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Existence and stability analysis of solutions for a new kind of boundary value problems of nonlinear fractional differential equations*

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Abstract. This research work is dedicated to an investigation for a new kind of boundary value problem of nonlinear fractional differential equation supplemented with general boundary condition. A full analysis of existence and uniqueness of positive solutions is respectively proved by Leray–Schauder nonlinear alternative theorem and Boyd–Wong's contraction principles. Furthermore, we prove the Hyers–Ulam (HU) stability and Hyers–Ulam–Rassias (HUR) stability of solutions. An example illustrating the validity of the existence result is also discussed.

Keywords: fixed point theorem, fractional differential equations, existence, uniqueness, stability.

1 Introduction

Fractional calculus (FC) has a history of more than 300 years, there are some applications of FC within various fields of mathematics itself. During the last few decades, FC has obtained vigorous development in the applied sciences and gained considerable popularity. Compared with classical integer-order models, fractional derivatives and integrals are more suitable to describe the memory and hereditary properties of various materials, fractional derivatives are more advantageous in simulating mechanical and electrical properties of real materials and describing rheological properties of rocks and many other fields. Based on the description of their properties in terms of fractional derivatives, fractional differential equations (FDEs) are generated naturally, and how to solve these equations is also very necessary. For example, some new models that involve FDEs have been applied successfully, e.g., in mechanics (theory of viscoelasticity and viscoplasticity [6, 24]), (bio-)chemistry (modelling of polymers and proteins [11]), electrical engineering (transmission of ultrasound waves [3, 28]), medicine (modelling of

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human tissue under mechanical load [31])... Accordingly, more and more researchers and scholars devote themselves to the study of various problem of FDEs. In particular, study of boundary value problems for nonlinear FDEs is particularly concerned among these problems.

The aim of this paper is to investigate the following fractional differential equation:

$$D_{0+}^{\alpha}\vartheta(x) + g\bigl(x, \vartheta(x), D_{0+}^{\beta}\vartheta(x), D_{0+}^{\gamma}\vartheta(x)\bigr) = 0, \quad 0 < x < 1, \tag{1}$$

subjected to boundary condition

$$\vartheta(0) = 0, \qquad \lambda_1 D_{0+}^{\beta}\vartheta(1) + \lambda_2 D_{0+}^{\gamma}\vartheta(1) = \sum_{i=1}^{m} b_i \int_{I_i} h_i(s)\vartheta(s)\,\mathrm{d}s, \tag{2}$$

where $1 < \alpha \leqslant 2$, $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha$. $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$, $b_i \geqslant 0$ $(i = 1, 2, \dots, m)$, $h_i \in C(I_i) \cap L^1(I_i)$ $(i = 1, 2, \dots, m)$ is nonnegative, $I_i \subset [0, 1]$ is measurable, $m \geqslant 1$ is an integer. g is singular at $x = 0$. D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α defined by

$$D_{0+}^{\alpha}\vartheta(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} \int_0^x (x-s)^{n-\alpha-1}\vartheta(s)\,\mathrm{d}s, \quad n = [\alpha] + 1,$$

where Γ denotes the Gamma function, and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$; see [14, 25].

When $\lambda_2 = 0$, $0 \leqslant \beta \leqslant 1$, there has been a great deal of literature on the fractional differential equation of such boundary conditions; see [2, 4, 9, 15, 16, 30]. As for $\beta = 0$, for example, in [2],

$$\begin{aligned} &D^{\alpha}u(t) + f\bigl(t, u(t), D^{\mu}u(t)\bigr) = 0, \quad \text{a.e. } 0 < t < 1, \\ &u(0) = u(1) = 0, \end{aligned}$$

where $1 < \alpha < 2$, $0 < \mu \leqslant \alpha - 1$, f satisfies the Carathéodory conditions on $[0, 1] \times \mathfrak{B}$, $\mathfrak{B} = (0, \infty) \times \mathbb{R}$, $f(t, x, y)$ is positive and singular at $x = 0$. Based on regularization and sequential techniques, the existence of a positive solution was obtained.

By Schauder fixed point theorem and the Banach contraction principle, Rehman et al. [26] investigated existence and uniqueness of solutions for a class of nonlinear multipoint boundary value problems for fractional differential equation

$$\begin{aligned} &D_t^{\alpha}y(t) = f\bigl(t, y(t), D_t^{\beta}y(t)\bigr), \\ &y(0) = 0, \qquad D_t^{\beta}y(1) - \sum_{i=1}^{m-2} \zeta_i D_t^{\beta}y(\xi_i) = y_0, \end{aligned}$$

where $1 < \alpha \leqslant 2$, $0 < \beta < 1$, $0 < \xi_i < 1$ $(i = 1, 2, \dots, m-2)$, $\zeta_i \geqslant 0$, $\gamma = \sum_{i=1}^{m-2} \zeta_i \xi_i^{\alpha-\beta-1} < 1$.

In [15], the author studied the nonlinear fractional differential equation

$$D_{0+}^{\alpha} u(t) = f\bigl(t, u(t), u'(t)\bigr), \quad \text{a.e. } t \in (0,1),$$

subjected with the boundary conditions $u(0) = u(1) = 0$. Under Carathéodory conditions, using the Leray–Schauder continuation principle, the existence of at least one solution was obtained.

KyuNam et al. [16] discussed the existence and uniqueness of solutions for a class of integral boundary value problems of nonlinear multiterm fractional differential equation

$$D_{0+}^{\alpha} y(x) = f\bigl(t, y(x), D_{0+}^{\beta_1} y(x), \dots, D_{0+}^{\beta_n} y(x)\bigr), \quad t \in (0,1),$$
$$y(0) = 0, \qquad y(1) = \int_0^1 g\bigl(s, y(s)\bigr)\,\mathrm{d}s,$$

where $1 < \alpha < 2$, $0 < \beta_1 < \dots < \beta_n < 1$, $\alpha - \beta_n \geqslant 1$, $f : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$, and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous. The existence results are established by the Banach fixed point theorem, and approximate solutions are determined by the Daftardar-Gejji and Jafari iterative method (DJIM) and the Adomian decomposition method (ADM).

In this article, we will deal with singular nonlinear fractional differential equation with a new boundary condition, which is a generalization of many previous researches. To the best of our knowledge, when $\lambda_2 \neq 0$, neither $\lambda_1 = 0$ nor $\lambda_1 \neq 0$ was studied like this type of boundary condition. Furthermore, the nonlinear term contained not only lower derivative of order β, but also another lower derivative of order γ. In comparison with the above literature, our results about the difference include both $\alpha - \beta \geqslant 1$ and $0 < \alpha - \gamma < 1$, this has never been seen before.

The stability of differential equations has grown to be one of the considerable areas in the field of mathematical analysis, and we find many different types of stability, such as exponential [7,17,23], Mittag-Leffler [20,27], Hyers–Ulam (HU) stability and other types of stability [13,18,19,21,22]. Among these kinds of stability, Hyers–Ulam stability and its various types provide a bridge between the exact and numerical solutions, so researchers devoted their work to the study of different kinds of HU stability for nonlinear fractional differential equation; see [5, 10, 12, 29].

The paper is organized as follows. In Section 2, we will present some useful lemmas and give some valuable preliminary results. In Section 3, we prove the existence and uniqueness of positive solution to problem (1), (2) by using Leray–Schauder nonlinear alternative theorem and Boyd–Wong's contraction principles. In Section 4, we investigate various kinds of HU stability of solutions.

2 Auxiliary results

To simplify our statements, we introduce the following spaces:

- $C[0,1]$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the norm defined by $\|z\|_\infty = \max\{|z(x)|,\, x \in [0,1]\}$.

- $C_{\alpha-\gamma}[0,1] = \{z : (0,1] \to \mathbb{R} \mid x^{1+\gamma-\alpha} z \in C[0,1]\}$ endowed the norm

$$\|z\|_* = \max_{x\in[0,1]} |x^{1+\gamma-\alpha} z(x)|.$$

- $L^p_{\alpha-\gamma}[0,1] = \{z : (0,1] \to \mathbb{R} \mid x^{1+\gamma-\alpha} z \in L^p[0,1]\}$ endowed the norm

$$\|z\|_{L^p_{\alpha-\gamma}} = \left(\int_0^1 \left|x^{1+\gamma-\alpha} z(x)\right|^p \mathrm{d}x \right)^{1/p},$$

  where $0 < 1/p < \alpha - \gamma$.

First, we introduce some fundamental facts of the fractional calculus theory, which are used in this paper; see [14, 25].

Property 1. *Let $\alpha, \beta > 0$.*

(i) *If $f \in L(0,1)$, then $I^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\alpha+\beta}_{0+} f(t)$, $D^{\alpha}_{0+} I^{\alpha}_{0+} f(t) = f(t)$;*
(ii) *If $n-1 < \alpha < n$, $f(x) \in L_1(a,b)$ has a summable derivative $D^{\alpha}_{a+} f$, then*

$$I^{\alpha}_{a+} D^{\alpha}_{a+} f = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} f^{(n-k-1)}_{n-\alpha}(a),$$

where $f_{n-\alpha}(x) = (I^{n-\alpha}_{a+} f)(x)$;
(iii) *$I^{\alpha}_{0+} D^{\alpha}_{0+} f(t) = f(t)$, $0 < \alpha < 1$, where $f \in C[0,1]$, $D^{\alpha}_{0+} f \in C(0,1) \cap L(0,1)$;*
(iv) *$D^{\gamma}_{0+} u(t) = D^1 D^{\gamma-1}_{0+} u(t)$ ($\gamma > 1$), where $D^1 = \mathrm{d}/\mathrm{d}t$, provided that $D^{\gamma}_{0+} u(t)$ exists;*
(v) *Let $\alpha > 0$. Assume that $\{f_k\}_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $[a,b]$ and that $D^{\alpha}_{a+} f_k$ exists for every k. Moreover, assume that $\{D^{\alpha}_{a+} f_k\}_{k=1}^{\infty}$ converges uniformly on $[a+\varepsilon, b]$ for every $\varepsilon > 0$. Then for every $x \in (a,b]$, we have*

$$\Bigl(\lim_{k\to\infty} D^{\alpha}_{a+} f_k\Bigr)(x) = \Bigl(D^{\alpha}_{a+} \lim_{k\to\infty} f_k\Bigr)(x).$$

Next, for the sake of readers' convenience, we present some necessary lemmas, which will be used in the main results that follow.

Lemma 1. *Suppose $\phi \in L^p_{\alpha-\gamma}[0,1]$ and $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2$,*

$$\varpi := \frac{\lambda_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} + \frac{\lambda_2 \Gamma(\alpha)}{\Gamma(\alpha-\gamma)} - \sum_{i=1}^{m} b_i \int_{I_i} h_i(s) s^{\alpha-1} \,\mathrm{d}s \neq 0.$$

Then the linear fractional boundary value problem

$$\begin{aligned} &D^{\alpha}_{0+} \vartheta(x) + \phi(x) = 0, \quad 0 < x < 1, \\ &\vartheta(0) = 0, \qquad \lambda_1 D^{\beta}_{0+} \vartheta(1) + \lambda_2 D^{\gamma}_{0+} \vartheta(1) = \sum_{i=1}^{m} b_i \int_{I_i} h_i(s) \vartheta(s) \,\mathrm{d}s \end{aligned} \tag{3}$$

has a unique solution given by $\vartheta(x) = \int_0^1 G(x,s)\phi(s)\,\mathrm{d}s$*, where*

$$G(x,s) = K(x,s) + \frac{x^{\alpha-1}}{\varpi}\sum_{i=1}^{m} b_i \int\limits_{I_i} K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau \tag{4}$$

and

$$K(x,s) = \begin{cases} x^{\alpha-1}\kappa(s) - \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant x \leqslant 1, \\ x^{\alpha-1}\kappa(s), & 0 \leqslant x \leqslant s < 1, \end{cases}$$

$$K_0(x,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-1}[\lambda_1(1-s)^{\alpha-\beta-1} + \lambda_2(1-s)^{\alpha-\gamma-1}] - (x-s)^{\alpha-1}, \\ \quad 0 \leqslant s \leqslant x \leqslant 1, \\ x^{\alpha-1}[\lambda_1(1-s)^{\alpha-\beta-1} + \lambda_2(1-s)^{\alpha-\gamma-1}], \\ \quad 0 \leqslant x \leqslant s < 1, \end{cases}$$

in which

$$\kappa(s) = \lambda_1\rho_2(1-s)^{\alpha-\beta-1} + \lambda_2\rho_1(1-s)^{\alpha-\gamma-1},$$

and

$$\rho_1 = \frac{1}{\Gamma(\alpha)} - \frac{\lambda_1}{\varpi}\left(\frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha-\gamma)}\right),$$
$$\rho_2 = \frac{1}{\Gamma(\alpha)} + \frac{\lambda_2}{\varpi}\left(\frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha-\gamma)}\right).$$

Proof. It is very well known that the equation $D_{0+}^{\alpha}\vartheta(x) + \phi(x) = 0$ is equivalent to the following integral equation:

$$\vartheta(x) = -I_{0+}^{\alpha}\phi(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2}.$$

The boundary condition $\vartheta(0) = 0$ implies that $c_2 = 0$. Using the property of Riemann–Liouville fractional derivative, we know

$$D_{0+}^{\beta}\vartheta(x) = -I_{0+}^{\alpha-\beta}\phi(x) + c_1\frac{\Gamma(\alpha)x^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)},$$
$$D_{0+}^{\gamma}\vartheta(x) = -I_{0+}^{\alpha-\gamma}\phi(x) + c_1\frac{\Gamma(\alpha)x^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}.$$

Combining the boundary condition in (3), it follows that

$$c_1 = \frac{1}{\varpi}\left[\lambda_1 I_{0+}^{\alpha-\beta}\phi(1) + \lambda_2 I_{0+}^{\alpha-\gamma}\phi(1) - \sum_{i=1}^{m} b_i \int\limits_{I_i} h_i(s) I_{0+}^{\alpha}\phi(s)\,\mathrm{d}s\right].$$

Therefore the unique solution of problem (3) is given by

$$
\begin{aligned}
\vartheta(x) = {} & -I_{0+}^{\alpha}\phi(x) + \int_0^1 \bigl[\lambda_1\rho_2 x^{\alpha-1}(1-s)^{\alpha-\beta-1} + \lambda_2\rho_1 x^{\alpha-1}(1-s)^{\alpha-\gamma-1}\bigr]\phi(s)\,\mathrm{d}s \\
& + \frac{\lambda_1 x^{\alpha-1}}{\varpi} I_{0+}^{\alpha-\beta}\phi(1) - \lambda_1\rho_2 x^{\alpha-1}\int_0^1 (1-s)^{\alpha-\beta-1}\phi(s)\,\mathrm{d}s \\
& + \frac{\lambda_2 x^{\alpha-1}}{\varpi} I_{0+}^{\alpha-\gamma}\phi(1) - \lambda_2\rho_1 x^{\alpha-1}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi(s)\,\mathrm{d}s \\
& - \frac{x^{\alpha-1}}{\varpi}\sum_{i=1}^m b_i \int_{I_i} h_i(s) I_{0+}^{\alpha}\phi(s)\,\mathrm{d}s \\
= {} & \int_0^1 K(x,s)\phi(s)\,\mathrm{d}s + \frac{\lambda_1 x^{\alpha-1}}{\varpi\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\phi(s)\,\mathrm{d}s \sum_{i=1}^m b_i \int_{I_i} h_i(s)s^{\alpha-1}\,\mathrm{d}s \\
& + \frac{\lambda_2 x^{\alpha-1}}{\varpi\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}\phi(s)\,\mathrm{d}s \sum_{i=1}^m b_i \int_{I_i} h_i(s)s^{\alpha-1}\,\mathrm{d}s \\
& - \frac{x^{\alpha-1}}{\varpi}\sum_{i=1}^m b_i \int_{I_i} h_i(s) I_{0+}^{\alpha}\phi(s)\,\mathrm{d}s \\
= {} & \int_0^1 K(x,s)\phi(s)\,\mathrm{d}s + \frac{x^{\alpha-1}}{\varpi}\int_0^1 \Biggl(\sum_{i=1}^m b_i \int_{I_i} K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau\Biggr)\phi(s)\,\mathrm{d}s. \qquad \square
\end{aligned}
$$

Lemma 2. *Let G be the Green function related to problem* (3)*, which is given by expression* (4)*. Then for $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2$, $\varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$, G, K, K_0 have the following properties*:

(i) *G, K, K_0 are nonnegative and continuous on* $[0,1]\times[0,1)$;

(ii) *$K(0,s) = K_0(0,s) = 0$ for all $s \in [0,1)$, $K(x,0) = K_0(x,0) = 0$ for all $x \in [0,1]$, and*

$$K(x,s), K_0(x,s) > 0 \quad \forall x \in (0,1],\ s \in (0,1);$$

(iii) *For $(x,s) \in [0,1]\times[0,1)$,*

$$\frac{\beta}{\Gamma(\alpha)}x^{\alpha-1}s(1-s)^{\alpha-\beta-1} \leqslant K(x,s), K_0(x,s) \leqslant \frac{x^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-\gamma-1};$$

(iv) *Let $\mathfrak{M} = 1 + (1/\varpi)\sum_{i=1}^m b_i \int_{I_i} h_i(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau$, then for $(x,s) \in [0,1]\times[0,1)$,*

$$\frac{\mathfrak{M}\beta}{\Gamma(\alpha)}x^{\alpha-1}s(1-s)^{\alpha-\beta-1} \leqslant G(x,s) \leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)}x^{\alpha-1}(1-s)^{\alpha-\gamma-1}.$$

Proof. From the condition $\varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$ we easily get

$$\rho_1 = \frac{1}{\Gamma(\alpha)} - \frac{\lambda_1}{\varpi}\left(\frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha-\gamma)}\right) > 0.$$

In addition, we also have a relation $\lambda_2\rho_1 + \lambda_1\rho_2 = 1/\Gamma(\alpha)$.

(i) For $0 \leqslant x \leqslant s < 1$, $K(x,s) \geqslant 0$ is obvious. For $0 \leqslant s \leqslant x \leqslant 1$,

$$\begin{aligned} K(x,s) &\geqslant \frac{x^{\alpha-1}}{\Gamma(\alpha)}\big((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\big) \\ &\quad + \lambda_2\rho_1 x^{\alpha-1}\big((1-s)^{\alpha-\gamma-1} - (1-s)^{\alpha-\beta-1}\big) \\ &\geqslant 0. \end{aligned}$$

In the same way, we get K_0 is nonnegative, then we have $G \geqslant 0$. It is obvious that G, K, K_0 are continuous on $[0,1]\times[0,1)$.

(ii) The conclusion is obvious, we omit it.

(iii) First, we introduce an inequality. For $\lambda, \mu \in (0,\infty)$ and $a, x \in [0,1]$, we have

$$\min\left\{1, \frac{\mu}{\lambda}\right\}\big(1 - ax^{\lambda}\big) \leqslant 1 - ax^{\mu} \leqslant \max\left\{1, \frac{\mu}{\lambda}\right\}\big(1 - ax^{\lambda}\big). \tag{5}$$

When $s \leqslant x$, using inequality (5), we obtain

$$\begin{aligned} K(x,s) &\geqslant \lambda_1\rho_2 x^{\alpha-1}\big((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\big) \\ &\quad + \lambda_2\rho_1 x^{\alpha-1}\big((1-s)^{\alpha-\gamma-1} - (1-s)^{\alpha-1}\big) \\ &\geqslant \beta\lambda_1\rho_2 x^{\alpha-1}(1-s)^{\alpha-\beta-1}s + \min\{\gamma,1\}\lambda_2\rho_1 x^{\alpha-1}(1-s)^{\alpha-\gamma-1}s \\ &\geqslant \beta\lambda_1\rho_2 x^{\alpha-1}s(1-s)^{\alpha-\beta-1} + \beta\lambda_2\rho_1 x^{\alpha-1}s(1-s)^{\alpha-\beta-1} \\ &= \frac{\beta}{\Gamma(\alpha)}x^{\alpha-1}s(1-s)^{\alpha-\beta-1}. \end{aligned}$$

When $x \leqslant s$,

$$\begin{aligned} K(x,s) &\geqslant \beta\lambda_1\rho_2 x^{\alpha-1}s(1-s)^{\alpha-\beta-1} + \beta\lambda_2\rho_1 x^{\alpha-1}s(1-s)^{\alpha-\beta-1} \\ &= \frac{\beta}{\Gamma(\alpha)}x^{\alpha-1}s(1-s)^{\alpha-\beta-1}. \end{aligned}$$

In addition,

$$\begin{aligned} K(x,s) &\leqslant \lambda_1\rho_2 x^{\alpha-1}(1-s)^{\alpha-\gamma-1} + \lambda_2\rho_1 x^{\alpha-1}(1-s)^{\alpha-\gamma-1} \\ &= \frac{1}{\Gamma(\alpha)}x^{\alpha-1}(1-s)^{\alpha-\gamma-1} \quad \forall (x,s) \in [0,1]\times[0,1). \end{aligned}$$

Again, we can get the property of K_0, that is,

$$\beta x^{\alpha-1}s(1-s)^{\alpha-\beta-1} \leqslant \Gamma(\alpha)K_0(x,s) \leqslant x^{\alpha-1}(1-s)^{\alpha-\gamma-1}.$$

Therefore,

$$\frac{\beta s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\int\limits_{I_i}\tau^{\alpha-1}h_i(\tau)\,\mathrm{d}\tau$$
$$\leqslant\int\limits_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau\leqslant\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}\int\limits_{I_i}\tau^{\alpha-1}h_i(\tau)\,\mathrm{d}\tau.$$

(iv) By substituting the two inequalities (iii) into formula (4) we naturally come to the conclusion. □

Consider the problem $\vartheta=\mathbf{G}\vartheta$, where operator $\mathbf{G}$ is defined by

$$\begin{aligned}\mathbf{G}\vartheta(x)&=-I_{0+}^{\alpha}g_\vartheta(x)+x^{\alpha-1}\int\limits_0^1\kappa(s)g_\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{x^{\alpha-1}}{\varpi}\int\limits_0^1\left(\sum_{i=1}^m b_i\int\limits_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau\right)g_\vartheta(s)\,\mathrm{d}s\\&=\int\limits_0^1G(x,s)g_\vartheta(s)\,\mathrm{d}s,\end{aligned}\tag{6}$$

where $g_\vartheta(s)=g(s,\vartheta(s),D_{0+}^{\beta}\vartheta(s),D_{0+}^{\gamma}\vartheta(s))$, G is defined in (4). In order to prove that problem (1), (2) has a solution, we just have to show that operator $\mathbf{G}$ has a fixed point. Take the fractional derivative of order β for $\mathbf{G}\vartheta$, we have

$$\begin{aligned}D_{0+}^{\beta}\mathbf{G}\vartheta(x)&=-I_{0+}^{\alpha-\beta}g_\vartheta(x)+\frac{\Gamma(\alpha)x^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int\limits_0^1\kappa(s)g_\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)\varpi}x^{\alpha-\beta-1}\int\limits_0^1\left(\sum_{i=1}^m b_i\int_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau\right)g_\vartheta(s)\,\mathrm{d}s\\&=\int\limits_0^1G_1(x,s)g_\vartheta(s)\,\mathrm{d}s,\end{aligned}\tag{7}$$

where

$$G_1(x,s)=K_1(x,s)+\frac{\Gamma(\alpha)x^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)\varpi}\sum_{i=1}^m b_i\int\limits_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau,\tag{8}$$

$$K_1(x,s)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\begin{cases}x^{\alpha-\beta-1}\kappa(s)-\frac{1}{\Gamma(\alpha)}(x-s)^{\alpha-\beta-1}, & 0\leqslant s\leqslant x\leqslant 1,\\ x^{\alpha-\beta-1}\kappa(s), & 0\leqslant x\leqslant s<1.\end{cases}\tag{9}$$

Similarly, we have

$$\begin{aligned}D_{0+}^{\gamma}\mathbf{G}\vartheta(x) &= -I_{0+}^{\alpha-\gamma}g_{\vartheta}(x) + \frac{\Gamma(\alpha)x^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\int_0^1 \kappa(s)g_{\vartheta}(s)\,\mathrm{d}s \\ &\quad + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)\varpi}x^{\alpha-\gamma-1}\int_0^1\Biggl(\sum_{i=1}^m b_i\int_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau\Biggr)g_{\vartheta}(s)\,\mathrm{d}s \\ &= \int_0^1 G_2(x,s)g_{\vartheta}(s)\,\mathrm{d}s,\end{aligned}$$

where

$$G_2(x,s) = K_2(x,s) + \frac{\Gamma(\alpha)x^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\varpi}\sum_{i=1}^m b_i\int_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau,$$

$$K_2(x,s) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}\begin{cases} x^{\alpha-\gamma-1}\kappa(s) - \frac{1}{\Gamma(\alpha)}(x-s)^{\alpha-\gamma-1}, & 0 \leqslant s < x \leqslant 1, \\ x^{\alpha-\gamma-1}\kappa(s), & 0 < x \leqslant s < 1. \end{cases} \tag{10}$$

Then

$$\begin{aligned}&x^{1+\gamma-\alpha}D_{0+}^{\gamma}\mathbf{G}\vartheta(x) \\ &\quad = \int_0^1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}\kappa(s)g_{\vartheta}(s)\,\mathrm{d}s - \frac{x^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)}\int_0^x (x-s)^{\alpha-\gamma-1}g_{\vartheta}(s)\,\mathrm{d}s \\ &\qquad + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)\varpi}\int_0^1\Biggl(\sum_{i=1}^m b_i\int_{I_i}K_0(\tau,s)h_i(\tau)\,\mathrm{d}\tau\Biggr)g_{\vartheta}(s)\,\mathrm{d}s.\end{aligned} \tag{11}$$

By simple deduction we can get the following properties.

Lemma 3. *Let* $0 \leqslant \beta \leqslant \alpha-1 \leqslant 1 < \gamma < \alpha \leqslant 2$, $\varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$. *The functions* K_i, G_i $(i=1,2)$ *defined in* (8), (9) *and* (10) *have the following properties*:

(i) K_1, G_1 *are nonnegative and continuous on* $[0,1]\times[0,1)$;

(ii) *For any* $(x,s) \in [0,1]\times[0,1)$,

$$\begin{aligned}&\frac{\min\{\gamma-\beta,1\}\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\lambda_2\rho_1 x^{\alpha-\beta-1}s(1-s)^{\alpha-\gamma-1} \\ &\quad \leqslant K_1(x,s) \leqslant \frac{x^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)};\end{aligned}$$

(iii) *For any* $(x,s) \in [0,1]\times[0,1)$,

$$\frac{\mathfrak{m}x^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}s(1-s)^{\alpha-\beta-1} \leqslant G_1(x,s) \leqslant \frac{\mathfrak{M}x^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(1-s)^{\alpha-\gamma-1},$$

where

$$\mathfrak{m} = \min\{\gamma - \beta,\, 1\}\Gamma(\alpha)\lambda_2\rho_1 + \frac{\beta}{\varpi}\sum_{i=1}^{m} b_i \int\limits_{I_i} h_i(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau,$$

and $\mathfrak{M}$ *is defined in Lemma* 2;

(iv) $K_2(x,s) > 0$, $0 < x \leqslant s < 1$, *and* $K_2(x,s) < 0$, $0 < s < x \leqslant 1$.

Lemma 4. *Integral operators* I_{0+}^{α}, $I_{0+}^{\alpha-\beta}$, $I_{0+}^{\alpha-\gamma}$ *have properties*:

(i) $I_{0+}^{\alpha}, I_{0+}^{\alpha-\beta} : L_{\alpha-\gamma}^{p}[0,1] \to C[0,1]$ *are continuous*;

(ii) $I_{0+}^{\alpha-\gamma} : L_{\alpha-\gamma}^{p}[0,1] \to C_{\alpha-\gamma}[0,1]$ *is continuous.*

Proof. (i) Since $1 \leqslant \alpha-\beta < \alpha \leqslant 2$, we only show the continuity of I_{0+}^{α}. First, we prove that $I_{0+}^{\alpha} : L_{\alpha-\gamma}^{p}[0,1] \to C[0,1]$ is well defined, that is, for any $f \in L_{\alpha-\gamma}^{p}[0,1]$, we have $I_{0+}^{\alpha} f \in C[0,1]$. Notice that $(\alpha-\gamma-1)q+1 > 0$ because of the condition $1/p < \alpha-\gamma$. For any $0 \leqslant x_1 \leqslant x_2 \leqslant 1$,

$$\begin{aligned}
&\bigl|I_{0+}^{\alpha} f(x_2) - I_{0+}^{\alpha} f(x_1)\bigr| \\
&\quad\leqslant \frac{\|f\|_{L_{\alpha-\gamma}^{p}}}{\Gamma(\alpha)}\Biggl[\Biggl(\int\limits_0^{x_1}\bigl((x_2-s)^{\alpha-1} - (x_1-s)^{\alpha-1}\bigr)^q s^{(\alpha-\gamma-1)q}\,\mathrm{d}s\Biggr)^{1/q} \\
&\qquad + \Biggl(\int\limits_{x_1}^{x_2}(x_2-s)^{(\alpha-1)q} s^{(\alpha-\gamma-1)q}\,\mathrm{d}s\Biggr)^{1/q}\Biggr] \\
&\quad\leqslant \frac{\|f\|_{L_{\alpha-\gamma}^{p}}}{\Gamma(\alpha)}\Biggl[(x_2-x_1)^{\alpha-1}\Biggl(\int\limits_0^{x_1} s^{(\alpha-\gamma-1)q}\,\mathrm{d}s\Biggr)^{1/q} + \Biggl(\int\limits_{x_1}^{x_2} s^{(\alpha-\gamma-1)q}\,\mathrm{d}s\Biggr)^{1/q}\Biggr] \\
&\quad= \frac{\|f\|_{L_{\alpha-\gamma}^{p}}}{\Gamma(\alpha)}\Biggl[(x_2-x_1)^{\alpha-1}\biggl(\frac{x_1^{(\alpha-\gamma-1)q+1}}{(\alpha-\gamma-1)q+1}\biggr)^{1/q} \\
&\qquad + \biggl(\frac{x_2^{(\alpha-\gamma-1)q+1} - x_1^{(\alpha-\gamma-1)q+1}}{(\alpha-\gamma-1)q+1}\biggr)^{1/q}\Biggr],
\end{aligned}$$

then we naturally get the continuity of $I_{0+}^{\alpha} f(x)$ as $x_2 \to x_1$.

Let $\{f_n\}$ be convergent sequence in the space $L_{\alpha-\gamma}^{p}[0,1]$, i.e., there exists a function $f_0 \in L_{\alpha-\gamma}^{p}[0,1]$ such that $\|f_n - f_0\|_{L_{\alpha-\gamma}^{p}} \to 0$ $(n \to \infty)$. In order to show that operator I_{0+}^{α} is continuous, we have to prove $\|I_{0+}^{\alpha} f_n - I_{0+}^{\alpha} f_0\|_{\infty} \to 0 (n \to \infty)$. In fact,

$$\begin{aligned}
&\max_{0\leqslant x\leqslant 1}\bigl|I_{0+}^{\alpha} f_n(x) - I_{0+}^{\alpha} f_0(x)\bigr| \\
&\quad\leqslant \max_{0\leqslant x\leqslant 1}\frac{1}{\Gamma(\alpha)}\Biggl(\int\limits_0^{x}(x-s)^{(\alpha-1)q} s^{(\alpha-\gamma-1)q}\,\mathrm{d}s\Biggr)^{1/q}\|f_n - f_0\|_{L_{\alpha-\gamma}^{p}} \\
&\quad= \max_{0\leqslant x\leqslant 1}\frac{\theta}{\Gamma(\alpha)} x^{\alpha-1+\alpha-\gamma-1/p}\|f_n - f_0\|_{L_{\alpha-\gamma}^{p}} = \frac{\theta}{\Gamma(\alpha)}\|f_n - f_0\|_{L_{\alpha-\gamma}^{p}},
\end{aligned}$$

where

$$\theta = \left(\frac{\Gamma((\alpha-1)q+1)\Gamma((\alpha-\gamma-1)q+1)}{\Gamma((2\alpha-\gamma-2)q+2)}\right)^{1/q}.$$

According to the inequality above, we have $\|I_{0+}^{\alpha} f_n - I_{0+}^{\alpha} f_0\|_\infty \to 0$ as $n \to \infty$.

(ii) For any $f \in L_{\alpha-\gamma}^p[0,1]$, denote $F(x) = x^{1+\gamma-\alpha} I_{0+}^{\alpha-\gamma} f(x), x \in [0,1]$.

$$\begin{aligned}
\left|F(x)\right| &\leqslant \frac{x^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_0^x (x-s)^{\alpha-\gamma-1}\left|f(s)\right| \mathrm{d}s \\
&\leqslant \frac{x^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \left(\int_0^x (x-s)^{(\alpha-\gamma-1)q} s^{(\alpha-\gamma-1)q} \,\mathrm{d}s\right)^{1/q} \|f\|_{L_{\alpha-\gamma}^p} \\
&= \frac{\varrho x^{\alpha-\gamma-1/p}}{\Gamma(\alpha-\gamma)} \|f\|_{L_{\alpha-\gamma}^p},
\end{aligned} \tag{12}$$

where

$$\varrho := \left(\frac{\Gamma((\alpha-\gamma-1)q+1)\Gamma((\alpha-\gamma-1)q+1)}{\Gamma(2(\alpha-\gamma-1)q+2)}\right)^{1/q}.$$

From this inequality we know F is continuous on $t=0$ if one supplies the definition of F on $t=0$: $F(0)=0$.

For any $0 < x_1 \leqslant x_2 \leqslant 1$,

$$\begin{aligned}
&\left|F(x_2) - F(x_1)\right| \\
&\quad \leqslant \left|x_2^{1+\gamma-\alpha} - x_1^{1+\gamma-\alpha}\right| \left|I_{0+}^{\alpha-\gamma} f(x_2)\right| + x_1^{1+\gamma-\alpha}\left|I_{0+}^{\alpha-\gamma} f(x_2) - I_{0+}^{\alpha-\gamma} f(x_1)\right| \\
&\quad \leqslant \frac{|x_2^{1+\gamma-\alpha} - x_1^{1+\gamma-\alpha}|}{\Gamma(\alpha-\gamma)} \int_0^{x_2} (x_2-s)^{\alpha-\gamma-1}\left|f(s)\right| \mathrm{d}s && (13) \\
&\qquad + \frac{x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_0^{x_1} \left((x_1-s)^{\alpha-\gamma-1} - (x_2-s)^{\alpha-\gamma-1}\right)\left|f(s)\right| \mathrm{d}s && (14) \\
&\qquad + \frac{x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_{x_1}^{x_2} (x_2-s)^{\alpha-\gamma-1}\left|f(s)\right| \mathrm{d}s. && (15)
\end{aligned}$$

Now, we will evaluate these formulae (13)–(15), respectively.

$$\begin{aligned}
&\frac{|x_2^{1+\gamma-\alpha} - x_1^{1+\gamma-\alpha}|}{\Gamma(\alpha-\gamma)} \int_0^{x_2} (x_2-s)^{\alpha-\gamma-1}\left|f(s)\right| \mathrm{d}s \\
&\quad \leqslant \frac{|x_2^{1+\gamma-\alpha} - x_1^{1+\gamma-\alpha}|}{\Gamma(\alpha-\gamma)} \varrho x_2^{\alpha-\gamma-1+\alpha-\gamma-1/p} \|f\|_{L_{\alpha-\gamma}^p} \\
&\quad \leqslant \frac{\varrho|1-(\frac{x_1}{x_2})^{1+\gamma-\alpha}|}{\Gamma(\alpha-\gamma)} \|f\|_{L_{\alpha-\gamma}^p} \to 0 \quad (x_2 \to x_1).
\end{aligned}$$

Choose a constant η satisfying $0 < \eta < \min\{1+\gamma-\alpha, \alpha-\gamma-1/p\}$, then

$$\begin{aligned}
&\frac{x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_0^{x_1} \left((x_1-s)^{\alpha-\gamma-1} - (x_2-s)^{\alpha-\gamma-1}\right)\left|f(s)\right| \mathrm{d}s \\
&\quad \leqslant \frac{x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_0^{x_1} \left(\frac{1}{x_1-s} - \frac{1}{x_2-s}\right)^{1+\gamma-\alpha} \left|f(s)\right| \mathrm{d}s \\
&\quad \leqslant \frac{x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \left(\int_0^{x_1} \left(\frac{1}{x_1-s} - \frac{1}{x_2-s}\right)^{(1+\gamma-\alpha-\eta)q} \left(\frac{x_2-x_1}{(x_2-s)(x_1-s)}\right)^{\eta q} s^{(\alpha-\gamma-1)q} \,\mathrm{d}s\right)^{1/q} \\
&\qquad \times \|f\|_{L^p_{\alpha-\gamma}} \\
&\quad \leqslant \frac{(x_2-x_1)^\eta x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \left(\int_0^{x_1} (x_1-s)^{(\alpha-\gamma-1-\eta)q} s^{(\alpha-\gamma-1)q} \,\mathrm{d}s\right)^{1/q} \|f\|_{L^p_{\alpha-\gamma}} \\
&\quad = \frac{(x_2-x_1)^\eta x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \kappa x_1^{\alpha-\gamma-1-\eta+\alpha-\gamma-1/p} \|f\|_{L^p_{\alpha-\gamma}} \\
&\quad \leqslant \frac{\kappa(x_2-x_1)^\eta}{\Gamma(\alpha-\gamma)} \|f\|_{L^p_{\alpha-\gamma}} \to 0 \quad (x_2 \to x_1),
\end{aligned}$$

where

$$\kappa = \left(\frac{\Gamma((\alpha-\gamma-\eta-1)q+1)\Gamma((\alpha-\gamma-1)q+1)}{\Gamma((2(\alpha-\gamma-1)-\eta)q+2)}\right)^{1/q}.$$

In the end, as for (15), let $x_2 \to x_1$, then

$$\frac{x_1^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_{x_1}^{x_2} (x_2-s)^{\alpha-\gamma-1}\left|f(s)\right| \mathrm{d}s \leqslant \frac{(x_2-x_1)^{\alpha-\gamma-1/p}}{\Gamma(\alpha-\gamma)((\alpha-\gamma-1)q+1)^{1/q}} \|f\|_{L^p_{\alpha-\gamma}} \to 0.$$

Taking all the conclusions above into (13)–(15), we can get $|F(x_2) - F(x_1)| \to 0$ as $x_2 \to x_1$, which implies that $F(x)$ is continuous on $[0,1]$ and $I_{0+}^{\alpha-\gamma} f \in C_{\alpha-\gamma}[0,1]$.

Combining (12), we easily infer the continuity of integral operator $I_{0+}^{\alpha-\gamma}$. □

We define a normed vector space

$$\mathbb{X} = \left\{\vartheta \in C[0,1] \;\middle|\; D_{0+}^\beta \vartheta \in C[0,1],\ D_{0+}^\gamma \vartheta \in C_{\alpha-\gamma}[0,1]\right\}$$

equipped with the norm $\|\vartheta\| = \max\{\|\vartheta\|_\infty, \|D_{0+}^\beta \vartheta\|_\infty, \|D_{0+}^\gamma \vartheta\|_*\}$.

Lemma 5. $(\mathbb{X}, \|\cdot\|)$ *is Banach space.*

Proof. Let $\{\vartheta_n\}$ be Cauchy sequence in $(\mathbb{X}, \|\cdot\|)$. Clearly, $\{\vartheta_n\}, \{D_{0+}^\beta \vartheta_n\}$ are also Cauchy sequence in the space $C[0,1]$. Therefore, $\{\vartheta_n\}, \{D_{0+}^\beta \vartheta_n\}$ converge to some $\vartheta, u \in C[0,1]$. By similar work of Su and Liu [30], we have $u = D_{0+}^\beta \vartheta$.

Let $U_n(t) = t^{1+\gamma-\alpha} D_{0+}^\gamma \vartheta_n(t)$. Evidently, $\{U_n(t)\}$ is Cauchy sequence in $C[0,1]$, then there exists $\mu \in C[0,1]$ such that $U_n(t) \to \mu(t)$ in $C[0,1]$. That is to say, for any

$\varepsilon > 0$, there is a positive integer N,

$$\left|U_n(t) - \mu(t)\right| = \left|t^{1+\gamma-\alpha} D_{0+}^{\gamma} \vartheta_n(t) - \mu(t)\right| < \varepsilon \quad \forall n > N,\ t \in [0,1].$$

Choose arbitrary positive number $a < 1$, we will prove $\{D_{0+}^{\gamma}\vartheta_n\}$ uniformly converges to $t^{\alpha-\gamma-1}\mu(t)$ on $[a,1]$. In fact, for all $n > N$, $t \in [a,1]$, we have

$$\left|D_{0+}^{\gamma}\vartheta_n(t) - t^{\alpha-\gamma-1}\mu(t)\right| = t^{\alpha-\gamma-1}\left|t^{1+\gamma-\alpha} D_{0+}^{\gamma}\vartheta_n - \mu(t)\right| \leqslant a^{\alpha-\gamma-1}\varepsilon.$$

By property of fractional calculus, for every $t \in (0,1]$, we have $(\lim_{n\to\infty} D_{0+}^{\gamma}\vartheta_n)(t) = (D_{0+}^{\gamma}\lim_{n\to\infty}\vartheta_n)(t)$, Hence, $t^{\alpha-\gamma-1}\mu(t) = D_{0+}^{\gamma}\vartheta(t)$ for all $t \in (0,1]$. So, $D_{0+}^{\gamma}\vartheta(t) \in C(0,1]$ and $\lim_{t\to 0+} t^{1+\gamma-\alpha} D_{0+}^{\gamma}\vartheta(t) = \lim_{t\to 0+}\mu(t) = \mu(0)$, i.e., $D_{0+}^{\gamma}\vartheta(t) \in C_{\alpha-\gamma}[0,1]$. The proof is completed. □

Let $\mathbb{K} = \{\vartheta \in \mathbb{X} \mid \vartheta(x) \geqslant 0,\ D_{0+}^{\beta}\vartheta(x) \geqslant 0,\ x \in [0,1]\}$. Apparently, $\mathbb{K}$ is a cone of $\mathbb{X}$.

3 Existence and uniqueness results

This section deals with existence and uniqueness of solutions for problem (1), (2). The nonlinear term g satisfies the following assumptions:

(C1) $g : (0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, $g(x,0,0,0)$ does not vanish on any compact interval of $(0,1]$. Furthermore, there exist nonnegative functions $\sigma_i \in L_{\alpha-\gamma}^{p}[0,1]$ $(i = 1,2,3)$ and continuous and nondecreasing functions $\vartheta_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1,2,3)$ such that for any $x \in (0,1]$, $u, v \in \mathbb{R}^+$, $w \in \mathbb{R}$,

$$g(x,u,v,w) \leqslant \sigma_1(x)\vartheta_1(u) + \sigma_2(x)\vartheta_2(v) + \sigma_3(x)\vartheta_3\bigl(x^{1+\gamma-\alpha}|w|\bigr).$$

(C2) There exists a positive number R such that

$$\max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}\right\}(\mathfrak{M}+1)\bigl[\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\bigr] < R,$$

where $\overline{\sigma_i} = \varrho\|\sigma_i\|_{L_{\alpha-\gamma}^{p}}$ $(i = 1,2,3)$.

(C3) There exist $L_i \in C_{\alpha-\gamma}[0,1]$ $(i = 1,2,3)$ and $\chi_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1,2,3)$ are upper semicontinuous from the right and nondecreasing such that for any $x \in (0,1]$, $u_i, v_i \in \mathbb{R}^+$, $w_i \in \mathbb{R}$ $(i = 1,2)$, we have

$$\begin{aligned}&\bigl|g(x,u_1,v_1,w_1) - g(x,u_2,v_2,w_2)\bigr| \\ &\quad \leqslant L_1(x)\chi_1\bigl(|u_1-u_2|\bigr) + L_2(x)\chi_2\bigl(|v_1-v_2|\bigr) + L_3(x)\chi_3\bigl(x^{1+\gamma-\alpha}|w_1-w_2|\bigr).\end{aligned}$$

(C4) Denote

$$\varPhi(x) := (\mathfrak{M}+1)\max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}\right\}\bigl(\overline{L_1}\chi_1(x) + \overline{L_2}\chi_2(x) + \overline{L_3}\chi_3(x)\bigr).$$

It satisfies $\varPhi(x) < x$ for all $x > 0$, where $\overline{L_i} = \varrho\|L_i\|_{L_{\alpha-\gamma}^{p}}$ $(i = 1,2,3)$.

(C4′) Denote

$$\varPhi(x) := \overline{L_1}\chi_1(x) + \overline{L_2}\chi_2(x) + \overline{L_3}\chi_3(x),$$

it satisfies $\varPhi(x) < x$ for all $x > 0$.

Lemma 6. *Suppose that* (C1) *holds, and* $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2$, $\varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$. *Then* $\mathbf{G} : \mathbb{K} \to \mathbb{K}$ *is completely continuous.*

Proof. Let us first show that the operator $\mathbf{G}$ is well defined. For any $\vartheta \in \mathbb{K}$, from (C1), since $\sigma_i \in L^p_{\alpha-\gamma}[0,1]$ $(i = 1,2,3)$, we know $g_\vartheta \in L^p_{\alpha-\gamma}[0,1]$. Whereupon, according to Hölder inequality and Lemma 2, we have $\int_0^1 (1-s)^{\alpha-\beta-1} g_\vartheta(s)\,\mathrm{d}s$, $\int_0^1 (1-s)^{\alpha-\gamma-1} \times g_\vartheta(s)\,\mathrm{d}s < \infty$ and $\int_0^1 (\sum_{i=1}^m b_i \int_{I_i} K_0(\tau,s) h_i(\tau)\,\mathrm{d}\tau) g_\vartheta(s)\,\mathrm{d}s < \infty$, and we deduce that $I^\alpha_{0+} g_\vartheta(x) \in C[0,1]$ from Lemma 4. Then by (6) we get $\mathbf{G}\vartheta \in C[0,1]$. Similarly, using Lemma 4 and formulae (7), (11), we get the conclusions: $D^\beta_{0+}\mathbf{G}\vartheta \in C[0,1]$, $D^\gamma_{0+}\mathbf{G}\vartheta \in C_{\alpha-\gamma}[0,1]$. Furthermore, we naturally get $\mathbf{G}\vartheta(x), D^\beta_{0+}\mathbf{G}\vartheta(x) \geqslant 0$, $x \in [0,1]$ from the fact that g and Green's functions G, G_1 are nonnegative. So, $\mathbf{G}\vartheta \in \mathbb{K}$.

Suppose that $\vartheta_n \to \vartheta_0 (n \to \infty)$ in cone $\mathbb{K}$, then there exists a constant $M > 0$ such that $\|\vartheta_n\| \leqslant M (n = 0,1,\dots)$. In order to get the conclusion that operator $\mathbf{G}$ is continuous, let us start with the fact that $g_{\vartheta_n} \to g_{\vartheta_0} (n \to \infty)$ in $L^P_{\alpha-\gamma}[0,1]$, where $g_{\vartheta_n}(x) = g(x, \vartheta_n(x), D^\beta_{0+}\vartheta_n(x), D^\gamma_{0+}\vartheta_n(x))$ $(n = 0,1,\dots)$. By condition (C1) it follows that

$$\left| g_{\vartheta_n}(x) - g_{\vartheta_0}(x) \right| \leqslant 2\bigl(\sigma_1(x)\vartheta_1(M) + \sigma_2(x)\vartheta_2(M) + \sigma_3(x)\vartheta_3(M)\bigr), \quad x \in (0,1].$$

Moreover, on the basis of the continuity of g, we deduce that $g_{\vartheta_n}(x) \to g_{\vartheta_0}(x)$, $n \to \infty$, for all $x \in (0,1]$. Taking advantage of Lebesgue dominated convergence theorem, we know $\int_0^1 |x^{1+\gamma-\alpha}(g_{\vartheta_n}(x) - g_{\vartheta_0}(x))|^p\,\mathrm{d}x \to 0$, $n \to \infty$. Thereupon, by Lemma 4, we have $I^\alpha_{0+} g_{\vartheta_n}$ convergence to $I^\alpha_{0+} g_{\vartheta_0}$ in $C[0,1]$. In addition, by Hölder inequality, we can also get $\int_0^1 (1-s)^{\alpha-\beta-1} |g_{\vartheta_n}(s) - g_{\vartheta_0}(s)|\,\mathrm{d}s$, $\int_0^1 (1-s)^{\alpha-\gamma-1} |g_{\vartheta_n}(s) - g_{\vartheta_0}(s)|\,\mathrm{d}s \to 0$ and $\int_0^1 (\sum_{i=1}^m b_i \int_{I_i} K_0(\tau,s) h_i(\tau)\,\mathrm{d}\tau) |g_{\vartheta_n}(s) - g_{\vartheta_0}(s)|\,\mathrm{d}s \to 0$ as $n \to \infty$. Synthesizing the above conclusions, from the expression of $\mathbf{G}\vartheta$ (6), we have $\mathbf{G}\vartheta_n \to \mathbf{G}\vartheta_0$ in $C[0,1]$. Analogously, we can deduce $D^\beta_{0+}\mathbf{G}\vartheta_n \to D^\beta_{0+}\mathbf{G}\vartheta_0$ in $C[0,1]$ and $D^\gamma_{0+}\mathbf{G}\vartheta_n \to D^\gamma_{0+}\mathbf{G}\vartheta_0$ in $C_{\alpha-\gamma}[0,1]$. To wit, $\mathbf{G}\vartheta_n \to \mathbf{G}\vartheta_0 (n \to \infty)$ in $\mathbb{K}$.

In the end, Ascoli–Arzela theorem guarantees operator $\mathbf{G} : \mathbb{K} \to \mathbb{K}$ is compact. That is to say, we can deduce that $\mathbf{G}(\mathbb{B})$ is bounded and equicontinuous for any bounded subset $\mathbb{B} \subseteq \mathbb{K}$. The proof can be obtained by the conventional procedure, so we omit this step.

From the above we conclude that the operator $\mathbf{G}$ is completely continuous. □

Theorem 1. (*See* [1].) *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume that U is relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous compact map. Then either*

(i) *A has a fixed point in $\overline{U}$ or*
(ii) *There exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda A u$.*

Theorem 2. *Assume that* (C1), (C2) *hold, and* $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2, \varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$. *Then BVP* (1), (2) *has at least one positive solution.*

Proof. By applying nonlinear alternative of Leray–Schauder-type fixed point theorem (Theorem 1), we will prove that $\mathbf{G}$ has a fixed point. Let $B_R = \{u \in \mathbb{K} \mid \|u\| < R\}$, R is given in condition (C2). Consider the following integral equation:

$$\vartheta(x) = \lambda \int_0^1 G(x,s) g\big(s, \vartheta(s), D_{0+}^{\beta}\vartheta(s), D_{0+}^{\gamma}\vartheta(s)\big)\,\mathrm{d}s, \tag{16}$$

where $\lambda \in (0,1)$. We claim that any solution of (16) for any $\lambda \in (0,1)$ must satisfies $\|\vartheta\| \neq R$. Otherwise, assume that ϑ is a solution of (16) for some $\lambda \in (0,1)$ such that $\|\vartheta\| = R$. Hence, from condition (C1) and Lemma 2(iv), for any $x \in [0,1]$, we have

$$\begin{aligned}
0 \leqslant \vartheta(x) &\leqslant \frac{\lambda \mathfrak{M} x^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} \big(\sigma_1(s)\vartheta_1\big(\vartheta(s)\big) + \sigma_2(s)\vartheta_2\big(D_{0+}^{\beta}\vartheta(s)\big) \\
&\quad + \sigma_3(s)\vartheta_3\big(s^{1+\gamma-\alpha}\big|D_{0+}^{\gamma}\vartheta(s)\big|\big)\big)\,\mathrm{d}s \\
&\leqslant \frac{\lambda \mathfrak{M}}{\Gamma(\alpha)} \big(\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\big),
\end{aligned}$$

so,

$$\|\vartheta\|_{\infty} \leqslant \frac{\lambda \mathfrak{M}}{\Gamma(\alpha)} \big(\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\big).$$

Similarly, in view of Lemma 3 and (7), we have

$$0 \leqslant D_{0+}^{\beta}\vartheta(x) \leqslant \frac{\lambda \mathfrak{M}}{\Gamma(\alpha-\beta)} \big(\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\big),$$

then

$$\|D_{0+}^{\beta}\vartheta\|_{\infty} \leqslant \frac{\lambda \mathfrak{M}}{\Gamma(\alpha-\beta)} \big(\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\big).$$

According to (11), we have

$$\begin{aligned}
&\big|x^{1+\gamma-\alpha} D_{0+}^{\gamma}\mathbf{G}\vartheta(x)\big| \\
&\quad \leqslant \lambda \int_0^1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \big(\lambda_1\rho_2(1-s)^{\alpha-\beta-1} + \lambda_2\rho_1(1-s)^{\alpha-\gamma-1}\big) g_{\vartheta}(s)\,\mathrm{d}s \\
&\qquad + \lambda \frac{x^{1+\gamma-\alpha}}{\Gamma(\alpha-\gamma)} \int_0^x (x-s)^{\alpha-\gamma-1} g_{\vartheta}(s)\,\mathrm{d}s \\
&\qquad + \lambda \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)\varpi} \int_0^1 \left(\sum_{i=1}^m b_i \int_{I_i} K_0(\tau,s) h_i(\tau)\,\mathrm{d}\tau\right) g_{\vartheta}(s)\,\mathrm{d}s \\
&\quad \leqslant \frac{\lambda(\mathfrak{M}+1)}{\Gamma(\alpha-\gamma)} \big(\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\big),
\end{aligned}$$

then

$$\|D_{0+}^{\gamma}\vartheta\|_{*} \leqslant \frac{\lambda(\mathfrak{M}+1)}{\Gamma(\alpha-\gamma)} \big(\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\big).$$

Therefore,

$$R = \|\vartheta\| \leqslant \lambda \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}\right\}(\mathfrak{M}+1)\left[\overline{\sigma_1}\vartheta_1(R) + \overline{\sigma_2}\vartheta_2(R) + \overline{\sigma_3}\vartheta_3(R)\right] < R.$$

This is a contradiction and the claim is proved. Leray–Schauder nonlinear alternative theorem guarantees that operator $\mathbf{G}$ has a fixed point $\vartheta \in \overline{B_R}$. Since $g(x,0,0,0)$ does not vanish on any compact interval of $(0,1)$, we know ϑ must be positive. □

Next, our uniqueness result for problem (1), (2) relies on Boyd–Wong's contraction principle [8].

Theorem 3. *Let X be a complete metric space and suppose $T : X \to X$ satisfies*

$$d(Tx, Ty) \leqslant \Phi\big(d(x,y)\big) \quad \text{for each } x, y \in X,$$

where $\Phi : [0,\infty) \to [0,\infty)$ is upper semicontinuous function from the right (i.e., $r_j \downarrow r \geqslant 0 \Rightarrow \limsup_{j\to\infty} \Phi(r_j) \leqslant \Phi(r)$, and for $x > 0$, $0 \leqslant \Phi(x) < x$ for $x > 0$). Then T has a unique fixed point $x \in X$.

Theorem 4. *Assume that* (C3)*,* (C4) *hold, and $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2, \varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$. Then BVP* (1)*,* (2) *has a unique positive solution.*

Proof. For any $\vartheta, y \in \mathbb{K}$ and $x \in [0,1]$, by using Lemma 2(iv) and condition (C3), we get

$$\begin{aligned}
&\left|\mathbf{G}\vartheta(x) - \mathbf{G}y(x)\right| \\
&\quad\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1}\Big(L_1(s)\chi_1\big(\left|x(s)-y(s)\right|\big) + L_2(s)\chi_2\big(\left|D^{\beta}_{0+}x(s) - D^{\beta}_{0+}y(s)\right|\big) \\
&\qquad + L_3(s)\chi_3\big(s^{1+\gamma-\alpha}\left|D^{\gamma}_{0+}\vartheta(s) - D^{\gamma}_{0+}y(s)\right|\big)\Big)\,\mathrm{d}s \\
&\quad\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)}\big(\overline{L_1}\chi_1\big(\|\vartheta-y\|\big) + \overline{L_2}\chi_2\big(\|\vartheta-y\|\big) + \overline{L_3}\chi_3\big(\|\vartheta-y\|\big)\big),
\end{aligned}$$

so

$$\|\mathbf{G}\vartheta - \mathbf{G}y\|_\infty \leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)}\big(\overline{L_1}\chi_1\big(\|\vartheta-y\|\big) + \overline{L_2}\chi_2\big(\|\vartheta-y\|\big) + \overline{L_3}\chi_3\big(\|\vartheta-y\|\big)\big).$$

Similarly,

$$\begin{aligned}
&\left\|D^{\beta}_{0+}\mathbf{G}\vartheta - D^{\beta}_{0+}\mathbf{G}y\right\|_\infty \\
&\quad\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha-\beta)}\big(\overline{L_1}\chi_1\big(\|\vartheta-y\|\big) + \overline{L_2}\chi_2\big(\|\vartheta-y\|\big) + \overline{L_3}\chi_3\big(\|\vartheta-y\|\big)\big), \\
&\left\|D^{\gamma}_{0+}\mathbf{G}\vartheta - D^{\gamma}_{0+}\mathbf{G}y\right\|_* \\
&\quad\leqslant \frac{\mathfrak{M}+1}{\Gamma(\alpha-\gamma)}\big(\overline{L_1}\chi_1\big(\|\vartheta-y\|\big) + \overline{L_2}\chi_2\big(\|\vartheta-y\|\big) + \overline{L_3}\chi_3\big(\|\vartheta-y\|\big)\big).
\end{aligned}$$

Synthesizing the above three inequalities and combining with condition (C4), we get

$$\begin{aligned}\|\mathbf{G}\vartheta-\mathbf{G}y\| &\leqslant (\mathfrak{M}+1)\max\left\{\frac{1}{\Gamma(\alpha)},\frac{1}{\Gamma(\alpha-\beta)}\right\}\bigl(\overline{L_1}\chi_1\bigl(\|\vartheta-y\|\bigr)\\ &\quad+\overline{L_2}\chi_2\bigl(\|\vartheta-y\|\bigr)+\overline{L_3}\chi_3\bigl(\|\vartheta-y\|\bigr)\bigr)\\ &=\Phi\bigl(\|\vartheta-y\|\bigr)<\|\vartheta-y\|.\end{aligned}$$

Then Boyd–Wong's contraction principle can be applied and $\mathbf{G}$ has a unique fixed point which is the unique solution of problem (1), (2). □

Example.

$$\begin{aligned}&D_{0+}^{7/4}\vartheta(x)+x^{1/4}\bigl[\vartheta(x)\bigr]^{1/2}+x^{1/3}\bigl[D_{0+}^{1/2}\vartheta(x)\bigr]^{1/3}+x\bigl[D_{0+}^{3/2}\vartheta(x)\bigr]^{2/3}=0,\\ &\quad 0<x<1,\\ &\lambda_1D_{0+}^{1/2}\vartheta(1)+\lambda_2D_{0+}^{3/2}\vartheta(1)\\ &\quad=0.1\int_0^{1/4}s^{-1/2}\vartheta(s)\,\mathrm{d}s+0.05\int_{1/4}^{3/4}s^{-1}\vartheta(s)\,\mathrm{d}s+0.1\int_{3/4}^{1}s^{-1/4}\vartheta(s)\,\mathrm{d}s\\ &\qquad+0.02\int_0^1 s^{1/4}\vartheta(s)\,\mathrm{d}s,\\ &\vartheta(0)=0,\end{aligned}\tag{17}$$

Let $\alpha=7/4$, $\beta=1/2$, $\gamma=3/2$, $I_1=[0,1/4]$, $I_2=[1/4,3/4]$, $I_3=[3/4,1]$, $I_4=[0,1]$, $b_1=0.1$, $b_2=0.05$, $b_3=0.1$, $b_4=0.02$, $h_1(s)=s^{-1/2}$, $h_2(s)=s^{-1}$, $h_3(s)=s^{-1/4}$, $h_4(s)=s^{1/4}$, $\lambda_1=0.9$, $\lambda_2=0.1$. A simple calculation yields

$$\varpi=\frac{\lambda_1\Gamma(\alpha)}{\Gamma(\alpha-\beta)}+\frac{\lambda_2\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}-\sum_{i=1}^m b_i\int_{I_i}h_i(s)s^{\alpha-1}\,\mathrm{d}s\approx 0.86084$$

and $1/\Gamma(\alpha-\beta)-1/\Gamma(\alpha-\gamma)\approx 0.82741$, then $\varpi>1/\Gamma(\alpha-\beta)-1/\Gamma(\alpha-\gamma)$.

Let $g(x,u,v,w)=x^{-1/2}(1+u^{1/2})+x^{-1/4}v^{1/4}+x^{-1/4}w^{2/3}$, then $g:(0,1]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is continuous, and $g(x,0,0,0)$ does not vanish on any compact interval of $(0,1]$. Let $\sigma_1(x)=x^{-1/2}$, $\sigma_2(x)=x^{-1/4}$, $\sigma_3(x)=x^{-3/4}$, $\vartheta_1(u)=1+u^{1/2}$, $\vartheta_2(v)=1+v^{1/4}$, $\vartheta_3(w)=w^{2/3}$, then for $x\in(0,1]$, $u,v\in\mathbb{R}^+$, $w\in\mathbb{R}$.

$$g(x,u,v,w)\leqslant\sigma_1(x)\vartheta_1(u)+\sigma_2(x)\vartheta_2(v)+\sigma_3(x)\vartheta_3\bigl(x^{3/4}|w|\bigr).$$

Furthermore, $\overline{\sigma_1}=8.5634$, $\overline{\sigma_2}=8.0337$, $\overline{\sigma_3}=9.8240$,

$$\mathfrak{M}=1+\frac{1}{\varpi}\sum_{i=1}^m b_i\int_{I_i}h_i(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau\approx 1.09028.$$

Take $R = 2^{18} = 262144$,

$$\max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}\right\}(\mathfrak{M}+1)\big[\sigma_1\vartheta_1(R) + \sigma_2\vartheta_2(R) + \sigma_3\vartheta_3(R)\big]$$
$$\approx 103328 < R.$$

Hence, conditions (C1), (C2) in Theorem 2 hold, then problem (17) has a positive solution.

4 Stability analysis

In this section, we consider the Banach space $\mathbb{X} = \{u \in C[0,1] \mid D^{\beta}_{0+}u \in C[0,1],\ D^{\gamma}u \in C_{\alpha-\gamma}[0,1]\}$ equipped with the norm $\|u\| = \|u\|_\infty + \|D^{\beta}_{0+}u\|_\infty + \|D^{\gamma}_{0+}u\|_*$. Let us introduce some definitions related to Ulam stability.

Suppose that function $H \in C_{\alpha-\gamma}[0,1]$ is nonnegative and $\epsilon > 0$. Consider the inequalities given below:

$$\big|D^{\alpha}_{0+}\vartheta(x) + g\big(t, \vartheta(x), D^{\beta}_{0+}\vartheta(x), D^{\gamma}_{0+}\vartheta(x)\big)\big| \leqslant \epsilon, \qquad x \in (0,1], \tag{18}$$

$$\big|D^{\alpha}_{0+}\vartheta(x) + g\big(x, \vartheta(x), D^{\beta}_{0+}\vartheta(x), D^{\gamma}_{0+}\vartheta(x)\big)\big| \leqslant H(x)\epsilon, \quad x \in (0,1], \tag{19}$$

$$\big|D^{\alpha}_{0+}\vartheta(x) + g\big(x, \vartheta(x), D^{\beta}_{0+}\vartheta(x), D^{\gamma}_{0+}\vartheta(x)\big)\big| \leqslant H(x), \quad x \in (0,1]. \tag{20}$$

Definition 1. We say that $\overline{\vartheta} \in \mathbb{X}$ is a solution of inequality (18): if there is $N_{\overline{\vartheta}} \in C(0,1]$ which depends on $\overline{\vartheta}$, such that $|N_{\overline{\vartheta}}(x)| \leqslant \epsilon$ and $D^{\alpha}_{0+}\overline{\vartheta}(x) + g_{\overline{\vartheta}}(x) = N_{\overline{\vartheta}}(x)$, meanwhile $\overline{\vartheta}(x)$ satisfies boundary condition (2), where

$$g_{\overline{\vartheta}}(x) = g\big(x, \overline{\vartheta}(x), D^{\beta}_{0+}\overline{\vartheta}(x), D^{\gamma}_{0+}\overline{\vartheta}(x)\big).$$

Remark 1. The solution of inequalities (19), (20) can be defined as well.

Definition 2. BVP (1), (2) is Hyers–Ulam stable: if there is a constant $C > 0$ such that for any $\epsilon > 0$ and for each solution $\overline{\vartheta} \in \mathbb{X}$ of inequality (18), there exists a unique solution $\vartheta \in \mathbb{X}$ of BVP (1), (2) satisfying

$$\big|\overline{\vartheta}(x) - \vartheta(x)\big| \leqslant C\epsilon, \quad x \in [0,1]. \tag{21}$$

Definition 3. BVP (1), (2) is generalized Hyers–Ulam stable: if there is a function $\Psi \in C([0,1], \mathbb{R}^+)$ with $\Psi(0) = 0$ such that for any $\epsilon > 0$ and for each solution $\overline{\vartheta} \in \mathbb{X}$ of inequality (18), there exists a unique solution $\vartheta \in \mathbb{X}$ of BVP (1), (2) with

$$\big|\overline{\vartheta}(x) - \vartheta(x)\big| \leqslant \Psi(x), \quad x \in [0,1].$$

Definition 4. BVP (1), (2) is Hyers–Ulam–Rassias (HUR) stable w.r.t. nonnegative function $H \in C_{\alpha-\gamma}[0,1]$: if there is a constant $C > 0$ such that for any $\epsilon > 0$ and for each solution $\overline{\vartheta} \in \mathbb{X}$ of inequality (19), there is a unique solution $\vartheta \in \mathbb{X}$ of BVP (1), (2) satisfying

$$\big|\overline{\vartheta}(x) - \vartheta(x)\big| \leqslant CH(x)\epsilon, \quad x \in (0,1].$$

Definition 5. BVP (1), (2) is generalized Hyers–Ulam–Rassias stable w.r.t. $H \in C((0,1], \mathbb{R}^+)$: if there is a constant $C > 0$ such that for each solution $\overline{\vartheta} \in \mathbb{X}$ of inequality (20), there is a unique solution $\vartheta \in \mathbb{X}$ of BVP (1), (2) satisfying

$$\left|\overline{\vartheta}(x) - \vartheta(x)\right| \leqslant CH(x), \quad x \in (0,1].$$

Theorem 5. *Assume that* (C3), (C4′) *hold, and* $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2, \varpi > 1/\Gamma(\alpha-\beta) - 1/\Gamma(\alpha-\gamma)$. *Let* $\Delta = \mathfrak{M}/\Gamma(\alpha) + \mathfrak{M}/\Gamma(\alpha-\beta) + \mathfrak{M} + 1/\Gamma(\alpha-\gamma)$. *If* $\Delta < 1$, *then BVP* (1), (2) *is HU stable.*

Proof. For any $\epsilon > 0$, suppose $\overline{\vartheta} \in \mathbb{X}$ be the solution of inequality (18), then $\overline{\vartheta}(x) = \int_0^1 G(x,s)(g_{\overline{\vartheta}}(s) - N_{\overline{\vartheta}}(s))\,\mathrm{d}s$, where $|N_{\overline{\vartheta}}(s)| \leqslant \epsilon$. Just like the proof method in Theorem 4, we also know that BVP (1), (2) has a unique solution $\vartheta \in \mathbb{X}$ under the new norm and ϑ can be expressed by $\vartheta(x) = \int_0^1 G(x,s) g_\vartheta(s)\,\mathrm{d}s$ by Lemma 1.

On the basis of Lemma 2, we have

$$\begin{aligned}
\left\|\overline{\vartheta} - \vartheta\right\|_\infty &\leqslant \max_{0\leqslant x\leqslant 1} \int_0^1 G(x,s)\left|g_{\overline{\vartheta}}(s) - g_\vartheta(s)\right| \mathrm{d}s + \max_{0\leqslant t\leqslant 1} \int_0^1 G(x,s)\left|N_{\overline{\vartheta}}(s)\right| \mathrm{d}s \\
&\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1}\Big(L_1(s)\chi_1\big(\left|\overline{\vartheta}(s) - \vartheta(s)\right|\big) \\
&\qquad + L_2(s)\chi_2\big(\left|D^\beta_{0+}\overline{\vartheta}(s) - D^\beta_{0+}\vartheta(s)\right|\big) \\
&\qquad + L_3(s)\chi_3\big(s^{1+\gamma-\alpha}\left|D^\gamma_{0+}\overline{\vartheta}(s) - D^\gamma_{0+}\vartheta(s)\right|\big)\Big)\,\mathrm{d}s \\
&\quad + \frac{\epsilon\mathfrak{M}}{\Gamma(\alpha)(\alpha-\gamma)} \\
&\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)}\Big(\overline{L_1}\chi_1\big(\|\overline{\vartheta} - \vartheta\|_\infty\big) + \overline{L_2}\chi_2\big(\left\|D^\beta_{0+}\overline{\vartheta} - D^\beta_{0+}\vartheta\right\|_\infty\big) \\
&\quad + \overline{L_3}\chi_3\big(\left\|D^\gamma_{0+}\overline{\vartheta} - D^\gamma_{0+}\vartheta\right\|_*\big)\Big) + \frac{\epsilon\mathfrak{M}}{\Gamma(\alpha)(\alpha-\gamma)} \\
&\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon\mathfrak{M}}{\Gamma(\alpha)(\alpha-\gamma)}.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
\left\|D^\beta_{0+}\overline{\vartheta} - D^\beta_{0+}\vartheta\right\|_\infty &\leqslant \frac{\mathfrak{M}}{\Gamma(\alpha-\beta)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon\mathfrak{M}}{\Gamma(\alpha-\beta)(\alpha-\gamma)}, \\
\left\|D^\gamma_{0+}\overline{\vartheta} - D^\gamma_{0+}\vartheta\right\|_* &\leqslant \frac{(\mathfrak{M}+1)}{\Gamma(\alpha-\gamma)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon(\mathfrak{M}+1)}{\Gamma(\alpha-\gamma)(\alpha-\gamma)}.
\end{aligned}$$

Hence, $\|\overline{\vartheta} - \vartheta\| \leqslant \Delta\|\overline{\vartheta} - \vartheta\| + (\epsilon/(\alpha-\gamma))\Delta$, then

$$\|\overline{\vartheta} - \vartheta\| \leqslant \frac{\epsilon\Delta}{(\alpha-\gamma)(1-\Delta)}. \tag{22}$$

Let $C = \Delta/((\alpha-\gamma)(1-\Delta))$, clearly,

$$\left|\overline{\vartheta}(x) - \vartheta(x)\right| \leqslant C\epsilon \quad \forall x \in [0,1].$$

In other words, conclusion (21) in Definition 2 is satisfied, i.e., BVP (1), (2) is Hyers–Ulam stable. □

Theorem 6. *Suppose that* (C3), (C4′) *hold, and* $0 \leqslant \beta \leqslant \alpha - 1 \leqslant 1 < \gamma < \alpha \leqslant 2, \varpi > 1/\Gamma(\alpha - \beta) - 1/\Gamma(\alpha - \gamma)$. *Let* $\varDelta = \mathfrak{M}/\Gamma(\alpha) + \mathfrak{M}/\Gamma(\alpha - \beta) + (\mathfrak{M} + 1)/\Gamma(\alpha - \gamma)$. *If* $\varDelta < 1$, *then BVP* (1), (2) *is generalized HU stable.*

Proof. For any $\epsilon > 0$, for convenience, let us assume $0 < \epsilon < 1$. Suppose $\overline{\vartheta} \in \mathbb{X}$ be the solution of inequality (18), $\vartheta \in \mathbb{X}$ be a unique solution of BVP (1), (2). Following the method of Theorem 5, we know that (22) holds. Thereupon, for any $x \in [0, 1]$,

$$\begin{aligned}\left|\overline{\vartheta}(x) - \vartheta(x)\right| &\leqslant \int_0^1 G(x,s)\left|g_{\overline{\vartheta}}(s) - g_{\vartheta}(s)\right| \mathrm{d}s + \int_0^1 G(x,s)\left|N_{\overline{\vartheta}}(s)\right| \mathrm{d}s \\ &\leqslant \frac{\mathfrak{M}x^{\alpha-1}}{\Gamma(\alpha)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon\mathfrak{M}x^{\alpha-1}}{\Gamma(\alpha)(\alpha-\gamma)} \\ &\leqslant \frac{\mathfrak{M}x^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\epsilon\varDelta}{(\alpha-\gamma)(1-\varDelta)} + \frac{\epsilon}{\alpha-\gamma}\right) \\ &\leqslant \frac{\mathfrak{M}x^{\alpha-1}}{\Gamma(\alpha)(\alpha-\gamma)(1-\varDelta)} := \varPsi(x).\end{aligned}$$

It is obvious that $\varPsi(0) = 0$. Then by Definition 3, BVP (1), (2) is generalized HU stable. □

Theorem 7. *Let the conditions of Theorem* 6 *be satisfied, moreover, there exists a nonnegative function* $H \in C_{\alpha-\gamma}[0,1]$ *satisfying* $x^{\alpha-1} \leqslant H(x)$ *for all* $x \in (0, 1]$. *Then BVP* (1), (2) *is HUR stable.*

Proof. For any $\epsilon > 0$, suppose $\overline{\vartheta} \in \mathbb{X}$ be the solution of inequality (19), then $\overline{\vartheta}(x) = \int_0^1 G(x,s)(g_{\overline{\vartheta}}(s) - N_{\overline{\vartheta}}(s))\,\mathrm{d}s$, where $|N_{\overline{\vartheta}}(x)| \leqslant H(x)\epsilon$. $\vartheta \in \mathbb{X}$ is a unique solution of BVP (1), (2). On the similar way of Theorem 5, one can prove

$$\|\overline{\vartheta} - \vartheta\|_\infty \leqslant \frac{\mathfrak{M}}{\Gamma(\alpha)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon\mathfrak{M}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}H(s)\,\mathrm{d}s,$$

$$\left\|D_{0+}^{\beta}\overline{\vartheta} - D_{0+}^{\beta}\vartheta\right\|_\infty \leqslant \frac{\mathfrak{M}}{\Gamma(\alpha-\beta)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon\mathfrak{M}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\gamma-1}H(s)\,\mathrm{d}s,$$

$$\left\|D_{0+}^{\gamma}\overline{\vartheta} - D_{0+}^{\gamma}\vartheta\right\|_* \leqslant \frac{\mathfrak{M}+1}{\Gamma(\alpha-\gamma)}\|\overline{\vartheta} - \vartheta\| + \frac{\epsilon(\mathfrak{M}+1)}{\Gamma(\alpha-\gamma)}\int_0^1 (1-s)^{\alpha-\gamma-1}H(s)\,\mathrm{d}s.$$

So, combining with these three inequalities, we have

$$\|\overline{\vartheta} - \vartheta\| \leqslant \frac{\epsilon\varDelta}{1-\varDelta}\int_0^1 (1-s)^{\alpha-\gamma-1}H(s)\,\mathrm{d}s. \tag{23}$$

For any $x \in (0,1]$, from (23), we get

$$\begin{aligned}\left|\overline{\vartheta}(x)-\vartheta(x)\right| &\leqslant \int_0^1 G(x,s)\left|g_{\overline{\vartheta}}(s)-g_{\vartheta}(s)\right|\mathrm{d}s+\int_0^1 G(x,s)\left|N_{\overline{\vartheta}}(s)\right|\mathrm{d}s\\ &\leqslant \frac{\mathfrak{M}x^{\alpha-1}}{\Gamma(\alpha)}\|\overline{\vartheta}-\vartheta\|+\frac{\epsilon\mathfrak{M}x^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\gamma-1}H(s)\,\mathrm{d}s\\ &\leqslant \frac{\epsilon\mathfrak{M}H(x)}{\Gamma(\alpha)(1-\varDelta)}\int_0^1(1-s)^{\alpha-\gamma-1}H(s)\,\mathrm{d}s\\ &\leqslant \frac{\epsilon\mathfrak{M}H(x)}{\Gamma(\alpha)(1-\varDelta)}\frac{\Gamma(\alpha-\gamma)\Gamma(\alpha-\gamma)}{\Gamma(2\alpha-2\gamma)}\|H\|_*.\end{aligned}$$

Let

$$C:=\frac{\mathfrak{M}\Gamma(\alpha-\gamma)\Gamma(\alpha-\gamma)}{(1-\varDelta)\Gamma(\alpha)\Gamma(2\alpha-2\gamma)}\|H\|_*,$$

then

$$\left|\overline{\vartheta}(x)-\vartheta(x)\right|\leqslant CH(x)\epsilon \quad \forall x\in(0,1].$$

By Definition 4, BVP (1), (2) is HUR stable. □

Remark 2. Under the condition of Theorem 7, imitating the process, we can prove that BVP (1), (2) is generalized HUR stable.

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