Cyclic k-Cycle Systems of Order $2kn + k$: A Solution of the Last Open Cases

Andrea Vietri
Dipartimento Me. Mo. Mat., Università Roma I, Via A. Scarpa, 16, 00161 Roma, Italia, E-mail: vietri@dmmm.uniroma1.it

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Abstract: We exhibit cyclic (K_v, C_k)-designs with $v > k$, $v \equiv k \pmod{2k}$, for k an odd prime power but not a prime, and for $k = 15$. Such values were the only ones not to be analyzed yet, under the hypothesis $v \equiv k \pmod{2k}$. Our construction avails of Rosa sequences and approximates the Hamiltonian case $(v = k)$, which is known to admit no cyclic design with the same values of k. As a particular consequence, we settle the existence question for cyclic (K_v, C_k)-designs with k a prime power. © 2004 Wiley Periodicals, Inc. J Combin Designs 12: 299–310, 2004

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1. INTRODUCTION

A *k-cycle system* of a graph $G = (V, E)$ is a (multi)set $\mathcal{B}$ of k-cycles whose edges partition E. $\mathcal{B}$ is said to be *cyclic* if $V = \mathbf{Z}_v$ for some v and $B = (b_0, b_1, \ldots\ldots, b_{k-1}) \in \mathcal{B}$ implies that $B + 1 \in \mathcal{B}$, where for each $z \in \mathbf{Z}_v$ the sum $B + z$ is defined in the obvious way as $(b_0 + z, b_1 + z, \ldots, b_{k-1} + z)$. In the sequel, the vertices of any graph will be considered as elements of $\mathbf{Z}_v$ for some fixed v.

Details about k-cycle systems may be found for example in Refs. [13] and [15]. Moreover, in Ref. [11] the decomposition of a graph G into k-cycles is considered as a particular case of the decomposition into copies of an assigned subgraph H, namely a (G, H)-*design*. Following this terminology, in the present paper we will be concerned with *cyclic* (K_v, C_k)*-designs*, where K_v is the complete graph on v vertices and C_k is the cycle of length k. In more details, we will prove the following

Theorem 1.1. *If* $v > k$, $v \equiv k(\text{mod } 2k)$ *and* k *is* 15 *or an odd prime power but not a prime, then there exists a cyclic* (K_v, C_k)*-design.*

The values of k, mentioned in the theorem, are the only ones which have been not investigated, so far, under the same hypothesis $v > k$, $v \equiv k \ (\text{mod } 2k)$. The other cases, and the further case $v = k$, have been analyzed by Buratti and Del Fra [4], who established the following result.

Theorem 1.2. *Let* $M = \{p^a \mid \text{p } odd\ prime, \text{ a} \geq 2\} \cup \{15\}$. *If* $v > k$, $v \equiv k(\text{mod 2k})$ *and* $k \in M$, *then there exists a cyclic* (K_v, C_k)*-design if and only if* $(v, k) \neq (9, 3)$. *If* $v = k$ *(the Hamiltonian case), then there exists a cyclic* (K_v, C_k)*-design if and only if* k *is an odd integer not belonging to* M.

The first part of the above theorem is a consequence of the second part (Hamiltonian case), which has required a careful proof. Furthermore, it is a short exercise to prove that no cyclic (K_9, C_3)-design exists.

We recall the following definitions.

Definition 1.1. *The type of a cycle* B *is the cardinality of the stabilizer of* B *under the action of* $\mathbf{Z}_v$ *defined by* $z(B) = B + z$.

Definition 1.2. *If* $B = (b_0, b_1, \ldots, b_{k-1})$ *is a* k*-cycle of type* d, *the list of partial differences from* B *is the multiset* $\partial B = \{\pm(b_{i+1} - b_i) : 0 \leq i < k/d\}$, *where* $b_k = b_0$. *More generally, if* $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ *is a set of* k*-cycles, the list of partial differences from* $\mathcal{F}$ *is the multiset* $\partial \mathcal{F} = \bigcup_i \partial S_i$.

Note that if B is a cycle of type 1, then ∂B is the list ΔB of differences from B in the usual sense. The following assertion, as elementary as fundamental, is a consequence of the theory developed in Ref. [7].

Proposition 1.1. *Let* $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ *be a set of* k*-cycles and let* d_i *be the type of* $S_i, i = 1, \ldots, n$. *If* $\partial \mathcal{F}$ *covers* $\mathbf{Z}_v \backslash \{0\}$ *exactly once, then the cycles* $\{S_i + z : 1 \leq i \leq n, 1 \leq z \leq v/d_i\}$ *form a cyclic* (K_v, C_k)*-design.*

A set $\mathcal{F}$ as in the above proposition will be called a (K_v, C_k)*-difference system.* Each S_i is called a *starter cycle*. Starter cycles can be constructed using particular sequences (see also Ref. [9], Proposition 1.2).

Definition 1.3. *Let* v, k, d *be positive integers such that* k *divides* v *and* d *divides both* v *and* k. *A sequence* $(c_0, c_1, \ldots, c_{k/d})$ *of integers has the* (v, k, d)*-property if*

(1) $c_i \not\equiv c_j (\text{mod } v/d)$ for $0 \leq i < j < k/d$.
(2) $c_{k/d} - c_0 = xv/d$ with $\gcd(x, d) = 1$.
(3) $c_i - c_{i-1} \not\equiv \pm(c_j - c_{j-1}) \ (\text{mod } v)$ for $1 \leq i < j \leq k/d$.

For example, assuming that $v = 45, k = 15$, and $d = 3$, the sequence $(0, -13, 3, 24, 6, 15)$ has the (45,15,3)-property. We do not give the short proof of the following basic fact.

Lemma 1.1. *Let the sequence* $(c_0, c_1, \ldots, c_{k/d})$ *have the* (v, k, d)*-property, and consider the extended sequence* $C = (c_0, c_1, \ldots, c_{k/d}, \ldots, c_{k-1})$ *such that*

$$c_i = q(c_{k/d} - c_0) + c_r,$$

where q *and* r *are, respectively, the quotient and the remainder of the euclidean division of* i *by* k/d*. Then, the pairs* C *is a cycle of type* d*, whose list of partial differences has no repetitions.*

The above cycle is also denoted by $[c_0, c_1, \ldots, c_{k/d}]_k$ (see Ref. [4]). Since in this case $\partial[c_0, c_1, \ldots, c_{k/d}]_k$ is not a multiset, in the present paper sequences having the (v, k, d)-property will be a basic ingredient for obtaining suitable families of cycles.

A further tool for our purposes will be *Rosa sequences* which, together with *Skolem sequences*, have been used for example in Refs. [3] and [9].

We recall that the existence problem for cyclic k-cycle systems of the complete graph K_v with $v \equiv 1 \pmod{2k}$ has been exhaustively settled in Ref. [3] and, independently, in [5,6,10].

Theorem 1.3. *There exists a cyclic* (K_{2kn+1}, C_k)*-design for any pair of positive integers* k, n*.*

The same has been done for cyclic k-cycle systems of the complete m-partite graph $K_{m\times k}$ with m and k odd (see Ref. [3] for details). Yet the existence problem restricted to the case $v \equiv 1 \pmod{2k}$ with k even was completely settled in the sixties, by Kotzig [12] and by Rosa [16,18]. Rosa also settled the cases $k = 3, 5, 7$ [17]. The earliest solution of the case $k = 3$ was given by Peltesohn [14]. The existence question for (K_v, C_k)-designs (not necessarily cyclic) has been exhaustively settled by Alspach and Gavlas [2] in the case of k odd (see also Ref. [8]) and by Šajna [19] in the even case.

In the Conclusion of the present paper, Theorems 1.2 and 1.3 will contribute to settle the existence question for cyclic (K_v, C_k)-designs, with k a prime power.

2. THE CASE $k = p^a$

In the sequel, each set of numbers of the form $\{\pm n_1, \ldots, \pm n_s\}$ will be denoted by $\pm\{n_1, \ldots, n_s\}$. The first part of Theorem 1.1 can be stated in the following alternative way.

Theorem 2.1 (Part I)**.** *For any prime* $p \geq 3$ *and any two integers* m, a *with* $m \geq 3$ *odd and* $a \geq 2$*, there exists a cyclic* (K_{mp^a}, C_{p^a})*-design.*

Proof. Having fixed p, m, and a, the following equality suggests the number of starter cycles to use, as well as the corresponding types.

$$\text{No. of cycles} = \frac{mp^a(mp^a - 1)}{2p^a} = \frac{m-1}{2} mp^a + \frac{p-1}{2} \sum_{i=0}^{a-1} mp^i.$$

We will therefore look for $(m-1)/2$ starter cycles of type 1, (each one generating $2p^a$ differences) and $(p-1)/2$ starter cycles of type p^{a-i} (each one generating $2p^i$

partial differences) where i ranges from 0 to $a-1$. The above types are clearly admissible.

Our construction can be summarized as follows. The cycles of type p^{a-i}, with $0 \leq i \leq a-1$, generate almost all the partial differences which are divisible by m. Moreover, their vertices consist of almost all the multiples of m. Though all the required cycles generate as many differences as the ones which are divisible by m (namely, $p^a - 1$), using all these multiples of m as differences and some multiples of m as vertices (with no further vertex) would be equivalent to finding a cyclic (K_{p^a}, C_{p^a})-design, as it can be quickly seen. This is a contradiction because no such design exists, according to Theorem 1.2. Therefore, we perform an approximation by allowing some extra difference, as well as some extra vertex. Subsequently, we define $(m-1)/2$ cycles of type 1, namely $\Gamma_1^*, \Gamma_2, \Gamma_3, \ldots, \Gamma_{(m-1)/2}$, which generate all the remaining differences. In particular, each set $\Delta\Gamma_i$ consists of almost all the differences of the form $mz \pm i$. The construction of Γ_i coincides with the one performed in Ref. [3] for obtaining the cycles of a cyclic $(K_{m\times p^a}, C_{p^a})$-design. Instead, Γ_1^* requires some more attention, as $\Delta\Gamma_1^*$ is supposed to contain all those multiples of m which are not generated by the above cycles of larger type. Thus, we slightly modify the cycle Γ_1, arising from the cited construction.

We begin with the cycles of largest type. For $1 \leq i \leq (p-1)/2$, let B_i stand for the k-cycle of type p^a defined by $B_i = (0, im, 2im, \ldots, (k-1)im)$.

Clearly,

$$\partial B_1 \cup \cdots \cup \partial B_{(p-1)/2} = \pm\{m, 2m, \ldots, (p-1)m/2\} \ . \tag{1}$$

Now we define the cycles of smaller type. Let us fix i,j such that $1 \leq i < a$, $0 \leq j \leq (p-3)/2$ and assume that $p^i \neq 3$ (the case $p^i = 3$ is postponed).

We consider the sequence $(c_{ij0}, c_{ij1}, \ldots, c_{ijp^i})$ defined by

$$c_{i,j,\ell} = \begin{cases} \frac{\ell}{2}m & \text{if } \ell \text{ is even}, 0 \leq \ell \leq \frac{p^i-1}{2}; \\ (\frac{p^i-\ell}{2} + (j+1)p^i)m & \text{if } \ell \text{ is odd}, 1 \leq \ell < \frac{p^i-1}{2}\ ; \\ (2j+2)p^i - 1 - \frac{\ell}{2})m & \text{if } \ell \text{ is even}, \frac{p^i-1}{2} < \ell < p^i - 1; \\ (\frac{p^i+\ell}{2} + jp^i)m & \text{if } \ell \text{ is odd}, \frac{p^i-1}{2} \leq \ell < p^i - 1; \end{cases}$$

$$c_{i,j,p^i-1} = -1 \text{ if } p^i \equiv 1 \pmod 4),\ \ c_{i,j,p^i-1} = (2j+2)p^i m + 1 \text{ if } p^i \equiv 3 \pmod 4);$$
$$c_{i,j,p^i} = (j+1)p^i m.$$

It is easy to check that the above sequence has the (mp^a, p^a, p^{a-i})-property.

In particular, the $2p^i$ partial differences from the resulting cycle (say C_{ij}) of type p^{a-i} are given by the formulas

$$\partial C_{ij} \cap m\mathbf{Z}_{mk} = \pm\left\{ \left(\frac{p^i-1}{2} + jp^i + s\right)m \colon 1 \leq s \leq p^i, s \neq \frac{p^i \pm 1}{2} + \varepsilon \right\}; \tag{2}$$

$$\partial C_{ij} \cap (\mathbf{Z}_{mk} - m\mathbf{Z}_{mk}) = \pm\{((j+1)p^i - 1 + \varepsilon)m + 1, ((j+1)p^i + \varepsilon)m + 1\}, \tag{3}$$

where $\varepsilon = 0$ if $p^i \equiv 1 (\text{mod } 4), \varepsilon = 1$ in the other case. Indeed, if $p^i \equiv 1$ (mod 4),

$$\left(\frac{p^i-1}{2}+jp^i+1\right)m = c_{i,j,(p^i+1)/2} - c_{i,j,(p^i-1)/2};$$
$$\left(\frac{p^i-1}{2}+jp^i+s\right)m = \begin{cases} (-1)^s(c_{i,j,p^i-s-1} - c_{i,j,p^i-s}) & \text{if } 2 \le s < \frac{p-1}{2}; \\ (-1)^s(c_{i,j,p^i-s} - c_{i,j,p^i-s+1}) & \text{if } \frac{p^i+1}{2} < s \le p^i; \end{cases}$$
$$((j+1)p^i-1)m+1 = c_{i,j,p^i-2} - c_{i,j,p^i-1}\ ; \quad (j+1)p^i m + 1 = c_{i,j,p^i} - c_{i,j,p^i-1}.$$

Otherwise, if $p^i \equiv 3$ (mod 4),

$$\left(\frac{p^i-1}{2}+jp^i+1\right)m = c_{i,j,(p^i-1)/2} - c_{i,j,(p^i-3)/2};$$
$$\left(\frac{p^i-1}{2}+jp^i+s\right)m = \begin{cases} (-1)^s(c_{i,j,p^i-s-1} - c_{i,j,p^i-s}) & \text{if } 2 \le s \le \frac{p-1}{2}; \\ (-1)^s(c_{i,j,p^i-s} - c_{i,j,p^i-s+1}) & \text{if } \frac{p^i+3}{2} < s \le p^i; \end{cases}$$
$$(j+1)p^i m + 1 = c_{i,j,p^i-1} - c_{i,j,p^i}; \quad ((j+1)p^i+1)m+1 = c_{i,j,p^i-1} - c_{i,j,p^i-2}$$

(notice that, due to the pigeon-hole principle, no further difference is produced).

In Figure 1, we have represented two sequences which give rise to a cycle $C_{2,0}$ (whose set of vertices is assumed to be $\mathbf{Z}_{5^a m}$, for some $a \ge 3$) and to a cycle $C_{3,0}$ (whose set of vertices is $\mathbf{Z}_{3^{a'} m'}$, for some $a' \ge 4$). Notice that only some vertices have been depicted, so as to remind the "approximated" Hamiltonian construction.

The case $p^i = 3$ is managed by defining a sequence $(c_{100}, c_{101}, c_{102}, c_{103})$ with the $(3^a m, 3^a, 3^{a-1})$-property as $(0, 2m+1, -m, 3m)$. Therefore, in this particular case we have

$$\partial C_{10} \cap m\mathbf{Z}_{mk} = \pm\{4m\}, \partial C_{10} \cap (\mathbf{Z}_{mk} - m\mathbf{Z}_{mk}) = \pm\{2m+1, 3m+1\}. \qquad (*)$$

Now we set

$$\mathcal{F} = \{B_1, \ldots, B_{(p-1)/2}\} \cup \{C_{ij}\colon 1 \le i < a, 0 \le j \le (p^i-3)/2\};$$
$$X = \pm\{(jp^i - 1 + \varepsilon)m, (jp^i + \varepsilon)m\colon 1 \le i < a, 1 \le j \le (p-1)/2\};$$
$$Y = \pm\{(jp^i - 1 + \varepsilon)m + 1, (jp^i + \varepsilon)m + 1\colon 1 \le i < a, 1 \le j \le (p-1)/2\},$$

where $\varepsilon = 0$ also if $p^i = 3$ (the index j has been given two different ranges, for convenience). Using (1), (2), (3), and $(*)$ we get

$$\partial\mathcal{F} \cap m\mathbf{Z}_{mk} = m\mathbf{Z}_{mk} - X, \quad \partial\mathcal{F} \cap (\mathbf{Z}_{mk} - m\mathbf{Z}_{mk}) = Y. \qquad (4)$$

Now we construct the remaining cycles. If $m \ge 5$ such objects will be defined with the help of Rosa sequences. The case $m = 3$ is postponed.

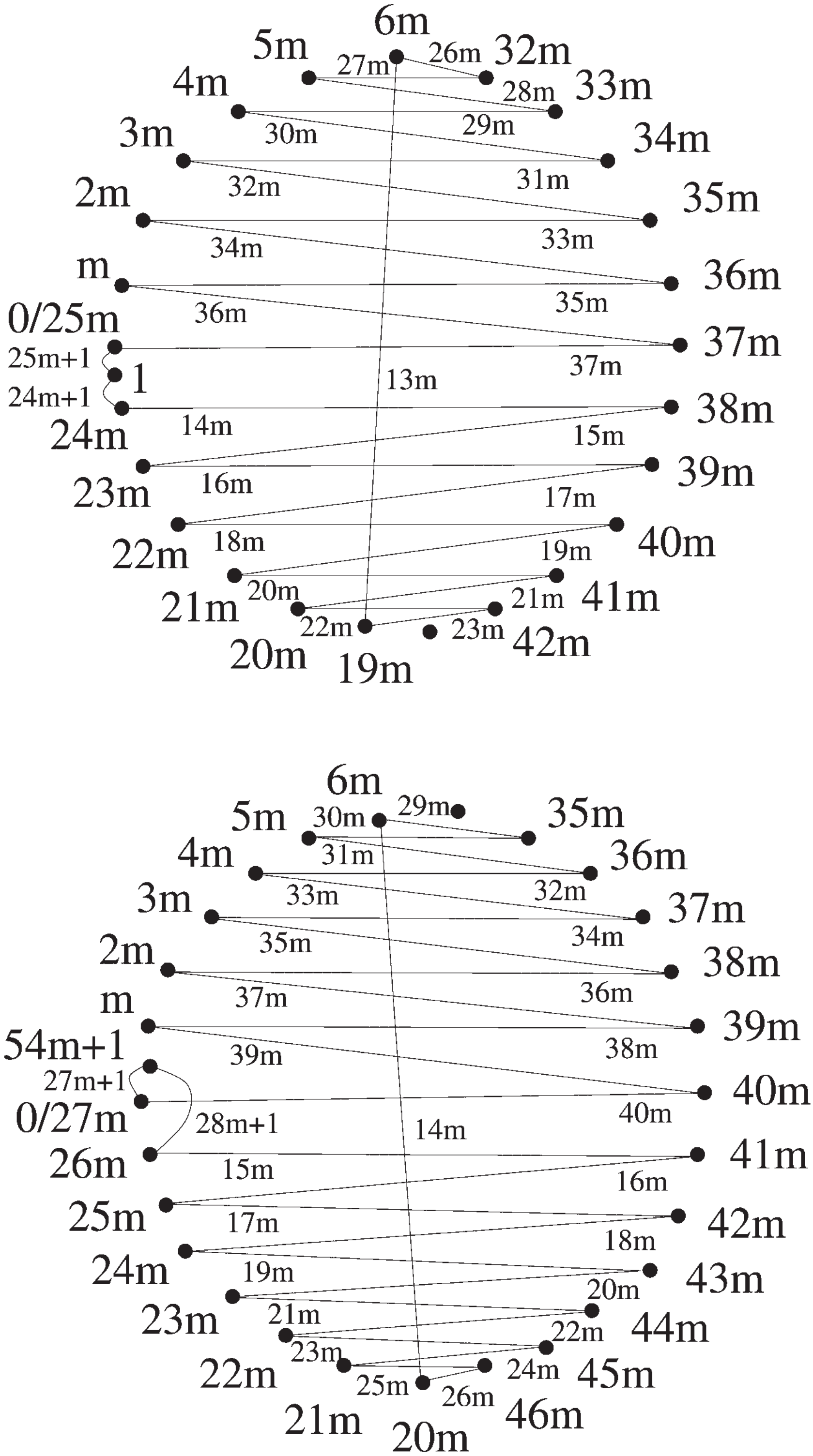

FIGURE 1. A cycle $C_{2,0}$ and a cycle $C_{3,0}$.

Definition 2.1. *Let n be a positive integer. A Rosa sequence of order n is a sequence $\{r_1, \dots, r_n\}$ of n integers such that*

$$\bigcup_{i=1}^{n} \{r_i, i + r_i\} = \{1, 2, \dots, 2n + 1, 2n + 2\} - \{n + 1, s\}$$

where $s = 2n + 1$ or $2n + 2$ according to whether $n \equiv 0, 3$ or $n \equiv 1, 2 \pmod 4$, respectively.

The following result is well known (see for example Ref. [1]).

Theorem 2.2. *A Rosa sequence of order n exists for every integer $n \geq 2$.*

A suitable Rosa sequence is utilized as follows. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_{(m-1)/2}$ denote the starter cycles of the cyclic $(K_{m\times k}, C_k)$-design, as constructed in Ref. [3] (see Theorem 3.2; actually, the present cycles have been renamed). More precisely, let us set $h = (k - 1)/2$, and define such cycles as $\Gamma_i = (\gamma_{i0}, \gamma_{i1}, \dots, \gamma_{i,k-1})$ with

$$\gamma_{i,\ell} = \begin{cases} \frac{\ell m}{2} & \text{if } \ell \neq k-1 \text{ is even;} \\ (h - \frac{\ell+1}{2})m - i & \text{if } \ell \text{ is odd;} \end{cases}$$
$$\gamma_{i,k-1} = r_i + hm - \frac{m+1}{2},$$

where $\{r_1, \dots, r_{(m-1)/2}\}$ is a Rosa sequence of order $(m - 1)/2$. We have

$$\partial\Gamma_1 \cup \cdots \cup \partial\Gamma_{(m-1)/2} = \Delta\Gamma_1 \cup \cdots \cup \Delta\Gamma_{(m-1)/2} = \mathbf{Z}_{mk} - m\mathbf{Z}_{mk}. \tag{5}$$

Now we define a k-cycle Γ_1^* by replacing certain vertices of Γ_1. We distinguish two kinds of vertex. If either $p^i \equiv 1 \pmod 4$ or $p^i = 3$, with $1 \leq i \leq a - 1$, then

$$\begin{array}{lll} (\mathbf{a}) & \gamma_{1,h+jp^i-1} \longrightarrow \gamma^*_{h+jp^i-1} & 1 \leq j \leq (p-1)/2, \\ (\mathbf{b}) & \gamma_{1,h-jp^i} \longrightarrow \gamma^*_{h-jp^i} & 1 \leq j \leq (p-1)/2, \end{array}$$

where

$$\gamma^*_{h+jp^i-1} = \begin{cases} \frac{h-jp^i}{2} m & \text{if } j+h \text{ is even;} \\ \frac{h+jp^i-1}{2} m - 1 & \text{if } j+h \text{ is odd;} \end{cases}$$
$$\gamma^*_{h-jp^i} = \begin{cases} \frac{h+3jp^i-2}{2} m - 2 & \text{if } j+h \text{ is even;} \\ \frac{h-3jp^i+1}{2} m + 1 & \text{if } j+h \text{ is odd.} \end{cases}$$

Otherwise, if $p^i \equiv 3 \pmod 4$ and $p^i \neq 3$, then

$$\begin{array}{lll} (\mathbf{a}') & \gamma_{1,h+jp^i} \longrightarrow \gamma^*_{h+jp^i} & 1 \leq j \leq (p-1)/2, \\ (\mathbf{b}') & \gamma_{1,h-jp^i-1} \longrightarrow \gamma^*_{h-jp^i-1} & 1 \leq j \leq (p-1)/2, \end{array}$$

where

$$\gamma^*_{h+jp^i} = \begin{cases} \frac{h+jp^i}{2}m - 1 & \text{if } j+h \text{ is even;} \\ \frac{h-jp^i-1}{2}m & \text{if } j+h \text{ is odd;} \end{cases}$$

$$\gamma^*_{h-jp^i-1} = \begin{cases} \frac{h-3jp^i-2}{2}m + 1 & \text{if } j+h \text{ is even;} \\ \frac{h+3jp^i+1}{2}m - 2 & \text{if } j+h \text{ is odd.} \end{cases}$$

Notice that the replacements $(\mathbf{a})$, $(\mathbf{a}')$ serve to decrease by 1 certain differences, so as to generate the remaining multiples of m. More precisely, if either $p^i \equiv 1 \pmod 4$ or $p^i = 3$, then the 4 differences (from Γ_1^*) related to $\gamma^*_{h+jp^i-1}$ are

$$\pm\{\gamma^*_{h+jp^i-1} - \gamma_{1,h+jp^i}, \gamma^*_{h+jp^i-1} - \gamma_{1,h+jp^i-2}\} = \pm\{(jp^i-1)m, jp^im\}.$$

Such numbers replace the 4 differences (from Γ_1) related to γ_{h+jp^i-1}, namely

$$\pm\{\gamma_{1,h+jp^i-1} - \gamma_{1,h+jp^i}, \gamma_{1,h+jp^i-1} - \gamma_{1,h+jp^i-2}\} = \pm\{(jp^i-1)m+1, jp^im+1\}.$$

In the case $p^i \equiv 3(\bmod 4), p^i \neq 3$, a similar analysis can show that the differences $\pm\{jp^im, (jp^i+1)m\}$ replace $\pm\{jp^im+1, (jp^i+1)m+1\}$. Moreover, it is not hard to see that Γ_1^* has all distinct vertices. Indeed, as the only replacements $(\mathbf{a})$, $(\mathbf{a}')$ produce some pairs of equal vertices, by the further replacements $(\mathbf{b})$, $(\mathbf{b}')$ one repeated vertex for each pair is changed into a new vertex, without altering the involved differences. It can be easily checked that every resulting vertex is generated by *interchanging* the two differences.

It is worth noting that the prescribed replacements are compatible with the differences arising from the Rosa sequence. Indeed, the largest difference to modify is smaller than every difference associated to the Rosa sequence, as the reader may check by solving an elementary inequality.

Evidently,

$$Y \subset \partial\Gamma_1 \text{ and } \partial\Gamma_1^* = (\partial\Gamma_1 - Y) \cup X. \tag{6}$$

Therefore, using (4), (5), and (6) we deduce that $\mathcal{F} \cup \{\Gamma_1^*, \Gamma_2, \ldots, \Gamma_{(m-1)/2}\}$ is the set of starter cycles of a cyclic (K_{mp^a}, C_{p^a})-design.

Now we manage the case $m = 3$. To this end, we first define the unique cycle $\check{\Gamma}_1 = (\check{\gamma}_{1,0}, \ldots, \check{\gamma}_{1,k-1})$ as

$$\check{\gamma}_{1,\ell} = \begin{cases} \frac{\ell m}{2} & \text{if } 0 \neq \ell \neq k-1 \text{ and } \ell \text{ is even;} \\ (h - \frac{\ell+1}{2})m - 1 & \text{if } \ell \neq k-2 \text{ and } \ell \text{ is odd;} \end{cases}$$

$$\check{\gamma}_{1,0} = -3\ ,\ \check{\gamma}_{1,k-2} = -5\ ,\ \check{\gamma}_{1,k-1} = \frac{k-3}{2} \cdot 3 - 4.$$

Subsequently we modify $\check{\Gamma}_1$ as in the general case, obtaining the cycle $\check{\Gamma}_1^*$ (say). The reader may check that the replacements $(\mathbf{a}), \ldots, (\mathbf{b}')$ are allowed in all cases except $k = 9$, and that all the required differences are generated.

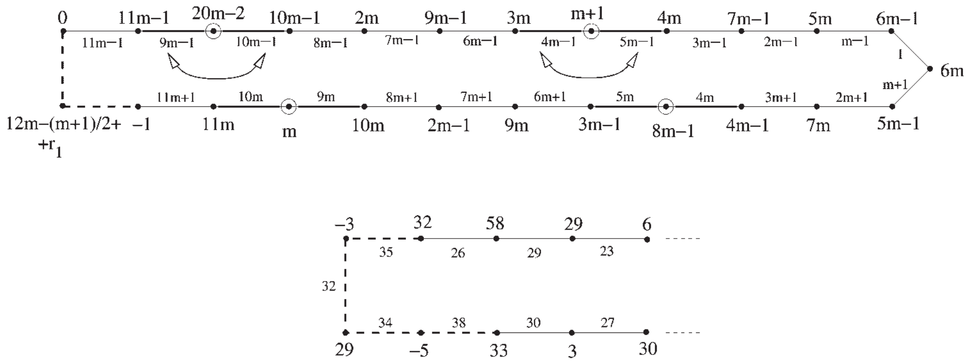

FIGURE 2. The cycles Γ_1^*, $\check{\Gamma}_1^*$ if $k = 25$.

Finally, if $k = 9$ we avail ourselves of the cyclic (K_{27}, C_9)-design, exhibited in Ref. [4], namely the one generated by the starter cycles

$$[0,6]_9,\ [0,3,25,9]_9,\ (0,1,26,3,22,4,21,8,20)$$

of type 9, 3, 1 respectively. □

In the upper side of Figure 2, we have represented Γ_1^* with $k = 25$. In the lower side, we have sketched the case $k = 25, m = 3$. In both cases, we have put in evidence some particular edges and vertices, according to the above constructions. In particular, the dashed edges are related to either $\gamma_{i,k-1}$ or $\check{\gamma}_{1,0}, \check{\gamma}_{1,k-2}, \check{\gamma}_{1,k-1}$.

3. THE CASE $k = 15$

The second part of Theorem 2.1 can be stated as follows.

Theorem 2.1. (Part II). *For any odd integer $m \geq 3$ there exists a cyclic (K_{15m}, C_{15})-design.*

Proof. In order to manage this case, we use the same techniques of the above section. As

$$\text{No. of cycles} = \frac{15m(15m-1)}{2 \cdot 15} = \frac{m-1}{2} 15m + 5m + m + m,$$

we will look for $(m-1)/2$ starter cycles of type 1, one of type 3 and two of type 15. The three cycles of types 3 and 15 will be used to generate the partial differences which are divisible by m, with some exception.

We postpone the case $m = 3$. Assuming that $m \geq 5$, let $\{r_1, \ldots, r_{(m-1)/2}\}$ be a Rosa sequence of order $(m-1)/2$. Due to the cited Theorem 3.2 in Ref. [3], the cycles $\Gamma_1 \Gamma_2, \ldots, \Gamma_{(m-1)/2}$ defined by

$$\Gamma_i = (0,\ 6m-i,\ m,\ 5m-i,\ 2m,\ 4m-i,\ 3m,\ 3m-i,$$
$$4m,\ 2m-i,\ 5m,\ m-i,\ 6m,\ -i,\ r_i + (13m-1)/2)$$

are the starter cycles of a $(K_{m\times 15}, C_{15})$-design. Evidently,

$$\partial\Gamma_1 \cup \cdots \cup \partial\Gamma_{(m-1)/2} = \Delta\Gamma_1 \cup \cdots \cup \Delta\Gamma_{(m-1)/2} = \mathbf{Z}_{15m} - m\mathbf{Z}_{15m}. \tag{7}$$

Let us consider the cycle Γ_1^* obtained from Γ_1 by replacing the vertex $6m-1$ with $-5m+1$ and the vertex $6m$ with $6m-1$:

$$\Gamma_1^* = (0,\ -\mathbf{5m}+1,\ m,\ 5m-1,\ 2m,\ 4m-1,\ 3m,\ 3m-1,$$
$$4m,\ 2m-1,\ 5m,\ m-1,\ \mathbf{6m-1},\ -1,\ r_1+(13m-1)/2).$$

We have

$$\Delta\Gamma_1^* = (\Delta\Gamma_1 - \pm\{5m+1, 6m+1\}) \cup \pm\{5m, 6m\}. \tag{8}$$

Furthermore, we introduce the following 15-cycles of types 15, 15, 3 respectively

$$A = [0, m]_{15},\ B = [0, 4m]_{15},\ C = [0, 2m, 14m, 4m+1, 13m, 5m]_{15}.$$

It is straightforward to check that

$$\partial A \cup \partial B \cup \partial C = \pm\{m, 2m, 3m, 4m, 5m+1, 6m+1, 7m\}. \tag{9}$$

Using (7), (8), (9) we can state that $\{A, B, C, \Gamma_1^*, \Gamma_2, \ldots, \Gamma_{(m-1)/2}\}$ is a set of starter cycles of a (K_{15m}, C_{15})-design.

Concerning the case $m = 3$, the 15-cycles (with vertices in $\mathbf{Z}_{45}$)

$$[0,3]_{15},\ [0,6]_{15},\ [0,32,3,24,6,15]_{15},$$
$$(0,1,42,4,39,24,36,8,34,12,37,6,40,3,43)$$

are easily seen to be the starter cycles (of types 15, 15, 3, 1 respectively) of a cyclic (K_{45}, C_{15})-design. Thus, the statement is true for $m = 3$ as well. □

4. CONCLUSION

Theorem 1.1, together with Theorem 1.2, enables us to state the following

Theorem 4.1. *There exists a cyclic (K_v, C_k)-design for every odd v, k such that $v \equiv k \pmod{2k}$, with the only definite exceptions: $(v,k) = (9,3)$; $v = k = 15$; $v = k = p^a$ with p prime and $a > 1$.*

As a consequence of the above theorem and Theorem 1.3, we can settle the existential question for all cyclic (K_v, C_k)-designs with k ranging over all prime powers.

Proposition 4.1. *If k is a prime power, then there exists a cyclic (K_v, C_k)-design for any admissible v with the only definite exceptions of $(v,k) = (9,3)$, and $v = k$ with k not a prime.*

Proof. The *admissible* values of v for which there exists a (K_v, C_k)-design with k a prime power are those satisfying the following conditions:

$$v \equiv 1 \text{ or } k (\text{mod } 2k) \text{ if } k \text{ is odd}; \quad v \equiv 1 \, (\text{mod } 2k) \text{ if } k \text{ is even}.$$

Then the result immediately follows from Theorem 4.1 and Theorem 1.3.

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