$ and β if $\xi = \beta + 1$. We have to show that τ_1 is well defined. First observe that the values of τ_1 can be taken to be in $\mathbb{N}$ since, by Lemma 3.19, we can assume an order preserving bijection between $T_{\omega_{d-1}^c, h(x)}$ and $2_{d-2}^{h(x)^{6c}}$:

$$\tau_1(x, y) < 2_{d-2}\big(h(x)^{6c}\big) = 2_{d-2}\left(\left(\sqrt[6c]{\frac{1}{3}}\ \sqrt[6c^2]{\log_{d-1}(x)}\right)^{6c}\right) = 2_{d-2}\left(\frac{1}{3}\sqrt[c]{\log_{d-1}(x)}\right).$$

In the following we will only use properties of values of τ_1 that can be inferred from this assumption.

Let $\xi = \min\{\alpha \in T_{\omega_{d-1}^c, h(x)}\colon\ y < B_\alpha(x)\}$. Suppose that the minimum ξ is a limit ordinal, call it λ. Then, by definition of the hierarchy, we have

$$B_\lambda(x) = B_{\lambda[h(x)]}(x) > y.$$

But $\lambda[h(x)] < \lambda$ and $\lambda[h(x)] \in T_{\omega_{d-1}^c, h(x)}$, against the minimality of λ.

Define a function τ_2 as follows:

$$\tau_2(x, y) := \begin{cases} 0 & \text{if } B_{\omega_{d-1}^c}(x) \leqslant y \text{ or } h(x) < 2, \\ k - 1 & \text{otherwise, where } B_{\tau_1(x,y)}^{k-1}(x) \leqslant y < B_{\tau_1(x,y)}^{k}(x). \end{cases}$$

If $\xi = \min\{\alpha \in T_{\omega_{d-1}^c, h(x)}\colon\ y < B_\alpha(x)\} = 0$, i.e., $B_0(x) > y$, then $\tau_2(x, y) = 0$. On the other hand, if $\xi > 0$ then one observes that $k - 1 < \epsilon \cdot \sqrt[c]{\log_{d-1}(x)}$ by definition of τ_1 and of B, so that τ_2 is f-regressive.

Define a function τ_3 as follows:

$$\tau_3(x, y) := \begin{cases} 0 & \text{if } B_{\omega_{d-1}^c}(x) \leqslant y \text{ or } h(x) < 2, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose H is as hypothesized. We show that τ_3 takes constant value 0. This implies the $B_{\omega_{d-1}^c}$-sparseness since $h(\min(H)) \geqslant 2$. Assume otherwise and let $x < y < z$ be in H. Note first that by the condition (c),

$$\min\big\{\alpha \in T_{\omega_{d-1}^c, h(x)}\colon\ y < B_\alpha(x)\big\} > 0 \quad \text{and hence } \tau_2(x, y) > 0.$$

By hypotheses on H, $\tau_1(x, y) = \tau_1(x, z)$, $\tau_2(x, y) = \tau_2(x, z)$, $\tau_1(x, z) \leqslant \tau_1(y, z)$. We have the following, by definition of τ_1, τ_2,

$$B_{\tau_1(x,z)}^{\tau_2(x,z)}(x) \leqslant y < z < B_{\tau_1(x,z)}^{\tau_2(x,z)+1}(x).$$

This implies that $B_{\tau_1(x,z)}^{\tau_2(x,z)+1}(x) \leqslant B_{\tau_1(x,z)}(y)$, by one application of $B_{\tau_1(x,z)}$.

We now show that $\tau_1(y,z) \rightarrow_{h(y)} \tau_1(x,z)$. We know $\tau_1 \in T_{\omega^c_{d-1},h(x)}$, i.e., $\omega^c_{d-1} \rightarrow_{h(x)} \tau_1(x,z)$. Since $x<y$ implies $h(x) \leqslant h(y)$ we have $\omega^c_{d-1} \rightarrow_{h(y)} \tau_1(x,z)$. But since $\tau_1(y,z) \in T_{\omega^c_{d-1},h(y)}$ and $\tau_1(y,z) \geqslant \tau_1(x,z)$ by hypotheses on H, we can conclude that $\tau_1(y,z) \rightarrow_{h(y)} \tau_1(x,z)$.

Hence, by Lemma 3.17 and Proposition 3.18(3), we have $B_{\tau_1(x,z)}(y) \leqslant B_{\tau_1(y,z)}(y)$, and we know that $B_{\tau_1(y,z)}(y) \leqslant z$ by definition of τ_1. So we reached the contradiction $z<z$. □

A comment about the utility of Proposition 3.21. If, assuming $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$, we are able to infer the existence of a set H satisfying the conditions of Proposition 3.21, then we can conclude that $R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1,\cdot)$ eventually dominates $B_{\omega^c_{d-1}}$. In fact, suppose that there exists an M such that for almost all x there exists a set H satisfying the conditions of Proposition 3.21 and such that $H \subseteq R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1,x+M)$, which means that such an H can be found as a consequence of $(\mathrm{KM})^{d+1}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}$. Also suppose that, for almost all x we can find such an H of cardinality $\geqslant x+2$. Then for such an $H=\{h_0,\dots,h_k\}$ we have $k \geqslant x+1$, $h_{k-1} \geqslant x$ and, by Proposition 3.21 $h_k \geqslant B_{\omega^c_{d-1}}(h_{k-1})$. Hence we can show that $R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1,\cdot)$ has eventually dominates $B_{\omega^c_{d-1}}$:

$$R^{\mathrm{reg}}_{\lfloor \sqrt[c]{\log_{d-1}} \rfloor}(d+1,x+M) \geqslant h_k \geqslant B_{\omega^c_{d-1}}(h_{k-1}) \geqslant B_{\omega^c_{d-1}}(x).$$

In the following we show how to obtain a set H as in Proposition 3.21 using the Regressive Ramsey Theorem for $(d+1)$-hypergraphs with parameter function $\lfloor \sqrt[c]{\log_{d-1}} \rfloor$.

3.4.2. Glueing and logarithmic compression of f-regressive functions

We here collect some tools that are needed to combine or glue distinct f-regressive functions in such a way that a min-homogeneous set (or a subset thereof) for the resulting function is min-homogeneous for each of the component functions. Most of these tools are straightforward adaptations of analogous results for regressive partitions from [6].

The first simple lemma (Lemma 3.22 below) will help us glue the partition ensuring sparseness obtained by the Sparseness Lemma 3.15 with some other relevant function introduced below. Observe that one does not have to go to an higher dimension if one is willing to give up one square root in the regressiveness condition.

Lemma 3.22. *Let $P:[\mathbb{N}]^n \to \mathbb{N}$ be $Q:[\mathbb{N}]^n \to \mathbb{N}$ be $\lfloor \sqrt[2c]{\log_k} \rfloor$-regressive functions. Let define $(P \otimes Q):[\mathbb{N}]^n \to \mathbb{N}$ as follows*:

$$(P \otimes Q)(x_1,\dots,x_n) := P(x_1,\dots,x_n) \cdot \left\lfloor \sqrt[2c]{\log_k(x_1)} \right\rfloor + Q(x_1,\dots,x_n).$$

Then $(P \otimes Q)$ is $\lfloor \sqrt[c]{\log_k} \rfloor$-regressive and if H is min-homogeneous for $(P \otimes Q)$ then H is min-homogenous for P and for Q.

Proof. We show that $(P \otimes Q)$ is $\sqrt[c]{\log_k}$-regressive:

$$\begin{aligned}(P \otimes Q)(\vec{x}) &= P(\vec{x}) \cdot \left\lfloor \sqrt[2c]{\log_k(x_1)} \right\rfloor + Q(\vec{x}) \\ &\leqslant \left(\sqrt[2c]{\log_k(x_1)} - 1\right) \cdot \sqrt[2c]{\log_k(x_1)} + \left(\sqrt[2c]{\log_k(x_1)} - 1\right) \\ &= \sqrt[c]{\log_k(x_1)} - 1 \\ &< \left\lfloor \sqrt[c]{\log_k(x_1)} \right\rfloor.\end{aligned}$$

We show that if H is min-homogeneous for $(P \otimes Q)$ then H is min-homogeneous for both P and Q. Let $x<y_2<\dots<y_n$ and $x<z_2<\dots<z_n$ be in H. Then $(P \otimes Q)(x,\vec{y}) = (P \otimes Q)(x,\vec{z})$. Then we show $a := P(x,\vec{y}) = P(x,\vec{z}) =: c$ and $c := Q(x,\vec{y}) = Q(x,\vec{z}) =: d$.

If $w := \lfloor \sqrt[2c]{\log_k(x_1)} \rfloor = 0$ then it is obvious since $a=b=0$. Assume now $w>0$. Then $a \cdot w + b = c \cdot w + d$. This, however, implies that $a=c$ and $b=d$, since $a,b,c,d<w$. □

The next two results are adaptations of Lemma 3.3 and Proposition 3.6 of Kanamori and McAloon [6] for f-regressiveness (for any choice of f). Lemma 3.23 is used in [6] for a different purpose, and it is quite surprising how well it fits in the present investigation. Essentially, it will be used to obtain, from a 2^f_{d-2}-regressive of dimension 2, an f-regressive function of dimension $d-2$ such that both have almost same min-homogeneous sets. Each iteration of the following lemma costs one dimension.

Lemma 3.23. *If $P:[\mathbb{N}]^n \to \mathbb{N}$ is f-regressive, then there is $\bar{P}:[\mathbb{N}]^{n+1} \to \mathbb{N}$, such that $\bar{P}$ is f-regressive and the following hold.*

(i) *$\bar{P}(s) < 2\log(f(\min(s))) + 1$ for all $s \in [\mathbb{N}]^{n+1}$, and*
(ii) *if $\bar{H}$ is min-homogeneous for $\bar{P}$, then $H = \bar{H} - (f^{-1}(7) \cup \{\max(\bar{H})\})$ is min-homogeneous for P.*

Proof. Write $P(s) = (y_0(s), \ldots, y_{d-1}(s))$ where $d = \log(f(\min(s)))$. Define $\bar{P}$ on $[N]^{n+1}$ as follows:

$$\bar{P}(x_0, \ldots, x_n) := \begin{cases} 0 & \text{if either } f(x_0) < 7 \text{ or } \{x_0, \ldots, x_n\} \\ & \text{is min-homogeneous for } P, \\ 2i + y_i(x_0, \ldots, x_{n-1}) + 1 & \text{otherwise, where } i < \log(f(x_0)) \\ & \text{is the least such that } \{x_0, \ldots, x_n\} \\ & \text{is not min-homogeneous for } y_i. \end{cases}$$

Then $\bar{P}$ is f-regressive and satisfies (i). We now verify (ii). Suppose that $\bar{H}$ is min-homogeneous for $\bar{P}$ and H is as described. If $\bar{P}|[H]^{n+1} = \{0\}$ then we are done, since then all $\{x_0, \ldots, x_n\} \in [H]^{n+1}$ are min-homogeneous for P. Suppose then that there are $x_0 < \cdots < x_n$ in H such that $\bar{P}(x_0, \ldots, x_n) = 2i + y_i(x_0, \ldots, x_{n-1}) + 1$. Given $s, t \in [\{x_0, \ldots, x_n\}]^n$ with $\min(s) = \min(t) = x_0$ we observe that

$$\bar{P}\big(s \cup \max(\bar{H})\big) = \bar{P}(x_0, \ldots, x_n) = \bar{P}\big(t \cup \max(\bar{H})\big)$$

by min-homogeneity. But then $y_i(s) = y_i(t)$, a contradiction. □

The next proposition allows one to glue together a finite number of f-regressive functions into a single f-regressive function. This operation costs one dimension.

Proposition 3.24. *There is a primitive recursive function $p:\mathbb{N} \to \mathbb{N}$ such that for any $n, e \in \mathbb{N}$, if $P_i:[\mathbb{N}]^n \to \mathbb{N}$ is f-regressive for every $i \leqslant e$ and $P:[\mathbb{N}]^{n+1} \to \mathbb{N}$ is f-regressive, there are $\rho_1:[\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive and $\rho_2:[\mathbb{N}]^{n+1} \to 2$ such that if $\bar{H}$ is min-homogeneous for ρ_1 and homogeneous for ρ_2, then*

$$H = \bar{H} \setminus \big(\max\{f^{-1}(7), p(e)\} \cup \{\max(\bar{H})\}\big)$$

is min-homogeneous for each P_i and for P.

Proof. Note that given any $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for all $x \geqslant m$,

$$\big(2\log\big(f(x)\big) + 1\big)^{k+1} \leqslant f(x).$$

Let $p(k)$ be the least such m.

For each P_i, let $\bar{P}_i$ be obtained by an application of Lemma 3.23. Define $\rho_2:[\mathbb{N}]^{n+1} \to 2$ as follows:

$$\rho_2(s) := \begin{cases} 0 & \text{if } \bar{P}_i(s) \neq 0 \text{ for some } i \leqslant e, \\ 1 & \text{otherwise.} \end{cases}$$

Define $\rho_1:[\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive as follows:

$$\rho_1(s) := \begin{cases} \langle \bar{P}_0(s), \ldots, \bar{P}_e(s) \rangle & \text{if } \rho_2(s) = 0 \text{ and } \min(s) \geqslant p(e), \\ P(s) & \text{otherwise.} \end{cases}$$

Observe that ρ_1 can be coded as an f-regressive function by choice of $p(\cdot)$.

Suppose $\bar{H}$ is as hypothesized and H is as described. If ρ_2 on $[H]^{n+1}$ were constantly 0, we can derive a contradiction as in the proof of the previous lemma. Thus ρ_2 is constantly 1 on $[H]^{n+1}$ and therefore $\rho_1(s) = P(s)$ for $s \in [H]^{n+1}$ and the proof is complete. □

The following proposition is an f-regressive version of Proposition 3.4 in Kanamori and McAloon [6]. It is easily seen to hold for any choice of f, but we include the proof for completeness. This proposition will allow us to find a min-homogeneous set on which τ_1 from Proposition 3.21 is weakly monotonic increasing on first arguments. The cost for this is one dimension.

Proposition 3.25. *If $P : [\mathbb{N}]^n \to \mathbb{N}$ is f-regressive, then there are $\sigma_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ f-regressive and $\sigma_2 : [\mathbb{N}]^{n+1} \to 2$ such that if H is of cardinality $> n+1$, min-homogeneous for σ_1 and homogeneous for σ_2, then $H \setminus \{\max(H)\}$ is min-homogeneous for P and for all $s, t \in [H]^n$ with $\min(s) < \min(t)$ we have $P(s) \leqslant P(t)$.*

Proof. Define $\sigma_1 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ as follows:

$$\sigma_1(x_0, \ldots, x_n) := \min\bigl(P(x_0, \ldots, x_{n-1}), P(x_1, \ldots, x_n)\bigr).$$

Obviously σ_1 is f-regressive since P is f-regressive. Define $\sigma_2 : [\mathbb{N}]^{n+1} \to \mathbb{N}$ as follows:

$$\sigma_2(x_0, \ldots, x_n) := \begin{cases} 0 & \text{if } P(x_0, \ldots, x_{n-1}) \leqslant P(x_1, \ldots, x_n), \\ 1 & \text{otherwise.} \end{cases}$$

Now let H be as hypothesized. Suppose first that σ_2 is constantly 0 on $[H]^{n+1}$. Then weak monotonicity is obviously satisfied. We show that $H \setminus \{\max(H)\}$ is min-homogeneous for P as follows. Let $x_0 < x_1 < \cdots < x_{n-1}$ and $x_0 < y_1 < \cdots < y_{n-1}$ be in $H \setminus \{\max(H)\}$. Since σ_2 is constantly 0 on H, we have $F(x_0, x_1, \ldots, x_{n-1}) \leqslant F(x_1, \ldots, x_{n-1}, \max(H))$, and $F(x_0, y_1, \ldots, y_{n-1}) \leqslant F(y_1, \ldots, y_{n-1}, \max(H))$. Since H is also min-homogeneous for σ_1, we have

$$\sigma_1\bigl(x_0, x_1, \ldots, x_{n-1}, \max(H)\bigr) = \sigma_1\bigl(x_0, y_1, \ldots, y_{n-1}, \max(H)\bigr).$$

Thus, $F(x_0, x_1, \ldots, x_{n-1}) = F(x_0, y_1, \ldots, y_{n-1})$.

Assume by way of contradiction that σ_2 is constantly 1 on $[H]^{n+1}$. Let $x_0 < \cdots < x_{n+1}$ be in H. Then, by two applications of σ_2 we have

$$F(x_0, \ldots, x_{n-1}) > F(x_1, \ldots, x_n) > F(x_2, \ldots, x_{n+1}),$$

so that $\sigma_1(x_0, \ldots, x_n) = F(x_1, \ldots, x_n)$ while $\sigma_1(x_0, x_2, \ldots, x_{n+1}) = F(x_2, \ldots, x_{n+1})$, against the min-homogeneity of H for σ_1. □

3.4.3. Putting things together

Now we have all ingredients needed for the lower bound part of the sharp threshold result.

Given f let $\bar{f}_k$ be defined as follows: $\bar{f}_0(x) := f(x)$, $\bar{f}_{k+1}(x) := 2\log(\bar{f}_k(x)) + 1$. Thus,

$$\bar{f}_k(x) := 2\log\bigl(2\log\bigl(\cdots\bigl(2\log\bigl(f(x)\bigr) + 1\bigr)\cdots\bigr) + 1\bigr) + 1,$$

with k iterations of $2\log(\cdot) + 1$ applied to f.

Let $f(x) = \lfloor \sqrt[c]{\log_{d-1}} \rfloor$ and $f'(x) = 2_\ell(1/3 \cdot f(x))$, $\ell = d-2$. Observe then that $\bar{f}'_\ell$ is eventually dominated by f, so that an $\bar{f}'_\ell$-regressive function is also f-regressive if the arguments are large enough. Let m be such that $\lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor \geqslant \bar{f}'_\ell(x)$ for all $x \geqslant m$. We have

$$R^{\text{reg}}_f(d+1, x+m) \geqslant R^{\text{reg}}_{\bar{f}'_\ell}(d+1, x).$$

We summarize the above argument in the following lemma.

Lemma 3.26. *If h eventually dominates g then*

$$R_h^{\mathrm{reg}}(d, x+m) \geqslant R_g^{\mathrm{reg}}(d, x),$$

where m is such that $h(x) \geqslant g(x)$ for all $x \geqslant m$.

Proof. (Sketch) If G is g-regressive then define G' on the same interval by letting $G'(i) = 0$ if $i \leqslant m$ and $G'(i) = G(i)$ otherwise. Then G' is h-regressive. If H' is min-homogeneous for G' and $\mathrm{card}(H') \geqslant x + m$ then $H = H' - \{\text{first } m \text{ elements of } H'\}$ is min-homogeneous for G and of cardinality $\geqslant x$. $\square$

The next theorem shows that $R_f^{\mathrm{reg}}(d+1, \cdot)$, with $f(x) = \lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor$, eventually dominates $B_{\epsilon,c,d-1,\omega_{d-1}^c}(x)$. As a consequence – using Lemma 3.6 – we will obtain the desired lower bound in terms of F_{ω_d}.

The following theorem is provable in Primitive Recursive Arithmetic ($\mathrm{I}\Sigma_1$).

Theorem 3.27. *Given $c, d \geqslant 2$ let $f(x) = \lfloor \sqrt[c]{\log_{d-1}(x)} \rfloor$. Then for all x,*

$$R_f^{\mathrm{reg}}\big(d+1, 12x + K(c,d)\big) > B_{\epsilon,2c,d-1,\omega_{d-1}^{2c}}(x),$$

where $\epsilon = \sqrt[12c]{1/3}$ and $K : \mathbb{N}^2 \to \mathbb{N}$ is a primitive recursive function.

Proof. Let $\hat{f}(x) := \lfloor \sqrt[2c]{\log_{d-1}(x)} \rfloor$ and $q(x) := 2_{d-2}(\frac{1}{3}\hat{f}(x))$. Then $\bar{q}_{d-2}$ is eventually dominated by $\hat{f}$, so there is a number r such that for all $x \geqslant r$ we have $\bar{q}_{d-2}(x) \leqslant \hat{f}(x)$. Let $D(c,d)$ be the least such r. Notice that $D : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive.

Let $h(x) := \lfloor \sqrt[12c]{1/3} \cdot \sqrt[24c^2]{\log_{d-1}(x)} \rfloor$. Now we are going to show that for all x,

$$R_f^{\mathrm{reg}}\big(d+1, 3\ell' - 1\big) > B_{\epsilon,2c,d-1,\omega_{d-1}^{2c}}(x),$$

where $\ell' = \ell + 4x + 4d + 4D(c,d) + 7$, $\ell = 2_{d-1}((n \cdot (n+1) + 1)^{2c})$, $n = 2^{d-1} \cdot m$, where m is the least number such that $m \geqslant 2(2c)^2$, and

$$\ell \geqslant \max\big(\big\{\hat{f}^{-1}(7), h^{-1}(2), p(0)\big\} \cup \big\{\bar{q}_k^{-1}(7) : k \leqslant d-3\big\}\big),$$

where $p(\cdot)$ is as in Proposition 3.24. The existence of such an m depends primitive recursively on c, d. Notice that the Sparseness Lemma 3.15 functions for any such m with respect to $\hat{f}$. We just remark that one should not wonder about how one comes to the exact numbers above. They just follow from the following construction of the proof.

Let τ_1, τ_2, τ_3 be the functions defined in Proposition 3.21 with respect to $\hat{f}$. Observe that τ_1 is $2_{d-2}(\frac{1}{3}\hat{f}(\cdot))$-regressive and τ_2 is $\hat{f}$-regressive.

Let σ_1, σ_2 be the functions obtained by Proposition 3.25 applied to τ_1. Observe that σ_1 is $2_{d-2}(\frac{1}{3}\hat{f}(\cdot))$-regressive, i.e., q-regressive.

Let $\sigma_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N}$ be the function obtained by applying Proposition 3.23 to σ_1 $d-2$ times. Observe that σ_1^* is *eventually* $\hat{f}$-regressive by the same argument as above.

Define $\hat{\sigma}_1^* : [\mathbb{N}]^{d+1} \to \mathbb{N}$ as follows:

$$\hat{\sigma}_1^* := \begin{cases} 0 & \text{if } x < D(c,d), \\ \sigma_1^*(x) & \text{otherwise.} \end{cases}$$

Then $\hat{\sigma}_1^*$ is $\hat{f}$-regressive such that if H is min-homogeneous for $\hat{\sigma}_1^*$ then

$$H \setminus \big\{\text{first } D(c,d) \text{ elements of } H\big\}$$

is min-homogeneous for σ_1^*.

Let ρ_1 and ρ_2 be the functions obtained by applying Proposition 3.24 to the $\hat{f}$-regressive functions $\hat{\sigma}_1^*$ and τ_2 (the latter trivially lifted to dimension d). Observe that ρ_1 is $\hat{f}$-regressive.

Now let $(P_{2c,d} \otimes \rho_1)$ be obtained, as in Lemma 3.22, from ρ_1 and the partition $P_{2c,d} : [\mathbb{N}]^{d+1} \to \mathbb{N}$ from the Sparseness Lemma 3.15 with respect to $\hat{f}$. Observe that, by Lemma 3.22, we have that $(P_{2c,d} \otimes \rho_1)$ is $\sqrt[c]{\log_{d-1}}$-regressive, i.e., f-regressive.

Now x be given. Let $H \subseteq R_f^{\mathrm{reg}}(d+1, 3\ell' - 1)$ be such that

$$\mathrm{card}(H) > 3\ell' - 1$$

and H is min-homogeneous for $(P_{2c,d} \otimes \rho_1)$ and homogeneous for ρ_2, for σ_2 and for τ_3. This is possible since the Finite Ramsey Theorem is provable in Primitive Recursive Arithmetic ($\mathrm{I}\Sigma_1$). Notice that H is then min-homogeneous for $P_{2c,d}$ and for ρ_1.

Now we follow the process just above in the reverse order to get a set which satisfies the conditions of the Capturing Proposition 3.21.

Define first H_0 and H_1 by:

$$H_0 := H \setminus \big(\{\text{first } \ell \text{ elements of } H\} \cup \{\text{last } \ell - 2 \text{ elements of } H\}\big),$$
$$H_1 := H_0/4.$$

Then for all $a, b \in H_1$ such that $a < b$ we have $2_{d-1}^{(\log_{d-1}(a))^{2c}} \leqslant b$ by Lemma 3.15. Notice that

$$\mathrm{card}(H_0) \geqslant \ell' + 1,$$
$$\mathrm{card}(H_1) \geqslant \big\lfloor (\ell' + 1)/4 \big\rfloor + 1.$$

Since H_1 is also min-homogeneous for ρ_1 (and ρ_2) we have by Proposition 3.24 that H_2 defined by

$$H_2 := H_1 \setminus \big(\max\big\{\hat{f}^{-1}(7), p(0)\big\} \cup \big\{\max(H_1)\big\}\big) = H_1 \setminus \big\{\max(H_1)\big\}$$

is min-homogeneous for $\hat{\sigma}_1^*$ and for τ_2, and

$$\mathrm{card}(H_2) \geqslant \big\lfloor (\ell' + 1)/4 \big\rfloor.$$

Let

$$H_3 := H_2 \setminus \big\{ \text{ first } D(c,d) \text{ elements of } H_2 \big\}.$$

Then H_3 is also min-homogeneous for σ_1^* (and obviously still min-homogeneous for τ_2, homogeneous for ρ_2, for σ_2 and for τ_3). Also, we have

$$\mathrm{card}(H_3) \geqslant \big\lfloor (\ell' + 1)/4 \big\rfloor - D(c,d).$$

By Lemma 3.23 we have that H_4 defined by

$$\begin{aligned} H_4 &:= H_3 \setminus \big(\max\big\{\bar{q}_k^{-1}(7)\colon k \leqslant d-3\big\} \cup \{\text{last } d-2 \text{ elements of } H_3\}\big) \\ &= H_3 \setminus \{\text{last } d-2 \text{ elements of } H_3\} \end{aligned}$$

is min-homogeneous for σ_1 (and σ_2), and

$$\mathrm{card}(H_4) \geqslant \big\lfloor (\ell' + 1)/4 \big\rfloor - D(c,d) - d + 2.$$

Now define H^* as follows:

$$H^* := H_4 \setminus \{\max H_4\}.$$

Notice that $\mathrm{card}(H_4) > 3$. Then by Proposition 3.25 H^* is min-homogeneous for τ_1 which is weakly monotonic on first arguments on $[H^*]^2$, and

$$\mathrm{card}\big(H^*\big) \geqslant \big\lfloor\big(\ell'+1\big)/4\big\rfloor - D(c,d) - d + 1 > x + 1.$$

The second inequality follows from the definition of ℓ'. Notice now that H^* satisfies all the conditions of the Capturing Proposition 3.21 with respect to $\hat{f}$.

Let $H^* = \{h_0, \ldots, h_k\}$ ($k \geqslant x+1$, so that $h_{k-1} \geqslant x$). Then, by Proposition 3.21, for all $a, b \in H^*$ such that $a < b$ we have $B_{\omega_{d-1}^c}(a) \leqslant b$,

$$R_f^{\mathrm{reg}}\big(d+1, 3\ell'-1\big) > h_k \geqslant B_{\epsilon,2c,d-1,\omega_{d-1}^{2c}}(h_{k-1}) \geqslant B_{\epsilon,2c,d-1,\omega_{d-1}^{2c}}(x),$$

where $\epsilon = \sqrt[12c]{1/3}$. The first inequality holds since we chose $H^* \subseteq R_f^{\mathrm{reg}}(d+1, \ell'-1)$. The second holds by Proposition 3.21. The third holds because $h_{k-1} \geqslant x$. $\square$

Let us restate Theorem 3.27 in a somewhat simplified form. Given $c, d \geqslant 2$ set, from now on,

$$\hat{g}_{c,d}(x) := \sqrt[c]{\log_{d-1}(x)}.$$

Theorem 3.28. *There are primitive recursive functions $h : \mathbb{N} \to \mathbb{N}$ and $K : \mathbb{N}^2 \to \mathbb{N}$ such that for all x and all $c, d \geqslant 2$,*

$$R_{\hat{g}_{c,d}}^{\mathrm{reg}}\big(d+1, h(x) + K(c,d)\big) \geqslant B_{\epsilon,c,d-1,\omega_{d-1}^c}(x),$$

where $\epsilon = \sqrt[6c]{1/3}$.

Proof. By inspection of the proof of Theorem 3.27, and by the fact that, as proved in Theorem 3.7, $B_{c,d,\alpha}$ and $B_{2c,d,\alpha}$ have the same growth rate. $\square$

Theorem 3.29. *Given $d \geqslant 2$ let $f(x) = \lfloor \sqrt[F_{\omega_d}^{-1}(i)]{\log_{d-1}(i)} \rfloor$. Then $R_f^{\mathrm{reg}}(d+1, \cdot)$ eventually dominates F_α for all $\alpha < \omega_d$.*

Proof. First remember that, by Lemma 3.6, there is a primitive recursive function $r : \mathbb{N}^2 \to \mathbb{N}$ such that

$$B_{\omega_{d-1}^c}\big(r(c,x)\big) \geqslant F_{\omega_{d-1}^c}(x).$$

On the other hand by Theorem 3.28, we have that for all x,

$$R_{\hat{g}_{c,d}}^{\mathrm{reg}}\big(d+1, h(x) + K(c,d)\big) > B_{\omega_{d-1}^c}(x)$$

for some primitive recursive functions h and K. Hence

$$R_{\hat{g}_{c,d}}^{\mathrm{reg}}\big(d+1, h\big(r(c,x)\big) + K(c,d)\big) > B_{\omega_{d-1}^c}\big(r(c,x)\big) > F_{\omega_{d-1}^c}(x).$$

We claim that

$$R_f^{\mathrm{reg}}\big(d+1, h\big(r(x,x)\big) + K(x,d)\big) > F_{\omega_d}(x)$$

for all x.

Assume it is false for some x and let

$$N(x) := R_f^{\mathrm{reg}}\big(d+1, h\big(r(x,x)\big) + K(x,d)\big).$$

Then for all $i \leqslant N(x)$ we have $F^{-1}_{\omega_d}(i) \leqslant x$ and so

$$f(i) = \sqrt[F^{-1}_{\omega_d}(i)]{\log_{d-1}(i)} \geqslant \sqrt[x]{\log_{d-1}(i)} = \hat{g}_x(i).$$

This implies that

$$\begin{aligned} R^{\mathrm{reg}}_f\big(d+1, h\big(r(x,x)\big) + K(x,d)\big) &\geqslant R^{\mathrm{reg}}_{\hat{g}_{x,d}}\big(d+1, h\big(r(x,x)\big) + K(x,d)\big) \\ &> F_{\omega^x_{d-1}}(x) \\ &= F_{\omega_d}(x). \end{aligned}$$

Contradiction! □

4. Concluding remarks

As a corollary of our main results one gets the following dichotomy.

Corollary 4.1. *Let $d, \ell \geqslant 1$.*

(1) *For all $n < d$, $R^{\mathrm{reg}}_{\lfloor \sqrt[\ell]{\log_n(\cdot)} \rfloor}(d+1, x)$ is primitive recursive in F_α for some $\alpha < \omega_d$ as a function of x.*
(2) *For all $n \geqslant d$, F_{ω_d} is primitive recursive in $R^{\mathrm{reg}}_{\lfloor \sqrt[\ell]{\log_n(\cdot)} \rfloor}(d+1, x)$ as a function of x.*

This also proves Lee's conjecture and closes the gap between $d-2$ and d left open in [10].

Our result can also be used to classify the threshold for the full Regressive Ramsey Theorem $(\forall d)\ (\mathrm{KM})^d_f$ with respect to F_{ε_0}.

Theorem 4.2.

(1) *For all $\alpha < \varepsilon_0$, $x \mapsto R^{\mathrm{reg}}_{|\cdot|_{F^{-1}_\alpha(\cdot)}}(x)$ is primitive recursive in some F_β, with $\beta < \varepsilon_0$.*
(2) *$x \mapsto R^{\mathrm{reg}}_{|\cdot|_{F^{-1}_{\varepsilon_0}(\cdot)}}(x)$ eventually dominates F_α for all $\alpha < \varepsilon_0$.*

Proof. The upper bound is established in Theorem 2.9. Now let $f(x) = |x|_{F^{-1}_{\varepsilon_0}(x)}$. Note first that it follows from the proof of Theorem 3.29 that

$$R^{\mathrm{reg}}_{|\cdot|_{d-1}}\big(d+1, s(c,d,x)\big) > F_{\omega^c_{d-1}}(x)$$

for some primitive recursive function s. This is because $\log_{d-1}$ and $|\cdot|_{d-1}$ have the same growth rate.

We claim that $R^{\mathrm{reg}}_f(d+1, s(d-1,d,d-1)) > F_{\omega_d}(d-1)$ for all $d > 0$. Assume otherwise. Then there is a $d > 0$ such that

$$N(d) := R^{\mathrm{reg}}_f\big(d+1, s(d-1,d,d-1)\big) \leqslant F_{\omega_d}(d-1) = F_{\omega^{d-1}_{d-1}}(d-1).$$

Then for all $i \leqslant N(d)$ we have $F^{-1}_{\omega_d}(i) \leqslant d-1$. Therefore

$$\begin{aligned} R^{\mathrm{reg}}_f\big(d+1, s(d-1,d,d-1)\big) &\geqslant R^{\mathrm{reg}}_{|\cdot|_{d-1}}\big(d+1, s(d-1,d,d-1)\big) \\ &> F_{\omega^{d-1}_{d-1}}(d-1). \end{aligned}$$

Contradiction! This implies the lower bound. □

References

[1] P. Erdős, A. Hajnal, A. Máté, R. Rado, Combinatorial Set Theory: Partition Relations for Cardinals, Stud. Logic Found. Math., vol. 106, North-Holland, 1984.
[2] P. Erdős, G. Mills, Some bounds for the Ramsey–Paris–Harrington numbers, J. Combin. Theory Ser. A 30 (1) (1981) 53–70.
[3] P. Erdős, R. Rado, Combinatorial theorems on classifications of subsets of a given set, Proc. Lond. Math. Soc. (3) 2 (1952) 417–439.
[4] K. Gödel, Über formal unentscheidebare Sätze der Principia Mathematica und verwandter Systeme, Monatsh. Math. Phys. 37 (1931) 173–198.
[5] R.L. Graham, B.L. Rotschild, J.H. Spencer, Ramsey Theory, Wiley, 1980.
[6] A. Kanamori, K. McAloon, On Gödel incompleteness and finite combinatorics, Ann. Pure Appl. Logic 33 (1) (1987) 23–41.
[7] J. Ketonen, R. Solovay, Rapidly growing Ramsey functions, Ann. of Math. (2) 113 (1981) 267–314.
[8] M. Kojman, G. Lee, E. Omri, A. Weiermann, Sharp thresholds for the phase transition between primitive recursive and Ackermannian Ramsey numbers, J. Combin. Theory Ser. A 115 (6) (2008) 1036–1055.
[9] M. Kojman, S. Shelah, Regressive Ramsey numbers are Ackermannian, J. Combin. Theory Ser. A 86 (1) (1999) 177–181.
[10] G. Lee, Phase transitions in axiomatic thought, PhD thesis, Münster, 2005, 121 pp., written under the supervision of A. Weiermann.
[11] J. Paris, L. Harrington, A mathematical incompleteness in Peano arithmetic, in: J. Barwise (Ed.), Handbook of Mathematical Logic, in: Stud. Logic Found. Math., vol. 90, North-Holland, 1977, pp. 1133–1142.
[12] F.P. Ramsey, On a problem of formal logic, Proc. Lond. Math. Soc. (2) 30 (1930) 338–384.
[13] H. Schwichtenberg, Eine Klassifikation der ε_0-rekursiven Funktionen, Z. Math. Logik Grundl. Math. 17 (1971) 61–74.
[14] S.S. Wainer, A classification of the ordinal recursive functions, Arch. Math. Logik Grundl. 13 (1970) 136–153.
[15] A. Weiermann, An application of graphical enumeration to PA, J. Symbolic Logic 68 (1) (2003) 5–16.
[16] A. Weiermann, A classification of rapidly growing Ramsey functions, Proc. Amer. Math. Soc. 132 (2) (2004) 553–561.
[17] A. Weiermann, Analytic combinatorics, proof-theoretic ordinals and phase transitions for independence results, Ann. Pure Appl. Logic 136 (2005) 189–218.