# Sailing towards, and then against, the Graceful Tree Conjecture: some promiscuous results

Andrea Vietri*
Università La Sapienza, Roma

## 1 Introduction

The results collected in the present paper, although of a different nature depending on the section, are eventually expected to raise some doubts on the well-known Graceful Tree Conjecture. We introduce this conjecture in the following lines.

Let $G = (V, E)$ be a connected graph and $\lambda : V \to \{0, 1, 2, ..., |E|\}$ be an injective vertex labelling.

**Definition 1.1.** The labelling $\lambda$ is termed *graceful* if the $|E|$ numbers $|\lambda(u) - \lambda(v)|$, over all edges $\{u, v\}$, make up the set $\{1, 2, ..., |E|\}$. A graph which admits a graceful labelling is termed *graceful* as well.

The paper [11] by Rosa can be considered the pioneering work on graceful labellings. Some applications of these labellings in real life were first described in [4]. While several classes of graphs have so far been proved to be graceful (see [8] for a thorough survey), trees still appear as rather unwilling to disclose their big secret on their being graceful in any case or not. For this reason, and at least for now, combinatorialists have to content themselves with the celebrated Graceful Tree Conjecture, which formalises a quite general feeling.

**Conjecture 1.2.** *Every tree is graceful.*

Such conjecture appeared first in Ringel's paper [10] dated 1964. It is therefore often specified as Ringel's conjecture, but also as either Kotzig's or Rosa's conjecture. Numerous papers written since that time are quoted in the above mentioned survey by Gallian, which is undoubtedly thorough and well written.

Although a number of partial results are in keeping with the conjecture, yet any reader will agree on the not so large amount of data supporting that position. In our opinion, the strongest result in the affirmative direction is the gracefulness of trees with at most 27 vertices, proved by Aldred and McKay (see [1]). We regard this result as a serious blow to

*Dipartimento Me.Mo.Mat., via A. Scarpa 16, 00161 Rome, Italy. http://www.dmmm.uniroma1.it/~ vietri/ vietri@dmmm.uniroma1.it .

that long standing mystery, because it is the very one to really manage the chaotic behaviour of trees. Unfortunately, 27 seems on one hand a large number, while on the other hand it appears minute (we also remark that the authors' proof had to rely on a computer).

A big amount of constructions of graceful labellings have been devised for trees which, more or less explicitly, have some features of regularity. The conjecture has been successfully tested even on irregular trees which, nonetheless, have a very simple structure (e.g. *caterpillars*, that is trees which reduce to paths once their pendent edges are removed – see [8, 11]; but the slightly larger class of *lobsters*, that is trees reducing to caterpillars once their pendent edges are removed, has not yet proved to be graceful).

It is conceivable that a powerful tool for attacking the conjecture is some induction argument. And actually, Stanton and Zarnke found in 1973 an elegant and effective way of putting something like a "yeast" into trees and make them "grow" by suitably attaching copies of a graceful tree to some other graceful tree. Their result is along the same line as our contribution in the next section, which is in fact devoted to the construction of graceful bipartite graphs[1] starting from smaller, graceful bipartite graphs. While refraining from going into details, we for now limit ourself to remark that both approaches have to take into account the labellings of the initial components, and that consequently the few prescribed ways of putting together the pieces weaken severely the induction "engine". In rough terms, if we split a tree into some graceful components, then we are not sure that induction will be applicable, because the location of labels in each component might force a different re-attachment of pieces.

Leaving aside sharp results connected to Conjecture 1.2, we mention a quite meaningful approximation result on graceful trees, namely that of Van Bussel, described in [13]. In that paper, among other things, the author proved that every tree $T$ on $m$ edges admits an injective vertex labelling in the range $0, \dots, 2m - diam(T)$, that produces distinct differences not exceeding the largest label (we speak of a *range-relaxed* graceful labelling). The import of such results brings us to the asymptotical aspects of graceful labellings, which constitute a not less interesting field of research (see e.g. [3, 5]).

The third section of the present paper should succeed in showing both sides of a medal, one glancing at the Graceful Tree Conjecture, the other turning its back on it. Such medal is the classification of all graceful labellings for two particular subclasses of trees. Whereas these trees are pretty elementary and can be easily shown to be graceful, yet from the knowledge of *all* possible labellings the reader will probably get an impression of richness but also of rigidity, amounting to the impossibility of assigning certain labels to certain vertices in some cases. The two pertinent corollaries warn about such rigidity and are expected to be followed, in the next future, by stronger results pointing out constraints on graceful labellings in a more intense way, and for more complicated trees.

In the fourth section we present a polynomial which should, we hope, help to find some nongraceful tree. Perhaps this polynomial is only the beginning of some definitely more accurate approach that uses tools from algebra and algebraic geometry.

[1] Clearly, a tree is in particular bipartite.

## 2 Graceful bipartite graphs

This section is devoted to the construction of graceful bipartite graphs by means of more elementary graceful graphs which make up the final graph if properly assembled – also with the help of a further graph to which they will "cling". Although the present method avails of *Skolem sequences* in an original way – as far as we know – yet the basic idea, amounting to the alteration of labels in each elementary component, dates back to [12]. The 30 year old technique leads however to a different kind of graceful labelling, and is applied only to copies of a given tree. It must be remarked that the results collected in the cited paper largely encompass – along certain directions – what claimed in our Corollary 2.5. We will show at due time some connections and differences between the two approaches.

Let us start with the definition of the basic component to which all the graceful graphs will be attached.

**Definition 2.1.** Let $q$ be a nonnegative integer. The *$q$-stem* $\mathcal{S}_q$ is a path $(\mathbf{r}, \mathbf{d}_1, \mathbf{u}_1, \mathbf{t})$ together with $q$ edges $\{(\mathbf{d}_1, \mathbf{d}_i), (\mathbf{u}_1, \mathbf{u}_i) : 2 \leq i \leq q+1\}$.

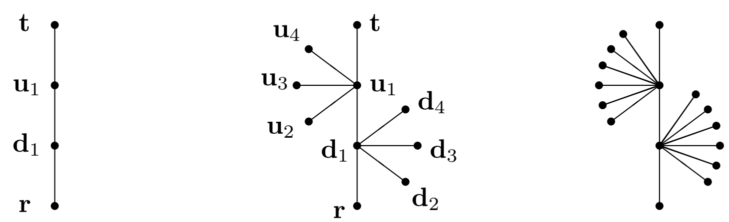


Figure 1: $q$-stems with $q = 0, 3, 6$

The following notion is the main arithmetical tool we shall utilise to produce a larger graceful graph from smaller ones.

**Definition 2.2.** Let $n$ be a positive integer. A *Skolem sequence of order* $n$ is a set of $n$ pairs $\{\{a_i, b_i\}, 1 \leq i \leq n\}$ such that $\bigcup_{1 \leq i \leq n} \{a_i, b_i\} = \{1, 2, ..., 2n\}$ and $\bigcup_{1 \leq i \leq n} \{b_i - a_i\} = \{1, 2, ..., n\}$.

Skolem sequences have been employed in many contexts (see for example [2, 7]). As the next result shows, the spectrum of all the possible orders for Skolem sequences is exhaustively known (see e.g. [2] for a proof of it).

**Theorem 2.3.** *A Skolem sequence of order $n$ exists if and only if either $n \equiv 0$ or $n \equiv 1$ (mod 4).*

We can now proceed with the main theorem on gracefulness in this section.

**Theorem 2.4.** *If $\{\{a_i < b_i\} : 1 \leq i \leq n\}$ is a Skolem sequence of order $n$ and $(G_0, \Gamma_0), (G_1, \Gamma_1), ..., (G_n, \Gamma_n)$ are gracefully labelled bipartite graphs, each one with the same number, $e$, of edges, then there* *exists a gracefully labelled bipartite graph* $(\mathcal{S}_{n-1} \cup H_0 \cup H_1 \cup K_1 \cup H_2 \cup K_2 \cup ... \cup H_n \cup K_n \ , \Gamma)$ *characterised as follows.*

$$\Gamma(\mathbf{r}) = 0 \ , \ \Gamma(\mathbf{d}_1) = (e+1)(2n+1) \ , \ \Gamma(\mathbf{t}) = 2n \ ,$$

$$\Gamma(\mathbf{d}_i) = i-1 \ (2 \leq i \leq n) \ , \ \Gamma(\mathbf{u}_i) = n-1+i \ (1 \leq i \leq n) \ ,$$

$$H_0 \cong G_0 \ , \ H_i \cong K_i \cong G_i \ (1 \leq i \leq n) \ , \ V(H_0) \cap V(\mathcal{S}_{n-1}) = \{\mathbf{d}_1\} \ ,$$

$$V(H_i) \cap V(\mathcal{S}_{n-1}) = \{\Gamma^{-1}(a_i)\} \ , \ V(K_i) \cap V(\mathcal{S}_{n-1}) = \{\Gamma^{-1}(b_i)\} \ (1 \leq i \leq n) \ ,$$

*where the* $2n+1$ *vertices of the graphs* $H_i$ *and* $K_i$ *appearing in the intersections correspond to those initially labelled* 0 *in the graphs* $G_i$*, and no other non-empty intersection occurs between vertices.*

*Proof.* The property of being bipartite for each graph $G_i$ is equivalent to the existence of $n+1$ bicolourings $\beta_i : V(G_i) \to \{0,1\}$. We can choose these maps in such a way that, for all $i$, the vertex of $G_i$ labelled 0 is sent to 0 by $\beta_i$. We then start by defining $\Gamma$ only on $H_0$ as $(2n+1)\,(e+1-\Gamma_0)$ (with a little abuse of notation, we shall apply $\Gamma_i$ directly to $H_i$). Therefore, on $H_i$ the labelling $\Gamma$ generates the set of differences $\Delta = \{t(2n+1) \colon 1 \leq t \leq e\}$. Now for any $i$ ranging in $\{1,2,...,n\}$ and any $v \in V(H_i)$ we define $\Gamma(v)$ as $(2n+1)\Gamma_i(v)+a_i$ if $\beta_i(v) = 0$, and as $(2n+1)\Gamma_i(v)+b_i$ if $\beta_i(v) = 1$. Similarly, for any $v \in V(K_i)$ we define $\Gamma(v)$ as $(2n+1)\Gamma_i(v)+b_i$ if $\beta_i(v) = 0$, and as $(2n+1)\Gamma_i(v)+a_i$ if $\beta_i(v) = 1$. It can then be checked with few difficulties that the restriction of $\Gamma$ to the graph $H_i \cup K_i$ generates the set of differences $\Delta_i = \{t(2n+1) \pm (b_i - a_i) \colon 1 \leq t \leq e\}$ for any $i$, and that no repetition of label occurs as $i$ varies. The Skolem sequence property now ensures that all the differences so far generated are distinct, and cover in fact the whole interval $[n+1, e(2n+1)+n] \cap \mathbf{N}$. Finally, the remaining differences $\{1,2,...,n\} \cup \{e(2n+1)+n+1, e(2n+1)+n+2, ..., (e+1)(2n+1)\}$ are generated by suitably connecting the graphs $H_0, H_1, ..., H_n, K_1, ..., K_n$ to the $(n-1)$-stem labelled as in the claim. The resulting graph is still bipartite because it contains no cycles save those already contained in the initial graph, which have all even length by assumption (bipartiteness is indeed also equivalent to the even parity of any cycle of the graph). □

A weaker version of the above theorem can be immediately obtained by specialising bipartite graphs to trees – this is for sure our closest leaning towards the Graceful Tree Conjecture in this paper. We avoid writing the corresponding claim. Instead, we limit ourselves to provide an even more specialised result, with $n = 1$ and the unique Skolem sequence $\{(1,2)\}$, just in order to give a glimpse of the matter.

**Corollary 2.5.** *For any graceful tree* $T$*, a graceful tree can be obtained by taking three copies of* $T$*, then adding a* 2*-path that connects the three vertices corresponding to the label 0, and finally adding a pendent edge to one of the two endvertices of the* 2*-path.*

We remark that the results on graceful trees obtained by Stanton and Zarnke in 1973 are incomparably more general than the above corollary, and could be easily extended to bipartite graphs. In fact, the quite effective and elegant method the authors used works well

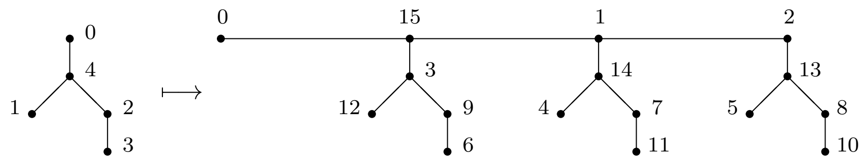


Figure 2: Applying the corollary

also in the more general context, if properly adjusted. In details, what claimed among other things in [12] is that $k$ copies $T_1, ..., T_k$ of any graceful tree on $n$ vertices can be attached to another graceful tree $U$, with $k$ vertices, provided each vertex of $U$ is identified with that vertex of $T_i$ initially labelled by $n$. It turns out that the modification of labels yielding the final graceful labelling does actually rely only on the bipartiteness, and is not prejudiced by considering any other bipartite graph different from a tree.

# 3 Classification of graceful labellings: an example

The main motivation for determining all possible graceful labellings of a given tree is, in our opinion, that of understanding the extent to which the combinatorial structure of a tree conditions the labelling of its vertices, whatever the graceful labelling. We believe, indeed, that a good knowledge of the "local" behaviour of a graceful tree – in particular with respect to nonadmissible labels for some vertices – may be helpful, in the future, to work out a larger tree whose gracefulness is prejudiced.

**Definition 3.1.** We denote by $\mathcal{C}^h_{m,n}$ the tree obtained by connecting the centres of an $m$-star and an $n$-star by a path of length $h$ (see for example Figure 3). The corresponding sequence of labels shall be denoted by $(a_1, a_2, ..., a_m, c_0, c_1, ..., c_h, b_1, b_2, ..., b_n)$, obtained by first writing the labels of the leaves of the $m$-star (increasingly), then passing to the centre of that star and moving along the path to the other centre, then finally writing the labels of the leaves of the $n$-star (increasingly).

We recall that any $\mathcal{C}^h_{m,n}$ is graceful because it is a caterpillar. In the sequel, by *label complementation* of a graceful labelling $\Gamma$ of a graph having $e$ edges we shall understand the switching from $\Gamma()$ to $e - \Gamma()$, which of course produces again a graceful labelling.

**Theorem 3.2.** *The graceful labellings of $\mathcal{C}^1_{m,n}$ are – up to label complementation and interchange of $m$ and $n$ – precisely all those satisfying one of the following (mutually distinct) conditions, where $S = m + n + 1$.*

$$(\mathbf{A}) \quad \begin{cases} m \text{ even} \\ c_0 = S, c_1 = 0 \\ 1 \leq a_1 < a_2 < ... < a_{m/2} < S/2 \\ a_{m-i+1} = S - a_i \quad (1 \leq i \leq m/2) \end{cases}$$

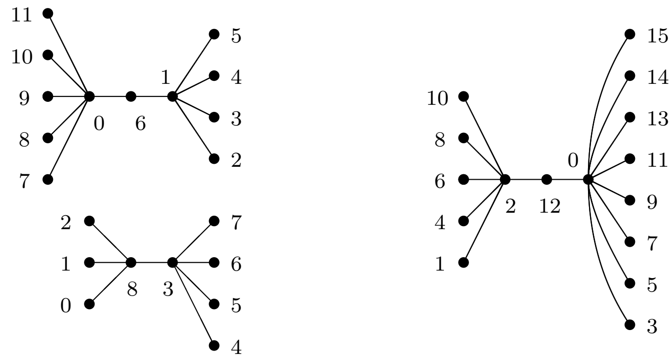


Figure 3: Graceful labellings of $\mathcal{C}^2_{5,4}$, $\mathcal{C}^1_{3,4}$, and $\mathcal{C}^2_{5,8}$

$$
(\mathbf{B}) \quad \begin{cases} m \text{ even} \\ c_1 = 0, b_n = S \\ c_0 > m \\ 1 \le a_1 < a_2 < ... < a_{m/2} < c_0/2 \\ a_{m+1-i} = c_0 - a_i \quad (1 \le i \le m/2) \end{cases}
$$

$$
(\mathbf{C}) \quad \begin{cases} m \text{ odd} \\ c_1 = 0, b_n = S \\ c_0 \text{ even}, c_0 > m \\ 1 \le a_1 < a_2 < ... < a_{(m-1)/2} < c_0/2 = a_{(m+1)/2} \\ a_{m+1-i} = c_0 - a_i \quad (1 \le i \le (m-1)/2) \end{cases}
$$

*Proof.* We split the proof into two main cases.

**Case A.** *No pendent edge generates the largest difference, $S$.*

Up to complementing the labelling we can assume that $c_0 = S$ and, consequently, that $c_1 = 0$. Because any label $b_i$ generates the difference $b_i$ itself, the set of labels $\{a_1, a_2, ..., a_m\}$ must coincide with the set of differences $\{S - a_1, S - a_2, ..., S - a_m\}$. Notice that the list in the first set is increasing. Therefore, assuming for the moment that $m$ is even, we deduce the following.

$$S - a_m = a_1 \, , \, S - a_{m-1} = a_2 \, , \, ... \, , \, S - a_{m/2+1} = a_{m/2} \; .$$

Hence we have to choose a sequence satisfying $1 \le a_1 < a_2 < ... < a_{m/2} < S/2$. Every such choice does in fact yield a solution of the form **(A)**, as it could be easily checked. Instead, if $m$ is odd, a similar family of equalities will also contain the condition $a_{(m+1)/2} = S/2$, which forces $n$ to be even. It is now enough to interchange $m$ and $n$ so as to reduce to the above case again. We have thus obtained Condition **(A)**.

**Case B-C.** *Some pendent edge generates the largest difference.*

By possibly interchanging $m$ and $n$ we can assume that such pendent edge belongs to the $n$-star. Therefore, up to complementation, we have that $c_1 = 0$ and $b_n = S$. Now $c_0$ must be larger than $a_m$, because otherwise the difference $a_m$ itself could not be generated. From the trivial inequality $a_m \geq m$ we thus deduce the necessary condition $c_0 > m$. Finally, reasoning as in the previous case we find that if $m$ is even all the sequences are precisely those satisfying $1 \leq a_1 < a_2 < ... < a_{m/2} < c_0/2$, while in the odd case the characterising condition is that $c_0$ be even and $1 \leq a_1 < a_2 < ... < a_{(m-1)/2} < c_0/2 = a_{(m+1)/2}$. We have thus met Conditions **(B)** or **(C)** . □

As a fruit of the above classification we can immediately deduce the following necessary condition.

(...)

**Theorem 3.3.** *The graceful labellings of $\mathcal{C}^2_{m,n}$ are – up to label complementation and interchange of $m$ and $n$ – precisely all those satisfying one of the following (mutually distinct) conditions, where $S = m + n + 2$.*

$$(\mathbf{A1}) \begin{cases} c_1 = S, c_2 = 0 \\ m \text{ even}, n \text{ even}, m \leq n \\ c_0 = S/2 \\ 1 \leq a_1 < a_2 < ... < a_{m/2} < c_0/2 \\ a_{m+1-i} = S/2 - a_i \quad (1 \leq i \leq m/2) \end{cases}$$

$$(\mathbf{A2}) \begin{cases} c_1 = S, c_2 = 0 \\ m \text{ odd}, n \equiv m + 2 \,(\text{mod } 4), \;\; m \leq n + 2 \\ c_0 = S/2 \\ 1 \leq a_1 < a_2 < ... < a_{(m-1)/2} < c_0/2 = a_{(m+1)/2} \\ a_{m+1-i} = S/2 - a_i \quad (1 \leq i \leq (m-1)/2) \end{cases}$$

$$(\mathbf{B1}) \begin{cases} c_1 = S, c_2 = 0 \\ m \text{ even}, n \text{ even} \\ c_0 \text{ even}, c_0 = S/(t+1) \text{ for some odd } t \geq 3, t \leq m + 1 \\ \{a_{m-t+2}, a_{m-t+3}, ..., a_m\} = \{2c_0, 3c_0, ..., tc_0\} \\ 1 \leq a_1 < a_2 < ... < a_{(m-t+1)/2} < c_0/2 \\ a_{m-t+2-i} = c_0 - a_i \quad (1 \leq i \leq (m-t+1)/2) \end{cases}$$

$$(\mathbf{B2}) \begin{cases} c_1 = S, c_2 = 0 \\ m \text{ odd} \\ c_0 = S/(t+1) \text{ for some even } t \geq 2, t \leq m + 1 \\ \{a_{m-t+2}, a_{m-t+3}, ..., a_m\} = \{2c_0, 3c_0, ..., tc_0\} \\ 1 \leq a_1 < a_2 < ... < a_{(m-t+1)/2} < c_0/2 \\ a_{m-t+2-i} = c_0 - a_i \quad (1 \leq i \leq (m-t+1)/2) \end{cases}$$

$$(\mathbf{C1})\ \begin{cases} c_1 = S, c_2 = 0 \\ m \text{ odd}, n \text{ odd} \\ c_0 \text{ even}, c_0 = S/(t+1) \text{ for some odd } t \geq 3, t \leq m \\ \{a_{m-t+2}, a_{m-t+3}, ..., a_m\} = \{2c_0, 3c_0, ..., tc_0\} \\ 1 \leq a_1 < a_2 < ... < a_{(m-t)/2} < c_0/2 = a_{m-t+1} \\ a_{m-t+2-i} = c_0 - a_i \quad (1 \leq i \leq (m-t)/2) \end{cases}$$

$$(\mathbf{C2})\ \begin{cases} c_1 = S, c_2 = 0 \\ m \text{ even} \\ c_0 = S/(t+1) \text{ for some even } t \geq 2, t \leq m \\ \{a_{m-t+2}, a_{m-t+3}, ..., a_m\} = \{2c_0, 3c_0, ..., tc_0\} \\ 1 \leq a_1 < a_2 < ... < a_{(m-t)/2} < c_0/2 = a_{m-t+1} \\ a_{m-t+1-i} = c_0 - a_i \quad (1 \leq i \leq (m-t)/2) \end{cases}$$

$$(\mathbf{D1})\ \begin{cases} c_2 = 0, b_n = S \\ m \text{ even} \\ m < c_0 < S/2, c_1 = 2c_0 \\ 1 \leq a_1 < a_2 < ... < a_{m/2} < c_0/2 \\ a_{m+1-i} = c_0 - a_i \quad (1 \leq i \leq m/2) \end{cases}$$

$$(\mathbf{D2})\ \begin{cases} c_2 = 0, b_n = S \\ m \text{ odd} \\ c_0 \text{ even}, m < c_0 < S/2, c_1 = 2c_0 \\ 1 \leq a_1 < a_2 < ... < a_{(m-1)/2} < c_0/2 = a_{(m+1)/2} \\ a_{m+1-i} = c_0 - a_i \quad (1 \leq i \leq (m-1)/2) \end{cases}$$

$$(\mathbf{E1})\ \begin{cases} c_2 = 0, b_n = S \\ c_1 = (t+1)c_0 \text{ for some } t \not\equiv m \pmod 2, 2 \leq t \leq m+1 \\ \{a_{m-t+2}, a_{m-t+3}, ..., a_m\} = \{2c_0, 3c_0, ..., tc_0\} \\ 1 \leq a_1 < a_2 < ... < a_{(m-t+1)/2} < c_0/2 \\ a_{m+2-t-i} = c_0 - a_i \quad (1 \leq i \leq (m-t+1)/2) \end{cases}$$

$$(\mathbf{E2})\ \begin{cases} c_2 = 0, b_n = S \\ c_0 \text{ even}, c_1 = (t+1)c_0 \text{ for some } t \equiv m \pmod 2, 2 \leq t \leq m \\ \{a_{m-t+2}, a_{m-t+3}, ..., a_m\} = \{2c_0, 3c_0, ..., tc_0\} \\ 1 \leq a_1 < a_2 < ... < a_{(m-t)/2} < c_0/2 = a_{m-t+1} \\ a_{m-t+1-i} = c_0 - a_i \quad (1 \leq i \leq (m-t)/2) \end{cases}$$

*Proof.* Let us also here consider two main cases, depending on the realisation of the largest difference.

**Case A-B-C.** *No pendent edge generates the largest difference, $S$.*

After possibly interchanging $m$ and $n$, and complementing the labels, we will find ourselves with $c_1 = S$ and $c_2 = 0$, which we can therefore assume to hold from the beginning. Consequently, the set of labels (and of differences) $\{c_0, a_1, a_2, ..., a_m\}$ must coincide with the set of differences $\{|c_0 - a_1|, |c_0 - a_2|, ..., |c_0 - a_m|, S - c_0\}$. We split the present case into two subcases, depending on the realisation of the difference $S - c_0$.

**- Subcase A.** $c_0 = S - c_0$

Since $2c_0 = m + n + 2$, the integers $m$ and $n$ have the same parity. In addition, $c_0$ must be larger than $a_m$, because otherwise no difference of the form $|c_0 - a_i|$ could be equal to $a_m$. As a consequence, all the above absolute value bars are superfluous. Now the $a_i$'s are increasing with $i$, whence the following holds.

$$\begin{array}{ll} \text{For}\, m\, \text{even}: & c_0 - a_m = a_1, c_0 - a_{m-1} = a_2, ..., c_0 - a_{m/2+1} = a_{m/2}\,; \\ \text{for}\, m\, \text{odd}: & c_0 - a_m = a_1, c_0 - a_{m-1} = a_2, ..., c_0 - a_{(m+1)/2} = a_{(m+1)/2}. \end{array}$$

In particular, if $m$ is odd then $c_0$ is even – due to the last equality – and consequently $m \not\equiv n \pmod 4$. Furthermore, from the trivial inequality $m \leq a_m$ we obtain $m < c_0$, that is, $m < n + 2$ (because $c_0 = (m + n + 2)/2$). Consequently, in the only case where $m$ is odd the above condition (mod 4) implies that $m \leq n - 2$. Once these elementary necessary conditions have been made explicit there are no further constraints for the choice of the $a_i$'s. And indeed, if $m$ is even every sequence satisfying $1 \leq a_1 < a_2 < ... < a_{m/2} < c_0/2$ works well – the remaining $a_i$'s being determined by the above conditions – while in the odd case any suitable sequence satisfies $1 \leq a_1 < a_2 < ... < a_{(m-1)/2} < c_0/2 = a_{(m+1)/2}$, this property being also sufficient. We have thus obtained Conditions **(A1)** and **(A2)**.

**- Subcase B-C.** $c_0 \neq S - c_0$.

Let $i$ be the index such that $a_i - c_0 = c_0$. The label, and difference, $a_i$ (i.e. $2c_0$) must be equal either to some difference $a_j - c_0$ or to $S - c_0$. In the former case $a_j$ (i.e. $3c_0$) is in its turn equal either to some $a_k - c_0$ or to $S - c_0$. By iterating this argument we will have, in the end, necessarily that $tc_0 = S - c_0$ for some $t \geq 2$ and that some $a_i$'s exist that are equal to $2c_0, 3c_0, ..., tc_0$. Now let $a_q$ be the largest label not yet examined. If $a_q = c_0/2$, then the difference $a_q$ is generated by the label $a_q$ itself. Otherwise, $a_q$ can be uniquely realised as $c_0 - a_r$ for some $a_r < a_q$, whence the difference $a_r$ is in its turn given by $c_0 - a_q$. By similarly reasoning until all leaves have been examined, we obtain either Conditions **(C1)**,**(C2)**, or Conditions **(B1)**,**(B2)**, according to whether $a_q = c_0/2$ or not.

**Case D-E.** *Some pendent edge generates the largest difference.*

In this case we can assume that $b_n = S$ and $c_2 = 0$. Let $\xi$ be the largest label (and difference) not equal to any $b_i$ nor to $c_1$. If the difference $\xi$ is realised as $|a_i - c_0|$ for some $a_i$ then a contradiction is reached, because $\xi$ would be smaller than either $c_0$ or $a_i$. It follows that necessarily $\xi = c_1 - c_0$ (the absolute value is superfluous, as $c_0$ cannot exceed $\xi$). Let us now consider two subcases, depending on the position of the label $\xi$.

**- Subcase D.** $\xi = c_0$ (equivalently, $c_1 = 2c_0$).

Due to the maximality property of $\xi$ we have that $c_0 > a_i$ for all $i$, whence $c_0 > m$. Reasoning as above, the reader can easily obtain both the equalities $c_0 = a_i + a_{m-i+1}$ for all $i$, and the claimed conditions, **(D1)** and **(D2)**.

**- Subcase E.** $\xi$ *is on a leaf of the* $m$*-star.*

In order to generate the difference $c_0$, some leaf must be labelled $2c_0$. If $2c_0 = \xi$ (equivalently, $c_1 = 3c_0$) the difference $2c_0$ is generated by $c_1$ and $c_0$. If instead $2c_0 \neq \xi$, some other leaf must be labelled $3c_0$ in order to generate the difference $2c_0$. By iterating this process we eventually find that $\xi = tc_0$ for some $t \geq 2$ (that is, $c_1 = (t+1)c_0$). In particular, all the so far involved leaves are labelled $2c_0, 3c_0, ..., tc_0 = \xi$. As to the remaining differences (only if $t < m+1$), the set they form must coincide with the set of labels still to be placed. According to the parity of $m - t$, an argument similar to the above one yields the two conditions **(E1)** and **(E2)**. □

# 4 Graceful labellings: a polynomial

In the present, short, section we introduce an algebraic tool by associating a given tree to a polynomial in several variables whose positive, integral, roots are related to the graceful labellings of that tree.

**Definition 4.1.** Let $T$ be a tree with $n+1$ vertices $v_0, v_1, ..., v_n$. We associate to $T$ the polynomial in $n+1$ variables

$$\mathcal{P}_T(x_0, x_1, ..., x_n) = \sum_{i=0}^{n} (deg(v_i) - 1)x_i^2 \ - \ 2 \left( \sum_{\{v_i, v_j\} \in E(T)} x_i x_j \right) .$$

The following provides the immediate connection between this polynomial and graceful labellings of trees.

**Lemma 4.2.** *If* $\lambda : V(T) \to \{0, 1, ..., n\}$ *is a graceful labelling of a tree* $T$*, then* $\mathcal{P}_T(\lambda(v_0), \lambda(v_1), ..., \lambda(v_n)) = 0$.

*Proof.* (...) □

At the current stage of our research, and as far as we know, this polynomial has not yet contributed to shed more light on the Graceful Tree Conjecture. As the next move we plan to investigate the following integer programming problem.

$$(*)_T \quad \begin{cases} \mathcal{P}_T(\underline{x}) = 0 \\ x_i \in [0, n] \cap \mathbf{N} \ \forall i \\ x_i \neq x_j \ \forall i \neq j \end{cases}$$

The reader can easily see that any graceful labelling of $T$ provides a solution to $(*)_T$, whence any unsolvable system of the form $(*)_T$, for some tree $T$, would disprove the conjecture "in one blow". Our future efforts could be therefore directed to finding some system $(*)_T$ that admits no solution. Such a question amounts to studying the integral points, in the hypercube $[0, n]^n$ deprived of the hyperplanes of equations $x_i - x_j = 0$, of the quadric defined by $\mathcal{P}_T(\underline{x})$. Because this polynomial is homogeneous, some tools from projective geometry might be resorted to.

## References


[1] R.E. Aldred and B.D. McKay, *Graceful and harmonious labellings of trees*, Bull. Inst. Combin. Appl. **23** (1998), pp. 69-72.

[2] I. Anderson, *Combinatorial Designs and Tournaments*, Oxford University Press, Oxford, 1997.

[3] B. Beavers, *Golomb Rulers and Graceful Graphs*, a research paper for Math7999 Communicating Mathematics, http://www.math.lsu.edu/~beavers/Golomb_Rulers_and_Graceful_Graphs.pdf.

[4] G.S. Bloom and S.W. Golomb, *Applications of Numbered Undirected Graphs*, Proc. IEEE **65(4)** (1977), pp. 562-570.

[5] B. Bollobás and O. Pikhurko, *Integer sets with prescribed pairwise differences being distinct*, Eur. J. Comb. **26(5)** (2005), pp. 607-616.

[6] V.W. Bryant, *Aspects of Combinatorics, a Wide-Ranging Introduction*, Cambridge University Press, Cambridge, 1993.

[7] C.J. Colbourn and J.H. Dinitz (eds.), *The CRC handbook of combinatorial designs*, Boca Raton, FL, CRC Press, 1996.

[8] J.A. Gallian, *A Dynamic Survey of Graph Labeling*, Electr. J. Comb. **DS6** (2002).

[9] F. Harari, *Graph Theory*, Addison-Wesley, 1969.

[10] G. Ringel, *problem* 25, in: *Theory of Graphs and its Applications*, Proc. Symp. Smolenice, 1963, G/A Czech. Acad. Sci. **162** (1964).

[11] A. Rosa, *On certain valuations of the vertices of a graph,* Theory of Graphs (Internat. Sympos. Rome, 1966) pp. 349-355. Eds. Gordon and Breach, New York; Dunod, Paris, 1967.

[12] R.G. Stanton and C.R. Zarnke, *Labelling of balanced trees*, Proc. Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing, pp. 479-495. Utilitas Math., Winnipeg, Man., 1973.

[13] F. Van Bussel, *Relaxed Graceful Labellings of Trees*, Electr. J. Comb. **9(1)** (2002).

[14] A. Vietri, *Graceful labellings for an infinite class of generalised Petersen graphs*, to appear in *Ars Combin.*.