Chapter

Hybrid Transforms

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Abstract

Hybrid transforms are constructed by associating the Wigner-Ville distribution (WVD) with widely-known signal processing tools, such as fractional Fourier transform, linear canonical transform, offset linear canonical transform (OLCT), and their quaternion-valued versions. We call them hybrid transforms because they combine the advantages of both transforms. Compared to classical transforms, they show better results in applications. The WVD associated with the OLCT (WVD-OLCT) is a class of hybrid transform that generalizes most hybrid transforms. This chapter summarizes research on hybrid transforms by reviewing a computationally efficient type of the WVD-OLCT, which has simplicity in marginal properties compared to WVD-OLCT and WVD.

Keywords: time-frequency analysis, Wigner-Ville distribution, offset linear canonical transform, hybrid transform, linear frequency modulated (LFM) signal

1. Introduction

The linear canonical transform (LCT) [1–4] and its generalization, the offset linear canonical transform (OLCT) [5, 6] are introduced to study non-stationary signals (audio, image, biomedical, linear frequency modulated (LFM) signals). OLCT has five degrees of freedom, and LCT has three degrees of freedom, which makes them more flexible than the well-known fractional Fourier transform (FrFT) [7] with one degree of freedom and the Fourier transform (FT) with no freedom. Various applications of LCT have been found in the different fields of optics and signal processing. In fact, the properties and applications of the OLCT are similar to the LCT, but they are more general than the LCT, thanks to its two extra parameters, which correspond to time-shift and frequency modulation. It is proven that the Wigner-Ville distribution (WVD) plays a major role in time-frequency signal analysis and processing.

The LFM signal is used in communications, radar and sonar systems. Consequently, LFM signal detection and estimation is one of the most important topics in engineering. The WVD and LCT/OLCT are used in LFM signal processing, but they have their disadvantages:

- WVD does not fully exploit the phase feature of LFM signal;
- LCT/OLCT cannot gather signal energy strongly like WVD.

This results in poor performance under a low signal-to-noise ratio for detection and estimation. Recently, for the purpose to improve the performance of LFM signal detection and estimation, several researchers have associated WVD with the FrFT, LCT, and OLCT, respectively [8–27]. Results show that such transforms exploit the advantages of both transforms, which is why we call them hybrid transforms. The aim of this chapter is to review and summarize research on hybrid transforms by studying WVD association with the OLCT (WVD-OLCT) definitions and properties.

2. Preliminaries

2.1 Wigner-Ville distribution

FT analysis originated long ago and is used in many areas of mathematics and engineering, including quantum mechanics, wave propagation, turbulence, signal analysis and processing. In spite of remarkable success, the FT analysis seems to be inadequate for studying some problems for the following reasons:

- There is no local information in the FT analysis since it does not reflect the change of frequency with time;
- The FT analysis investigates problems either in the time domain or in the frequency domain, but not simultaneously in both domains.

Therefore, we see that FT is sufficient to study signals that are statistically invariant over time, e.g. stationary signals. Naturally, we are surrounded by many signals: audio, video, radar, biomedical signals, etc., all those signals are non-stationary. FT is insufficient to do a complete analysis for such signals because it requires both time-frequency representations of the signal. So it was necessary to define a single transformation of time and frequency domains.

Historically, Eugene Paul Wigner, the 1963 Nobel Prize winner in physics, in 1932 first introduced a fundamental nonlinear transformation to study quantum corrections for classical statistical mechanics in the form [28].

$$\mathcal{W}_{\psi}(x,p)=\frac{1}{h}\int_{\mathbb{R}}\psi\left(x-\frac{\tau}{2}\right)\overline{\psi}\left(x+\frac{\tau}{2}\right)\exp\left(\frac{ip\tau}{\hbar}\right)d\tau, \tag{1}$$

where the wave function $\psi(x)$ satisfies the one-dimensional Schrödinger equation, the quantum mechanical position x and momentum p are independent variables, and $h = 2\pi\hbar$ is the Planck constant. The Wigner distribution $\mathcal{W}_{\psi}(x,p)$ has many important properties and is found to behave as a distribution function defined on a phase space consisting of points (x,p). The most remarkable properties of the Wigner distribution include the marginal integrals in the position and momentum domains as follows [29, 30].

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{W}_{\psi}(x,p)dx &= |\varphi(p)|^2,\\ \int_{\mathbb{R}}\mathcal{W}_{\psi}(x,p)dp &= |\psi(x)|^2,\end{aligned} \tag{2}$$

and the total energy of the wave function ψ in the (x, p) space

$$\int_{\mathbb{R}^2} \mathcal{W}_\psi(x, p)dxdp = \int_{\mathbb{R}} |\psi(x)|^2 dx = \|\psi\|. \tag{3}$$

In the context of non-stationary signal analysis, in 1948 Jean-Andre Ville independently re-derived the Wigner distribution given in Eq. (1) as a quadratic representation of the local time-frequency energy of a signal [31]. Besides linear time-frequency representations of a signal like the Gabor transform, the Zak transform, and the short-time Fourier transform, the WVD (or Wigner-Ville transform (WVT)) occupies a central position in the field of quadratic time-frequency representations and it is recognized as a valuable method/tool for time-frequency of time-varying signals and non-stationary random processes.

With its remarkable structure and properties, the WVD has been regarded as the main distribution of all the time-frequency distributions and used as the classical and fundamental time-frequency analysis tool in different areas of physics and engineering. Particularly, it has been used for instantaneous frequency estimation, spectral analysis of random signals, detection and classification, algorithms for computer implementation, and has a wide range of applications in vision, X-ray diffraction of crystals, pattern recognition, radar, and sonar. Additionally, it has been applied to the analysis of seismic data, speech, and phase distortions in audio engineering problems.

Definition 1 (WVD). If f belong to the Hilbert space $L^2(\mathbb{R})$, the WVD $\mathcal{W}_f$ of signal f is defined as [3, 29, 30].

$$\mathcal{W}_f(t, u) = \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right)\overline{f\left(t - \frac{\tau}{2}\right)} e^{-iu\tau} d\tau. \tag{4}$$

It is easy to see that the WVD is the FT of the instantaneous autocorrelation function

$$R_f(t, \tau) = f\left(t + \frac{\tau}{2}\right)\overline{f\left(t - \frac{\tau}{2}\right)} \tag{5}$$

with respect to τ.

Some main properties of WVD are summarized in **Table 1**. For some recent works and surveys on the WVD, we refer readers to [3, 29, 30] and the references therein.

2.2 Linear canonical transform

The LCT is a four-parameter (a, b, c, d) integral transform that was introduced in the 1970s by Collins, and Moshinsky and Quesne to analyze optical systems and solve differential equations [1, 2]. After the fast algorithm for calculating the discrete LCT was proposed in [32], the LCT was widely used to process non-stationary signals. It has been applied in radar system analysis, filter design, watermarking, phase retrieval, pattern recognition, signal synthesis, and in other areas of engineering sciences. With intensive research, many properties of the LCT are well studied. Transforms and operations, such as the FT, FrFT, Fresnel transform (FRST), Laplace transform, fractional Laplace transform, time scaling, and chirp operations are the special cases of the LCT.

Property	Formulation
Conjugation symmetry	$\mathcal{W}_f(t,u) = \overline{\mathcal{W}_f(t,u)}$
Time shifting (Translation)	$\mathcal{W}_{f'}(t,u) = \mathcal{W}_f(t-\lambda,u),\ f'(t) = f(t-\lambda)$
Frequency shifting (Modulation)	$\mathcal{W}_{f'}(t,u) = \mathcal{W}_f(t,u-u_0),\ f'(t) = f(t)e^{iu_0 t}$
Time marginal	$\int_{\mathbb{R}} \mathcal{W}_f(t,u)du = \lvert f(t)\rvert^2$
Frequency marginal	$\int_{\mathbb{R}} \mathcal{W}_f(t,u)dt = \left\lvert \hat{f}(u)\right\rvert^2$
Energy distribution	$\int_{\mathbb{R}^2} \mathcal{W}_f(t,u)dtdu = \int_{\mathbb{R}} \lvert f(t)\rvert^2 dt = \langle f(t), f(t)\rangle$
Moyal's formula	$\int_{\mathbb{R}^2} \mathcal{W}_f(t,u)\overline{[\mathcal{W}_g(t,u)]}dtdu = \lvert\langle f, g\rangle\rvert^2$

Table 1.
Properties of the WVD.

In some works, the LCT is known under different names as the Collins formula, Moshinsky and Quesne integrals, extended fractional Fourier transform, quadratic-phase integral or quadratic-phase system, generalized Fresnel transform, generalized Huygens integral [33], ABCD transform [34], and affine Fourier transform [35], etc.

Definition 2 (LCT). The LCT $\mathcal{L}_{\mathbf{A}}$ of a signal $f(t)$ with matrix $\mathbf{A} = (a, b, c, d)$, where $a,b,c,d \in \mathbb{R}$ are real parameters and $\det(\mathbf{A}) = ad - bc = 1$, is defined as [2–4]

$$\mathcal{L}_{\mathbf{A}}\{f(t)\}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(t)\frac{1}{\sqrt{i2\pi b}} e^{i\left(\frac{a}{2b}t^2 - \frac{1}{b}tu + \frac{d}{2b}u^2\right)}dt, & b \neq 0, \\ \sqrt{d}e^{\frac{i}{2}cdu^2} f(du), & b = 0. \end{cases} \tag{6}$$

From the definition of LCT, we can see that, when the parameter $b = 0$, the LCT is a scaling transformation coupled with amplitude and quadratic phase modulation and it is of no particular interest to our object. Therefore, without loss of generality, in this chapter we always assume $b \neq 0$.

A detailed and comprehensive view of LCT can be found in [2, 3] and the references therein.

2.3 Offset linear canonical transform

The OLCT is a six-parameter $(a, b, c, d, u_0, \omega_0)$ integral transform, which has been shown as a powerful tool and received much attention in signal processing and optics. It is a time-shifted and frequency-modulated version of the LCT. In some works OLCT called the special affine Fourier transform [35–37] and the inhomogeneous canonical transform [38].

Definition 3 (OLCT). The OLCT $\mathcal{O}_{\mathbf{A}}$ of a signal $f(t)$ with real parameters of matrix $\mathbf{A} = (a, b, c, d, u_0, \omega_0)$, where $a,b,c,d,u_0,\omega_0 \in \mathbb{R}$ are real parameters and $\det(\mathbf{A}) = 1$, is defined as [6, 19]

$$\mathcal{O}_{\mathbf{A}}\{f(t)\}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}} f(t)K_{\mathbf{A}}(t, u)dt, & b \neq 0, \\ \sqrt{d}e^{i\frac{cd}{2}(u-u_0)^2 + j\omega_0 u} f(d(u-u_0)), & b = 0. \end{cases} \tag{7}$$

where $K_{\mathbf{A}}(t, u)$ is the OLCT kernel and expressed as

$$K_{\mathbf{A}}(t, u) = \frac{1}{\sqrt{i2\pi b}} e^{i\left(\frac{a}{2b}t^2 - \frac{1}{b}t(u-u_0) + \frac{d}{2b}\left(u^2+u_0^2\right) - \frac{u}{b}(du_0 - b\omega_0)\right)}. \tag{8}$$

From Eq. (7) it can be seen that for case $b = 0$ the OLCT is simply a time scaled version of f multiplied by a linear chirp. Therefore, from now we restrict our attention to OLCT for the case $b \neq 0$. And without loss of generality, we assume $b > 0$ in the following sections of this chapter.

A number of widely known classical transforms and mathematical operations related to signal processing and optics are special cases of the OLCT. The OLCT converts to its special cases when taking different parameters of matrix $\mathbf{A}$. For example, the OLCT with parameters $(a, b, c, d, u_0, \omega_0) = (a, b, c, d, 0, 0)$ reduces to LCT; when $\mathbf{A} = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, 0, 0)$, it becomes the FrFT; when $\mathbf{A} = (0, 1, -1, 0, 0, 0)$, the OLCT becomes FT; when $\mathbf{A} = (1, b, 0, 1, 0, 0)$, it becomes FRST; and when $\mathbf{A} = \left(d^{-1}, 0, 0, d, 0, 0\right)$, it becomes time scaling operation. Multiplication by Gaussian or chirp function is obtained with an $\mathbf{A} = (1, 0, \tau, 1, 0, 0)$ [1]. The offset Fourier transform $\mathbf{A} = (0, 1, -1, 0, u_0, \omega_0)$, offset fractional Fourier transform $\mathbf{A} = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, u_0, \omega_0)$, frequency modulation $\mathbf{A} = (1, 0, 0, 1, 0, \omega_0)$, and time shifting $\mathbf{A} = (1, 0, 0, 1, u_0, 0)$ are also special cases of the OLCT. The OLCT is able to extend their properties and applications and can solve some problems that cannot be solved well by these operations. In fact, offset versions of FT, FrFT, and LCT are similar to the classical FT, FrFT, and LCT, but they are more flexible than the classical ones, and mainly useful for analyzing optical systems with prisms or shifted lenses. The OLCT has a close relationship with its special cases. So it is practically useful to develop relevant theorems for OLCT. By developing theories for OLCT, we can gain a deeper understanding of its special cases and transfer knowledge from one subject to another. As a generalization of many other linear transforms, the OLCT has found wide applications in applied mathematics, signal processing, and optical system modeling [5, 6, 19, 34, 35, 37].

2.4 Previous results

With the development of the FrFT, Lohmann in [8] and Almeida in [9] investigated the relationship between the WVD and the FrFT. They show that the WVD of the FrFTed signal can be seen as a rotation of the WVD in the time-frequency plane. In this direction, based on the properties of the FrFT, the LCT, and the WVD, Pei and Ding [10] investigated and discussed the relations between the common fractional and canonical operators. The WVD associated with the LCT, named LCWD, denoted as $\mathcal{WD}_{\mathbf{A}}$, given in [10] is useful for the separation of multi-component signals. It is defined as [10, 18].

$$\mathcal{WD}_{\mathbf{A}}(u, v) = \int_{\mathbb{R}} \mathcal{L}_{\mathbf{A}}\left(u + \frac{\tau}{2}\right) \overline{\mathcal{L}_{\mathbf{A}}\left(u - \frac{\tau}{2}\right)} e^{-iv\tau} d\tau, \tag{9}$$

where $\mathcal{L}_{\mathbf{A}}(u)$ is the LCT of signal $f(t)$ with parameter matrix $\mathbf{A} = (a, b, c, d)$.

Unlike the definition of LCWD, Bai et al. obtained generalized type of WVD in the LCT domain, named WVD-LCT (or WDL), denoted as $\mathcal{WDL}_f$, by substituting FT kernel $e^{-iu\tau}$ with LCT kernel $\frac{1}{\sqrt{i2\pi b}} e^{i\left(\frac{a}{2b}\tau^2 - \frac{1}{b}\tau u + \frac{d}{2b}u^2\right)}$ [11].

$$\mathcal{WDL}_f(t,u) = \int_{\mathbb{R}} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)}\frac{1}{\sqrt{i2\pi b}}e^{i\left(\frac{a}{2b}\tau^2-\frac{1}{b}\tau u+\frac{d}{2b}u^2\right)}d\tau. \tag{10}$$

The WVD-LCT generalizes the LCWD and WVD. It is easy to see that the WVD-LCT is the LCT of the instantaneous autocorrelation function $R_f(t, \tau)$ with respect to τ

$$\mathcal{WDL}_f(t,u) = \mathcal{L}_{\mathbf{A}}\{R_f(t,\ \tau)\}. \tag{11}$$

Also, in [11] authors derived the main properties and applications of the WVD-LCT in the LFM signal detection. Uncertainty principles for the WVD-LCT were studied in [13, 25]. Song et al. presented WVD-LCT applications for quadratic frequency modulated signal parameter estimation in [14]. Convolution and correlation theorems for WVD-LCT are obtained in [16]. In [26] authors proposed a new method of instantaneous frequency estimation by associating the WVD with the LCT, which has a higher capacity for anti-noise and a higher estimation accuracy than WVD. Zhang unified LCWD and WVD-LCT [20], and then presented its special cases with less parameters [21, 22]. Urynbassarova et al. presented the WVD associated with the instantaneous autocorrelation function in the LCT domain, named WL, which has elegance and simplicity in marginal properties and affine transformation relationships compared to the WVD [17]. Similar to this in [27] Xin and Li proposed a new definition of WVD associated with LCT, and its integration form, which estimates two phase coefficients of LFM signal simultaneously and effectively suppresses cross terms for multi-component LFM signal. In [19] introduced the WVD association with the OLCT (WVD-OLCT), which is a generalization of the WVD-LCT and its special cases. Recently, in order to study higher dimensions, WVD associations with the quaternion LCT/OLCT were studied in [39–42], and WVD in the framework of octonion LCT was proposed by Dar and Bhat [43].

3. Definition

The WVD given in Eq. (4) can be re-written as

$$\mathcal{W}_f(t,u) = \int_{\mathbb{R}} f_{\mathcal{F}}\left(t+\frac{\tau}{2}\right)\overline{f_{\mathcal{F}}\left(t-\frac{\tau}{2}\right)}d\tau, \tag{12}$$

where $f_{\mathcal{F}}$ equals to $f(t)$ multiplied with FT kernel e^{-iut}. By substituting FT kernel e^{-iut} with OLCT kernel (Eq. (8)), we will get the following definition of the WVD in the OLCT domain, named WOL, denoted as $\mathcal{WOL}_f$, which is the type of the WVD-OLCT.

Definition 4 (WOL). The WOL $\mathcal{WOL}_f$ of signal f for the parameter matrix $\mathbf{A} = (a, b, c, d, u_0, \omega_0)$ is defined as follows [18]

$$\mathcal{WOL}_f(t,u) = \frac{1}{2\pi|b|}\int_{\mathbb{R}} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)}e^{\frac{ia}{b}\tau t}e^{\frac{i}{b}\tau(u_0-u)}d\tau.$$

The WOL is reduced to the WL, when $\mathbf{A} = (a, b, c, d, 0, 0)$,

$$\mathcal{WOL}_f^{(a,\ b,\ c,\ d,\ 0,\ 0)}(t,u) = \mathcal{WL}_f(t,u). \tag{13}$$

Obviously, when the parameter matrix has the special form $\mathbf{A} = (0, 1, -1, 0, 0, 0)$, the WOL is reduced to the WVD

$$\mathcal{WOL}_f^{(0,\ 1,\ -1,\ 0,\ 0,\ 0)}(t, u) = \mathcal{W}_f(t, u). \tag{14}$$

It is clear from Eq. (13) and Eq. (14) that the WOL is a generalization of the WL and the WVD.

4. Properties

Bellow we list some basic properties of the WOL.

Conjugation symmetry property.

The conjugation symmetry property of the WOL is expressed as

$$\mathcal{WOL}_f(t, u) = \overline{\mathcal{WOL}_f(t, u)}. \tag{15}$$

Proof. From the Definition 4, we have

$$\begin{aligned}\overline{\mathcal{WOL}_f(t, u)} &= \overline{\int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right)\overline{f\left(t - \frac{\tau}{2}\right)} e^{\frac{ia}{b}\tau t} e^{\frac{i}{b}\tau(u_0 - u)} d\tau} \\ &= \int_{\mathbb{R}} \overline{f\left(t + \frac{\tau}{2}\right)} f\left(t - \frac{\tau}{2}\right) e^{\frac{ia}{b}(-\tau)t} e^{\frac{i}{b}(-\tau)(u_0 - u)} d\tau,\end{aligned} \tag{16}$$

let $-\tau = \tau'$, then we will arrive at

$$\begin{aligned}\overline{\mathcal{WOL}_f(t, u)} &= \int_{\mathbb{R}} f\left(t + \frac{\tau'}{2}\right)\overline{f\left(t - \frac{\tau'}{2}\right)} e^{\frac{ia}{b}\tau' t} e^{\frac{i}{b}\tau'(u_0 - u)} d\tau' \\ &= \mathcal{WOL}_f(t, u). \blacksquare\end{aligned} \tag{17}$$

This property shows that the WOL is always a real number.

Time marginal property.

The time marginal property of the WOL is given as

$$\int_{\mathbb{R}} \mathcal{WOL}_f(t, u) du = |f(t)|^2. \tag{18}$$

Proof.

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{WOL}_f(t, u) du &= \frac{1}{2\pi|b|} \int_{\mathbb{R}^2} f\left(t + \frac{\tau}{2}\right)\overline{f\left(t - \frac{\tau}{2}\right)} e^{\frac{ia}{b}\tau t} e^{\frac{i}{b}\tau(u_0 - u)} d\tau du \\ &= \frac{1}{2\pi|b|} \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right)\overline{f\left(t - \frac{\tau}{2}\right)} e^{\frac{ia}{b}\tau t} e^{\frac{i}{b}u_0\tau} \left(\int_{\mathbb{R}} e^{-\frac{i}{b}u\tau} du\right) d\tau \\ &= \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right)\overline{f\left(t - \frac{\tau}{2}\right)} e^{\frac{ia}{2b}\tau^2} e^{\frac{i}{b}u_0\tau} \delta(\tau) d\tau \\ &= |f(t)|^2. \ \blacksquare\end{aligned} \tag{19}$$

Frequency marginal property.
The frequency marginal property of the WOL is given by

$$\int_{\mathbb{R}} \mathcal{WOL}_f(t,u)dt = \left|\hat{f}(u)\right|^2. \tag{20}$$

Proof.

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{WOL}_f(t,u)dt &= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)} e^{\frac{ia}{b}\tau t} e^{-\frac{i}{b}(u-u_0)\tau} d\tau dt \\ &= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)} e^{\frac{ia}{2b}\left(t^2+t\tau+\frac{\tau^2}{4}-t^2+t\tau-\frac{\tau^2}{4}\right)} e^{-\frac{i}{b}(u-u_0)\left(t+\frac{\tau}{2}+\frac{\tau}{2}-t\right)} d\tau dt \\ &= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)} e^{\frac{ia}{2b}\left(t+\frac{\tau}{2}\right)^2} e^{-\frac{ia}{2b}\left(t-\frac{\tau}{2}\right)^2} e^{-\frac{i}{b}(u-u_0)\left(t+\frac{\tau}{2}+\frac{\tau}{2}-t\right)} d\tau dt.\end{aligned} \tag{21}$$

Let $\omega = t + \frac{\tau}{2}$ and let $v = t - \frac{\tau}{2}$, then above equation reduces to the final result

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{WOL}_f(t,u)dt &= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f(\omega)\overline{f(v)} e^{\frac{ia}{2b}\omega^2} e^{-\frac{ia}{2b}v^2} e^{\frac{i}{b}u_0(\omega-v)} e^{-\frac{i}{b}u(\omega-v)} d\omega dv \\ &= \left|\hat{f}(u)\right|^2. \blacksquare\end{aligned} \tag{22}$$

Energy distribution property.
The energy distribuition property of the WOL is given as

$$\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)dtdu = \int_{\mathbb{R}} |f(t)|^2 dt. \tag{23}$$

Proof.

$$\begin{aligned}\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)dtdu &= \frac{1}{2\pi|b|}\int_{\mathbb{R}^3} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)} e^{\frac{ia}{b}\tau t} e^{-\frac{i}{b}(u-u_0)\tau} d\tau dt du \\ &= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)} e^{\frac{ia}{b}\tau t} e^{\frac{i}{b}u_0\tau}\left(\int_{\mathbb{R}} e^{-\frac{i}{b}u\tau} du\right) d\tau dt \\ &= \int_{\mathbb{R}} |f(t)|^2 dt. \blacksquare\end{aligned} \tag{24}$$

Moyal's formula.
The Moyal's formula of the WOL is presented as

$$\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)\overline{\mathcal{WOL}_g(t,u)}dtdu = |\langle f, g\rangle|^2. \tag{25}$$

Proof.

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)\overline{\mathcal{WOL}_g(t,u)}dtdu = \\
&= \frac{1}{2\pi|b|}\int_{\mathbb{R}^4} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)}e^{\frac{ia}{b}\tau t}e^{\frac{i}{b}u_0\tau}e^{-\frac{i}{b}u\tau} \\
&\times \frac{1}{2\pi|b|}\overline{g\left(t+\frac{\tau'}{2}\right)}g\left(t-\frac{\tau'}{2}\right)e^{-\frac{ia}{b}\tau' t}e^{-\frac{i}{b}u_0\tau}e^{\frac{i}{b}u\tau'}d\tau d\tau' dtdu \\
&= \frac{1}{2\pi|b|}\int_{\mathbb{R}^3} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)}e^{\frac{ia}{b}\tau t}e^{\frac{i}{b}u_0\tau}e^{-\frac{i}{b}u\tau}d\tau \\
&\times \frac{1}{2\pi|b|}\int_{\mathbb{R}}\overline{g\left(t+\frac{\tau'}{2}\right)}g\left(t-\frac{\tau'}{2}\right)e^{-\frac{ia}{b}\tau' t}e^{-\frac{i}{b}u_0\tau}e^{\frac{i}{b}u\tau'}d\tau' dtdu \\
&= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)}e^{\frac{ia}{b}\tau t}e^{\frac{i}{b}u_0\tau}d\tau \\
&\times \int_{\mathbb{R}}\overline{g\left(t+\frac{\tau'}{2}\right)}g\left(t-\frac{\tau'}{2}\right)e^{-\frac{ia}{b}\tau' t}e^{-\frac{i}{b}u_0\tau}d\tau' dt\,\frac{1}{2\pi|b|}\int_{\mathbb{R}} e^{\frac{i}{b}u(\tau'-\tau)}du \\
&= \frac{1}{2\pi|b|}\int_{\mathbb{R}^2} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)}e^{\frac{ia}{b}\tau t}e^{\frac{i}{b}u_0\tau}d\tau\int_{\mathbb{R}}\overline{g\left(t+\frac{\tau'}{2}\right)}g\left(t-\frac{\tau'}{2}\right)e^{-\frac{ia}{b}\tau' t}e^{-\frac{i}{b}u_0\tau}\delta(\tau-\tau')d\tau' dt \\
&= \frac{1}{2\pi|b|}\int_{\mathbb{R}}\left[\int_{\mathbb{R}} f\left(t+\frac{\tau}{2}\right)\overline{f\left(t-\frac{\tau}{2}\right)g\left(t+\frac{\tau}{2}\right)}g\left(t-\frac{\tau}{2}\right)dt\right]d\tau.
\end{aligned} \tag{26}
$$

Now, we make the change of variable $\mu = t - \frac{\tau}{2}$, and come to

$$
\begin{aligned}
\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)\overline{\mathcal{WOL}_g(t,u)}dtdu &= \frac{1}{2\pi|b|}\int_{\mathbb{R}} f(\mu+\tau)\overline{g(\mu+\tau)}d\tau\overline{\left[\int_{\mathbb{R}} f(\mu)\overline{g(\mu)}d\mu\right]} \\
&= \frac{1}{2\pi|b|}|\langle f,\ g\rangle|^2. \blacksquare
\end{aligned} \tag{27}
$$

Property	Formulation
Conjugation symmetry	$\mathcal{WOL}_f(t,u) = \overline{\mathcal{WOL}_f(t,u)}$
Time shifting	$\mathcal{WOL}_{f'}(t,u) = \mathcal{WOL}_f(t-\lambda, u-a\lambda),\ f'(t) = f(t-\lambda)$
Frequency shifting	$\mathcal{WOL}_{f'}(t,u) = \mathcal{WOL}_f(t, u-u_1 b),\ f'(t) = f(t)e^{iu_1 t}$
Time marginal	$\int_{\mathbb{R}} \mathcal{WOL}_f(t,u)du = \lvert f(t)\rvert^2$
Frequency marginal	$\int_{\mathbb{R}} \mathcal{WOL}_f(t,u)dt = \left\lvert \hat{f}(u)\right\rvert^2$
Energy distribution	$\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)dtdu = \int_{\mathbb{R}} \lvert f(t)\rvert^2 dt$
Moyal's formula	$\int_{\mathbb{R}^2} \mathcal{WOL}_f(t,u)\overline{[\mathcal{WOL}_g(t,\ u)]}dtdu = \frac{1}{2\pi\lvert b\rvert}\lvert\langle f,\ g\rangle\rvert^2$

Table 2.
Properties of the WOL.

Some main properties of WOL are summarized in **Table 2**. The comprehensive view on the WOL can be seen in [17, 18].

5. Conclusion

In this chapter, we thoroughly revised research on hybrid transforms, which are constructed by associating WVD with well-known signal processing tools, such as FrFT, LCT, and OLCT. The WVD-OLCT generalizes most hybrid transforms, and the WOL is its special type. It is proven that hybrid transforms have better output in detection and estimation applications. Since the idea of associating two transforms is novel, it needs deep theoretical analysis and lacks diverse applications. Interested readers can develop hybrid transforms into quaternion and octonion algebra. These studies may be helpful in color image processing.

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