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The complexity of arc-colorings for directed hypergraphs

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Abstract

We address some complexity questions related to the arc-coloring of directed hypergraphs. Such hypergraphs arise as a generalization of digraphs, by allowing the tail of each arc to consist of more than one node. The related arc-coloring extends the notion of digraph arc-coloring, which has been studied by diverse authors. Using two classical results we easily prove that the optimal coloring of a digraph, as well as the 2-coloring test for every directed hypergraph, require polynomial time. Instead, the k-colorability problem for some fixed degree d is shown to be NP-complete if $k \geqslant d \geqslant 2$ and $k \geqslant 3$, even if the input is restricted to the so-called *non-overlapping* hypergraphs. We also describe a sub-class of hypergraphs for which the 3-colorability test is polynomially decidable. Some results are rephrased and proved using suitable adjacency matrices, namely *walls*.

MSC: 05B99; 05C15; 05C20; 68Q17

Keywords: Arc-coloring; Brick-coloring; Complexity; Directed hypergraph; Wall

1. Introduction

Directed hypergraphs are not univocally defined, though in any case such structures generalize digraphs. In this paper we follow the definition given in [1,2] (see Fig. 1). A different approach is provided in [7], whereas in [5] the definition of directed hypergraph closely resembles the present one (in fact, the latter is a slight generalization of the former, in that it allows directed loops).

Definition 1.1. A *directed hypergraph* $H=(V(H),E(H))$ consists of a set V, the *nodes*, together with a set $E \subseteq \mathscr{P}(V)\times V$. Each element $e=(A,z)$ of E is a *hyperarc* (or simply an *arc*). A and z are, respectively, the *tail* and the *head* of e. If $v\in V$, the *degree* of v is $\delta(v)=\max(|\{(A,z)\in E: v\in A\}|, |\{(A,z)\in E: v=z\}|)$. The *degree* of H is $\Delta(H)=\max_{v\in V}(\delta(v))$.

The following notion of arc-coloring for directed hypergraphs has been introduced in [10]. In the case of digraphs, such coloring coincides with the arc-coloring defined and studied in [4,9].

Definition 1.2. An *arc-coloring* of H is a map $\gamma: E(H)\to \mathbf{N}$ such that

(i) $((A,x)\in E,(B,y)\in E, A\cap B\neq\emptyset) \Rightarrow \gamma(A,x)\neq\gamma(B,y)$
(ii) $((C,z)\in E,(D,z)\in E) \Rightarrow \gamma(C,z)\neq\gamma(D,z)$ provided $(A,x)\neq(B,y)$ and $(C,z)\neq(D,z)$.

If k colors are enough for coloring the arcs of H, then H is said *k-colorable*. The (*directed*) *chromatic index* of H, denoted by $q(H)$, is the least number k such that H is k-colorable. Every coloring of H in $q(H)$ colors is termed *optimal*.

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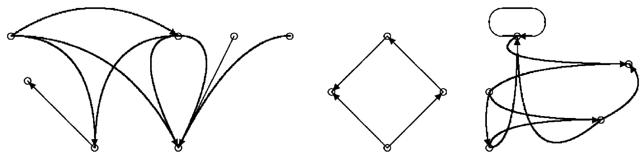
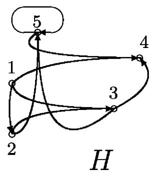


Fig. 1. Directed hypergraphs.



$$E(H) = \{(\{1\},2),(\{1,2\},3), (\{1,5\},4),(\{3\},4), (\{2,3\},5),(\{5\},5)\}$$

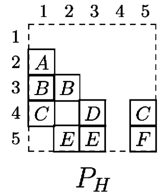


Fig. 2. A directed hypergraph and the related wall.

In simple words, to obtain a legal coloring we must use distinct colors for any two arcs having either intersecting tails or the same head. Several properties of this coloring have been pointed out in [10].

In the present paper, we analyze the arc-coloring from the point of view of complexity. In Section 2 we firstly show that digraphs are optimally colorable in polynomial time, by adapting a classical result due to Gabow. Further, by invoking the well-known polynomial complexity of the 2-coloring test for the vertices of a graph, we easily prove that the 2-colorability of a hypergraph of degree 2 is polynomially decidable as well. Finally, we use some results established in [10] to show that the 3-coloring is polynomially decidable for a particular sub-class of *interval* hypergraphs of degree 2. This sub-class consists of *non-overlapping* interval hypergraphs.

Definition 1.3. A directed hypergraph H is an *interval* (*directed*) *hypergraph* if there exists a linear ordering $\leqslant$ of its nodes, such that every tail of H is a closed interval with respect to $\leqslant$.

Definition 1.4. A directed hypergraph H is *non-overlapping* if there exists no pair of arcs of H sharing the same head and some node of the tails. Otherwise, H is termed *overlapping* and any such pair of arcs is termed *overlapping* as well.

We observe that it is rather easy to exhibit non-overlapping hypergraphs which are not interval hypergraphs, and conversely.

In Section 3 we show that the k-coloring of a non-overlapping hypergraph with fixed degree d is an NP-complete problem whenever $k \geqslant d \geqslant 2$ and $k \geqslant 3$. The starting point of our proof is Holyer's theorem [8], on the NP-completeness of the 3-coloring problem for the edges of a cubic graph (actually, for our purposes it suffices the NP-completeness of the 3-coloring problem over *all* graphs).

It seems convenient to formulate many of the above questions using a different language than the graph-theoretical one, namely, by means of the rephrasing introduced in [10]. In that paper non-overlapping hypergraphs have been represented by suitable adjacency matrices (walls) which can be regarded as a generalization of the intuitive concept of a wall made of bricks.

Definition 1.5. A *wall* P is a partial chessboard whose squares have been labelled under the condition that every label occurs in a unique row of P. Its *degree*, $\delta(P)$, is the greatest number of distinct labels in the same row or column. Every maximal set of squares having the same label is a *brick*.

In Fig. 2 we show how to associate a non-overlapping hypergraph H to the corresponding wall, usually denoted by P_H. Notice that the nodes of H need to be numbered, and that the representation depends on the given numbering. In the sequel we will assume that any directed hypergraph is endowed with a numbering of the nodes. Also notice that if H is a digraph, P_H reduces to the transposed adjacency matrix of H.

The reader can easily check that $\Delta(H) = \delta(P_H)$ for every non-overlapping hypergraph H. Using the language of walls, the arc-coloring of a non-overlapping hypergraph translates as follows.

Definition 1.6. A (*brick*)-*coloring* of P is the assignment of a symbol (*color*) to each square of P in such a way that: (1) *All the squares of any fixed brick are given the same color*; (2) *Any two squares of distinct bricks lying in the same row are given different colors*; (3) *Any two squares lying in the same column are given different colors*. If k colors are enough for coloring the bricks of P, then we say that P is k-*colorable*. The *chromatic number* of P, denoted by $\rho(P)$, is the least number k such that P is k-colorable. Every coloring of P in $\rho(P)$ colors is termed *optimal*.

Using the above definition it is straightforward to see that $q(H) = \rho(P_H)$.

2. Some coloring problems in P

In the rest of this paper the input size of a directed hypergraph is understood to be the positive integer $size(H) = \sum_{(A,z)\in E(H)} |A|$.

Firstly, we deal with *digraphs*. For this sub-class of directed hypergraphs, a classical result proved by Gabow [6] guarantees the existence of a polynomial algorithm for optimally coloring the arcs. We state the cited result.

Theorem 2.1. *An optimal edge-coloring of a bipartite graph G can be performed in time* $O(|E(G)|\sqrt{|V(G)|}\log(|V(G)|) + |V(G)|)$.

Applying the above theorem to digraphs yields

Corollary 2.2. *An optimal arc-coloring of a digraph can be performed in polynomial time with respect to its size.*

Proof. A digraph D is equivalent to a bipartite graph G_D whose vertices are partitioned into two copies of the nodes of D, two vertices v, w being the end-points of some edge of G_D if and only if there exists an arc from v to w in the digraph. It is easily seen that any fixed arc-coloring of the digraph translates to some edge-coloring of the bipartite graph. As $size(D) = |E(G_D)|$, we can prove the assertion using Theorem 2.1 and the inequality $|V(G_D)| \leqslant 2|E(G_D)|$ (without loosing generality, we have assumed that no isolated node occurs in D). □

In the general case (that is, by allowing tails to consist of more than one node) the following result can be easily obtained.

Proposition 2.3. *There exists a polynomial algorithm which decides the* 2*-colorability of a given directed hypergraph of degree* 2.

Proof. As the reader may easily check, a polynomial reduction can be performed from our problem to the 2-coloring problem for the vertices of a suitable non-directed graph (in details, the vertices of this graph are put in 1-1 correspondence with the arcs of the hypergraph, two vertices being adjacent if and only if the corresponding arcs have either the same head or intersecting tails). Since the 2-coloring problem for the vertices of graphs is solvable in polynomial time (see for example [3]), the assertion is proved. □

As mentioned in the Introduction, a further example of polynomial complexity involves non-overlapping interval hypergraphs. In what follows we interpret such structures as particular walls.

Definition 2.4. A wall P is termed *coherent* if every brick of P consists of adjacent squares. More generally, P is termed *pre-coherent* if for some chessboard containing P there exists a permutation of the columns of the chessboard which yields a coherent wall.

Lemma 2.5. *The wall* P_H *is pre-coherent if and only if* H *is a* (*non-overlapping*) *interval hypergraph.*

Proof. If P_H is pre-coherent, we can use the chessboard of P_H as a suitable one for obtaining a coherent wall. Let π be a permutation of the columns giving rise to a coherent wall, say Q. By also applying π to the rows of Q we obtain a still coherent wall, say R. As a result, both rows and columns of P_H have been permuted using π, which is equivalent to reordering the nodes of H according to π. It follows that every fixed tail of H, whose nodes have been renumbered, is an interval; indeed, it corresponds to some connected brick of R. The converse assertion can be proved in a similar fashion. □

The following result—established in [10]—is the basic ingredient for deducing the subsequent claim.

Proposition 2.6. *If P is a pre-coherent wall, then*

$$\rho(P) \leqslant 2\delta(P) - 1.$$

Proof. Since ρ and δ are not altered by any permutation of columns, we can assume that P is coherent. The claimed inequality is then proved by induction on the length of the smallest chessboard containing P, the basic case trivially reducing to $\rho(P) = \delta(P)$. In the induction step, by removing the rightmost column of P we obtain a wall which is colorable in $2\delta(P) - 1$ colors. Any fixed optimal coloring can be legally extended to all the squares of the removed column which are not bricks. The remaining squares can be recursively colored without exceeding the prescribed number of colors, because all the bricks lying in the same row or sharing the column of any such square are at most $2\delta(P)-2$. □

Now we can prove the

Proposition 2.7. *Let H be a non-overlapping interval hypergraph having degree* 2. *Then, there exists a polynomial algorithm which evaluates $q(H)$.*

Proof. Lemma 2.5 and Proposition 2.6 imply that $q(H) \in \{2, 3\}$. Therefore, finding $q(H)$ is equivalent to deciding whether H is 2-colorable or not, and Proposition 2.3 ensures that such question can be answered in polynomial time. □

3. Some NP-complete problems

The main result of this section is that the k-colorability of a non-overlapping hypergraph of degree d is an NP-complete problem if $k \geqslant d \geqslant 2$ and $k \geqslant 3$. The following proposition is a preparatory tool.

Proposition 3.1. *The* 3-*colorability of the arcs of a given non-overlapping hypergraph having degree* 2 *is an* NP-*complete problem.*

Before proving the above assertion we state a theorem due to Holyer [8]. Such result is crucial for our purposes.

Theorem 3.2. *The* 3-*colorability of the edges of a cubic graph* (*that is, a graph whose vertices have all degree* 3) *is an* NP-*complete problem.*

In his proof, the author provides a polynomial reduction from the 3-SAT problem to the edge-coloring problem specified in the statement. We are now ready for the

Proof of Proposition 3.1. The problem is easily seen to be in NP. We will then give a polynomial reduction from the edge-coloring of graphs (not necessarily cubic) to our problem. If G is a given graph with t edges $e_1, \ldots, e_t$, we define a non-overlapping directed hypergraph $\mathscr{G}$ as follows. Firstly, we introduce $2t$ nodes $n_1, \ldots, n_{2t}$ and t arcs $(\{n_1\}, n_2), (\{n_3\}, n_4), \ldots, (\{n_{2t-1}\}, n_{2t})$. Further, for every intersecting edges e_i, e_j we introduce a new node n_{ij} and add it to the two tails containing n_{2i-1} and n_{2j-1}, respectively. By doing so, it can be easily checked that any fixed 3-coloring of $E(\mathscr{G})$ corresponds to some 3-coloring of $E(G)$, and conversely. Furthermore $\Delta(\mathscr{G}) = 2$, and the reduction is clearly achievable in polynomial time with respect to the input size. □

As a first consequence, we obtain the

Proposition 3.3. *For every integer $k \geqslant 3$, the k-colorability of the arcs of a given non-overlapping hypergraph having degree* 2 *is an* NP-*complete problem.*

Proof. Clearly, the problem is in NP. The NP-hardness will be proved by induction on k. The basis is Proposition 3.1. Let us now fix some integer $k \geqslant 3$. We use the rephrasing by means of walls. If we are given a wall R of degree 2, then we construct—in polynomial time—a wall S of degree 2, such that R is k-colorable if and only if S is $(k+1)$-colorable. The new wall is obtained by adding some suitable squares to R, as illustrated in Fig. 3.

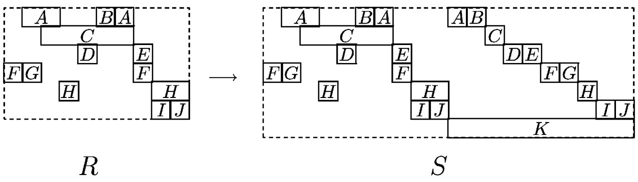


Fig. 3. $\delta(R)=\delta(S)=2$, $\rho(R)\leqslant k \Leftrightarrow \rho(S)\leqslant k+1$.

More formally, in each row of R containing two bricks we introduce two differently labelled squares, so as to extend the already present bricks; furthermore, in the remaining rows we extend the unique brick by introducing one further square. All the above squares must be arranged in such a way that no two of them lie in the same column. Finally, we introduce one further brick whose squares range over all the columns containing the previously added squares. It is not difficult to check that this construction fulfills our requirement. □

Now we establish the

Proposition 3.4. *For all integers $k \geqslant d \geqslant 3$, the k-colorability of the arcs of a given non-overlapping hypergraph having degree d is an* NP-*complete problem.*

Proof. The problem is clearly in NP. Now, for some fixed integers k, d with $k \geqslant d \geqslant 3$, a polynomial reduction from the k-colorability-degree 2 problem to the k-colorability-degree d problem can be easily provided. Indeed, using walls in place of hypergraphs again, let R denote some k-colorable wall of degree 2. It suffices to modify R by introducing a new row containing d differently labelled squares, with the further condition that each square lies alone in its column (using the graph-theoretical language, the above construction can be performed, for example, by introducing a new node as a tail, and connecting it to d new, further nodes). It can be checked with no difficulties that the k-colorability of the resulting wall is equivalent to the k-colorability of R. □

Putting together the above propositions, we obtain the announced result:

Theorem 3.5. *For all integers k, d such that $k \geqslant d \geqslant 2$ and $k \geqslant 3$, the k-colorability of the arcs of a given non-overlapping hypergraph having degree d is an* NP-*complete problem.*

We observe, trivially, that the NP-completeness of any problem concerning non-overlapping hypergraphs still holds when the input is extended to all possible directed hypergraphs.

As a conclusive remark, we do not know whether the analogue of Theorem 3.5 for non-overlapping *interval* hypergraphs holds (in this case, the input degree must be greater than 2, due to Proposition 2.7). Therefore, we leave the above question as an open problem.

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