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On the Posets $\left(\mathcal{W}_2^k, <\right)$ and their Connections with Some Homogeneous Inequalities of Degree 2

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Abstract. A class of ranked posets $\{(D_h^k, \ll)\}$ has been recently defined in order to analyse, from a combinatorial viewpoint, particular systems of real homogeneous inequalities between monomials. In the present paper we focus on the posets D_2^k, which are related to systems of the form $\{x_a x_b *_{abcd} x_c x_d: 0 \leq a, b, c, d \leq k, *_{abcd} \in \{<, >\}, 0 < x_0 < x_1 < \ldots < x_k\}$. As a consequence of the general theory, the logical dependency among inequalities is adequately captured by the so-defined posets $\left(\mathcal{W}_2^k, <\right)$. These structures, whose elements are all the D_2^k's incomparable pairs, are thoroughly surveyed in the following pages. In particular, their order ideals – crucially significant in connection with logical consequence – are characterised in a rather simple way. In the second part of the paper, a class of antichains $\left\{\mathcal{P}_k \subseteq \mathcal{W}_2^k\right\}$ is shown to enjoy some arithmetical properties which make it an efficient tool for detecting incompatible systems, as well as for posing some compatibility questions in a purely combinatorial fashion.

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1. Introduction

Certain monomial homogeneous inequalities involving sequences of real, positive numbers ($x_0 < x_1 < \ldots < x_k$) were studied from a combinatorial viewpoint in [7], so as to adequately capture the notions of logical consequence and system satisfiability. In particular, for any fixed integer $h \geq 2$ all possible satisfiable systems of the form $\{x_0^a x_2^b *_{a,b} x_1^{a+b}: a, b \in \mathbf{N}^+, a + b \leq h, *_{a,b} \in \{<, >\}\}$ were listed. Such result, as well as other achievements in the cited paper, were obtained by analysing the so-defined posets $(D_h^2, \ll)$ and $\left(\mathcal{W}_2^k, <\right)$, deeply related to the above homogeneous inequalities. In the present paper, we will instead focus on the posets $\left(\mathcal{W}_2^k, <\right)$. The general definitions of $(D_h^k, \ll)$ and $\left(\mathcal{W}_h^k, <\right)$ are the following (the canonical basis of $\mathbf{R}^{k+1}$ will be denoted by $\{\underline{e}_0 = (1, 0, \ldots, 0), \underline{e}_1, \ldots, \underline{e}_k = (0, \ldots, 0, 1)\}$; when using, any symbol $\underline{e}_i$, the dimension should be clear from the context).

DEFINITION 1.1. If $h \in \mathbf{N}^+$ and $k \in \mathbf{N}$, the symbol D_h^k stands for the set of $(k+1)$ -tuples $\underline{u} = (u_0, \ldots, u_k)$ such that $u_i \in \mathbf{N}$ for all i and $\sum_i u_i = h$. A partial ordering $\ll$ over D_h^k is then defined as the reflexive and transitive closure of the relation $\ll^*$ such that $\underline{u} \ll^* \underline{u}' \Leftrightarrow \exists i < k$: $u_i = u_i' + 1$, $u_{i+1} = u_{i+1}' - 1$ and $u_j = u_j'$ otherwise. Furthermore, $\mathcal{W}_h^k$ stands for the set $\{(\underline{u}, \underline{v}) \in D_\theta^k \times D_\theta^k : 2 \leq \theta \leq h$, $\underline{u} \not\ll \underline{v} \not\ll \underline{u}$, $u_i v_i = 0 \forall i\}$. It is endowed with a partial ordering $<$, namely the reflexive and transitive closure of $<^*$ defined through : $(\underline{u}, \underline{v}) <^* (\underline{u}', \underline{v}') \Leftrightarrow \exists i < k : (\underline{u} - \underline{u}', \underline{v} - \underline{v}') \in \{(\underline{e}_i - \underline{e}_{i+1}, \underline{0}), (\underline{0}, -\underline{e}_i + \underline{e}_{i+1}), (\underline{e}_i, \underline{e}_{i+1}), (-\underline{e}_{i+1}, -\underline{e}_i)\}$.

Among other things, it turns out that the ordering $\ll$ is the intersection of all the weight orders $<_{r_0 < r_1 < \ldots < r_k}$ restricted to D_h^k (notable connections between *monomial orders* and weight orders have been pointed out in [4]. See e.g. [1] for a detailed account of monomial orders as employed in Grobner bases theory). Each poset D_h^k appears to be a suitable environment for effectively managing all inequalities of the form $\prod_{0 \leq i \leq k} x_i^{u_i} < \prod_{0 \leq i \leq k} x_i^{v_i}$, with $\underline{u}, \underline{v} \in D_h^k$, from the viewpoint of logical relationships. Following [7], in the sequel we will shortly denote the above inequality by $\beta_{(\underline{u}, \underline{v})}(\underline{x})$ or simply $\beta_{(\underline{u}, \underline{v})}$. Since $\underline{u} \ll \underline{v} \neq \underline{u}$ if and only if $\beta_{(\underline{u}, \underline{v})}(q_0, \ldots, q_k)$ is true for all positive real numbers $q_0 < q_1 < \ldots < q_k$ ([7], Theorem 2.3), it is possible to define the following class of linearisations of $(D_h^k, \ll)$.

DEFINITION 1.2. A β-linearisation of $(D_h^k, \ll)$ is an extension of $\ll$ to a total ordering, obtained by defining for each incomparable pair $(\underline{u}, \underline{v})$

$$\underline{u} \ll \underline{v} \Leftrightarrow \prod_{0 \leq i \leq k} q_i^{u_i} < \prod_{0 \leq i \leq k} q_i^{v_i},$$

where $\underline{q}$ is a fixed increasing sequence of positive real numbers, yielding strict inequalities for all incomparable pairs.

Therefore, a β-linearisation can be interpreted as a consistent selection of one inequality between $\beta_{(\underline{u}, \underline{v})}$ and $\beta_{(\underline{v}, \underline{u})}$ for every $(\underline{u}, \underline{v}) \in \mathcal{W}_h^k$. We observe that a concept similar to the β-linearisation was introduced and studied by Maclagan [3] in connection with binary strings.

Essentially because $\mathcal{W}_h^2$ splits into two disjoint totally ordered sets, it is not hard to obtain a complete description of all β-linearisations of $(D_h^2, \ll)$ for any h. The pertinent result runs as follows.

THEOREM 1.3 ([7], Corollaries 3.2, 3.3). The β-linearisations of $(D_h^2, \ll)$ are indexed by the rational numbers of the form b/a, with $b \geq 0$, $a > 0$, $a + b \leq h - 1$. In particular, there are $2 \sum_{1 \leq i \leq h-1} \phi(i)$ such linearisations, where ϕ is the Euler function. The system associated to any fixed b/a is $\{x_0^a x_2^b < x_1^{a+b}, x_0^A x_2^B > x_1^{A+B}\}$, where B/A is the smaller admissible number following b/a in $(\mathbf{Q}^+ \cup \{\infty\}, <)$. The infinite case means that $A = 0$, $B = 1$ and, consequently, that the second inequality becomes trivial.

Evidently, the question settled in [7] is a particular case of the classification problem for systems $\{x_0^{u_0}x_1^{u_1}\cdots x_k^{u_k} *_{(\underline{u},\underline{v})} x_0^{v_0}x_1^{v_1}\cdots x_k^{v_k} : (\underline{u},\underline{v}) \in \mathcal{W}_h^k, *_{\underline{u},\underline{v}} \in \{<,>\}, x_0 < x_1 < \ldots < x_k\}$ for any fixed h, $k \geq 2$. The general case seems much harder to deal with. The best relevant result achieved in [7] is a characterisation of logic consequence between any two given inequalities, in terms of a relation (not necessarily an ordering) defined over $\mathcal{W}_h^k$. As the next definition shows, such relation generalises the partial ordering <. Anyway, in some cases it collapses to <.

DEFINITION 1.4. Let L, $M \in \mathcal{W}_h^k$. L *is weakly preceding* M ($L <^w M$) if there exist $2n$ positive integers $\{a_i, b_i\colon 0 \leq i \leq n-1\}$ and $n+1$ elements $\{L = L_0, L_1, \ldots, L_n = M\}$ such that $a_i L_i < b_i L_{i+1}$ for all $i < n$, in some $(\mathcal{W}_H^k, <)$ large enough.

The announced characterisation is the following.

THEOREM 1.5 ([7], Theorem 4.3). $\beta_M \Rightarrow \beta_L$ if and only if $L <^w M$.

If $k = 2$ the relation $<^w$ coincides with <. As already mentioned, it partitions $\mathcal{W}_h^2$ into two chains, thus allowing a complete understanding of all logical dependencies and all compatible systems for any fixed h. Also in the present case, although the structure of $(\mathcal{W}_2^k, <^w)$ is not as elementary as in the $k = 2$ cases, $<^w$ does reduce to <, and < itself can be fully described. This is indeed the main concern of Section 2. In particular, $\mathcal{W}_2^k$ is shown to split into two isomorphic posets $+\mathcal{W}_2^k$, $-\mathcal{W}_2^k$ which can be thoroughly understood. Every order ideal of $(\mathcal{W}_2^k, <)$ is subsequently detected, which is equivalent to detecting all logical consequences of any fixed inequality β_L. Although the β-linearizations of D_2^k remain unprobed, the satisfactory knowledge of $\mathcal{W}_2^k$ with all its order ideals seems encouraging.

The results of Section 2 pave the way for the analysis of compatibility issues related to subsets of any fixed $(\mathcal{W}_2^k, <)$. Availing of this opportunity, in Section 3 we consider a particular maximal antichain $\mathcal{P}_k \subset +\mathcal{W}_2^k$ and the corresponding sub-antichains $\mathcal{P}_u = \mathcal{P}_k \cap (+\mathcal{W}_2^u / +\mathcal{W}_2^{u-1})$ with $2 \leq u \leq k$. These sets possess a quite natural combinatorial meaning and an uncommon arithmetical expressiveness. A basic tool for our investigation is the splitting of $\mathcal{P}_k$ into two subsets $\mathcal{Q}_k = \cup_u \mathcal{Q}_u$, $\overline{\mathcal{Q}}_k = \cup_u \overline{\mathcal{Q}}_u$, with respect to the odd or even position of the elements in a prescribed lexicographical ordering over $\mathcal{P}_k$. We also provide a formula which characterises the parity of any given element. By arithmetically manipulating the $\mathcal{P}_k$'s in two specific instances, we show that certain inequalities are implied by some other inequalities and, consequently, that certain systems are not satisfiable. In order to account more formally and technically for the above facts, we start Section 4 by proving four similar properties of the antichains $\{P_u\}$. In each of the four claims − which refers to a particular congruence class of u (mod 4) − it is shown that the alternate sign summation, with respect to the position parity, over all elements of some fixed level P_u produces a particular pattern depending on the congruence class. This pattern can be easily displayed using certain summations of vector pairs. The above properties are subsequently exploited to prove, among other things, an incompatibility statement involving $P_{8\gamma-3}$ and $P_{8\gamma}$ for any $\gamma \geq 2$. In more details, we construct an incompatible system by collecting all the inequalities related to $P_{8\gamma-3} \cup P_{8\gamma}$ (reversing all-inequalities related to $Q_{8\gamma-3} \cup \overline{Q}_{8\gamma}$), and adding some prescribed sets of other inequalities of the form β_{L}, with $L \in \mathcal{W}_2^{8\gamma}$.

We recall that inference rules involving *linear* inequalities (some of which are the logarithmic analogue of the homogeneous inequalities studied in this paper) have been extensively surveyed over the last 50 years. Nevertheless, specific research areas such as the present one seem – as far as we know – beyond the reach of classical results on linear inequalities (see for example the pioneering paper of Kuhn [2] and the books [6, 9]). Finally, we remark that Snellman [5] has analysed similar posets from a different viewpoint.

2. The Structure of $\left(\mathcal{W}_2^k, <\right)$

In the present section we provide a detailed description of $\left(\mathcal{W}_2^k,\ <^w\right)$ and of its order ideals. Once proved the structure results (Proposition 2.1 and Lemma 2.3) we will characterise all order ideals in a cheap way.

PROPOSITION 2.1. *Let* $L = (\underline{e}_r + \underline{e}_s,\ \underline{e}_t + \underline{e}_u)$, $M = (\underline{e}_{r'} + \underline{e}_{s'},\ \underline{e}_{t'} + \underline{e}_{u'})$ *be elements of* $\mathcal{W}_2^k$. *Then,* $L <^w M$ *if and only if* $r \leq r'$, $s \leq s'$, $t \geq t'$, $u \geq u'$. *In particular,* $<^w$ *reduces to* $<$ *and L, M are incomparable whenever* $(r-t)(r'-t')$ < 0.

Proof. The *if* part can be easily obtained by the very definition of $<^w$. The *only if* part is proved as follows. Let $\underline{e}_t + \underline{e}_u - (\underline{e}_r + \underline{e}_s) = \sum_{0 \leq i \leq k-1} a_i(\underline{e}_{i+1} - \underline{e}_i)$ and $\underline{e}_{t'} + \underline{e}_{u'} - (\underline{e}_{r'} + \underline{e}_{s'}) = \sum_{0 \leq i \leq k-1} a_i'(\underline{e}_{i+1} - \underline{e}_i)$ for some integers $\{a_i\}$, $\{a_i'\}$. Since $\beta_M \Rightarrow \beta_L$, Lemma 4.6 of [7] guarantees the existence of some $q \in \mathbf{Q}^+$ such that $qa_i \geq a_i'$ for all i. Notice that $(a_0, \ldots, a_{k-1})$ and $(a_0', \ldots, a_{k-1}')$ may have only the two forms (possibly the same)

$$(0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0, -1, -1, \ldots, -1, 0, 0, \ldots, 0),$$
$$(0, 0, \ldots, 0, -1, -1, \ldots, -1, 0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0),$$

where each of the six sequences of zeroes needs not occur. Using the inequalities $\{qa_i \geq a_i'\}$ we deduce that if the vector $\underline{a}$ is of the first form, then $\underline{a}'$ is of the first form as well, whence the four claimed inequalities follow with few difficulties. Analogously, if $\underline{a}$ is of the second form then also $\underline{a}'$ is of the second form, and the inequalities can be obtained as in the above case. The relation defined by these inequalities is a partial ordering contained in or equal to $<$. Therefore, it must coincide with $<$. Finally, $(r-t)$ $(r'-t')$ is negative if and only if $\underline{a}$, $\underline{a}'$ have distinct forms, and the above discussion implies that the forms must coincide if L and M are comparable. □

Let $+\mathcal{W}_2^k$ and $-\mathcal{W}_2^k$ denote the sub-posets of $\mathcal{W}_2^k$ made up of those elements $(\underline{e}_r + \underline{e}_s, \underline{e}_t, \underline{e}_u)$ having $r > t$ and $r < t$ respectively. These two posets are isomorphic, while on account of Proposition 2.1 $\mathcal{W}_2^k$ is precisely the union of $\left(+\mathcal{W}_2^k, <\right)$ and $\left(-\mathcal{W}_2^k, <\right)$. The first result that concerns the above sub-posets is the following (anyway, in this paper we will never invoke it).

COROLLARY 2.2. *Let β_L be a logical consequence of $\beta_{L_1} \wedge \beta_{L_2} \wedge \ldots \wedge \beta_{L_n}$, with $L \in \mathcal{W}_2^k$ and $L_i \in +[-]\mathcal{W}_2^k$ for all i. Then, $L \in +[-]\mathcal{W}_2^k$.*

Proof. In the + case there exists at least one inequality $\neg\beta_{L_i}$ which is a logical consequence of $\neg\beta_L$ (indeed, $\neg\beta_L \Rightarrow (\neg\beta_{L_1} \vee \neg\beta_{L_2} \vee \ldots \vee \neg\beta_{L_n})$). A continuity argument could easily show that the same property holds if we turn the two non-strict inequalities $\neg\beta_L$, $\neg\beta_{L_i}$ into strict inequalities. Now Theorem 1.5 implies that L and L_i are weakly comparable, whence they belong to the same connected component of the poset, namely $+\mathcal{W}_2^k$. The $-$ case is treated analogously. □

The following terminology prepares the ground for an accurate description of $+\mathcal{W}_2^k$ and – as a straightforward consequence – of the whole $\mathcal{W}_2^k$ for any k. Let g, h, v be positive integers and $(R_g, <_g)$ be the poset represented on the left side of Figure 1. For more clearness we will denote the generic element $(x, y) \in R_g$ also by $(x, y)_g$. Let us define the poset $(S_h, <^h)$ as $\sqcup_{1 \le g \le h} R_g$ endowed with the transitive closure of the relation $(x, y)_g \dot{<}^h (x', y')_{g'} \Leftrightarrow \left(g = g', (x, y) <_g (x', y')\right) \vee \left(g' = g + 1, x' = x + 1, y' = y\right)$. We will possibly add the superscript h to the generic element $(x, y)_g \in S_h$. Let us finally define the Poset $(C_v, \prec)$ as $\sqcup_{1 \le h \le v} S_h$ endowed with the transitive closure of the relation $(x, y)_g^h \dot{\prec} (x', y')_{g'}^{h'} \Leftrightarrow \left(h = h', (x, y)_g <^h (x', y')_{g'}\right) \vee (h' = h - 1, g' = g - 1, x' = x,$

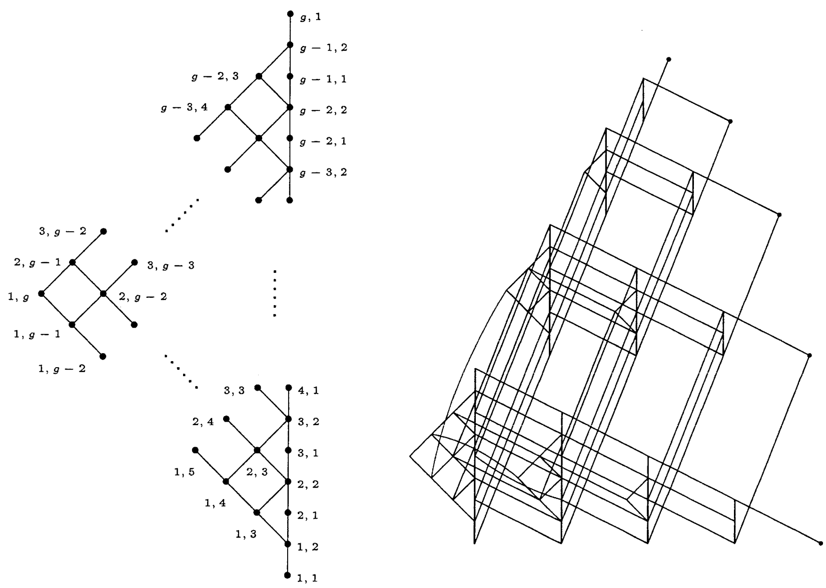


Figure 1. $(R_g, <_g)$, and $(C_5, \prec)$ - with some missing lines.

$y' = y$). The posets C_v (one of which has been depicted almost entirely on the right side of Figure 1) play a basic role, according to the

LEMMA 2.3. *For every* $k \geq 2, \left(+\mathcal{W}_2^k, <\right)$ *is isomorphic to* $(C_{k-1}, \prec)$.

Proof. A bijection from the former set to the latter is obtained by sending $(\underline{e}_r + \underline{e}_s, \underline{e}_t + \underline{e}_u)$ to $(r, s-r+1)_{u-t-1}^{u-1}$. We omit the routine argument which proves that the above bijection is actually an isomorphism of posets. □

Notice that, as an easy consequence of the general definition, it turns out that every $\mathcal{W}_h^k$ is ranked. Therefore, our representation of $\mathcal{W}_2^6$ in Figure 1 fails to take account of the rank property – although it may be hopefully appreciated for other reasons.

We proceed to describe the order ideals of $\mathcal{W}_2^k$. The first step concerns R_g. For our purposes, non-empty subsets of $\{1, 2, \ldots, g\}$ will be also regarded as sequences $(g \geq i_1 > i_2 > \ldots > i_\omega \geq 1)$. In keeping with the standard terminology, we recall that $J(P)$ stands for the set of order ideals of a poset $(P, <)$ and that, for each $p \in P$, Λ_p (resp. V_p) denotes the principal order ideal (resp. principal dual order ideal, or principal filter) $\{q \in P : q \leq p\}$ (resp. $\{q \in P : q \geq p\}$).

PROPOSITION 2.4. *Let the elements of some fixed* R_g *be labelled as in* Figure 1. *The map* $\Gamma_g : \mathcal{P}(\{1, 2, \ldots, g\}) \to J(R_g)$ *defined through*

$$(i_1 > i_2 > \ldots > i_\omega) \mapsto \bigcup_{e=1}^{\omega} \Lambda_{(e,\, i_e)}, \quad \emptyset \mapsto \emptyset$$

is a bijection. In particular, R_g *has* 2^g *order ideals.*

Proof. First we prove surjectivity. If $I \in J(R_g)$, let us consider the antichain made up of the maximal elements of I. Such elements can be easily arranged so as to form a unique sequence $((u_1, i_{u_1}), \ldots, (u_m, i_{u_m}))$ with $u_e < u_{e'}$ if $e < e'$ and $i_{u_e} > i_{u_{e+1}} + u_{e+1} - u_e$ for all $e < m$. If $u_m > m$, let us set $u_0 = 0$, $\omega = u_m$ and extend the above sequence to: $((1, i_1), (2, i_2), \ldots, (\omega, i_\omega))$ by defining $i_z = i_{u_{e+1}} + u_{e+1} - z$ for every z, e such that $u_e < z < u_{e+1}$. Having possibly extended the sequence, it can now be checked with few difficulties that $\Gamma_g((i_1, \ldots, i_\omega)) = I$. Injectivity is proved as follows. If $\Gamma_g((i_1, \ldots, i_\omega)) = \Gamma_g((i'_1, \ldots, i'_{\omega'})) = K$ with $\omega \neq \omega'$, the contradiction $K \supseteq \{(\omega, i_\omega), (\omega', i'_{\omega'})\} K$ is reached. Otherwise, if $\omega = \omega'$ we consider any integer u such that $i_u \neq i'_u$, thus obtaining the contradiction $K \supseteq \{(u, i_u), (u, i'_u)\} \nsubseteq K$. □

The above result enables us to list all the order ideals of $+\mathcal{W}_2^k$. To accomplish this goal we require some more notions. Let Θ_s denote the set of sequences of non-negative integers $(i_1, i_2, \ldots, i_s)$ such that $i_1 \leq s$ and that $i_q > i_{q+1}$ for every $q < s$, unless $i_q = i_{q+1} = 0$ (any such sequence is therefore obtained by possibly adding a number of zeroes to some sequence $(i_1 > i_2 > \ldots > i_\omega) \in \mathcal{P}(\{1, 2, \ldots, s\})$). For any positive integer v, the symbol $\sum_v$ stands for a particular set of

0			
2, 0	1		
2, 1, 0	2, 0	0	
4, 3, 2, 0	3, 2, 1	2, 1	1

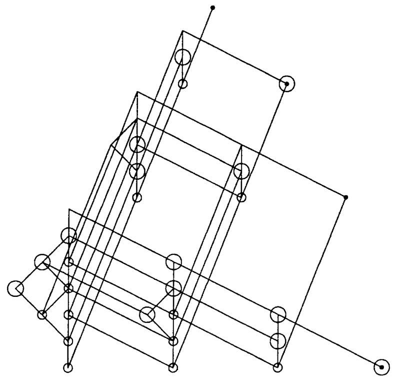


Figure 2. An element of $\sum_4$ and the corresponding order ideal of $+\mathcal{W}_2^5$ (circles).

lower triangular $v \times v$ matrices with the following properties: if $x \geq y \geq 1$, the (x, y)-entry of some fixed matrix of $\sum_v$ is a *sequence* $(i_1^{x,y}, i_2^{x,y}, \ldots, i_{x-y+1}^{x,y}) \in \Theta_{x-y+1}$; furthermore, sequences of any fixed matrix satisfy $i_z^{x,y} \leq i_z^{x+1,y}$, $i_{z+1}^{x,y} \leq i_z^{x,y+1}$ for all admissible choices of x, y, z (see the left side of Figure 2).

THEOREM 2.5. *For any integer $k \geq 2$, there exists a bijection from $J\left(+\mathcal{W}_2^k\right)$ to $\sum_{k-1}$ (which is definable by means of a formula).*

Proof. If $1 \leq y \leq x \leq k-1$, let Sx denote the sub-poset of C_{k-1} as defined before Lemma 2.3, and let R_{yx} stand for the corresponding sub-poset of S_x isomorphic to R_y. Using the isomorphism provided by Lemma 2.3, for any given $I \in J\left(+\mathcal{W}_2^k\right) = J(C_{k-1})$ we consider the set $I_{yx} = I \cap R_{yx} \in J(R_{yx})$. In accordance with Proposition 2.4, we associate I_{yx} to the sequence $\Gamma_y^{-1}(I_{yx})$ and possibly add as many zeroes as needed to obtain an element of Θ_y. With this element we fill the $(x, x-y+1)$-square of the matrix under construction. By repeating the above procedure for all y, x with $y \leq x$ we obtain an element of $\sum_{k-1}$, because the two schemes of inequalities follow easily by the structural features of C_{k-1}. Proposition 2.4 implies that the above correspondence is injective, whereas the inequalities related to each matrix of $\sum_{k-1}$ guarantee surjectivity. □

3. Specular Elements and Incompatibility Issues

Every poset $\left(+\mathcal{W}_2^k, <\right)$ contains a maximal antichain which seems to be of interest, both combinatorially and arithmetically.

DEFINITION 3.1. The generic element$(\underline{e}_r + \underline{e}_s, \underline{e}_t + \underline{e}_u) \in \mathcal{W}_2^k$ is briefly denoted by $(r, s; t, u)$. The set of specular elements of $+\mathcal{W}_2^k$ is $\mathcal{P}_k = \cup_{2 \leq u \leq k} P_u$, with $P_u = \{(r, s; t, u)\colon r - t = u - s > 0\}$.

PROPOSITION 3.2. *For any $k \geq 2$, $\mathcal{P}_k$ is a maximal antichain of $\left(+\mathcal{W}_2^k, <\right)$. Furthermore, $|\mathcal{P}_k| = k^3/12 + k^2/8 - k/12 - \alpha$, with $\alpha = 0$ if k is even, $\alpha =$ 1/8 otherwise.*

Proof. First, we show that any two specular elements are incomparable. If $(r, s; t, u) \leq (r', s'; t', u')$ then $r \leq r'$, $s \leq s'$, $t \geq t'$, $u \geq u'$ by Proposition 2.1. It follows that $r' - t' \geq r - t = u - s \geq u' - s'$, which implies $r' - t' = r - t$ because $r' - t' = u' - s'$. Thus, we have that $r = r'$, $t = t'$, and finally that $s = s'$, $u = u'$. Maximality is now proved by showing that any element outside $\mathcal{P}_k$ is comparable with some specular element. Indeed, if $(r, s; t, u)$ is such that $r - t < u - s$, then $(r, s; t, u) < (r, t + u - r; t, u)$, whereas the assumption $r - t > u - s$ yields $(r, s; t, u) > (t + u - s, s; t, u)$. We proceed to count the specular elements of $+\mathcal{W}_2^k$. Using the notation of Theorem 2.5, we regard any fixed P_u as a subset of S_{u-1} and notice that, for any admissible y, $P_u \cap R_{y, u-1}$ consists of $\lfloor (y+1)/2 \rfloor$ elements (precisely, those lying in the middle level of $R_{y, u-1}$). As $\sum_{1 \leq y \leq u-1} \lfloor (y+1)/2 \rfloor$ is equal to either $u^2/4$ or $(u^2 - 1)/4$, according to whether u is respectively even or odd, we are eventually led to the following cases.

$$k \text{ even } : |\mathcal{P}_k| = \sum_{\sigma=1}^{k/2} \sigma^2 + \sum_{\tau=1}^{(k-2)/2} \tau^2 + \tau \,;$$

$$k \text{ odd } : |\mathcal{P}_k| = \sum_{\sigma=1}^{(k-1)/2} \sigma^2 + \sum_{\tau=1}^{(k-1)/2} \tau^2 + \tau.$$

With the help of the well-known formulas which evaluate the sums of the first n natural numbers and the sums of their squares, the above cases can be easily handled so as to obtain the claimed equalities. □

In the sequel we will contract the two above formulas for $\sum_{1 \leq y \leq u-1} \lfloor (y+1)/2 \rfloor$ into the unique formula $\lfloor u/2 \rfloor \lceil u/2 \rceil$. As we begin to point out in the following lines, the arithmetical properties of specular elements may lead to interesting calculations and, in particular, to simple proofs of incompatibility. At the end of this section we will relate these preliminary results to the combinatorics of specular elements.

Firstly, let us consider the three elements of $\mathcal{P}_3$. We will write them as vector pairs, thus using the original notation. From $((0, 2, 0, 0), (1, 0, 1, 0)) - ((0, 1, 1, 0), (1, 0, 0, 1)) + ((0, 0, 2, 0), (0, 1, 0, 1)) = (\underline{z}, \underline{z})$ with $\underline{z} = (0, 1, 1, 0)$ it follows that the implication $\beta_{(1,1;0,2)} \wedge \beta_{(2,2;1,3)} \Rightarrow \beta_{(1,2;0,3)}$ is true. Similarly, the four elements in P_4 give rise to the equality $-((0, 1, 0, 1, 0), (1, 0, 0, 0, 1)) + ((0, 0, 2, 0, 0), (1, 0, 0, 0, 1)) - ((0, 0, 1, 1, 0), (0, 1, 0, 0, 1)) + ((0, 0, 0, 2, 0), (0, 0, 1, 0, 1)) = (\underline{z}', \underline{z}')$ with $\underline{z}' = (0, -1, 1, 0, 0)$. Hence, the implication $\beta_{(1,3;0,4)} \wedge \beta_{(0,4;2,2)} \wedge \beta_{(2,3;1,4)} \Rightarrow \beta_{(3,3;2,4)}$ is true. Although in both examples the antecedent conjunctions are not proved to be satisfiable, what can be deduced for certain is the

PROPERTY 3.3. *Each of the systems* $\{\beta_{(1,1;0,2)}, \beta_{(2,2;1,3)}, \neg\beta_{(1,2;0,3)}\}$, $\{\beta_{(1,3;0,4)}, \beta_{(0,4;2,2)}, \beta_{(2,3;1,4)}, \beta_{(3,3;2,4)}\}$ *is unsatisfiable.*

The reader can easily check that, in these two examples, the negative summands refer to all the elements in *even* position with respect to the lexicographical ordering on the triples $(u,\ t,\ r)$ corresponding to the elements $(r,\ s;\ t,\ u) \in \mathcal{P}_4$ (see Figure 3). In order to account more closely and extensively for the above arithmetical properties, we introduce some further notions. Let us endow $\mathcal{P}_k$ with the total ordering B defined through $(r, s; t, u) \lhd (r', s'; t', u') \Leftrightarrow (u < u') \vee (u = u',\ t < t') \vee (u = u',\ t = t',\ r < r')$ − therefore, $\lhd$ is the lexicographical ordering mentioned above. Furthermore, let us denote by $\mathcal{Q}_k = \cup_{2 \leq u \leq k} Q_u$ (resp. by $\overline{\mathcal{Q}}_k = \cup_{2 \leq u \leq k} \overline{Q}_u$) the set of specular elements of $\mathcal{P}_k$ in odd (resp. even) position with respect to $\lhd$, and by $\tilde{\mathcal{Q}}_k = \cup_{2 \leq u \leq k} \tilde{Q}_u$ the subset of $\mathcal{Q}_k$ whose generic element is of the form $(r, s; 0, u)$ (we assume that $Q_u,\ \overline{Q}_u,\ \tilde{Q}_u \subseteq P_u$ for all u). In Figure 3 bold elements form $\mathcal{Q}_7$ and an initial segment of Q_8. In particular, all elements belonging to $\tilde{\mathcal{Q}}_8$ are placed before the zig-zag line.

Now we have all the ingredients for managing more complex cases than the initial two. For example, as it can be patiently checked, by performing the same alternating sign summation over $(P_5 \cup P_6 \cup P_7 \cup P_8, \lhd)$ we obtain a pair of the form $\left(2\underline{e}_2 + 2\underline{e}_7 + \underline{z}'',\ 2\underline{e}_4 + 2\underline{e}_5 + \underline{z}''\right)$. Because $(4, 5; 2, 7)$ is in even position, we can remove $(\underline{e}_2 + \underline{e}_7,\ \underline{e}_4 + \underline{e}_5)$ from the alternate summation and also subtract it from the vector pair, thus obtaining the

PROPERTY 3.4. *The inequality* $x_2x_7 < x_4x_5$ *is a logical consequence of* $\{x_rx_s < x_tx_u : (r,\ s;\ t,\ u) \in (Q_5 \cup Q_6 \cup Q_7 \cup Q_8)\} \cup \{x_rx_s > x_tx_u : (r,\ s;\ t,\ u) \in \overline{Q}_5 \cup \overline{Q}_6 \cup (\overline{Q}_7 \backslash \{(4,\ 5;\ 2,\ 7)\}) \cup \overline{Q}_8\}$.

Again, while the above result says nothing about the simultaneous consistency of the two sets of inequalities, it can nonetheless be adapted to yield the following incompatibility statement.

PROPERTY 3.5. *The inequalities* $\{x_rx_s < x_tx_u : (r,\ s;\ t,\ u) \in Q_5 \cup Q_6 \cup Q_7 \cup Q_8\}$, $\{x_rx_s > x_tx_u : (r,\ s;\ t,\ u) \in \overline{Q}_5 \cup \overline{Q}_6 \cup (\overline{Q}_7 \backslash \{(4,\ 5;\ 2,\ 7)\}) \cup \overline{Q}_8\}$ *and* $x_2x_7 \geq x_4x_5$ *make up an incompatible system.*

It is clear that without removing (2, 7; 4, 5) the two resulting properties would become utterly trivial. Bearing in mind also the behaviour of $\mathcal{P}_3$ and P_4, it seems worth asking whether the above phenomena may be anyhow generalised. As

1,1 ; 0,2											
1,2 ; 0,3	**2,2 ; 1,3**										
1,3 ; 0,4	**2,2 ; 0,4**	2,3 ; 1,4	**3,3 ; 2,4**								
1,4 ; 0,5	**2,3 ; 0,5**	2,4 ; 1,5	**3,3 ; 1,5**	3,4 ; 2,5	**4,4 ; 3,5**						
1,5 ; 0,6	**2,4 ; 0,6**	3,3 ; 0,6	**2,5 ; 1,6**	3,4 ; 1,6	**3,5 ; 2,6**	4,4 ; 2,6	**4,5 ; 3,6**	5,5 ; 4,6			
1,6 ; 0,7	2,5 ; 0,7	**3,4 ; 0,7**	2,6 ; 1,7	**3,5 ; 1,7**	4,4 ; 1,7	**3,6 ; 2,7**	4,5 ; 2,7	**4,6 ; 3,7**	5,5 ; 3,7	**5,6 ; 4,7**	6,6 ; 5,7
1,7 ; 0,8	2,6 ; 0,8	**3,5 ; 0,8**	4,4 ; 0,8	**2,7 ; 1,8**	3,6 ; 1,8	**4,5 ; 1,8**	. . .				

Figure 3. $\mathcal{P}_8$ suitably displayed.

seemingly natural environments we propose $P_{2^c+1} \cup P_{2^c+2} \cup \ldots \cup P_{2^{c+1}}$ and $P_{4c+1} \cup P_{4c+2} \cup P_{4c+3} \cup P_{4c+4}$. Accordingly, in Section 4 we will shortly consider the latter case (the former will not be dealt with in the present paper). Some key results for our purposes – and for similar-questions concerning alternate summations of specular elements – will be collected in the same section. In the following lines, instead, we provide a comparatively fast method for deciding whether or not a given specular element belongs to some Q_u. The main theorem is preceded by a more restricted result.

PROPOSITION 3.6. *$(r, s; 0, r+s) \in \tilde{Q}_{r+s}$ if and only if $r + \lfloor (r+s+1)/4 \rfloor$ is odd.*

Proof. We use an inductive argument over $(\cup_{u\geq 2}\{(r, s; 0, u) \in P_u\}, \lhd)$. For the sake of simplicity we will denote by (r, s) the quadruple $(r, s; 0, r+s)$. As (1, 1) satisfies the claimed property, the induction basis holds. Let us assume that $r + \lfloor (r+s+1)/4 \rfloor$ is odd and $(r, s) \neq (1,1)$. Firstly we manage the case $r + s \not\equiv 3 \pmod 4$. If $r \geq 2$, the induction hypothesis implies that $(r-1, s) \in \overline{Q}_{r+s-1}$ because $r - 1 + \lfloor ((r+s)/4 \rfloor = r - 1 + \lfloor (r+s+1)/4 \rfloor$, which is even. It is therefore enough to check that the interval $[(r-1, s), (r, s)] \subseteq (\mathcal{P}_{r+s}, \lhd)$ has even size. With few difficulties we have that $|\,[(r-1, s), (r, s)]| = 2 + \lfloor (r+s-1)/2 \rfloor \lceil (r+s-1)/2 \rceil$ (Figure 3 may be helpful. In particular, notice that any two elements $(r-1, s)$, (r, s) lie consecutively along the *NW–SE* direction). If $r = 1$ then $s \geq 6$ and we can use a similar argument, with $(r+1, s-2)$ in place of $(r-1, s)$. The conclusion follows by $|\,[(r+1, s-2), (r, s)]| = \lfloor (r+s-1)/2 \rfloor \lceil (r+s-1)/2 \rceil$. Finally, if $r + s \equiv 3 \pmod 4$ then $r + \lfloor (r + (s-1) + 1)/4 \rfloor$ is even and $|[(r, s-1), (r, s)]| = 1 + \lfloor (r+s-1)/2 \rfloor \lceil (r+s-1)/2 \rceil$, where the last summand is now odd. Conversely, under the same induction hypothesis let us now assume that $(r, s; 0, r+s) \in \tilde{Q}_{r+s}$. If $r > 1$, the element immediately preceding (r, s) is $(r-1, s+1)$, and the above discussion implies that $(r-1) + \lfloor ((r-1) + (s+1) + 1/4 \rfloor$ is even. Otherwise, if $r = 1$ then $s \geq 6$ and a similar argument applies to $(r+1, s-1)$ in place of $(r-1, s+1)$. □

Now we are in a position to settle the general case.

THEOREM 3.7. *Let $g(t, u)$ be equal to 1 if $u - (t - 4\lfloor t/4 \rfloor) < 2 + 4n \leq u$ for some integer n; otherwise, let g be equal to 0. Then, $(r, s; t, u) \in Q_u$ if and only if $r - t + \lfloor (r+s-2t+1)/4 \rfloor + \lfloor t/4 \rfloor + g(t, u)$ is odd.*

Proof. First, we observe that if $t \geq 1$ the interval $[(r-1, s-1; t-1, u-1), (r, s; t, u)]$ has even size if and only if $u \equiv 2 \pmod 4$. Indeed, as $[(r-1, s-1; t-1, u-1), (u-2, u-2; u-3, u-1)]$ is equinumerous to $[(r, s; t, u), (u-1, u-1; u-2, u)]$ (a bijection can be obtained by adding (1, 1; 1, 1) to every element of the former interval), the above number is easily seen to be equal to $\lfloor u/2 \rfloor \cdot \lceil u/2 \rceil + 1$. Furthermore, it is not hard to realize that by iteratively subtracting (1, 1; 1, 1) from $(r, s; t, u)$, until $(r-t, s-t; 0, u-t)$ is reached, $\lfloor t/4 \rfloor + g(t, u)$ elements of the form $(a, b, c, 2+4n)$ with $c > 0$ are generated. As a consequence, we have that $(r, s; t, u) \in Q_u$ if and only if either $(r-t, s-t; 0, u-t) \in \tilde{Q}_{u-t}$

and $\lfloor t/4 \rfloor + g(t, u)$ is even, or $(r-t,\ s-t;\ 0,\ u-t) \notin \tilde{Q}_{u-t}$ and $\lfloor t/4 \rfloor + g(t, u)$ is odd. In both cases Proposition 3.6 leads to the required characterisation. □

As previously mentioned, we conclude this section with a look at the combinatorial meaning of specular elements in connection with systems of inequalities. The basic fact we allow for is that every $\mathcal{P}_k$ is an antichain. Because of this property, the combinatorial structure of D_2^k cannot provide any immediate information about the unsatisfiability of a given system $\{\beta_{(r,\,s;t,\,u)}\colon (r, s; t, u) \in A\} \wedge \{\beta_{(t,\,u;r,\,s)} : (r, s;t, u) \in B\}$ with $A \cap B = \emptyset$, $A \cup B \subseteq \mathcal{P}_k$. In fact, the choice of either $\beta_{(r,\,s;t,\,u)}$ or $\beta_{(t,\,u;r,\,s)}$ for each $(r,\ s;\ t,\ u)$ results in an overall choice of mutually compatible order ideals in $\mathcal{W}_2^k$. More precisely – reasoning in $+\mathcal{W}_2^k$ – the principal filter corresponding to some reversed inequality does not intersect the principal ideal corresponding to any non-reversed inequality. This implies that no pair of inequalities of the form $(a < b,\ b < a)$ can logically follow by the initial set of inequalities.

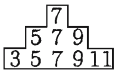
Although the above remark fits any antichain of $+\mathcal{W}_2^k$, we believe that the arithmetical structure of $\mathcal{P}_k$ makes it a rather interesting object of study. In the next section we will accordingly select certain order ideals of $\mathcal{W}_2^k$, by actually selecting an ideal or a filter of $+\mathcal{W}_2^k$ for each specular element of some prescribed levels P_u. Equivalently, we will decide whether or not to reverse any fixed inequality which refers to the chosen levels. The arithmetics lying behind the chosen inequalities will then give rise to some incompatibility results.

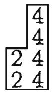
4. Working with Specular Elements

The first part of this section is devoted to showing that the alternate sum over each P_u yields essentially the same pattern for all u congruous to some fixed U (mod 4). By virtue of this property we will then have comparatively few difficulties in summing up elements over the union of some prescribed P_u's (using alternate signs as above). In particular, we will be able to generalise the examples of the previous section, as well as to pose some relevant questions and to provide new examples and suggestions. The following notation will considerably shorten both the claim and the proof of the main results.

NOTATION 4.1. *If ℓ is a positive integer, the symbols ψ_ℓ^x stands for* $(\ell + 1)/2 - |x - (\ell + 1)/2|$. *The function ψ_ℓ^- has a particular effect on the sequence* $(1, 2, \ldots, \ell)$. *For example ψ_5 transforms* $(1, 2, 3, 4, 5)$ *into* $(1, 2, 3, 2, 1)$*, whereas* $\psi_8((1, 2, 3, 4, 5, 6, 7, 8)) = (1, 2, 3, 4, 4, 3, 2, 1)$*, and so forth. Without rigorously defining the symbol χ_ℓ^x, where* W *is odd, we define the effect of χ_ℓ^- on two sequences whose lengths are odd and not congruous (mod 4), as follows:* $\chi_9((1, 2, \ldots, 9)) = (1, 1, 2, 2, 3, 2, 2, 1, 1)$, $\chi_{11}((1, 2, \ldots, 11)) = (1, 1, 2, 2, 3, 3, 3, 2, 2, 1,1)$. For more clearness, $\chi_7((1, 2, \ldots, 7)) = (1, 1, 2, 2, 2, 1, 1)$, $\chi_{13}((1, 2, \ldots, 3)) = (1, 1, 2, 2, 3, 3, 4, 3, 3, 2, 2, 1, 1)$.

In some cases, summations such as $\sum_{\omega=1}^{a} b\psi_a^{\omega}\underline{e}_{\xi(\omega)}$ or $\sum_{\omega=1}^{c} d\omega\underline{e}_{\eta(\omega)}$ could be interpreted as particular stair-like diagrams. For instance, if $a = 5$, $b = 1$, $c = 1$ $d = 2$, $\xi = 2\omega + 1$, $\eta = 2\omega$, the related diagrams might look like these:





A similar remark applies to χ_{ℓ}^{-}. We believe that such diagrams are a useful tool for carrying out calculations and visualising certain general properties (e.g. cancellation rules) which are more or less hidden by the formalism. As a little example, the identity $\sum_{\omega=1}^{10} \psi_{10}^{\omega}\underline{e}_{2\omega+1} = \sum_{\omega=1}^{9} \psi_{9}^{\omega}\underline{e}_{2\omega+3} + \sum_{\omega=1}^{5} \underline{e}_{2\omega+1}$ has a rather natural interpretation in terms of these diagrams.

PROPOSITION 4.2. *For every non-negative* $u \equiv 0 \pmod 4$ *there exists a vector* $\underline{z} \in \mathrm{N}^{u+1}$ such that

$$\sum_{(r,s;t,u)\in Q_u} (r,s;t,u) + \sum_{(r,s;t,u)\in \overline{Q}_u} (t,u;r,s) - (\underline{z},\underline{z}) =$$

$$= (-1)^{\frac{u}{4}} \left(\sum_{\omega=1}^{u/2-2} \psi_{u/2-2}^{\omega}\underline{e}_{2\omega} + 2\sum_{\omega=1}^{u/4-1} \omega\underline{e}_{u/2+1+2\omega}, \quad \sum_{\omega=1}^{u/2-2} \psi_{u/2-2}^{\omega}\underline{e}_{1+2\omega} + 2\sum_{\omega=1}^{u/4-1} \omega\underline{e}_{u/2+2\omega} \right).$$

Proof. We interpret the ordered elements $(r, s; t, u)$ as columns (r, s, t) of a suitable matrix which, to increase readability, has been split into four lines as follows.

r:	$\underline{1}$	$\mathbf{2}$	.....	$\mathbf{u/2}$	2	3	.....	$\mathbf{u/2}$	3	4	.....	$u/2+1$
s:	$u-1$	$u-2$	.....	$u/2$	$\mathbf{u-1}$	$\mathbf{u-2}$	.....	$\mathbf{u/2+1}$	$\mathbf{u-1}$	$\mathbf{u-2}$	.....	$\mathbf{u/2+1}$
t:	$\mathbf{0}$	$\mathbf{0}$	.....	$\mathbf{0}$	$\underline{1}$	1	.....	$\mathbf{1}$	2	$\mathbf{2}$	.....	2

4	5	.....	$u/2+1$	5	$\mathbf{6}$	.....	$\mathbf{u/2+2}$	$\mathbf{6}$	$\mathbf{7}$	.....	$\mathbf{u/2+2}$
$u-1$	$u-2$	.....	$u/2+2$	$u-1$	$u-2$	.....	$u/2+2$	$\mathbf{u-1}$	$\mathbf{u-2}$	.....	$\mathbf{u/2+3}$
$\mathbf{3}$	$\mathbf{3}$	.....	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{4}$	.....	$\mathbf{4}$	$\underline{5}$	$\mathbf{5}$	.....	$\mathbf{5}$

7	8	.....	$u/2+3$	8	9	.....	$u/2+3$		
$\mathbf{u-1}$	$\mathbf{u-2}$	.....	$\mathbf{u/2+3}$	$u-1$	$u-2$	.....	$u/2+4$	.....	.....
6	$\mathbf{6}$	.....	$\mathbf{6}$	7	$\mathbf{7}$	.....	$\mathbf{7}$		

		$u-5$	$u-4$	$u-3$	$u-4$	$u-3$	$\underline{u-3}$	$\mathbf{u-2}$	$\mathbf{u-2}$	$u-1$
.....	.....	$\mathbf{u-1}$	$\mathbf{u-2}$	$\mathbf{u-3}$	$u-1$	$u-2$	$u-1$	$u-2$	$\mathbf{u-1}$	$\mathbf{u-1}$
		$u-6$	$\mathbf{u-6}$	$\mathbf{u-6}$	$\mathbf{u-5}$	$\mathbf{u-5}$	$\mathbf{u-4}$	$\mathbf{u-4}$	$\underline{u-3}$	$u-2$

Because $|P_u$ is odd if and only if $u \equiv 2 \pmod 4$, it can be easily deduced that the minimum of P_u belongs to Q_u precisely when $\lfloor (u+1)/4 \rfloor$ is even (this property actually holds for all u, regardless of the congruence class (mod 4) they belong to). We will prove the proposition by assuming that the above quantity is even; the odd sub-case will automatically follow by reversing all signs. Notice that – as it also happens in the other two cases with $u \not\equiv 2 \pmod 4$ – the component u is inessential because it gives rise to the pair $1/2\lfloor u/2 \rfloor \lceil u/2 \rceil\ (\underline{e}_u, \underline{e}_u)$ (instead, in the remaining case the last component produces either $(\underline{e}_u, \underline{0})$ or $(\underline{0}, \underline{e}_u)$, according to whether $(u-2)/4$ is respectively odd or even – see Proposition 4.4). Now we analyse the above matrix. Let us group columns into maximal sequences of consecutive columns having constant t. The bold numbers in each row can be neglected because their overall contribute is of the form $(\underline{z}', \underline{z}')$. Indeed, the positions of bold, equal numbers of adjacent sequences in rows r and s have different parities, while any sequence of bold, equal numbers in row t has even size. Also the underlined numbers of rows r, t (forming two copies of $\{\rho : \rho \equiv 1 \pmod 4,\ 1 \le \rho \le u-3\}$) can be neglected. As the reader may check, the contribute of the remaining numbers in row r is

$$\left(2\sum_{\omega=1}^{u/2-3} \chi^{\omega}_{u/2-3}\underline{e}_{2+2\omega},\quad \sum_{\omega=1}^{u/2-1} \psi^{\omega}_{u/2-1}\underline{e}_{1+2\omega}\right).$$

Instead, the contribute of row s is

$$\left(\sum_{\omega=1}^{u/4}\underline{e}_{u/2-1+2\omega} + 2\sum_{\omega=1}^{u/4-1}\omega\underline{e}_{u/2+1+2\omega},\quad \sum_{\omega=1}^{u/4}\underline{e}_{u/2-2+2\omega} + 2\sum_{\omega=1}^{u/4-1}\omega\underline{e}_{u/2+2\omega}\right).$$

Finally, the remaining numbers of row t produce the summand $\left(\sum_{1\le\omega\le u/4} \underline{e}_{4\omega-2},\ \underline{0}\right)$. Now the claimed assertion is easily established using the equalities (at this stage, stair-like diagrams can really help)

$$2\sum_{\omega=1}^{u/2-3}\chi^{\omega}_{u/2-3}\underline{e}_{2+2\omega} - \sum_{\omega=1}^{u/4}\underline{e}_{u/2-2+2\omega} + \sum_{\omega=1}^{u/4}\underline{e}_{4\omega-2} = \sum_{\omega=1}^{u/2-2}\psi^{\omega}_{u/2-2}\underline{e}_{2\omega},$$

$$\sum_{\omega=1}^{u/2-1}\psi^{\omega}_{u/2-1}\underline{e}_{1+2\omega} - \sum_{\omega=1}^{u/4}\underline{e}_{u/2-1+2\omega} = \sum_{\omega=1}^{u/2-2}\psi^{\omega}_{u/2-2}\underline{e}_{1+2\omega}.$$

□

An argument similar to the above one yields the corresponding results for $u \equiv 1, 2, 3 \pmod 4$. We just state them and display the related matrices as a sketchy proof.

PROPOSITION 4.3. *For every* $u \equiv 1$ *(mod 4) with* $u \geq 5$ *there exists a vector* $\underline{z} \in \mathrm{N}^{u+1}$ *such that*

$$\sum_{(r,s;t,u)\in Q_u} (r,s;t,u) + \sum_{(r,s;t,u)\in \overline{Q}_u} (t,u;r,s) - (\underline{z},\underline{z})$$

$$= (-1)^{\frac{u-1}{4}} \Bigg(\sum_{\omega=1}^{(u-1)/2} \psi^{\omega}_{(u-1)/2} \underline{e}_{2\omega-1} + 2 \sum_{\omega=1}^{(u-1)/4} \omega \underline{e}_{(u-1)/2+2\omega},$$

$$\sum_{\omega=1}^{(u-1)/2} \psi^{\omega}_{(u-1)/2} \underline{e}_{2\omega} + 2 \sum_{\omega=1}^{(u-1)/4} \omega \underline{e}_{(u-3)/2+2\omega} \Bigg).$$

$$\begin{array}{lcccccccccccc}
r: & 1 & 2 & ..... & (u-1)/2 & 2 & 3 & ..... & (\mathbf{u}+\mathbf{1})/\mathbf{2} & \mathbf{3} & \mathbf{4} & ..... & (\mathbf{u}+\mathbf{1})/\mathbf{2} \\
s: & u-1 & u-2 & ..... & (u+1)/2 & u-1 & u-2 & ..... & (u+1)/2 & \mathbf{u-1} & \mathbf{u-2} & ..... & (\mathbf{u}+\mathbf{3})/\mathbf{2} \\
t: & \mathbf{0} & \mathbf{0} & ..... & \mathbf{0} & \mathbf{1} & \mathbf{1} & ..... & 1 & \underline{2} & 2 & ..... & 2
\end{array}$$

$$\begin{array}{cccccccccccc}
4 & 5 & ..... & (u+3)/2 & 5 & 6 & ..... & (u+3)/2 & 6 & 7 & ..... & (\mathbf{u}+5)/2 \\
\mathbf{u-1} & \mathbf{u-2} & ..... & (\mathbf{u}+3)/2 & u-1 & u-2 & ..... & (u+5)/2 & u-1 & u-2 & ..... & (u+5)/2 \\
3 & 3 & ..... & \mathbf{3} & \mathbf{4} & \mathbf{4} & ..... & \mathbf{4} & \mathbf{5} & \mathbf{5} & ..... & \mathbf{5}
\end{array}$$

$$\begin{array}{cccccccccc}
\mathbf{7} & \mathbf{8} & ..... & (\mathbf{u}+5)/\mathbf{2} & 8 & 9 & ..... & (u+7)/2 & & \\
\mathbf{u-1} & \mathbf{u-2} & ..... & (\mathbf{u}+\mathbf{7})/\mathbf{2} & \mathbf{u-1} & \mathbf{u-2} & ..... & (\mathbf{u}+\mathbf{7})/\mathbf{2} & ..... & ..... \\
\underline{6} & \mathbf{6} & ..... & \mathbf{6} & \mathbf{7} & \mathbf{7} & ..... & \mathbf{7} & &
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & u-5 & u-4 & u-3 & u-4 & u-3 & \underline{u-3} & \mathbf{u-2} & \mathbf{u-2} & u-1 \\
..... & ..... & \mathbf{u-1} & \mathbf{u-2} & \mathbf{u-3} & u-1 & u-2 & u-1 & u-2 & \mathbf{u-1} & \mathbf{u-1} \\
 & & u-6 & \mathbf{u-6} & \mathbf{u-6} & \mathbf{u-5} & \mathbf{u-5} & \mathbf{u-4} & \mathbf{u-4} & \underline{u-3} & u-2
\end{array}$$

PROPOSITION 4.4. *For every positive* $u \equiv 2$ *(mod 4) there exists a vector* $\underline{z} \in \mathbf{N}^{u+1}$ *such that*

$$\sum_{(r,s;t,u)\in Q_u} (r,s;t,u) + \sum_{(r,s;t,u)\in \overline{Q}_u} (t,u;r,s) - (\underline{z},\underline{z})$$

$$= (-1)^{\frac{u-2}{4}} \Bigg(\sum_{\omega=1}^{(u-1)/2} \psi^{\omega}_{u/2} \underline{e}_{2\omega-1} + 2 \sum_{\omega=1}^{(u-6)/4} \omega \underline{e}_{u/2+1+2\omega} + \sum_{\omega=u/2}^{u-1} \underline{e}_{2\omega}$$

$$\sum_{\omega=1}^{u/2} \psi^{\omega}_{u/2} \underline{e}_{2\omega-2} + 2 \sum_{\omega=1}^{(u-2)/4} \omega \underline{e}_{u/2+2\omega} + \underline{e}_u \Bigg).$$

$$\begin{array}{lcccccccccccc}
r: & 1 & 2 & ..... & ..... & u/2 & 2 & \underline{3} & ..... & u/2 & \underline{3} & 4 & ..... & \mathbf{u/2+1} \\
s: & \mathbf{u-1} & \mathbf{u-2} & ..... & \mathbf{u/2+1} & u/2 & \mathbf{u-1} & \mathbf{u-2} & ..... & \mathbf{u/2+1} & u-1 & u-2 & ..... & u/2+1 \\
t: & 0 & \mathbf{0} & ..... & ..... & \mathbf{0} & \mathbf{1} & \mathbf{1} & ..... & \mathbf{1} & 2 & 2 & ..... & 2
\end{array}$$

$$\begin{array}{ccccccccccccc}
\mathbf{4} & \mathbf{5} & ..... & \mathbf{u/2+1} & 5 & 6 & ..... & ..... & u/2+2 & 6 & \underline{7} & ..... & u/2+2 \\
u-1 & u-2 & ..... & u/2+2 & \mathbf{u-1} & \mathbf{u-2} & ..... & \mathbf{u/2+3} & u/2+2 & \mathbf{u-1} & \mathbf{u-2} & ..... & \mathbf{u/2+3} \\
3 & \mathbf{3} & ..... & \mathbf{3} & 4 & \mathbf{4} & ..... & ..... & \mathbf{4} & \mathbf{5} & \mathbf{5} & ..... & \mathbf{5}
\end{array}$$

$$\begin{array}{cccccccccc}
\underline{7} & \mathbf{8} & ..... & \mathbf{u/2+3} & \mathbf{8} & \mathbf{9} & ..... & \mathbf{u/2+3} & & \\
u-1 & u-2 & ..... & u/2+3 & u-1 & u-2 & ..... & u/2+4 & ..... & ..... \\
\mathbf{6} & \mathbf{6} & ..... & \mathbf{6} & 7 & \mathbf{7} & ..... & \mathbf{7} & &
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & u-5 & u-4 & u-3 & u-4 & \underline{u-3} & \underline{u-3} & \mathbf{u-2} & \mathbf{u-2} & u-1 \\
..... & ..... & \mathbf{u-1} & \mathbf{u-2} & u-3 & \mathbf{u-1} & \mathbf{u-2} & u-1 & u-2 & u-1 & u-1 \\
 & & u-6 & \mathbf{u-6} & \mathbf{u-6} & \mathbf{u-5} & \mathbf{u-5} & \mathbf{u-4} & \mathbf{u-4} & u-3 & u-2
\end{array}$$

PROPOSITION 4.5. *For every positive* $u \equiv 3$ *(mod 4) there exists a vector* $\underline{z} \in \mathrm{N}^{u+1}$ *such that*

$$\sum_{(r,s;t,u)\in Q_u} (r,s;t,u) + \sum_{(r,s;t,u)\in \overline{Q}_u} (t,u;r,s) - (\underline{z},\underline{z})$$

$$= (-1)^{\frac{u+1}{4}} \left(\sum_{\omega=1}^{(u-1)/2} \psi^{\omega}_{(u-1)/2} \underline{e}_{2\omega-1} + 2 \sum_{\omega=1}^{(u-3)/4} \omega \underline{e}_{(u+1)/2+2\omega} + \sum_{\omega=1}^{(u+1)/4} \underline{e}_{2\omega-1}, \right.$$

$$\left. \sum_{\omega=1}^{(u-1)/2} \psi^{\omega}_{(u-1)/2} \underline{e}_{2\omega} + 2 \sum_{\omega=1}^{(u-3)/4} \omega \underline{e}_{(u-1)/2+2\omega} + \sum_{\omega=1}^{(u+1)/4} \underline{e}_{2\omega-2} \right).$$

$$\begin{array}{ccccccccccccc}
r: & 1 & 2 & \ldots\ldots & (u-1)/2 & 2 & 3 & \ldots\ldots & (u+1)/2 & 3 & 4 & \ldots\ldots & (u+1)/2 \\
s: & \mathbf{u-1} & \mathbf{u-2} & \ldots\ldots & (\mathbf{u+1})/\mathbf{2} & \mathbf{u-1} & \mathbf{u-2} & \ldots\ldots & (\mathbf{u+1})/2 & u-1 & u-2 & \ldots\ldots & (u+3)/2 \\
t: & 0 & \mathbf{0} & \ldots\ldots & \mathbf{0} & 1 & \mathbf{1} & \ldots\ldots & \mathbf{1} & \mathbf{2} & \mathbf{2} & \ldots\ldots & \mathbf{2}
\end{array}$$

$$\begin{array}{cccccccccccc}
\underline{4} & 5 & \ldots\ldots & (\mathbf{u}+3)/2 & \mathbf{5} & \mathbf{6} & \ldots\ldots & (\mathbf{u+3})/\mathbf{2} & 6 & 7 & \ldots\ldots & (u+5)/2 \\
u-1 & u-2 & \ldots\ldots & (u+3)/2 & \mathbf{u-1} & \mathbf{u-2} & \ldots\ldots & (\mathbf{u+5})/\mathbf{2} & \mathbf{u-1} & \mathbf{u-2} & \ldots\ldots & (\mathbf{u+5})/\mathbf{2} \\
3 & 3 & \ldots\ldots & 3 & \underline{4} & \mathbf{4} & \ldots\ldots & \mathbf{4} & 5 & 5 & \ldots\ldots & \mathbf{5}
\end{array}$$

$$\begin{array}{cccccccccccccc}
7 & 8 & \ldots\ldots & (u+5)/2 & 8 & 9 & \ldots\ldots & (\mathrm{u}+7)/2 & 9 & 10 & \ldots\ldots & (\mathbf{u+7})/\mathbf{2} & & \\
u-1 & u-2 & \ldots\ldots & (u+7)/2 & u-1 & u-2 & \ldots\ldots & (u+7)/2 & \mathbf{u}-1 & \mathbf{u}-2 & \ldots\ldots & (\mathbf{u+9})/\mathbf{2} & \ldots\ldots & \ldots\ldots \\
\mathbf{6} & \mathbf{6} & \ldots\ldots & \mathbf{6} & \mathbf{7} & \mathbf{7} & \ldots\ldots & \mathbf{7} & \underline{8} & \mathbf{8} & \ldots\ldots & \mathbf{8} & &
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & u-5 & u-4 & u-3 & u-4 & u-3 & \underline{u-3} & \mathbf{u}-2 & u-2 & u-1 \\
\ldots\ldots & \ldots\ldots & \mathbf{u-1} & \mathbf{u-2} & \mathbf{u-3} & u-1 & u-2 & u-1 & u-2 & \mathbf{u-1} & \mathbf{u-1} \\
 & & u-6 & \mathbf{u-6} & \mathbf{u-6} & \mathbf{u-5} & \mathbf{u-5} & \mathbf{u-4} & \mathbf{u-4} & \underline{u-3} & u-2
\end{array}$$

As a first fruit of the above four propositions, it turns out easily that all adjacent levels of the form $P_{2\alpha}, P_{2\alpha+1}$ enjoy some cancellation properties which make the corresponding alternate summations rather simple. The proof of the relevant corollary consists of a routine calculation, which is omitted.

COROLLARY 4.6. *For every integer* $c \geq 0$ *there exist two vectors* $\underline{z}$, $\underline{z}'$ *such that*

$$\sum_{(r,s;t,u)\in Q_{4c+2}\cup Q_{4c+3}} (r,s;t,u) + \sum_{(r,s;t,u)\in \overline{Q}_{4c+2}\cup \overline{Q}_{4c+3}} (t,u;r,s) - (\underline{z},\underline{z}) =$$

$$= (-1)^{c+1} \left(\sum_{\omega=1}^{c+1} \underline{e}_{2\omega-1} + 2c\underline{e}_{4c+2}, \quad \sum_{\omega=2c+1}^{4c+1} \underline{e}_{\omega} + \sum_{\omega=1}^{c} \underline{e}_{2c+2\omega} \right),$$

$$\sum_{(r,s;t,u)\in Q_{4c+4}\cup Q_{4c+5}} (r,s;t,u) + \sum_{(r,s;t,u)\in \overline{Q}_{4c+4}\cup \overline{Q}_{4c+5}} (t,u;r,s) - (\underline{z}',\underline{z}') =$$

$$= (-1)^{c+1} \left(\sum_{\omega=1}^{c+1} \underline{e}_{2\omega-1} + (2c+1)\underline{e}_{4c+4}, \ \underline{e}_{2c+2} + 2 \sum_{\omega=1}^{c} \underline{e}_{2c+2+2\omega} + \sum_{\omega=1}^{c+1} \underline{e}_{2c+1+2\omega} \right).$$

With the above results at hand it becomes quite easy to show that the example of the previous section, concerning $P_5 \cup \ldots \cup P_8$, is rather peculiar when compared with the general case $P_{4c+1} \cup \ldots \cup P_{4c+4}$. Indeed, as stated in the next claim, the corresponding summations seem not so easy to interpret in terms of logical consequences (the routine proof, involving Propositions 4.2, 4.3 and the first part of Corollary 4.6, is omitted).

PROPERTY 4.7. *For every integer $c \geq 1$ there exists a vector $\underline{z}$ such that*

$$\sum_{(r,s;t,u)\in\bigcup_{1\leq i\leq 4}Q_{4c+i}} (r,s;t,u) + \sum_{(r,s;t,u)\in\bigcup_{1\leq i\leq 4}\overline{Q}_{4c+i}} (t,u;r,s) - (\underline{z},\underline{z}) =$$
$$= (-1)^{c+1}\left(2\sum_{\omega=1}^{2c}\psi_{2c}^{\omega}\underline{e}_{2\omega} + 2\sum_{\omega=1}^{c}\omega\underline{e}_{2c-1+2\omega} + 2\sum_{\omega=1}^{c}\omega\underline{e}_{2c+3+2\omega},\right.$$
$$\left. 2\sum_{\omega=1}^{2c}\psi_{2c}^{\omega}\underline{e}_{1+2\omega} + 4\sum_{\omega=1}^{c}\omega\underline{e}_{2c+2\omega}\right).$$

For example, by applying the above property in the case $c = 2$ we obtain

$$-((0,0,2,0,4,2,4,4,2,2,0,4,0),\ (0,0,0,2,0,4,4,4,8,2,0,0,0)).$$

By subtracting a suitable pair we then obtain

$$-((0,0,2,0,4,0,0,0,0,0,0,4,0),\ (0,0,0,2,0,2,0,0,6,0,0,0,0)).$$

If $c = 3$, subtracting a suitable pair from the resulting pair yields

$$((0,0,2,0,4,0,6,0,2,0,0,4,0,2,0,6,0),\ (0,0,0,2,0,4,0,4,0,2,4,0,10,0,0,0,0))$$

We leave it as an open question to provide a sensible interpretation of the two above results – as well as of the general phenomenon – in terms of logical dependencies among inequalities. Notice that, as also shown in the above examples, the formula of Property 4.7 is not optimal in that some coordinates may be different from zero in both sides. Instead, we conclude this section by providing an optimal formula, which allows a more successful analysis of some particular alternate summations. Let us consider the levels P_{4c+1}, P_{4c+4}, where c is any positive integer. Differently from the preceding cases, the roles of Q_{4c+1} and $\overline{Q}_{4c+1}$ are interchanged:

PROPERTY 4.8. *For every integer $c > 0$ there exists a vector $\underline{z}$ such that*

$$\sum_{(r,s;t,u)\in\overline{Q}_{4c+1}\cup Q_{4c+4}} (r,s;t,u) + \sum_{(r,s;t,u)\in Q_{4c+1}\cup\overline{Q}_{4c+4}} (t,u;r,s) - (\underline{z},\underline{z}) =$$
$$= (-1)^{c+1}\left(\sum_{\omega=1}^{c}\underline{e}_{2\omega-1} + 2\sum_{\omega=1}^{c}\underline{e}_{2c+2\omega} + (2c-3)\underline{e}_{4c+1} + 2c\underline{e}_{4c+3},\right.$$
$$\left. 2\underline{e}_{2c+1} + 5\sum_{\omega=1}^{c-1}\underline{e}_{2c+1+2\omega} + 2c\underline{e}_{4c+2}\right).$$

As previously done under similar circumstances, we leave the proof to the reader (the ingredients are Propositions 4.2 and 4.3). By neglecting $(\underline{z}, \underline{z})$ and suitably manipulating the formula in the case $c \geq 3$ and odd, we obtain the following incompatibility result.

THEOREM 4.9. *Let γ be an integer greater than* 1. *The inequalities*

$$\left\{x_r x_s < x_t x_u : (r, s; t, u) \in \overline{Q}_{8\gamma-3} \cup Q_{8\gamma}\right\},\ \left\{x_t x_u < x_r x_s : (r, s; t, u) \in Q_{8\gamma-3} \cup \overline{Q}_{8\gamma}\right\},$$

$$\left\{x^2_{4\gamma-1+2i} < x_{5+4i} x_{8\gamma-3} : 0 \leq i \leq \gamma - 2\right\},\ \left\{x^2_{4\gamma-1+2i} < x_{3+4i} x_{8\gamma-1} : 1 \leq i \leq \gamma - 2\right\},$$

$$\left\{x^2_{6\gamma-1+2i} < x_{4\gamma+4i} x_{8\gamma-2} : 0 \leq i \leq \gamma - 2\right\},\ \left\{x^2_{6\gamma-1+2i} < x_{4\gamma+2+4i} x_{8\gamma-4} : 0 \leq i \leq \gamma - 2\right\},$$

$$\left\{x^2_{6\gamma-3} < x_{4\gamma} x_{6\gamma-2},\ x^2_{6\gamma-3} < x_1 x_{8\gamma-2},\ x^2_{8\gamma-5} < x_3 x_{8\gamma-2},\ x^2_{8\gamma-2} < x_{8\gamma-3} x_{8\gamma-1}\right\},$$

subject to $x_i < x_j$ if $i < j$, make up an incompatible system.

Proof. Using the last five sets of inequalities, we perform a number of changes which altogether transform the resulting pair of Property 4.8 (corresponding to the product of all inequalities of the first two sets divided by the tautology $\prod x_i^{z_i} = \prod x_i^{z_i}$) into a pair of distinct vectors $(\underline{v}, \underline{v}')$ with $\underline{v} \gg \underline{v}'$. This procedure will yield a contradiction. Indeed, Theorem 1.5 ensures that $\beta_{(\underline{v}, \underline{v}')}(\underline{x})$ does not hold, whereas at each change the resulting pair $(\underline{w}, \underline{w}')$ is such that $\beta_{(\underline{w}, \underline{w}')}(\underline{x})$ holds, and each deduction from one inequality to the next is shown to be logically correct.

We begin with $2\gamma - 1$ iterated subtractions of the pair $(8\gamma - 3, 8\gamma - 1; 8\gamma - 2, 8\gamma - 2)$ from the initial pair

$$\begin{array}{ccccccccccccccccccccc} (0 & 1 & 0 & 1 & \ldots & 0 & 1 & 0 & 0 & \overset{4\gamma}{2} & 0 & 2 & 0 & 2 & \ldots & 0 & \overset{8\gamma-4}{2} & 4\gamma-5 & 0 & 4\gamma-2 & 0, \\ ,0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 5 & 0 & \ldots & 5 & 0 & 0 & 4\gamma-2 & 0 & 0). \end{array}$$

Since the inequality $x^2_{8\gamma-2} < x_{8\gamma-3} x_{8\gamma-1}$ is assumed, to hold, the corresponding inequality that results from the above subtractions is a logical consequence of the initial inequality. We have therefore obtained the pair

$$\begin{array}{ccccccccccccccccccccc} (0 & 1 & 0 & 1 & \ldots & 0 & 1 & 0 & 0 & \overset{4\gamma}{2} & 0 & 2 & 0 & 2 & \ldots & 0 & \overset{8\gamma-4}{2} & 2\gamma-4 & 0 & 2\gamma-1 & 0, \\ ,0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 5 & 0 & \ldots & 5 & 0 & 0 & 0 & 0 & 0). \end{array}$$

Let us assume that $\gamma \geq 3$ (the case $\gamma = 2$ is postponed). By exploiting the third and fourth set of inequalities, we now subtract the pairs $(5, 8\gamma - 3; 4\gamma - 1, 4\gamma - 1)$, $(7, 8\gamma - 1; 4\gamma + 1, 4\gamma + 1)$, $(9, 8\gamma - 3; 4\gamma + 1, 4\gamma + 1)$, $(11, 8\gamma - 1; 4\gamma + 3, 4\gamma + 3)$, $(13, 8\gamma - 3; 4\gamma + 3, 4\gamma + 3), \ldots, (4\gamma - 5, 8\gamma - 1; 6\gamma - 5, 6\gamma - 5)$, $(4\gamma - 3, 8\gamma - 3; 6\gamma - 5, 6\gamma - 5)$.

The resulting pair is

$$\begin{array}{l}(0\ 1\ 0\ 1\ 0\ 0\ 0\ \dots\ 0\ \overset{4\gamma}{2}\ 0\ 2\ 0\ \dots\ 0\ 2\ \overset{6\gamma-3}{0}\ 2\ 0\ 2\ \dots\ 0\ \overset{8\gamma-4}{2}\ \gamma-3\ 0\ \gamma+1\ 0,\\ ,0\ 0\ 0\ 0\ 0\ 0\ 0\ \dots\ 0\ 0\ 1\ 0\ 1\ \dots\ 1\ 0\ 5\ 0\ 5\ 0\ \dots\ 5\ 0\ 0\ 0\ 0\ 0).\end{array}$$

Notice that the above element is preceded – with respect to the partial ordering $<$ over $\mathcal{W}^{8\gamma}_{6\gamma-2}$ – by the pair whose left vector ends with $\gamma-1$, 0, $\gamma+1$, 0, 0 instead of 2, $\gamma-3$, 0, $\gamma+1$, 0 and whose right vector is unchanged. Theorem 1.5 implies that the latter pair represents a still valid inequality, with which we now replace the above inequality. Subsequently, using the remaining hypotheses we modify the last inequality by subtracting the pairs $(4\gamma, 8\gamma-2; 6\gamma-1, 6\gamma-1)$, $(4\gamma+2, 8\gamma-4; 6\gamma-1, 6\gamma-1)$, $(4\gamma+4, 8\gamma-2; 6\gamma+1, 6\gamma+1)$, $(4\gamma+6, 8\gamma-4; 6\gamma+1, 6\gamma+1), \dots, (8\gamma-8, 8\gamma-2; 8\gamma-5, 8\gamma-5)(8\gamma-6, 8\gamma-4; 8\gamma-5, 8\gamma-5)$. The outcome is

$$\begin{array}{l}(0\ 1\ 0\ 1\ 0\ 0\ 0\ \dots\ 0\ \overset{4\gamma}{1}\ 0\ 1\ 0\ \dots\ 0\ 1\ \overset{6\gamma-3}{0}\ 1\ 0\ 1\ 0\ \dots\ 1\ 0\ 0\ 0\ \overset{8\gamma-2}{2}\ 0\ 0,\\ ,0\ 0\ 0\ 0\ 0\ 0\ 0\ \dots\ 0\ 0\ 1\ 0\ 1\ \dots\ 1\ 0\ 5\ 0\ 1\ 0\ 1\ \dots\ 0\ 1\ 0\ 0\ 0\ 0\ 0).\end{array}$$

Finally, by subtracting $(4\gamma, 6\gamma-2; 6\gamma-3, 6\gamma-3)$, $(1, 8\gamma-2; 6\gamma-3, 6\gamma-3)$, $(3, 8\gamma-2; 6\gamma-3, 8\gamma-5)$ we obtain the pair

$$\begin{array}{l}(0\ 0\ \dots\ 0\ \overset{4\gamma+1}{0}\ 1\ 0\ 1\ \dots\ 0\ 1\ \overset{6\gamma-3}{0}\ 0\ 0\ 1\ 0\ 1\ \dots\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0,\\ ,0\ 0\ \dots\ 0\ 1\ 0\ 1\ 0\ \dots\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ \dots\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0).\end{array}$$

As anticipated at the beginning of the proof, we have reached contradiction. Indeed, the corresponding inequality should correctly read in the reverse way.

If $\gamma = 2$, to prevent the 13th coordinate of the left vector becoming negative $(\gamma-3)$ we add $(\underline{e}_{13}, \underline{e}_{13})$ to the initial pair, before starting with the subtraction procedure (the reader may have noticed that this mending is not strictly necessary, because also negative entries have consistent meaning when related to inequalities). □

Unfortunately, the above argument fails to yield an incompatibility result if $\gamma = 1$. Notice that in this case the sets of inequalities reduce to the first two and the last. Moreover, the inequalities $x_3^2 < x_4^2$ and $x_3^2 < x_3x_6$ of the last set are trivial. In Figure 4 we have emphasized all pairs of $+\mathcal{W}_8$ that are involved in the system. Many lines of the Hasse diagram are missing; some others, are only dotted. Full circles represent actual pairs of $+\mathcal{W}_8$, while empty circles refer to elements of $-\mathcal{W}_8$. The stars correspond to the remaining two inequalities, and to the two non-trivial consequences (smaller stars) of one inequality.

One may wonder if any inequalities exist, different from the above two, which lead to an incompatibility proof if added to the first two sets (a single inequality

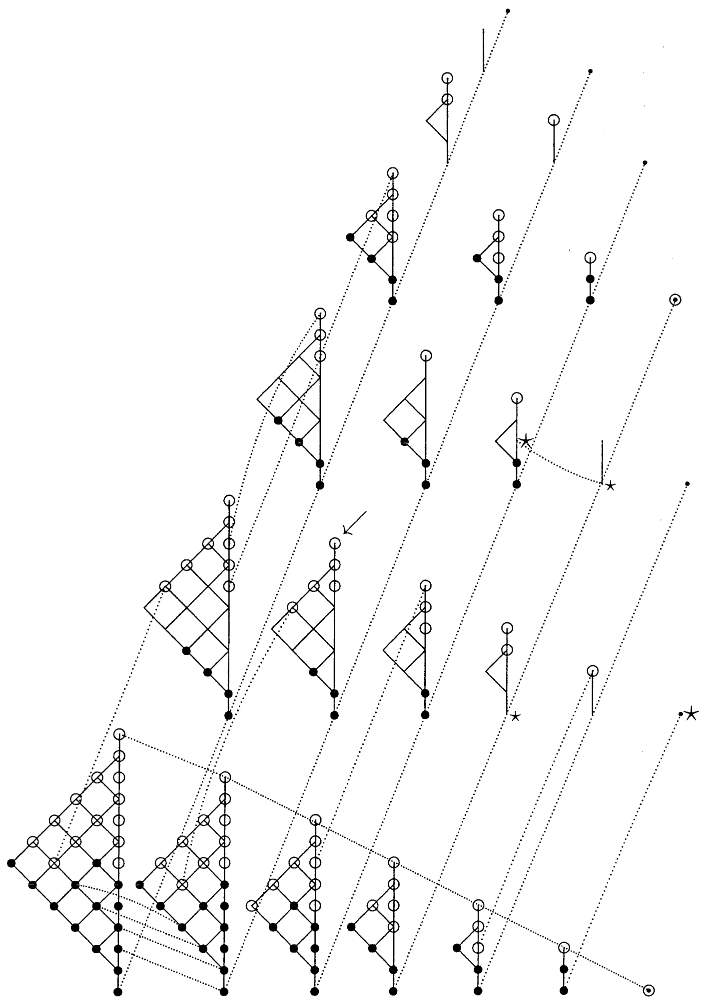


Figure 4. Representing some inequalities in $+\mathcal{W}_8$.

might possibly suffice). For example a quick glance to the depicted poset should warn the reader that the inequality $x_6^2 < x_1x_7$ – whose addition makes possible a straightforward incompatibility proof – cannot be sensibly added but in the reverse way. Indeed, the mentioned inequality (see the arrow) would contradict a basic requirement, namely that the union of all principal ideals (related to non-reversed inequalities) be disjoint from the union of all principal filters (related to reversed inequalities). The same figure should help realizing that many other inequalities, which cause incompatibility if added, are available only in the undesired reversed way. Adding any of them in the uncorrect way would indeed produce a trivial incompatibility at the root. On the contrary – as it could be shown with few difficulties – all the inequalities employed in the general proof of Theorem 4.9 do comply with the ideal-filter condition. Therefore, at least the

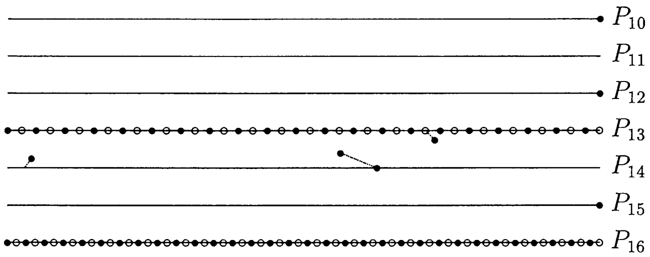


Figure 5. A sketch of the incompatible system when $\gamma = 2$.

very elementary combinatorial substratum of $+\mathcal{W}_8$ agrees with the seven involved sets of inequalities, and a real proof of incompatibility is needed. Figure 5 deals – sketchy – with the case $\gamma = 2$ of Theorem 4.9.

For the general even case ($c = 2\gamma$) a result essentially similar to Theorem 4.9 could be established, using the same reasoning as above.

5. Conclusive Remarks

We feel that the combinatorial expressiveness of specular elements has been unveiled to a rather small degree in the present paper. Presumably, pictures like the one in Figure 4 still have much to say, if properly examined. For example, with some efforts one might try to reduce the number of inequalities which altogether cause incompatibility in Theorem 4.9, so as to hopefully obtain a minimal incompatible system. Notice that the third and fourth set of inequalities in the claim, together with the first three inequalities of the last set, do not generally refer to specular elements. Finding an incompatible system made up of only specular elements (besides the first two sets) might be a challenging problem. On the other hand, a careful analysis of the mere combinatorial constraints in $+\mathcal{W}_8$ – more generally, in every $+\mathcal{W}_k$ – is expected to yield some *compatibility* results, with no need of an explicit numeric solution. In particular, we would welcome the discovery of some key properties that change the role of ideals and filters, namely from elementary indicators of incompatibility (as in the above case $\gamma = 1$) to reliable tools for proving compatibility.

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