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OPTIMAL QUANTIZATION FOR SOME TRIADIC UNIFORM CANTOR DISTRIBUTIONS WITH EXACT BOUNDS

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Abstract. Let $\{S_j : 1 \leq j \leq 3\}$ be a set of three contractive similarity mappings such that $S_j(x) = rx + \frac{j-1}{2}(1-r)$ for all $x \in \mathbb{R}$, and $1 \leq j \leq 3$, where $0 < r < \frac{1}{3}$. Let $P = \sum_{j=1}^{3} \frac{1}{3} P \circ S_j^{-1}$. Then, P is a unique Borel probability measure on $\mathbb{R}$ such that P has support the Cantor set generated by the similarity mappings S_j for $1 \leq j \leq 3$. Let $r_0 = 0.1622776602$, and $r_1 = 0.2317626315$ (which are ten digit rational approximations of two real numbers). In this paper, for $0 < r \leq r_0$, we give a general formula to determine the optimal sets of n-means and the nth quantization errors for the triadic uniform Cantor distribution P for all positive integers $n \geq 2$. Previously, Roychowdhury gave an exact formula to determine the optimal sets of n-means and the nth quantization errors for the standard triadic Cantor distribution, i.e., when $r = \frac{1}{5}$. In this paper, we further show that $r = r_0$ is the greatest lower bound, and $r = r_1$ is the least upper bound of the range of r-values to which Roychowdhury formula extends. In addition, we show that for $0 < r \leq r_1$ the quantization coefficient does not exist though the quantization dimension exists.

1. Introduction

Let P be a Borel probability measure on $\mathbb{R}^d$, where $d \geq 1$. For a finite set $\alpha \subset \mathbb{R}^d$, write

$$V(P;\alpha) = \int \min_{a \in \alpha} \|x-a\|^2 dP(x), \text{ and } V_n := V_n(P) = \inf\Big\{V(P;\alpha) : \alpha \subset \mathbb{R}^d, \ \text{card}(\alpha) \leq n\Big\},$$

where $\|\cdot\|$ represents the Euclidean norm on $\mathbb{R}^d$. Then, $V(P;\alpha)$ is called the *cost* or *distortion error* for P with respect to the set α, and V_n is called the nth quantization error for P with respect to the squared Euclidean distance. A set $\alpha \subset \mathbb{R}^d$ is called an *optimal set of n-means* for P if $V_n(P) = V(P;\alpha)$. It is well-known that for a continuous Borel probability measure an optimal set of n-means contains exactly n-elements (see [4]). To see some work in the direction of optimal sets of n-means, one is referred to [2, 5, 16]. For theoretical results in quantization we refer to [4, 6–8, 11], and for its promising application see [12, 13]. For a finite set $\alpha \subset \mathbb{R}^d$ and $a \in \alpha$, by $M(a|\alpha)$ we denote the set of all elements in $\mathbb{R}^d$ which are nearest to a among all the elements in α, i.e.,

$$M(a|\alpha) = \{x \in \mathbb{R}^d : \|x-a\| = \min_{b \in \alpha} \|x-b\|\}.$$

$M(a|\alpha)$ is called the *Voronoi region* generated by $a \in \alpha$. On the other hand, the set $\{M(a|\alpha) : a \in \alpha\}$ is called the *Voronoi diagram* or *Voronoi tessellation* of $\mathbb{R}^d$ with respect to the set α.

Definition 1.1. *A set $\alpha \subset \mathbb{R}^d$ is called a centroidal Voronoi tessellation (CVT) with respect to a probability distribution P on $\mathbb{R}^d$, if it satisfies the following two conditions:*

(i) $P(M(a|\alpha) \cap M(b|\alpha)) = 0$ for $a, b \in \alpha$, and $a \neq b$;

(ii) $E(X : X \in M(a|\alpha)) = a$ for all $a \in \alpha$,

where X is a random variable with distribution P, and $E(X : X \in M(a|\alpha))$ represents the conditional expectation of the random variable X given that X takes values in $M(a|\alpha)$.

A Borel measurable partition $\{A_a : a \in \alpha\}$ is called a *Voronoi partition* of $\mathbb{R}^d$ with respect to the probability distribution P, if P-almost surely $A_a \subset M(a|\alpha)$ for all $a \in \alpha$. Let us now state the following proposition (see [3, 4]).

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Proposition 1.2. *Let α be an optimal set of n-means, $a \in \alpha$, and $M(a|\alpha)$ be the Voronoi region generated by $a \in \alpha$, i.e., $M(a|\alpha) = \{x \in \mathbb{R}^d : \|x - a\| = \min_{b\in\alpha} \|x - b\|\}$. Then, for every $a \in \alpha$, (i) $P(M(a|\alpha)) > 0$, (ii) $P(\partial M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$.*

The number $D(P) := \lim\limits_{n\to\infty} \frac{2\log n}{-\log V_n(P)}$, if it exists, is called the *quantization dimension* of the probability measure P. On the other hand, for $s \in (0, +\infty)$, the number $\lim\limits_{n\to\infty} n^{\frac{2}{s}} V_n(P)$, if it exists, is called the s-dimensional *quantization coefficient* for P. To know details about the quantization dimension and the quantization coefficient one is referred to [4].

Let $\{S_j : 1 \leq j \leq 3\}$ be a set of three contractive similarity mappings such that $S_j(x) = rx + \frac{j-1}{2}(1 - r)$ for all $x \in \mathbb{R}$, where $0 < r < \frac{1}{3}$ and $1 \leq j \leq 3$. For any positive integer n, if $\sigma := \sigma_1\sigma_2\cdots\sigma_n \in \{1, 2, 3\}^n$, then we say that σ is a word of length n. By $\{1, 2, 3\}^*$, we denote the set of all words including the empty word $\emptyset$. The empty word $\emptyset$ has length zero. For $\sigma := \sigma_1\sigma_2\cdots\sigma_n \in \{1, 2, 3\}^n$, by S_σ it is meant that $S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}$, and by $a(\sigma)$, we mean $a(\sigma) := S_\sigma(\frac{1}{2})$. For the empty word $\emptyset$, by $S_\emptyset$ it is meant the identity mapping on $\mathbb{R}$. For $\sigma := \sigma_1\sigma_2\cdots\sigma_n \in \{1, 2, 3\}^n$, set $J_\sigma := S_\sigma([0, 1])$. For the empty word $\emptyset$, write $J := J_\emptyset = S_\emptyset([0, 1]) = [0, 1]$. Then, the set $C := \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in\{1,2,3\}^n} J_\sigma$ is known as the *Cantor set* generated by the mappings S_j, and equals the support of the probability measure P given by $P = \sum_{j=1}^3 \frac{1}{3} P \circ S_j^{-1}$. Notice that C satisfies the invariance equality $C = \mathop{\cup}\limits_{j=1}^{3} S_j(C)$ (see [10]). In this paper a Cantor set C, which is generated by a set of three contractive similarity mappings, is called a *triadic Cantor set*, and a probability measure P which has support the triadic Cantor set, is called a *triadic Cantor distribution*. For words $\beta, \gamma, \cdots, \delta$ in $\{1, 2, 3\}^*$, we write

$$a(\beta, \gamma, \cdots, \delta) := E(X|X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta\cup\cdots\cup J_\delta} x dP(x),$$

where X is a random variable with probability distribution P, and $E(X)$ and $V := V(X)$ represent the expectation and the variance of the random variable X. Notice that for any $\omega \in \{1, 2, 3\}^*$, the similarity mapping S_ω is an injective mapping on $\mathbb{R}$; on the other hand, for any discrete subset A of $\mathbb{R}$, the set $S_\omega(A)$ represents the set of values obtained by applying S_ω to each of the elements in A. Let us now give the following two definitions.

Definition 1.3. *For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\beta_2 := \{a(1), a(2, 3)\}$ and $\beta_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\beta_n := \beta_n(I)$ as follows:*

$$\beta_n(I) = \left\{\begin{array}{ll} \{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \bigcup \mathop{\cup}\limits_{\omega\in I} S_\omega(\beta_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\ \{S_\omega(\beta_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \bigcup \mathop{\cup}\limits_{\omega\in I} S_\omega(\beta_3) & \text{if } 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}, \end{array}\right.$$

where $I \subset \{1, 2, 3\}^{\ell(n)}$ is arbitray with $card(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$; and $card(I) = n - 2 \cdot 3^{\ell(n)}$ if $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$.

Definition 1.4. *For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\gamma_2 := \{a(1, 21), a(22, 23, 3)\}$ and $\gamma_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\gamma_n := \gamma_n(I)$ as follows:*

$$\gamma_n(I) = \left\{\begin{array}{ll} \{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \bigcup \mathop{\cup}\limits_{\omega\in I} S_\omega(\gamma_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\ \{S_\omega(\gamma_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \bigcup \mathop{\cup}\limits_{\omega\in I} S_\omega(\gamma_3) & \text{if } 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}, \end{array}\right.$$

where $I \subset \{1, 2, 3\}^{\ell(n)}$ is arbitrary with $card(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$; and $card(I) = n - 2 \cdot 3^{\ell(n)}$ if $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$.

Remark 1.5. In the paper there are several decimal numbers, they are rational approximations of some real numbers up to ten decimal places.

Roychowdhury showed that if $r = \frac{1}{5}$, then the sets γ_n given by Definition 1.3, determine the optimal sets of n-means for all positive integers $n \geq 2$ (see [15]). Proposition 2.5 implies that γ_n forms a CVT if $\frac{1}{79}\left(21 - 2\sqrt{51}\right) \leq r \leq \frac{1}{41}\left(2\sqrt{31} - 1\right)$, i.e., if $0.08502712839 \leq r \leq 0.2472080177$. Thus, we see that the range of r values for which the sets γ_n form the optimal sets of n-means is bounded below by $\frac{1}{79}\left(21 - 2\sqrt{51}\right)$, and bounded above by $\frac{1}{41}\left(2\sqrt{31} - 1\right)$. But, the greatest lower bound and the least upper bound of the range of r values for which the sets γ_n form the optimal sets of n-means were not known. In this paper, in Theorem 5.1 we give an answer of it.

Remark 1.6. Notice that if $r = 0$, then $S_1(x) = 0$, $S_2(x) = \frac{1}{2}$, and $S_3(x) = 1$ for all $x \in \mathbb{R}$, and then the probability measure P becomes a discrete uniform distribution with support $\{0, \frac{1}{2}, 1\}$. Because of that in our study we are assuming that the contractive ratios r are positive.

The arrangement of the paper is as follows: In Section 2, we give the basic preliminaries. In Section 3, we show that the sets β_n form the optimal sets of n-means if $r = \frac{1}{25}$. In Section 4, we prove the following theorem:

Theorem 1.7. *Let $\gamma_n := \gamma_n(I)$ be the set for arbitrary I as defined by Definition 1.4. Let $r_0, r_1 \in (0, \frac{1}{3})$ be the unique real numbers satisfying, respectively, the equations*

$$-\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r+1)} = -\frac{3r^3 - 3r^2 + r - 1}{24(r+1)},$$

$$-\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r+1)} = -\frac{3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121}{2184(r+1)}.$$

Then, $r_0 = 0.1622776602$, and $r_1 = 0.2317626315$. Then, for all $n \geq 3$, the sets γ_n form the optimal sets of n-means for $r = r_0$ and $r = r_1$.

In Theorem 5.1, we show that the sets β_n form the optimal sets of n-means if $0 < r \leq r_0$, and the sets γ_n form the optimal sets of n-means if $r_0 \leq r \leq r_1$. Thus, Theorem 5.1 implies the fact that the greatest lower bound, and the least upper bound of r for which the sets γ_n form the optimal sets of n-means are, respectively, given by $r = r_0$ and $r = r_1$. Notice that for $r = r_0$ both the sets β_n and γ_n form the optimal sets of n-means for P. In addition, in Theorem 5.2, we show that the quantization coefficient for $0 < r \leq r_1$ does not exist though the quantization dimension exists.

2. Preliminaries

As defined in the previous section, let S_j for $1 \leq j \leq 3$ be the contractive similarity mappings on $\mathbb{R}$ given by $S_j(x) = rx + \frac{j-1}{2}(1-r)$ for all $x \in \mathbb{R}$, and $1 \leq j \leq 3$, where $0 < r < \frac{1}{3}$. For $\sigma := \sigma_1\sigma_2\cdots\sigma_k \in \{1,2,3\}^k$ and $\tau := \tau_1\tau_2\cdots\tau_\ell \in \{1,2,3\}^\ell$, by $\sigma\tau := \sigma_1\cdots\sigma_k\tau_1\cdots\tau_\ell$ we mean the word obtained from the concatenation of the words σ and τ. For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \{1,2,3\}^*$, $n \geq 0$, write $p_\sigma := \frac{1}{3^n}$ and $s_\sigma := \frac{1}{r^n}$. Recall that if C is the Cantor set, then $C := \bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\{1,2,3\}^n} J_\sigma$. For $n \geq 1$, the intervals J_σ, where $\sigma \in \{1,2,3\}^n$, are called the *nth level basic intervals* of the Cantor set C.

The following two lemmas are well-known and easy to prove (see [5, 15]).

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$, and P be the probability measure on $\mathbb{R}$ given by $P = \sum_{j=1}^3 \frac{1}{3} P \circ S_j^{-1}$. Then,*

$$\int f(x)dP(x) = \sum_{\sigma\in\{1,2,3\}^k} \frac{1}{3^k} \int f \circ S_\sigma(x)dP(x).$$

Lemma 2.2. *Let X be a random variable with the probability distribution P. Then,*

$$E(X) = \frac{1}{2} \text{ and } V := V(X) = \frac{1-r}{6(r+1)}, \text{ and } \int (x - x_0)^2 dP(x) = V(X) + (x_0 - \frac{1}{2})^2,$$

where $x_0 \in \mathbb{R}$.

The following corollary is useful to obtain the distortion errors.

Corollary 2.3. Let $\sigma \in \{1,2,3\}^k$ for $k \geq 1$, and $x_0 \in \mathbb{R}$. Then,

$$\int_{J_\sigma} (x-x_0)^2 dP(x) = \frac{1}{3^k}\Big(r^{2k}V + (S_\sigma(\frac{1}{2}) - x_0)^2\Big).$$

Proof. By induction, $P = \frac{1}{3}\sum_{j=1}^3 P \circ S_j^{-1}$ implies $P = \sum_{\sigma \in \{1,2,3\}^k} p_\sigma P \circ S_\sigma^{-1}$. Using this fact, Lemma 2.1 and Lemma 2.2, the proof of the corollary follows. □

Proposition 2.4. *Let $\beta_n(I)$ be the set given by Definition 1.3. Then, $\beta_n(I)$ forms a CVT if $0 < r \leq 2-\sqrt{3}$, i.e., if $0 < r \leq 0.2679491924$. Moreover, if $3^{\ell(n)} \leq n \leq 2\cdot 3^{\ell(n)}$, then*

$$V(P, \beta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)}\Big((2\cdot 3^{\ell(n)} - n)V + (n - 3^{\ell(n)})V(P;\beta_2)\Big),$$

and if $2\cdot 3^{\ell(n)} \leq n < 3^{\ell(n)+1}$, then

$$V(P, \beta_n(I)) = \frac{1}{3^{\ell(n)}} \cdot r^{2\ell(n)}\Big((3^{\ell(n)+1} - n)V(P;\beta_2) + (n - 2\cdot 3^{\ell(n)})V(P;\beta_3)\Big).$$

Proof. By the definition, we have $\beta_2 = \{a(1), a(2,3)\}$ and $\beta_3 = \{a(1), a(2), a(3)\}$. Recall that $\beta_n := \beta_n(I)$ is defined for $n \geq 3$, where $I \subset \{1,2,3\}^{\ell(n)}$ with $\text{card}(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2\cdot 3^{\ell(n)}$; and $\text{card}(I) = n - 2\cdot 3^{\ell(n)}$ if $2\cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$. Notice that for $n \geq 3$, if $n \neq 3^{\ell(n)}$ or $n \neq 2\cdot 3^{\ell(n)}$, the subset I can be chosen more than one way. This leads to the fact that if $n \neq 3^{\ell(n)}$ or $n \neq 2\cdot 3^{\ell(n)}$, the sets β_n can be chosen multiple ways. Let us take

$$\begin{aligned}
&\beta_4 = \{a(1), a(2), a(31), a(32,33)\} \text{ (by choosing } I = \{3\}),\\
&\beta_5 = \{a(1), a(21), a(22,23), a(31), a(32,33)\} \text{ (by choosing } I = \{2,3\}),\\
&\beta_6 = \{a(11), a(12,13), a(21), a(22,23), a(31), a(32,33)\} \text{ (where } I = \{1,2,3\}),\\
&\beta_7 = \{a(11), a(12), a(13), a(21), a(22,23), a(31), a(32,33)\} \text{ (by choosing } I = \{1\}).
\end{aligned}$$

Since similarity mappings preserve the ratio of the distances of a point from any other two points, $\beta_n(I)$ will form a CVT if we can show that β_2, β_3, β_4, β_5, β_6, β_7 form a CVT. Recall that $a(1) = E(X : X \in J_1)$ and $a(2,3) = E(X : X \in J_2 \cup J_3)$, and also recall the Definition 1.1. Thus, β_2 will form a CVT if

$$P(M(a(1)|\beta_2) \cap M(a(2,3)|\beta_2)) = 0. \tag{1}$$

Since the basic intervals in the first level are $J_1 := [S_1(0), S_1(1)]$, $J_2 := [S_2(0), S_2(1)]$, and $J_3 := [S_3(0), S_3(1)]$, the relation (1) will be true if

$$S_1(1) \leq \frac{1}{2}\left(a(1) + a(2,3)\right) \leq S_2(0).$$

Similarly, β_3 will form a CVT if $S_i(1) < \frac{1}{2}(a(i) + a(i+1)) < S_{i+1}(0)$ for $i = 1,2$; β_4 will form a CVT if

$$\begin{aligned}
S_1(1) < \frac{1}{2}(a(1) + a(2)) < S_2(0) < S_2(1) < \frac{1}{2}(a(2) + a(31)) < S_{31}(0) < S_{31}(1)\\
< \frac{1}{2}(a(31) + a(32,33)) < S_{32}(0).
\end{aligned}$$

Similarly, we can obtain the inequalities for which β_5, β_6, and β_7 will form a CVT. Due to similarity, combining all the inequalities, we see that they will be true if the following inequalities are true:

$$S_1(1) \leq \frac{1}{2}\left(a(1)+a(2,3)\right) \leq S_2(0),$$
$$S_1(1) \leq \frac{1}{2}\left(a(1)+a(21)\right) \leq S_{21}(0),$$
$$S_{13}(1) \leq \frac{1}{2}(a(12,13)+a(21)) \leq S_{21}(0),$$
$$S_{13}(1) \leq \frac{1}{2}\left(a(13)+a(21)\right) \leq S_{21}(0).$$

Upon some simplification, we see that the above inequalities are true if $0<r\leq 2-\sqrt{3}$, i.e., if $0<r\leq 0.2679491924$. If $3^{\ell(n)}\leq n\leq 2\cdot 3^{\ell(n)}$, then

$$\begin{aligned} V(P;\beta_n(I)) &= \sum_{\sigma\in\{1,2,3\}^{\ell(n)}\setminus I}\int_{J_\sigma}(x-a(\sigma))^2dP+\sum_{\sigma\in I}\int_{J_\sigma}\min_{a\in S_\sigma(\beta_2)}(x-a)^2dP \\ &= \frac{1}{3^{\ell(n)}}r^{2\ell(n)}\Big(\sum_{\sigma\in\{1,2,3\}^{\ell(n)}\setminus I}V+\sum_{\sigma\in I}V(P;\beta_2)\Big) \\ &= \frac{1}{3^{\ell(n)}}\cdot r^{2\ell(n)}\Big((2\cdot 3^{\ell(n)}-n)V+(n-3^{\ell(n)})V(P;\beta_2)\Big). \end{aligned}$$

Similarly, if $2\cdot 3^{\ell(n)}\leq n<3^{\ell(n)+1}$, then

$$V(P,\beta_n(I))=\frac{1}{3^{\ell(n)}}\cdot r^{2\ell(n)}\Big((3^{\ell(n)+1}-n)V(P;\beta_2)+(n-2\cdot 3^{\ell(n)})V(P;\beta_3)\Big).$$

Thus, the proof of the proposition is complete. □

Proposition 2.5. *Let $\gamma_n(I)$ be the set given by Definition 1.4. Then, $\gamma_n(I)$ forms a CVT if $\frac{1}{79}\left(21-2\sqrt{51}\right)\leq r\leq\frac{1}{41}\left(2\sqrt{31}-1\right)$, i.e., if $0.08502712839\leq r\leq 0.2472080177$. Moreover, if $3^{\ell(n)}\leq n\leq 2\cdot 3^{\ell(n)}$, then*

$$V(P,\gamma_n(I))=\frac{1}{3^{\ell(n)}}\cdot r^{2\ell(n)}\Big((2\cdot 3^{\ell(n)}-n)V+(n-3^{\ell(n)})V(P;\gamma_2)\Big),$$

and if $2\cdot 3^{\ell(n)}\leq n<3^{\ell(n)+1}$, then

$$V(P,\gamma_n(I))=\frac{1}{3^{\ell(n)}}\cdot r^{2\ell(n)}\Big((3^{\ell(n)+1}-n)V(P;\gamma_2)+(n-2\cdot 3^{\ell(n)})V(P;\gamma_3)\Big).$$

Proof. By the definition, we have $\gamma_2=\{a(1,21),a(22,23,3)\}$ and $\gamma_3=\{a(1),a(2),a(3)\}$. For $n\geq 3$, if $n\neq 3^{\ell(n)}$ or $n\neq 2\cdot 3^{\ell(n)}$, the subset I can be chosen more than one way. This leads to the fact that if $n\neq 3^{\ell(n)}$ or $n\neq 2\cdot 3^{\ell(n)}$, the sets γ_n can be chosen multiple ways. Proceeding in the similar way, as Proposition 2.4, let us choose

$$\begin{aligned} &\gamma_4=\{a(1),a(2),a(31,321),a(322,323,33)\} \\ &\gamma_5=\{a(1),a(21,221),a(222,223,23),a(31,321),a(322,323,33)\} \\ &\gamma_6=\{a(11,121),a(122,123,13),a(21,221),a(222,223,23),a(31,321),a(322,323,33)\} \\ &\gamma_7=\{a(11),a(12),a(13),a(21,221),a(222,223,23),a(31,321),a(322,323,33)\}. \end{aligned}$$

Due to the same reasoning as described in the proof of Proposition 2.4, to show $\gamma_n(I)$ forms a CVT, it is enough to prove that the following inequalities are true:

$$\begin{aligned} S_{21}(1) &\leq \frac{1}{2}\left((a(1,21)+a(22,23,3)\right) \leq S_{22}(0),\\ S_1(1) &\leq \frac{1}{2}\left(a(1)+a(21,221)\right) \leq S_{21}(0),\\ S_{13}(1) &\leq \frac{1}{2}(a(122,123,13)+a(21,221)) \leq S_{21}(0),\\ S_{13}(1) &\leq \frac{1}{2}\left(a(13)+a(21,221)\right) \leq S_{21}(0). \end{aligned}$$

Upon some simplification, we see that the above inequalities are true if $\frac{1}{79}\left(21-2\sqrt{51}\right) \leq r \leq \frac{1}{41}\left(2\sqrt{31}-1\right)$, i.e., if $0.08502712839 \leq r \leq 0.2472080177$. The rest of the proof follows in the similar way as it is given for $V(P;\beta_n)$ in Proposition 2.4. Thus, the proof of the proposition is complete. □

Definition 2.6. *For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\delta_2 := \{a(1,21,221), a(222,223,23,3)\}$ and $\delta_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\delta_n := \delta_n(I)$ as follows:*

$$\delta_n(I) = \left\{\begin{array}{ll} \{a(\omega) : \omega \in \{1,2,3\}^{\ell(n)} \setminus I\} \bigcup \underset{\omega\in I}{\cup} S_\omega(\delta_2) & \text{if } 3^{\ell(n)} \leq n \leq 2\cdot 3^{\ell(n)},\\ \{S_\omega(\delta_2) : \omega \in \{1,2,3\}^{\ell(n)} \setminus I\} \bigcup \underset{\omega\in I}{\cup} S_\omega(\delta_3) & \text{if } 2\cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}, \end{array}\right.$$

where $I \subset \{1,2,3\}^{\ell(n)}$ with $card(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2\cdot 3^{\ell(n)}$; and $card(I) = n - 2\cdot 3^{\ell(n)}$ if $2\cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$.

Proposition 2.7. *Let $\delta_n(I)$ be the set given by Definition 2.6. Then, $\delta_n(I)$ forms a CVT if $0.1845020699 \leq r \leq 0.2705731187$. Moreover, if $3^{\ell(n)} \leq n \leq 2\cdot 3^{\ell(n)}$, then*

$$V(P,\delta_n(I)) = \frac{1}{3^{\ell(n)}}\cdot r^{2\ell(n)}\Big((2\cdot 3^{\ell(n)}-n)V + (n-3^{\ell(n)})V(P;\delta_2)\Big),$$

and if $2\cdot 3^{\ell(n)} \leq n < 3^{\ell(n)+1}$, then

$$V(P,\delta_n(I)) = \frac{1}{3^{\ell(n)}}\cdot r^{2\ell(n)}\Big((3^{\ell(n)+1}-n)V(P;\delta_2) + (n-2\cdot 3^{\ell(n)})V(P;\delta_3)\Big).$$

Proof. By the definition, we have $\delta_2 = \{a(1,21,221), a(222,223,23,3)\}$ and $\delta_3 = \{a(1), a(2), a(3)\}$. For $n \geq 3$, if $n \neq 3^{\ell(n)}$ or $n \neq 2\cdot 3^{\ell(n)}$, the subset I can be chosen more than one way. This leads to the fact that if $n \neq 3^{\ell(n)}$ or $n \neq 2\cdot 3^{\ell(n)}$, the sets δ_n can be chosen multiple ways. Proceeding in the similar way, as Proposition 2.4, let us choose

$$\begin{aligned} \delta_4 &= \{a(1), a(2), a(31,321,3221), a(3222,3223,323,33)\}\\ \delta_5 &= \{a(1), a(21,221,2221), a(2222,2223,223,23),\\ &\qquad a(31,321,3221), a(3222,3223,323,33)\}\\ \delta_6 &= \{a(11,121,1221), a(1222,1223,123,13), a(21,221,2221), a(2222,2223,223,23),\\ &\qquad a(31,321,3221), a(3222,3223,323,33)\}\\ \delta_7 &= \{a(11), a(12), a(13), a(21,221,2221), a(2222,2223,223,23),\\ &\qquad a(31,321,3221), a(3222,3223,323,33)\}. \end{aligned}$$

Due to the same reasoning as described in the proof of Proposition 2.4, to show $\delta_n(I)$ forms a CVT, it is enough to prove that the following inequalities are true:

$$S_{221}(1) \leq \frac{1}{2}\left(a(1,21,221) + a(222,223,23,3)\right) \leq S_{222}(0),$$
$$S_1(1) \leq \frac{1}{2}\left(a(1) + a(21,221,2221)\right) \leq S_{21}(0),$$
$$S_{13}(1) \leq \frac{1}{2}(a(1222,1223,123,13) + a(21,221,2221)) \leq S_{21}(0),$$
$$S_{13}(1) \leq \frac{1}{2}\left(a(13) + a(21,221,2221)\right) \leq S_{21}(0).$$

The above inequalities are true if $0.1845020699 \leq r \leq 0.2705731187$. The rest of the proof follows in the similar way as it is given for $V(P;\beta_n(I))$ in Proposition 2.4. Thus, the proof of the proposition is complete. □

The following proposition is useful to establish Lemma 3.1, and Lemma 4.1.

Proposition 2.8. *Let $\kappa := \{a_1, a_2\}$, where $a_1 := E(X : X \in [0, \frac{1}{2}])$, and $a_2 := E(X : X \in [\frac{1}{2}, 1])$. Then, $a_1 = \frac{r+1}{6-2r}$, and $a_2 = \frac{5-3r}{6-2r}$, and the corresponding distortion error is given by*

$$V(P;\kappa) = \frac{-7r^3 + 13r^2 - 9r + 3}{6(r-3)^2(r+1)}.$$

Proof. By the hypothesis, we have

$$a_1 = E(X : X \in [0, \frac{1}{2}]) = E\Big(X : X \in J_1 \cup J_{21} \cup J_{221} \cup \cdots\Big), \text{ and}$$
$$a_2 = E(X : X \in [\frac{1}{2}, 1]) = E\Big(X : X \in J_3 \cup J_{23} \cup J_{223} \cup \cdots\Big),$$

yielding

$$a_1 = 2\sum_{n=1}^{\infty} \frac{1}{3^n}\frac{1}{2}(-r^{n-1} + r^n + 1) = \frac{r+1}{6-2r}, \text{ and } a_2 = 2\sum_{n=1}^{\infty} \frac{1}{3^n}\frac{1}{2}(r^{n-1} - r^n + 1) = \frac{5-3r}{6-2r},$$

and the corresponding distortion error is given by

$$V(P;\kappa) = 2\int_{J_1\cup J_{21}\cup J_{221}\cup J_{2221}\cdots} \Big(x - \frac{r+1}{6-2r}\Big)^2 dP$$

implying

$$V(P;\kappa) = 2\Big(\sum_{n=1}^{\infty} \frac{r^{2n}}{3^n}V + \sum_{n=1}^{\infty} \frac{1}{3^n}\Big(\frac{1}{2}\left(-r^{n-1} + r^n + 1\right) - \frac{r+1}{6-2r}\Big)^2\Big) = \frac{-7r^3 + 13r^2 - 9r + 3}{6(r-3)^2(r+1)}.$$

Thus, the proposition is yielded. □

3. Optimal sets of n-means and the nth quantization errors for $r = \frac{1}{25}$

Let β_n be the set given by Definition 1.3. In this section, we show that for all $n \geq 2$, the sets β_n form the optimal sets of n-means for $r = \frac{1}{25}$. To calculate the distortion errors we will frequently use the formula given by Corollary 2.3. Notice that by Lemma 2.2, in this case, we have $E(X) = \frac{1}{2}$ and $V := V(X) = \frac{1-r}{6(r+1)} = \frac{2}{13}$.

Lemma 3.1. *The set $\beta := \{a(1), a(2,3)\}$ forms the optimal set of two-means, and the corresponding quantization error is given by $V_2 = \frac{314}{8125} = 0.0386462$.*

Proof. Let $\beta := \{a_1, a_2\}$ be an optimal set of two-means. Since the points in an optimal set are the conditional expectations in their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < 1$. Let us consider the set $\kappa := \{a(1), a(2,3)\}$. The distortion error due to the set κ is given by

$$V(P;\kappa) = \int_{J_1} (x-a(1))^2 dP + \int_{J_2\cup J_3} (x-a(2,3))^2 dP = 0.0386462. \tag{2}$$

Since V_2 is the quantization for two-means, we have $V_2 \leq 0.0386462$. Assume that $0.38 < a_1$. Then,

$$V_2 \geq \int_{J_1} (x-0.38)^2 dP = 0.0432821 > V_2,$$

which is a contradiction. Hence, $a_1 \leq 0.38$. Similarly, $0.62 \leq a_2$. Since $\frac{1}{2}(a_1+a_2) \leq \frac{1}{2}(0.38+1) = 0.69 < S_3(0) = 0.96$, the Voronoi region of a_1 does not contain any point from J_3. Similarly, the Voronoi region of a_2 does not contain any point from J_1. Since the union of the Voronoi regions of a_1 and a_2 covers $J_1 \cup J_2 \cup J_3$, without any loss of generality, we can assume that the Voronoi region of a_2 contains points from J_2, and $\frac{1}{2}(a_1+a_2) \leq \frac{1}{2}$. If $\frac{1}{2}(a_1+a_2) = \frac{1}{2}$, then substituting $r = \frac{1}{25}$, by Proposition 2.8, we have

$$V_2 = \frac{866}{17797} = 0.0486599 > V_2,$$

which leads to a contradiction. Hence, we can conclude that $\frac{1}{2}(a_1+a_2) < \frac{1}{2}$. Using the similar technique as it is given in the proof of Lemma 3.1 in [15], we can show that $S_1(1) \leq \frac{1}{2}(a_1+a_2) \leq S_2(0)$ yielding the fact that $a_1 = a(1)$, $a_2 = a(2,3)$, and $V_2 = \frac{314}{8125} = 0.0386462$. Hence, the proof of the lemma is complete. ☐

Lemma 3.2. *The set $\beta := \{a(1), a(2), a(3)\}$ forms an optimal set of three-means, and the corresponding quantization error is given by $V_3 = \frac{2}{8125} = 0.000246154$.*

Proof. Consider the set of three points $\kappa := \{a(1), a(2), a(3)\}$. The distortion error due to the set κ is given by

$$V(P;\kappa) = \sum_{j=1}^{3} \int_{J_j} (x-a(j))^2 dP = \frac{2}{8125} = 0.000246154.$$

Since V_3 is the quantization error for three-means, we have $V_3 \leq 0.000246154$. Let $\beta := \{a_1, a_2, a_3\}$, where $0 < a_1 < a_2 < a_3 < 1$, be an optimal set of three-means. If $S_1(1) = \frac{1}{25} < \frac{1}{23} < a_1$, then

$$V_3 \geq \int_{J_1} (x-\frac{1}{23})^2 dP = \frac{13709}{51577500} = 0.000265794 > V_3,$$

which gives a contradiction. Thus, we can assume that $a_1 \leq \frac{1}{23}$. Similarly, $\frac{22}{23} \leq a_3$. Suppose that $\beta \cap J_1 = \emptyset$. Then, due to symmetry, we can assume that $\beta \cap J_3 = \emptyset$, and then

$$V_3 \geq 2\int_{J_1} (x-a_1)^2 dP = 2\int_{J_1} (x-S_1(1))^2 dP = \frac{7}{16250} = 0.000430769 > V_3,$$

which leads to a contradiction. So, we can assume that $\beta \cap J_1 \neq \emptyset$, i.e., $a_1 < S_1(1)$. Similarly, $\beta \cap J_3 \neq \emptyset$, i.e., $S_3(0) < a_3$. Now, we show that $\beta \cap J_2 \neq \emptyset$. Suppose that $\beta \cap J_2 = \emptyset$. Then, either $a_2 < \frac{12}{25} = S_2(0)$, or $\frac{13}{25} = S_2(1) < a_2$. First, assume that $a_2 < S_2(0)$. Then, notice that $S_2(1) = \frac{13}{25} < \frac{1}{2}(S_2(0)+S_3(0)) < S_3(0)$ yielding the fact that the Voronoi region of $S_2(0)$ contains J_2. Hence,

$$V_3 \geq \int_{J_2} (x-S_2(0))^2 dP + \int_{J_3} (x-a(3))^2 dP = \frac{29}{97500} = 0.000297436 > V_3,$$

which is a contradiction. Similarly, we can show that a contradiction arises if $\frac{13}{25} = S_2(1) < a_2$. Thus, we can assume that $\beta \cap J_2 \neq \emptyset$. Now, if the Voronoi region of a_1 contains points from J_2, we have $\frac{1}{2}(a_1 + a_2) > \frac{12}{25} = S_2(0)$ implying $a_2 > \frac{24}{25} - a_1 \geq \frac{24}{25} - \frac{1}{25} = \frac{23}{25} > S_2(1)$, which is a contradiction as $\beta \cap J_2 \neq \emptyset$. Hence, we can assume that the Voronoi region of a_1 does not contain any point from J_2, and so from J_3. Similarly, we can show that the Voronoi region of a_2 does not contain any point from J_1 and J_3, and the Voronoi region of a_3 does not contain any point from J_2, and so from J_1. Thus, by Proposition 1.2, we conclude that $a_1 = a(1)$, $a_2 = a(2)$, and $a_3 = a(3)$, and the corresponding quantization error is given by $V_3 = \frac{2}{8125} = 0.000246154$, which is the lemma. □

Proposition 3.3. *Let β_n be an optimal set of n-means for any $n \geq 3$. Then, $\beta_n \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$, and β_n does not contain any point from the open intervals $(S_1(1), S_2(0))$ and $(S_2(1), S_3(0))$. Moreover, the Voronoi region of any point in $\beta_n \cap J_j$ does not contain any point from J_i, where $1 \leq i \neq j \leq 3$.*

Proof. By Lemma 3.2, the proposition is true for $n = 3$. Let us prove the lemma for $n \geq 4$. Let $\beta_n := \{a_1, a_2, \cdots, a_n\}$ be an optimal set of n-means for $n \geq 4$. Since the points in an optimal set are the conditional expectations in their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < \cdots < a_n < 1$. Consider the set of four elements $\kappa := S_1(\beta_2) \cup \{a(2), a(3)\}$. Then,

$$V(P;\kappa) = \int_{J_1} \min_{a \in S_1(\beta_2)} (x-a)^2 dP + \int_{J_2} (x-a(2))^2 dP + \int_{J_3} (x-a(3))^2 dP = \frac{938}{5078125} = 0.000184714.$$

Since V_n is the quantization error for n-means for $n \geq 4$, we have $V_n \leq V_4 \leq 0.000184714$. Suppose that $S_1(1) \leq a_1$. Then,

$$V_n \geq \int_{J_1} (x - S_1(1))^2 dP = \frac{7}{32500} = 0.000215385 > V_n,$$

which is a contradiction. So, we can assume that $a_1 < S_1(1)$, i.e., $\beta_n \cap J_1 \neq \emptyset$. Similarly, $\beta_n \cap J_3 \neq \emptyset$. We now show that $\beta_n \cap J_2 \neq \emptyset$. For the sake of contradiction, assume that $\beta_n \cap J_2 = \emptyset$. Let $a_j := \max\{a_i : a_i < S_2(0) \text{ for } 1 \leq i \leq n-1\}$. Then, $a_j < S_2(0)$. As $\beta_n \cap J_2 = \emptyset$, we have $S_2(1) < a_{j+1}$. If $a_j < \frac{1}{2}(S_1(1) + S_2(0)) = \frac{13}{50}$, then as $\frac{1}{2}(a_j + a_{j+1}) < \frac{1}{2}(\frac{13}{50} + S_2(1)) = \frac{39}{100} < \frac{12}{25} = S_2(0)$, we have

$$V_n \geq \int_{J_2} (x - S_2(1))^2 dP = \frac{7}{32500} = 0.000215385 > V_n,$$

which leads to a contradiction. So, we can assume that $\frac{13}{50} \leq a_j < S_2(0)$. Then, by Proposition 1.2, we have $\frac{1}{2}(a_{j-1} + a_j) < \frac{1}{25}$ implying $a_{j-1} < \frac{2}{25} - a_j \leq \frac{2}{25} - \frac{13}{50} = -\frac{9}{50} < 0$, which gives a contradiction as $\beta_n \cap J_1 \neq \emptyset$. Hence, we can conclude that $\beta_n \cap J_2 \neq \emptyset$. Notice that $(S_1(1), S_2(0)) = (\frac{1}{25}, \frac{12}{25})$. Suppose that β_n contains a point from the open interval $(\frac{1}{25}, \frac{12}{25})$. Let $a_j := \max\{a_i : a_i < \frac{1}{25} \text{ for } 1 \leq i \leq n-2\}$. Then, due to Proposition 1.2, $a_{j+1} \in (\frac{1}{25}, \frac{12}{25})$, and $a_{j+2} \in J_2$. The following cases can arise:

Case 1. $\frac{1}{25} < a_{j+1} \leq \frac{13}{50}$.

Then, $\frac{1}{2}(a_{j+1} + a_{j+2}) > \frac{12}{25}$ implying $a_{j+2} > \frac{24}{25} - a_{j+1} \geq \frac{24}{25} - \frac{13}{50} = \frac{35}{50} > S_2(1)$, which leads to a contradiction because $a_{j+2} \in J_2$.

Case 2. $\frac{13}{50} \leq a_{j+1} < \frac{12}{25}$.

Then, $\frac{1}{2}(a_j + a_{j+1}) < \frac{1}{25}$ implying $a_j \leq \frac{2}{25} - a_{j+1} \leq \frac{2}{25} - \frac{13}{50} = -\frac{9}{50}$, which is a contradiction because $a_j > 0$.

Thus, by Case 1 and Case 2, we can conclude that β_n does not contain any point from the open interval $(S_1(1), S_2(0))$. Reflecting the situation with respect to the point $\frac{1}{2}$, we can conclude that β_n does not contain any point from the open interval $(S_2(1), S_3(0))$ as well. To prove the last part of the proposition, we proceed as follows: Let $a_j := \max\{a_i : a_i < \frac{1}{25} \text{ for } 1 \leq i \leq n-2\}$. Then, a_j is the rightmost element in $\beta_n \cap J_1$, and $a_{j+1} \in \beta_n \cap J_2$. Suppose that the Voronoi region of a_j

contains points from J_2. Then, $\frac{1}{2}(a_j+a_{j+1}) > \frac{12}{25}$ implying $a_{j+1} > \frac{24}{25}-a_j \geq \frac{24}{25}-\frac{1}{25} = \frac{23}{25} > S_2(1)$, which yields a contradiction as $a_{j+1} \in J_2$. Thus, the Voronoi region of any point in $\beta_n \cap J_1$ does not contain any point from J_2, and J_3 as well. Similarly, we can prove that the Voronoi region of any point in $\beta_n \cap J_2$ does not contain any point from J_1 and J_3, and the Voronoi region of any point in $\beta_n \cap J_3$ does not contain any point from J_1 and J_2. Thus, the proof of the proposition is complete. □

The following lemma is a modified version of Lemma 4.5 in [5], and the proof follows similarly. One can also see Lemma 3.5 in [15].

Lemma 3.4. *Let $n \geq 3$, and let β_n be an optimal set of n-means such that $\beta_n \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$, and β_n does not contain any point from the open intervals $(S_1(1), S_2(0))$ and $(S_2(1), S_3(0))$. Further assume that the Voronoi region of any point in $\beta_n \cap J_j$ does not contain any point from J_i, where $1 \leq i \neq j \leq 3$. Set $\kappa_j := \beta_n \cap J_j$, and $n_j :=$ card (κ_j) for $1 \leq j \leq 3$. Then, $S_j^{-1}(\kappa_j)$ is an optimal set of n_j-means, and $V_n = \frac{1}{1875}(V_{n_1} + V_{n_2} + V_{n_3})$.*

Let us now state and prove the following theorem which gives the optimal sets of n-means for all $n \geq 3$, where $r = \frac{1}{25}$.

Theorem 3.5. *Let P be the probability measure on $\mathbb{R}$ with support the Cantor set C generated by the three contractive similarity mappings S_j for $j = 1, 2, 3$. Let $n \in \mathbb{N}$ with $n \geq 3$. Take $r = \frac{1}{25}$. Then, the sets $\beta_n := \beta_n(I)$ given by Definition 1.3 form the optimal sets of n-means for P with the corresponding quantization error $V_n := V(P; \beta_n(I))$, where $V(P; \beta_n(I))$ is given by Proposition 2.4.*

Proof. We will proceed by induction on $\ell(n)$. If $n = 3$, then by Lemma 3.2, the theorem is true. Now, we show that the theorem is true if $n = 4$. Let $\kappa_j := \beta_n \cap J_j$, and $n_j :=$ card (κ_j) for $1 \leq j \leq 3$. Since $S_j^{-1}(\kappa_j)$ is an optimal set of n_j-means for $1 \leq j \leq 3$, and for $n = 4$ the possible choices for the triplet (n_1, n_2, n_3) are $(2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$, by Proposition 3.3 and Lemma 3.4, the set β_4 forms an optimal set of four-means with quantization error $V(P; \beta_4)$ given by Proposition 2.4. Remember that for a given n, among all the possible choices of the triplets (n_1, n_2, n_3), the triplets (n_1, n_2, n_3) which give the smallest distortion error will give the optimal sets of n-means. Notice that for $n = 5$, the possible choices of the triplets are $(3, 1, 1)$, $(1, 3, 1)$, $(1, 1, 3)$, $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$ among which $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$ give the smallest distortion error. Hence, the optimal sets of five-means are $\{a(1)\} \cup S_2(\beta_2) \cup S_3(\beta_2)$, $S_1(\beta_2) \cup \{a(2)\} \cup S_3(\beta_2)$, and $S_1(\beta_2) \cup S_2(\beta_2) \cup \{a(3)\}$ which are the sets β_5 given by Definition 1.3. Similarly, we can calculate the optimal sets of six- and seven-means. Thus, the theorem is true for $\ell(n) = 1$. Let us assume that the theorem is true for all $\ell(n) < m$, where $m \in \mathbb{N}$ and $m \geq 2$. We now show that the theorem is true if $\ell(n) = m$. Let us first assume that $3^m \leq n \leq 2 \cdot 3^m$. Let β_n be an optimal set of n-means for P such that $3^m \leq n \leq 2 \cdot 3^m$. Let card $(\beta_n \cap J_j) = n_j$ for $j = 1, 2, 3$, and then by Lemma 3.4, we have

$$V_n = \frac{1}{1875}(V_{n_1} + V_{n_2} + V_{n_3}). \tag{3}$$

Without any loss of generality, we can assume that $n_1 \geq n_2 \geq n_3$. Let $u, v, w \in \mathbb{N}$ be such that

$$3^u \leq n_1 \leq 2 \cdot 3^u,\ 3^v \leq n_2 \leq 2 \cdot 3^v,\ \text{and } 3^w \leq n_3 \leq 2 \cdot 3^w. \tag{4}$$

Proceeding in the similar lines as the proof of Theorem 3.6 in [15], we can show that $u = v = w = m - 1$. Since by Lemma 3.4, for $S_j^{-1}(\beta_n \cap J_j)$ is an optimal set of n_j means where $3^{m-1} \leq n_j \leq 2 \cdot 3^{m-1}$, we have

$$S_j^{-1}(\beta_n \cap J_j) = \{a(\omega) : \omega \in \{1, 2, 3\}^{m-1} \setminus I_j\} \cup \left(\cup_{\omega \in I_j} S_\omega(\beta_2)\right),$$

where $I_j \subseteq \{1, 2, 3\}^{m-1}$ with card $(I_j) = n_j - 3^{m-1}$ for $1 \leq j \leq 3$. Hence,

$$\beta_n := \beta_n(I) = \bigcup_{j=1}^{3} S_j^{-1}(\beta_n \cap J_j) = \{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \left(\cup_{\omega \in I} S_\omega(\beta_2)\right),$$

where $I \subseteq \{1,2,3\}^m$ with card $(I) = n - 3^m$, is an optimal set of n-means. The corresponding quantization error is

$$V_n = \frac{1}{3^m} r^{2m} \left((2 \cdot 3^m - n)V + (n - 3^m)V_2\right) = V(P; \beta_n(I)),$$

where $V(P; \beta_n(I))$ is given by Proposition 2.4. Thus, the theorem is true if $3^m \leq n \leq 2 \cdot 3^m$. Similarly, we can prove that the theorem is true if $2 \cdot 3^m < n < 3^{m+1}$. Hence, by the induction principle, the proof of the theorem is complete. □

4. Optimal sets of n-means and the nth quantization errors for $r = r_0$ and $r = r_1$

In this section, we give the proof of Theorem 1.7. First, we prove the following two lemmas.

Lemma 4.1. *Let r_0 and r_1 be the real numbers given by Theorem 1.7. Then, the set $\gamma := \{a(1,21), a(22,23,3)\}$ for $r = r_0$ and $r = r_1$ form the optimal sets of two-means, and the corresponding quantization errors are, respectively, given by $V_2 = 0.0324042$, and $V_2 = 0.026897$.*

Proof. First, we prove that γ forms an optimal set of two-means for $r = r_0$. Let $\gamma := \{a_1, a_2\}$ be an optimal set of two-means. Since, the points in an optimal set are the expected values of their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < 1$. Let us consider the set $\kappa := \{a(1,21), a(22,23,3)\}$. The distortion error due to the set κ is given by

$$V(P;\kappa) = \int_{J_1} (x - a(1,21))^2 dP + \int_{J_2 \cup J_3} (x - a(22,23,3))^2 dP = 0.0324042. \tag{5}$$

Since V_2 is the quantization error for two-means, we have $V_2 \leq 0.0324042$. Assume that $0.39 < a_1$. Then,

$$V_2 \geq \int_{J_1} (x - 0.39)^2 dP = 0.0328529 > V_2,$$

which is a contradiction. Hence, $a_1 \leq 0.39$. Similarly, $0.61 \leq a_2$. Since $\frac{1}{2}(a_1+a_2) \leq \frac{1}{2}(0.39+1) = 0.695 < S_3(0) = 0.837722$, the Voronoi region of a_1 does not contain any point from J_3. Similarly, the Voronoi region of a_2 does not contain any point from J_1. Since the union of the Voronoi regions of a_1 and a_2 covers $J_1 \cup J_2 \cup J_3$, without any loss of generality, we can assume that the Voronoi region of a_2 contains points from J_2, and $\frac{1}{2}(a_1 + a_2) \leq \frac{1}{2}$. If $\frac{1}{2}(a_1 + a_2) = \frac{1}{2}$, then substituting $r = 0.1622776602$, by Proposition 2.8, we have

$$V(P;\kappa) = 0.0329779,$$

which contradicts (5). Hence, we can conclude that $\frac{1}{2}(a_1 + a_2) < \frac{1}{2}$. Using the similar technique as it is given in the proof of Lemma 3.1 in [15], we can show that either $\frac{1}{2}(a_1+a_2) = \frac{1}{2}(a(1,21)+ a(22,23,3)) = 0.466886$, or $\frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a(1) + a(2,3)) = 0.395285$, i.e., either $S_{21}(1) < \frac{1}{2}(a_1 + a_2) < S_{22}(0)$, or $S_1(1) < \frac{1}{2}(a_1 + a_2) < S_2(0)$. Notice that if $S_{21}(1) < \frac{1}{2}(a_1 + a_2) < S_{22}(0)$, then γ_2, given by Definition 1.4, forms the optimal set of two-means. On the other hand, if $S_1(1) < \frac{1}{2}(a_1 + a_2) < S_2(0)$, then β_2, given by Definition 1.3, forms the optimal set of two-means. In fact, later we will see that $V(P;\gamma_2) = V(P;\beta_2) = 0.0324042$ for $r = 0.1622776602$. Thus, γ_2 forms the optimal set of two-means for $r = r_0$ with quantization error $V_2 = 0.0324042$. Similarly, we can show that γ_2 forms the optimal set of two-means if $r = r_1$ with quantization error $V_2 = 0.026897$. Hence, the lemma is yielded. □

The following lemma is true analogously as Lemma 3.3 in [15].

Lemma 4.2. *The set $\gamma_3 := \{a(1), a(2), a(3)\}$ for $r = r_0$, and $r = r_1$ form the optimal sets of three-means, and the corresponding quantization errors are, respectively, given by $V_3 = 0.00316342$, and $V_3 = 0.00558347$.*

The following proposition is true analogously as Proposition 3.5 in [15].

Proposition 4.3. *Let $n \geq 3$, and let γ_n be an optimal set of n-means for $r = r_0$, and $r = r_1$. Then, $\gamma_n \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$, and γ_n does not contain any point from the open intervals $(S_1(1), S_2(0))$ and $(S_2(1), S_3(0))$. Moreover, the Voronoi region of any point in $\gamma_n \cap J_j$ does not contain any point from J_i, where $1 \leq i \neq j \leq 3$.*

The following remark is true due to Proposition 4.3.

Remark 4.4. Let $n \geq 3$, and let γ_n be an optimal set of n-means for $r = r_0$, and $r = r_1$. Set $\kappa_j := \gamma_n \cap J_j$, and $n_j :=$ card (κ_j) for $1 \leq j \leq 3$. Then, $S_j^{-1}(\kappa_j)$ is an optimal set of n_j-means, and for $r = r_0$ and $r = r_1$, respectively, we have $V_n = \frac{1}{3} r_0^n (V_{n_1} + V_{n_2} + V_{n_3})$ and $V_n = \frac{1}{3} r_1^n (V_{n_1} + V_{n_2} + V_{n_3})$.

Proof of Theorem 1.7. We proceed to prove it by induction on $\ell(n)$. By Lemma 4.2, we see that the theorem is true for $n = 3$. Proceeding in the similar way, as mentioned in the proof of Theorem 3.5, we can show that for $n = 4, 5, 6, 7$, the sets γ_n form the optimal sets of n-means for $r = r_0$ and $r = r_1$. Thus, the theorem is true if $\ell(n) = 1$. Let us assume that the theorem is true for all $\ell(n) < m$, where $m \in \mathbb{N}$ and $m \geq 2$. We now show that the theorem is true if $\ell(n) = m$. Let us first assume that $3^m \leq n \leq 2 \cdot 3^m$. Let γ_n be an optimal set of n-means for P such that $3^m \leq n \leq 2 \cdot 3^m$. Let card $(\gamma_n \cap J_j) = n_j$ for $j = 1, 2, 3$, and then by Remark 4.4, we have

$$V_n = \frac{1}{3} r_0^n (V_{n_1} + V_{n_2} + V_{n_3}) \text{ for } r = r_0, \text{ and } V_n = \frac{1}{3} r_1^n (V_{n_1} + V_{n_2} + V_{n_3}) \text{ for } r = r_1.$$

The rest of the proof for $r = r_0$ and $r = r_1$ follow in the similar way as the proof of Theorem 3.5. Thus, we complete the proof of the theorem. □

5. Main results

The two theorems in this section, state and prove the main results of the paper.

Theorem 5.1. *Let $r_0, r_1 \in (0, \frac{1}{3})$ be the unique real numbers satisfying, respectively, the equations*

$$-\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r+1)} = -\frac{3r^3 - 3r^2 + r - 1}{24(r+1)},$$
$$-\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r+1)} = -\frac{3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121}{2184(r+1)}.$$

Then, $r_0 = 0.1622776602$, and $r_1 = 0.2317626315$. Let the sets β_n and γ_n be, respectively, given by Definition 1.3, and Definition 1.4. Then, β_n form the optimal sets of n-means for $0 < r \leq r_0$, and γ_n forms the optimal sets of n-means for $r_0 \leq r \leq r_1$.

Proof. By Proposition 2.4, Proposition 2.5, and Proposition 2.7, we see that both β_n and γ_n form CVTs if $0.08502712839 \leq r \leq 0.2472080177$; both γ_n and δ_n form CVTs if $0.1845020699 \leq r \leq 0.2472080177$; both β_n and δ_n form CVTs if $0.1845020699 \leq r \leq 0.2679491924$. Again, $V(P; \beta_3) = V(P; \gamma_3) = V(P; \delta_3)$. Thus, for any $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$, from the aforementioned propositions, in the case of $V(P; \beta_n(I))$ and $V(P; \gamma_n(I))$, we see that $V(P; \beta_n(I)) > V(P; \gamma_n(I))$, $V(P; \beta_n(I)) = V(P; \gamma_n(I))$, and $V(P; \beta_n) < V(P; \gamma_n)$ will be true if $V(P; \beta_2) > V(P; \gamma_2)$, $V(P; \beta_2) = V(P; \gamma_2)$, and $V(P; \beta_2) < V(P; \gamma_2)$, respectively. Similarly, it hold in the case of $V(P; \beta_n)$ and $V(P; \delta_n)$, and in the case of $V(P; \gamma_n)$ and $V(P; \delta_n)$. Next, we have

$$V(P; \beta_2) = -\frac{3r^3 - 3r^2 + r - 1}{24(r+1)},$$
$$V(P; \gamma_2) = -\frac{3r^5 + 15r^4 + 6r^3 - 42r^2 + 31r - 13}{240(r+1)},$$
$$V(P; \delta_2) = -\frac{3r^7 + 15r^6 + 60r^5 + 66r^4 + 18r^3 - 324r^2 + 283r - 121}{2184(r+1)}.$$

After some calculation, we observe that $V(P;\beta_2) < V(P;\gamma_2)$ is true if $0.08502712839 \leq r < 0.1622776602$; $V(P;\beta_2) = V(P;\gamma_2)$ if $r = 0.1622776602$, and $V(P;\beta_2) > V(P;\gamma_2)$ if $0.1622776602 < r \leq 0.2472080177$. Again, $V(P;\beta_2) > V(P;\delta_2)$ if $0.1701473031 < r \leq 0.2679491924$ and $V(P;\beta_2) = V(P;\delta_2)$ if $r = 0.1701473031$. Recall that the sets β_n form CVTs if $0 < r \leq 0.2679491924$. Hence, we can say that the sets β_n do not form the optimal sets of n-means if $0.1622776602 < r \leq 0.2679491924$. In Theorem 1.7, we have seen that the sets β_n form the optimal sets of n-means if $r = \frac{1}{25}$. Using the similar technique, we can show that the sets β_n form the optimal sets of n-means if $0 < r \leq \frac{1}{25}$. Since $V(P;\beta_2) = V(P;\gamma_2)$ if $r = r_0$; and by Theorem 1.7, the sets γ_n form the optimal sets of n-means if $r = r_0$, we can say that the sets β_n also form the optimal sets of n-means if $r = r_0$. Again, $V(P;\beta_2)$ is strictly decreasing in the closed interval $[0, r_0]$. Hence, the sets β_n form the optimal sets of n-means for $0 < r \leq r_0$.

To prove the remaining part of the theorem, we see that

(*i*) $V(P;\beta_2) < V(P;\gamma_2)$ if $0.08502712839 \leq r < 0.1622776602$; $V(P;\beta_2) = V(P;\gamma_2)$ if $r = 0.1622776602$, and $V(P;\beta_2) > V(P;\gamma_2)$ if $0.1622776602 < r \leq 0.2472080177$.

(*ii*) $V(P;\delta_2) < V(P;\gamma_2)$ if $0.2317626315 < r \leq 0.2472080177$; $V(P;\delta_2) = V(P;\gamma_2)$ if $r = 0.2317626315$, and $V(P;\delta_2) > V(P;\gamma_2)$ if $0.1845020699 \leq r < 0.2317626315$.

Thus, the sets γ_n do not form the optimal sets of n-means if $0.08502712839 \leq r < 0.1622776602$, or if $0.2317626315 < r \leq 0.2472080177$; in other words, the range of r values for which the sets γ_n form the optimal sets of n-means is bounded below by $r_0 = 0.1622776602$ and bounded above by $r_1 = 0.2317626315$. By Theorem 1.7, we see that the sets γ_n form the optimal sets of n-means if $r = r_0$, and $r = r_1$. Again, $V(P;\gamma_2)$ is strictly decreasing in the closed interval $[r_0, r_1]$. Hence, the precise range of r values for which the sets γ_n form the optimal sets of n-means is given by $r_0 \leq r \leq r_1$. Thus, the proof of the theorem is complete. □

Since the Cantor set C under investigation satisfies the strong separation condition, with each S_j having contracting factor of r, the Hausdorff dimension of the Cantor set is equal to the similarity dimension. Hence, from the equation $3(r)^\beta = 1$, we have $\dim_{\rm H}(C) = \beta = -\frac{\log 3}{\log r}$. By Theorem 14.17 in [4], the quantization dimension $D(P)$ exists and is equal to β. In Theorem 5.2, we show that β dimensional quantization coefficient for P does not exist.

Theorem 5.2. *The β-dimensional quantization coefficient for $0 < r \leq r_1$ does not exist.*

Proof. We have $3^{\frac{1}{\beta}} = \frac{1}{r}$. Notice that $\left\{\left(3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{3^{\ell(n)}}(P)\right\}$ and $\left\{\left(2 \cdot 3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{2\cdot 3^{\ell(n)}}(P)\right\}$ are two different subsequences of the sequence $\left\{n^{\frac{2}{\beta}} V_n(P)\right\}$. First, assume that $0 < r \leq r_0$. Then, by Theorem 5.1, β_n is an optimal set of n-means for $0 < r \leq r_0$. Recall Proposition 2.4. Then, we have

$$\lim_{n\to\infty} \left(3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{3^{\ell(n)}}(P) = \lim_{n\to\infty} \frac{1}{r^{2\ell(n)}} \frac{1}{3^{\ell(n)}} r^{2\ell(n)} 3^{\ell(n)} V = V, \tag{6}$$

and

$$\lim_{n\to\infty} \left(2 \cdot 3^{\ell(n)}\right)^{\frac{2}{\beta}} V_{2\cdot 3^{\ell(n)}}(P) = \lim_{n\to\infty} 2^{\frac{2}{\beta}} \frac{1}{r^{2\ell(n)}} \frac{1}{3^{\ell(n)}} r^{2\ell(n)} 3^{\ell(n)} V(P;\beta_2) = 2^{\frac{2}{\beta}} V(P;\beta_2). \tag{7}$$

By (6) and (7), we see that $\left\{n^{\frac{2}{\beta}} V_n(P)\right\}$ has two different subsequences having two different limits, and so $\lim_{n\to\infty} n^{\frac{2}{\beta}} V_n(P)$ does not exist. Due to Theorem 5.1, and Proposition 2.5, similarly, we can show that if $r_0 \leq r \leq r_1$, then $\lim_{n\to\infty} n^{\frac{2}{\beta}} V_n(P)$ does not exist. Thus, we show that the β-dimensional quantization coefficient for $0 < r \leq r_1$ does not exist, which completes the proof of the theorem. □

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