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Abstract: This paper presents a computer algorithm that assists us in our research on non-perfect colorings of plane crystallographic patterns.

Introduction

A **periodic** or **repeating pattern** in the plane is a design having the following property: There exists a finite region and two linearly independent translations such that the set of all images of the region when acted on by the group generated by these translations produces the original design. In addition, it is assumed that there is a translation vector of minimum length that maps the pattern onto itself. In addition to translations, a periodic pattern may be mapped onto itself by any of the other plane isometries: rotations, reflections or glide reflections. The **symmetry group** of the pattern is the set of all isometries which map the pattern onto itself. The classification of periodic patterns according to their symmetry groups is the two-dimensional counterpart of the system used by crystallographers to classify crystals. Hence, these groups are called two-dimensional or **plane crystallographic groups**.

In color symmetry, one of the problems that remain of interest today is the study and classification of colorings associated with plane crystallographic patterns. There are two types of such colorings. Given a plane crystallographic group G as the symmetry group of a pattern with the colors disregarded, the pattern is said to be **perfectly colored** if every element of G effects a permutation of the colors of the pattern. In certain instances when not all elements of G permute the colors of the pattern, we obtain a **non-perfectly colored** pattern.

Perfect colorings have been completely characterized in [12]. However it has just been recently that non-perfect colorings have been studied closely with the framework for their study provided for in the paper "On Imperfect Colorings of Symmetrical Patterns" by R. P. Felix and F. C. Gorospe. The method provided by Felix and Gorospe determines for a given plane crystallographic group G which is the symmetry group of an uncolored pattern, all colorings where a subgroup H of G permutes the colors and a subgroup K of G fixes the colors or is the symmetry group of the colored pattern.

With this theory of studying non-perfect colorings available, a classification of non-perfect colorings associated with plane crystallographic patterns may be done systematically.

In this paper, we develop a computer algorithm to facilitate our classification of non-perfect colorings of plane crystallographic patterns. There are altogether seventeen(17) plane crystallographic group types that arise from infinitely repeating designs and patterns in the plane. To study in a meaningful manner non-perfect colorings associated with each of these plane crystallographic group types, it is important to consider the plane crystallographic group G which is the symmetry group G of the uncolored pattern, the subgroup H of G consisting of elements which effect color permutations and the subgroup K of G consisting of elements which fix all colors or is the symmetry group of the colored pattern. Now, there are too many different subgroups of varying structures in the group G depending on the index of each subgroup in the group. To complete the classification of the non-perfect colorings would entail studying as many examples of colorings associated with each of the 17 plane crystallographic groups, which we take as the group G, as well as looking at the colorings obtained by specifying particular subgroups H, K in G. A computer program can surely make the enormous task of listing such colorings manageable, and will allow us to consider as many examples of colorings as possible.

In this paper, we give the computer algorithm that
(i) determines the colorings that arise from plane crystallographic groups for which the elements of a given subgroup H of the symmetry group G of the uncolored pattern permute the colors and the elements of a given subgroup K of G fix the colors; and
(ii) determines for the colorings obtained in (i) the subgroup H' of G consisting of all elements permuting the colors ($H \leq H' \leq G$) and the subgroup K' of G consisting of all elements fixing the colors ($K \leq K'$), which are essential in the classification of non-perfect colorings.

A Setting for Coloring Symmetrical Patterns

Let G be a plane crystallographic group or a subgroup of a plane crystallographic group. Consider a subset S of a fundamental region for G. The set $\{g(S) = g \in G\}$ is called the G-orbit of S. Our assumption is that the given pattern can be obtained as the G-orbit of some subset S of a fundamental domain for G. This G-orbit of S and G are in one-to-one correspondence under the rule $g(S) \longleftrightarrow g$ for each $g \in G$, so that each element of the G-orbit may be labeled by g. By assigning a color to each element of G, we assign a color to each $g(S)$. This assignment of colors is a **coloring** of the pattern. This results in a partition P of G where a set in P consists of elements assigned the same color, thus, a coloring may be treated as simply a partition of G. Suppose H is a subgroup of G. A partition P of G is said to be H**-invariant** if P goes to itself under multiplication on the left by $h \in H$.

The following example illustrates the coloring of a pattern we described above. The uncolored pattern in Figure 1a has symmetry group $G = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, the group of all isometries of the Euclidean plane which send the square to itself where a is a $90°$ counterclockwise rotation about the center of the square, and b is a reflection in the horizontal line through the center of the square. If S is the triangular region labeled "e" in Figure 1b, then for each $g \in G$, the triangular region $g(S)$ is labeled "g". If we partition G into the sets $\{a^3, b, a^3b\}$, $\{a^2, a^2b\}$ and $\{e, a, ab\}$, and assign the respective colors red, white and blue to these sets, we obtain the coloring in Figure 1c.

There are three groups that play a vital role in the analysis of a given colored pattern: G the symmetry group of the uncolored pattern, H the subgroup of G consisting of elements which effect color permutations, and K the symmetry group of the colored pattern or subgroup of elements of G which fix all colors.

We will assume that $[G : K]$ is finite and K is not the trivial group. It is in this sense that we will consider the coloring of the pattern as a symmetrical coloring. The groups G, H, K are such that $K \leq H \leq G$. If the group G permutes the colors of the pattern, that is, $H = G$, then the coloring is perfect. To analyze non-perfect colorings we will look at the case $[G : H] > 1$. In our study, we will give particular attention to the case where $[G : H] = 2$ or 3. Given a color, its stabilizer in G will lie between H and K. Since H acts on the set C of colors of the pattern, this action induces a homomorphism $f : H \to A(C)$, where $A(C)$ is the group of permutations of the set C of colors of the pattern. For $h \in H$, $f(h)$ is the permutation of the colors that h induces. An element h is in the kernel of f if and only if $f(h)$ is the identity permutation, that is, h fixes all the colors. Thus the kernel of f is K and the resulting group of color permutations $f(h)$ is isomorphic to H/K. Consequently, K is a normal subgroup of H.

If we consider the coloring as a decomposition of G resulting in the partition $P = \{P_i : i \in I\}$, then $H = \{h \in G : \text{for each } i \in I, hP_i = P_j \text{ for some } j \in I\}$ and $K = \{k \in G : kP_i = P_i \text{ for all } i \in I\}$.

A Method for Analyzing and Enumerating Colored Patterns

A coloring is perfect if and only if it is a coloring using left cosets of a subgroup S in G. By such a coloring we mean a decomposition of G into the left cosets of S in G, $G = \bigcup_{g \in G} gS$ where each left coset is given a unique color.

A perfect coloring is transitive and the resulting group of color permutations is isomorphic to $G/core\ S$. Consequently, perfect colorings having a subgroup K of G as the symmetry group of the corresponding colored pattern are the colorings using left cosets of a subgroup J of G such that $core\ J = K$.

To obtain more general colored patterns, we look at colorings using right cosets of a subgroup S in G given by $G = \underset{g\in G}{\cup} Sg$. Generally, these colorings are non-perfect unless S is a normal subgroup of G. The normalizer of S in G, $N_G(S)$ is the group of elements of G which permute the colors for such colorings.

Theorem 1 *Let G be a group and S a subgroup of G. Then $a \in G$ permutes the set $S\backslash G$ of right cosets of S in G by left multiplication if and only if $a \in N_G(S)$, the normalizer of S in G. The resulting action of $N_G(S)$ on $S\backslash G$ induces the homomorphism f on $N_G(S)$ where for each $a \in N_G(S)$, $f(a)$ is the permutation sending Sg to $aSg = Sag$. The image of f is isomorphic to $N_G(S)/S$.*

We mention in the previous section that given the symmetry group G of an uncolored pattern, the subgroup H of G which effects color permutations contains K as a normal subgroup of elements of H which fix the colors. Consequently, $H \leq N_G(K)$. Now, the above theorem tells us that $N_G(K)$ acts on the set $K\backslash G$ of right cosets of K in G by left multiplication. This means that H also acts on $K\backslash G$ suggesting that $K\backslash G$ plays an important role in the analysis of a colored pattern. In fact, we see in the next theorem that in a coloring or decomposition $G = \cup P_i$ if K fixes all the colors or $kP_i = P_i$ for all $k \in K$, then P_i is a union of right cosets of K in G.

Theorem 2 *Let G be a group, X a non-empty subset of G and K a subgroup of G. Then $kX = X$ for all k in K if and only if X is a union of right cosets of K in G.*

Through the subsequent theorems, we pave the way to formulate a systematic approach to the study of non-perfect colorings. The action of H on $K\backslash G$ is described and this action enables us to analyze a coloring better by looking at the H-orbits of $K\backslash G$ of which there are $[G : H]$. Moreover, we can see that the corresponding decomposition of $K\backslash G, K\backslash G = \overset{s}{\underset{i=1}{\cup}} B_i$ is determined by the G-invariant partition $\{B_i : i = 1, ...s\}$ where each B_i consists of right cosets of K in G whose union is a left coset of J in G and $s = [G : J]$.

Theorem 3 *Let G be a group and H, K subgroups of G such that $K \leq H \leq N_G(K)$. Then H acts on the set of right cosets of K in G by left* *multiplication. Moreover, this action results in a group of permutations of $K\backslash G$ which is isomorphic to H/K. An orbit of the action consists of right cosets of K in G whose union is a right coset of H in G. The number of orbits is $[G : H]$ each of size $[H : K]$ and for any orbit, the action of H on the orbit is equivalent to the action of H on H/K by left multiplication.*

Theorem 4 *Let G be a group and K a normal subgroup of G. Let G act on G/K by left multiplication. Then (i) If J is a subgroup of G containing K, then the left cosets of J in G determine a G-invariant partition $\{B_i : i = 1, ..., s\}$ of G/K where the set of left cosets of K in J form one block B ($B = B_i$ for some i) and the other blocks are the sets gB where $g \in G$. (ii) If $\{B_i : i = 1, ..., s\}$ is a G-invariant partition of G/K and K is in a block B, then the union of the left cosets of K in block B is a subgroup J of G containing K.*

Theorem 5 *Let G be a group and K a subgroup of G. Let $K\backslash G = \bigcup_{i=1}^{s} B_i$ be a decomposition of $K\backslash G$ into non-empty disjoint subsets B_i. For any $H, K \leq H \leq N_G(K)$, the following are equivalent. (i) $\{B_i : i = 1, ..., s\}$ is H-invariant under left multiplication by elements of H. (ii) For all $h \in H$, if $Ka \in B_i$ and $hKa \in B_j$, then $Kc \in B_i$ implies $hKc \in B_j$. Moreover, if the orbits of H are O_k, $k = 1, 2, ..., [G : H]$, and (i) and (ii) hold, then for each k, $\{B_i \cap O_k : 1, ..., s\}$ is an H-invariant partition of O_k.*

Thinking of the B_i as colors, statement (i) is equivalent to saying that H permutes the colors while statement (ii) says that if two right cosets are assigned the same color, then their images under $h \in H$ should have the same color. With this interpretation, it is clear why (i) and (ii) are equivalent.

Based on the above theorems(proofs of which are found in [4]) , we now give the method of Felix and Gorospe that determines all H- invariant partitions of $K\backslash G$ or colorings for which H permutes the colors:

Let G be a group and H, K subgroups of G with $K \leq H \leq N_G(K)$.

1. Get the orbits of H under its action on $K\backslash G$ by left multiplication.

2. Color each orbit using an H-invariant partition of H/K under the action of H on H/K by left multiplication. If the orbit is $\{Khg : h \in H\}$ where $g \in G$, then Khg is given the color given to Kh in the partition of H/K that is used. Any H-invariant partition of H/K may be used as long as the set of colors used for the orbit is disjoint from the set of colors used in the other orbits.

3. If a color to be used in an orbit O'has been used in another orbit O, then the assignment of colors in O' is completely determined by the assignment of colors in O, i.e., if $Ka \in O$ and $Ka' \in O'$ have the same color, then if $h \in H$, hKa and hKa' should have the same color.

Algorithm and Implementation

In this section, we give the algorithm we used to classify non-perfect colorings.

Initial condition: Given a plane crystallographic group G which is the symmetry group of an uncolored pattern

Input: Select H, K subgroups of G from menu ($K \leq H \leq N_G(K)$)

Output: All colorings with H permuting the colors and K fixing the colors of the corresponding colored pattern. Moreover, for each colored pattern, we have:

1. The elements of the group G belonging to the same right coset of K in G may be determined. Each right coset of K in G is assigned a unique number from $N = \{0, 1, ..., [G : K] - 1\}$ by the program. Every element of G is accorded the number assigned to the right coset of K in G to which it belongs. The program labels each triangular region corresponding to an element g of G with the number assigned to g. Consequently, the regions corresponding to the elements of G belonging to the same right coset of K in G are given the same number.

2. The subgroup H' permuting the set of colors of the corresponding colored pattern may be determined. Since $[G : H] = 2$ or 3 the program determines if $H' = G$(perfect coloring) or $H' = H$(non-perfect coloring).

3. The subgroup K' fixing the colors of the current colored pattern may be arrived at. The program labels the triangular region of the colored pattern corresponding to the element $g \in G$ that fixes the colors.

We describe the algorithm as follows relevant to 1 - 3 above:

Step 1: Compute right cosets of K in G, $KCos(i)$, $i = 0, 1, ..., [G : K] - 1$. Assign a color j, $j \in \{0, 1, ..., 15\}$ to each right coset of $KCos(i)$ in G.This coloring is based on the method of Felix and Gorospe mentioned in the previous section. There are at most 16 colors for this program, each color is given a code j.

Step 2: Determine for each $KCos(i)$, $i = 0, 1, ..., [G : K] - 1$ a right coset representative n_i.

Step 3: For every $g \in G$, determine the coset no. i indicating the right coset $KCos(i)$ to which g belongs. Given $g \in G$, compute $h = gn_i^{-1}$

for every $n_i (i = 0, 1, ..., [G : K] - 1)$ obtained in Step 2. If $h \in K$, then g has coset no. i.

Step 4: Fix the vertices $pt1$, $pt2$, $pt3$ of a triangular fundamental region S. For every $g \in G$, apply g to $pt1$, $pt2$, $pt3$ giving $ppt1$, $ppt2$, $ppt3$, vertices of $g(S)$. Draw image $g(S)$ and label it with the coset no. i of g obtained in Step 3.

Step 5: **a)** For every generator a of G, initialize color code array, $CC[j] = -1$; $j = 0, 1, ..., 15$.

b) Take a generating region given by $g'(S)$, $g' \in G'$, $G' \subseteq G$. Get the coset no. i of g' obtained in Step 3. Consider the color j, $j \in \{0, 1, ..., 15\}$ that is assigned to $KCos(i)$ in Step 1. Apply a to $g'(S)$ to get the transformed image $g''(S)$ where $g'' = ag' \in G$. Now look at coset no. k of g'' also obtained in Step 3 and determine the color j', $j' \in \{0, 1, ..., 15\}$ assigned to $KCos(k)$.

c) If $CC[j] = -1$ then let $CC[j] = j'$ and go to **b)**. If $CC[j] \neq -1$ check if $CC[j] \neq j'$. If yes write $H' = H$ and go to Step 6. If no, then proceed.

d) Determine if you have considered every generating region $g'(S)\ \forall g' \in G'$. If no, then go to **b)**. If yes, proceed.

e) Check if you have exhausted all generators of G. If yes, write $H' = G$. Otherwise go to **a)**.

Step 6: **a)** Consider an element g of G.

b) Take a generating region given by $g'(S)$, $g' \in G'$, $G' \subseteq G$. Apply g to $g'(S)$ getting a transformed image $g''(S)$ where $g'' = gg'$. In a similar process mentioned in Step 5 b), obtain respective colors j, $j' \in \{0, 1, ..., 15\}$ corresponding to the elements g' and g'' of G.

c) If $j \neq j'$, go to **d)**. If $j = j'$ then determine if you have considered all generating regions $g'(S)\ \forall g' \in G$. If yes, label the region corresponding to g with the no. obtained in Step 3 and go to **d)**. Otherwise, go to **b)**.

d) Check if you have tested every element of G. If no, go to **a)**. Otherwise, end routine.

The algorithm above was implemented in the C programming language. The program is designed to run on an IBM PC-based computer with a VGA monitor. It uses the following header files which contain fuction prototypes, data and constant definitions: *group.h*, *key.h*, *menu.h*, *mouse.h*, and the following C source files: color2.c, colorings.c, menu.c and *mouse.c* with auxillary data files: *tables.inc* and *colorings.dat*.

The main program file is *colorings.c* and the files *menu.c* and *mouse.c* are for the graphical interface support routines. The file *color2.c* is an auxillary routine to change the colors of the pattern currently shown in the screen. t*ables.inc* Refer to the color tables depending on specified subgroups H, K in G. *colorings.dat* Contain specifications regarding characteristics for the subgroups H and K in G. (e.g. index of H, K in G)

We describe briefly the computer representation of plane crystallographic groups as defined in header file *group.h*. For more information about details of the computer program see [2].

The header file *group.h* defines the elements of a group as a geometric transformation matrix of dimension 3 X 3 in homogeneous coordinates, viz.

typedef double mat3x3[3][3];

A *rotation* or *reflection/glide reflection* specified in this form may be interpreted as the respective matrices.

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ x & y & 1 \end{bmatrix} \qquad \begin{bmatrix} cos(2\theta) & -sin(2\theta) & 0 \\ sin(2\theta) & cos(2\theta) & 0 \\ x & y & 1 \end{bmatrix}$$

The upper 2 X 2 submatrix in the upper left correspond to a rotation or reflection of the point group and the lower (x, y) submatrix represent the translation component in the x, y axis.

The identity isometry is represented by a matrix with 1.0 in the main diagonal and zero elsewhere.

A *point* in a plane is represented as a row vector

typedef double point[3]; /* [x y 1.0] */

where the third element is always 1.0.

A *group* is represented as an aggregate data type whose structure is shown below.

```
struc grouprec
     mat3x3   PG[16];            /* point group elements */
     mat3x3   gen1,gen2;         /* point group generators */
     int      order1, order2;    /* order of the generators */
     int      nGen;              /* number of generators */
     int      nPg;               /* number of elements in the point group */
     double   Ux, Uy,            /* translation vectors */
              Vx, Vy;
     double   invUx, invUy,      /* inverse of the translation vectors */
              invVx, invVy
```

The point group PG is the finite group consisting of rotations and mirror reflections having at least the origin of the plane unmoved.

Now, we give the following example that illustrates the output of the computer program that implements our algorithm. We choose the largest plane crystallographic group $p6m$ in terms of its number of subgroups. It takes .54 seconds to generate approximately 240 colored elements of a crystallographic group $p6m$(1coloring).

Input: G : plane crystallographic group $p6m$ with generators a,60° counterclockwise rotation about the indicated point p, b, a reflection in a horizontal line through p, and x, y translations whose vectors are indicated. (See Figure 2)

H : plane crystallographic group $p31m$ with generators a^2, b, x, y

K :plane crystallographic group $p3$ with generators a^2, x^3, xy

Output: 28 colorings with H permuting the colors and K fixing the colors of the corresponding colored pattern. (We give in Figure 2, 4 out of the 28 colorings)

The computer program also gives the following information for these 4 colorings:

Coloring	**H′**	**K′**
1	H	$< a^2, x, y > \cong p3$
2	G	$< a^2, x^3, xy > \cong p3$
3	H	$< a^2, b, x^3, xy > \cong p3m1$
4	H	$< a^2, x^3, xy > \cong p3$

In addition, we give in the table below the number of non-perfect colorings corresponding to a few other subgroups H, K of the plane crystallographic group $p6m$. These have also been obtained by the computer program. Notice that non-perfect colorings corresponding to * include the colorings 1, 3 and 4 in the table above.

Subgroup H of G : Generators	$[G:H]$	Subgroup K of H : Generators	$[H:K]$	Non-Perfect Colorings
$^*p31m : a^2, b, x, y$	2	$p3 : a^2, x, y$	2	2
		$p3m1 : a^2, b, x^3, xy$	3	2
		$^*p3 : a^2, x^3, xy$	6	8
$cmm : a^3, b, x, y$	3	$pgg : a^3, ya^3b, x, y^2$	2	26
		$pmg : ya^3, a^3b, x, y^2$	2	26
		$pmm : a^3, b, x, y^2$	2	26
		$cm : b, x, y$	2	25
		$cm : a^3b, x, y$	2	25
		$pg : yb, x, y^2$	4	236
		$pg : ya^3b, x, y^2$	4	236
		$pm : b, x, y^2$	4	236
		$pm : a^3b, x, y^2$	4	236
		$p2 : a^3, x, y^2$	4	236
		$p2 : ya^3, x, y^2$	4	236

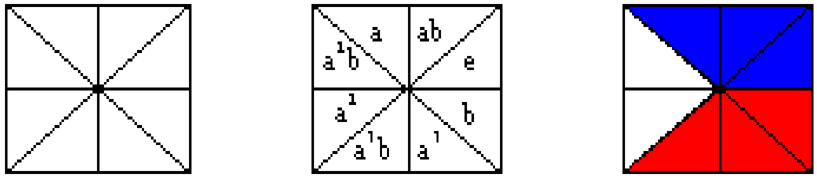
Results and Conclusion

We have used the computer as a convenient tool to determine for an uncolored pattern with symmetry group G and subgroups H, K of G with $K \leq H \leq N_G(K)$, colorings where the elements of H permute the colors and the elements of K fix the colors of the corresponding colored pattern. The computer algorithm we have written to determine H' and K' consisting of elements of G permuting and fixing the colors respectively, we have applied to a specific plane crystallographic group G. Consequently, this has helped us solve the problem of determining H' and K' for the cases where $[G : H] = 2$ and 3. The results of which are presented in [3]. For our future studies, we would like to develop a general formula for H' and K' for the more general cases where $[G : H] > 3$. It would still take some time for non-perfectly colored patterns to be enumerated and classified completely. However, with the direction provided here using computers, the task of doing so seems manageable.

Finally, it has to be mentioned that the computer program we have developed here can also be a useful tool in teaching abstract group theory. By merely working with perfect/non-perfect colorings generated by the program, students can identify symmetry groups, subgroups and their cosets, determine which ones are normal subgroups and find the corresponding permutation representation of the group using color symmetry. They can also gain valuable insight into conjugacy, group extensions and other algebra concepts.

References

[1] E. P. Adorio, Computer Algorithms for Perfectly Coloring Plane Crystallographic Patterns, MS Appl. Math./Comp. Sci. Thesis, U.P.Diliman(1994).
[2] M. L. A. De Las Peñas, On Non-Perfect Colorings of Plane Crystallographic Patterns with Hexagonal Symmetry, Ph.D. Dissertation, U.P Diliman(1996)
[3] M. L. A. De Las Peñas and R. P. Felix, Colorings of patterns where the isometries permuting the colors form a subgroup of index at most 3 in the symmetry group of the patterns, Matimyas Matematika Vol. 20, No. 2, 1-14(1997).
[4] R. P. Felix and F. Gorospe, On imperfect colorings of symmetrical patterns, Symmetry: Culture and Science Vol. 7, No. 1, 57-58(1996).
[5] T. M. Jarso, Computer Tessellations of the Euclidean Plane Using the Seventeen Plane Crystallographic Groups, MS Appl. Math./Comp. Sci. Thesis, U.P. Diliman(1994).
[6] T. Rapanut, Subgroups, Conjugate Subgroups and N-Color Groups of the Seventeen Plane Crystallographic Groups, Ph.D. Dissertation, U.P. Diliman(1988).
[7] R. Roth, Color symmetry and group theory. Discrete Mathematics 38, 273-296(1982).
[8] R. Roth, Coloring p4m with two, four, and six colors, Acta Crystallogr. A41, 484-490(1985).
[9] D. Schattschneider, The Plane Symmetry Groups: Their Recognition and Notation, The American Mathematical Monthly, Vol. 85, No. 6, June - July 1978.
[10] R. L. E. Schwarzenberger, Colour symmetry, Bull. Lond. Math. Soc.16, 209-240(1984).
[11] M. Senechal, Color symmetry, Comp. and Maths. with Appls. Vol. 16, No. 5-8, 545-553(1988).
[12]B. L. Van der Waerden and J. J. Burckhardt, Farbgruppen, Z.Kristallogr. 115, 231-234(1961).



(a) (b) (c)

Figure 1

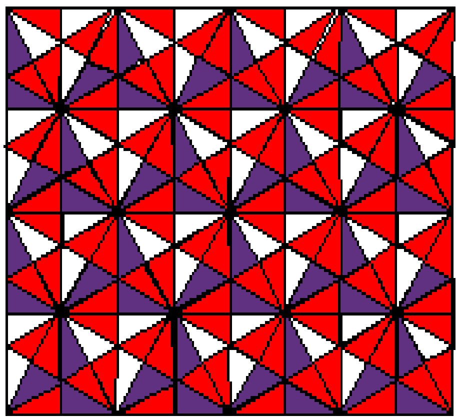
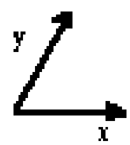


Figure 2 a



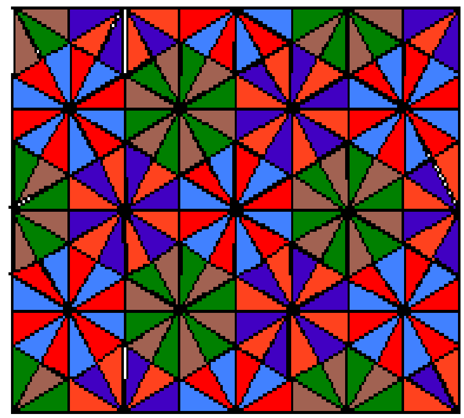


Figure 2 b

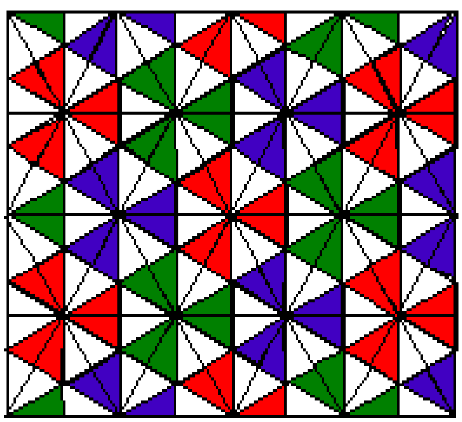


Figure 2 c

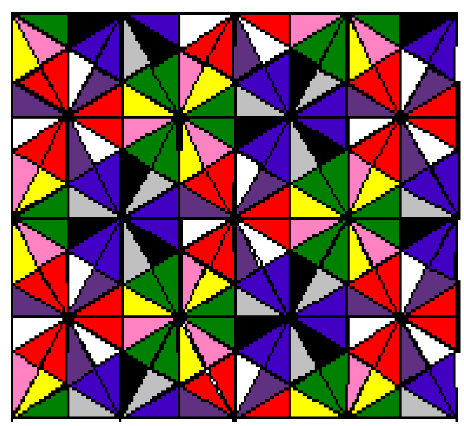


Figure 2 d