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# Comparison Between the Homotopy Perturbation Method and Variational Iteration Method for Fuzzy Differential Equations

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## Abstract

In this article, the authors discusses the numerical simulations of higher-order differential equations under a fuzzy environment by using Homotopy Perturbation Method and Variational Iteration Method. The fuzzy parameter and variables are represented by triangular fuzzy convex normalized sets. Comparison of the results are obtained by the homotopy perturbation method with those obtained by the variational iteration method. Examples are provided to demonstrate the theory.

**Keywords:** Fuzzy number; Fuzzy differential equation; Homotopy perturbation method; Variational iteration method; Approximate solutions

**MSC 2010 No.:** 34A07, 34A30, 03E72

## 1. Introduction

Fuzzy differential equations are a significant aspect of the fuzzy analytic theory, and a valuable instrument to describe a dynamical phenomenon when the information about it is vague and its nature is under uncertainty. The rise is in the modeling of the real-world problems, when there is impreciseness, for example, population models, physics, medicine, robotics, aircraft dynamics, electrical circuits, power systems, aerospace engineering, chemical processing, robotics, aircraft dynamics, biological systems, and time-series analysis. Chang and Zadeh (1972) were the first to establish the fuzzy derivative concept, which was followed by Dubois and Prade (1982), who utilized the extension principle. The measurability and integrability of fuzzy set-valued mappings of a real variable with values that are normal, convex, upper semicontinuous, and compactly supported by fuzzy set were discussed (Kaleva (1987); Kaleva (1990)). In Abbasbandy and Allahviranloo (2002), Allahviranloo et al. (2009), and Mansouri and Ahmady (2012), numerical techniques for solving fuzzy differential equations are proposed. Two analytical methods for solving $n^{th}$ order linear differential equations with fuzzy initial conditions were addressed by Buckley and Feuring (2001). The first technique involved fuzzifying the crisp solution and then determining whether it satisfied the differential equation with fuzzy initial conditions, whereas the second method involved solving the fuzzy initial value problem first and then determining whether it defined a fuzzy function. Bede (2008) found exact solutions to fuzzy differential equations. The study of fuzzy differential and integral equations, which has attracted interest for some time, especially concerning fuzzy processes, has progressed in recent years as a result of their applications in a wide range of domains (Chalco-Cano and Roman-Flores (2009); Salahshour and Allahviranloo (2013); Liu et al. (2020); Esmi et al. (2021); Paripour et al. (2015); Mosleh and Otadi (2015); Ahmadian et al. (2016); Gasilov et al. (2018)).

Several analytical approaches for solving linear and nonlinear differential equations have been developed in the recent phenomenon. Some of these techniques include the Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), Differential Transform Method (DTM), Adomian Decomposition Method (ADM), and Homotopy Analysis Method (HAM). The purpose of this study is to provide a numerical simulation for linear fuzzy differential equations using the HPM. Ji-Huan He was the first to introduce HPM in He (1999a), He (2000a), He (2003), and He (2009). He proposed a homotopy perturbation methodology for the solution of algebraic equations that combines the introduction of homotopy in topology with the classic perturbation method. Homotopy is generated using the homotopy procedure with an embedding parameter as a small parameter. This technique yields a summation of an infinite series with easily computed terms that converges quickly to the solution of the problem. Many researchers have used this technique to solve a wide range of linear and non-linear differential equations in science and engineering applications. Ghanbari (2009) investigates using HPM to approximate the solution of fuzzy initial value problems with generalized differentiability. In Bota and Caruntu (2017) and Roul and Meyer (2011), the authors investigated analytical and approximate solutions of nonlinear differential and integrodifferential equations by using HPM.

He also developed the VIM (He (1999b); He (2000b); He and Wu (2007)) which produces rapidly convergent consecutive approximations of the exact solution. In recent years, researchers have analyzed mathematical modelling by using the variational iteration method (Hetmaniok et al. (2011); Jafari (2014); Mungkasi (2021); Wang et al. (2020)). The equations are first approximated with possible unknowns in this manner. A general Lagrange multiplier, which can be found efficiently via variational theory, establishes a correction functional. The approach provides successive approximations of the precise solution that are fast converging. There are no limitations or unrealistic assumptions in the VIM, such as linearization or a nonlinear operator with minimal parameters.

The VIM is capable of handling both linear and nonlinear problems. The concept of convergence has been proven.

The main focus of this research is to demonstrate how HPM and VIM can be used to compare numerical simulations of higher-order fuzzy differential equations with fuzzy initial conditions.

## 2. Preliminaries

This section provides some fundamental definitions, concepts related to fuzzy numbers, which are essential to analyze fuzzy differential equations.

**Definition 2.1.**

A fuzzy number is a convex normalized fuzzy set $\widetilde{\mathcal{A}}$ of the real line $\mathbb{R}$ such that,

$$\mu_{\widetilde{\mathcal{A}}}(x) : \mathbb{R} \to [0,1] \;\; \forall x \in \mathbb{R},$$

where $\mu_{\widetilde{\mathcal{A}}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

**Definition 2.2.**

The $\alpha-$cut or $\alpha-$level cut of fuzzy set $\widetilde{\mathcal{A}}$ is a set consisting of those elements of the universe $\mathcal{X}$ whose membership values exceed the threshold level $\alpha$,

$$\widetilde{\mathcal{A}}_\eta = x/\mu_{\widetilde{\mathcal{A}}}(x) \geq \alpha.$$

**Definition 2.3.**

A triangular fuzzy number $\widetilde{\mathcal{A}}$ is a convex normalized fuzzy set $\widetilde{\mathcal{A}}$ of the real line $\mathbb{R}$ such that:

(i) There exist exactly one $r_0 \in \mathbb{R}$ with $\mu_{\widetilde{\mathcal{A}}}(x_0)$=1 ($x_0$ is called the mean value of $\widetilde{\mathcal{A}}$), where $\mu_{\widetilde{\mathcal{A}}}$ is called the membership function of the fuzzy set.
(ii) $\mu_{\widetilde{\mathcal{A}}}(x)$ is piecewise continuous.

The triangular fuzzy number $\widetilde{\mathcal{A}} = (a_1, a_2, a_3)$. The membership function $\mu_{\widetilde{\mathcal{A}}}$ of $\widetilde{\mathcal{A}}$ is defined as follows,

$$\mu_{\widetilde{\mathcal{A}}}(x) = \begin{cases} 0, & \text{if } \; x \leq a, \\ \frac{x-a_1}{a_2-a_1}, & \text{if } \; a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & \text{if } \; a_2 \leq x \leq a_3, \\ 0, & \text{if } \; x \geq a_3. \end{cases}$$

Any arbitrary triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$. It can be represented with an ordered pair of functions through $r-$cut approach,

$$[\underline{v}(r), \overline{v}(r)] = [(a_2 - a_1)r + a_1, (a_3 - a_2)r + a_3], \;\; r \in [0,1].$$

The triangular fuzzy numbers the left and right bound of the fuzzy numbers satisfies the following conditions:

(i) $\underline{v}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
(ii) $\overline{v}(r)$ is a bounded right continuous non-increasing function over $[0, 1]$.
(iii) $\underline{v}(r) \leq \overline{v}(r)$, $0 \leq$ r $\leq 1$.

**Lemma 2.1.**

If $\widetilde{w}(t) = (x(t), y(t), z(t))$ is fuzzy triangular number valued function and if $\widetilde{w}$ is Hukuhara differential, then $\widetilde{w}\prime(t) = (x\prime(t), y\prime(t), z\prime(t))$. Use this property to solve the fuzzy initial value problem.

***Proof:***

Now, consider the following fuzzy initial value problem

$$\widetilde{y}\prime = f(t, \widetilde{y}),$$
$$\widetilde{y}(t_0) = \widetilde{y}_0,$$

with,

$$\widetilde{y}_0 = (\underline{y}_0, y_0^c, \overline{y}_0) \in \mathbb{R},$$
$$\widetilde{y}(t) = (\underline{w}, w^c, \overline{w}) \in \mathbb{R},$$
$$f \ : \ [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R},$$
$$f(t, (\underline{w}, w^c, \overline{w})) = (\underline{f}(t, \underline{w}, w^c, \overline{w}), f^c(t, \underline{w}, w^c, \overline{w}), \overline{f}(t, \underline{w}, w^c, \overline{w})).$$

Insert the above equations into the following system of ordinary differential equations as below:

$$\underline{w} = \underline{f}(t, \underline{w}, w^c, \overline{w}),$$
$$w^c = f^c(t, \underline{w}, w^c, \overline{w}),$$
$$\overline{w} = \overline{f}(t, \underline{w}, w^c, \overline{w}).$$

$$\underline{w}(0) = \underline{y}_0, \quad \underline{w}^c(0) = y_0^c, \quad \overline{w}(0) = \overline{y}_0.$$

■

## 3. Analysis of Homotopy Perturbation Method

Consider the general non-linear differential equation shown below,

$$\mathcal{D}(u) - h(z) = 0, z \in \Psi, \tag{1}$$

with boundary condition

$$\mathcal{E}\left(u, \frac{\partial u}{\partial z}\right) = 0, z \in \Lambda, \tag{2}$$

where $\mathcal{D}-$ General differential operator, $\mathcal{E}-$ Boundary operator, $h(z)-$ Analytical function, and $\Lambda$- Boundary of the domain $\Psi$.

After dividing $\mathcal{D}$ into two parts, the equation $(1)$ can be written as,

$$\mathcal{K}(u) + \mathcal{M}(u) - h(z) = 0, \quad z \in \Lambda, \tag{3}$$

where $\mathcal{K}-$ linear and $\mathcal{M}-$ nonlinear.

By the homotopy technique (He (1999a); He (2000a)),

$$\mathcal{U}(z,q) : \Psi \times [0,1] \to \mathcal{R},$$

which satisfies

$$\left.\begin{array}{ll} \mathcal{G}(\mathcal{U},\mathcal{Q}) & = (1-q)[\mathcal{K}(\mathcal{U}) - \mathcal{K}(u_0)] + q[\mathcal{D}(\mathcal{U}) - h(z)] = 0, q \in [0,1], z \in \Lambda \\ \quad\text{or} & \\ \mathcal{G}(\mathcal{U},\mathcal{Q}) & = \mathcal{K}(\mathcal{U}) - \mathcal{K}(u_0) + q\mathcal{K}(u_o) + q[\mathcal{M}(\mathcal{U}) - h(z)] = 0 \end{array}\right\}, \tag{4}$$

where $u_0$ is an initial approximation of the equation $(1)$ that satisfies the boundary conditions, and $q \in [0,1]$ is an embedding parameter.

From the equations $(3)$ and $(4)$, we have

$$\mathcal{G}(\mathcal{U},0) = \mathcal{K}(\mathcal{U}) - \mathcal{K}(u_0) = 0, \tag{5}$$

$$\mathcal{G}(\mathcal{U},1) = \mathcal{D}(\mathcal{U}) - h(z) = 0. \tag{6}$$

The process of changing $q$ from $0$ to $1$, that is, $\mathcal{U}(z,q)$ from $u_0(z)$, is known as deformation in topology, and $\mathcal{K}(\mathcal{U}) - \mathcal{K}(u_0), \mathcal{D}(\mathcal{U}) - h(z)$ are homotopic. To proceed, use the embedding parameter $q$ as a small parameter in HPM, and assume that the solutions of $(4)$ and $(5)$ can be represented as a power series in $q$,

$$\mathcal{U} = u_0 + qu_1 + q^2u_2 + q^3u_3 + q^4u_4 + q^5u_5 + q^6u_6 + \dots, \tag{7}$$

and the exact solution is obtained as follows:

$$u = \lim_{q\to 1}\mathcal{U} = \lim_{q\to 1}\left(u_0 + qu_1 + q^2u_2 + q^3u_3 + q^4u_4 + q^5u_5 + q^6u_6 + \dots\right) = \sum_{j=0}^{\infty} u_j\ , \tag{8}$$

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3 + \mathcal{U}_4 + \mathcal{U}_5 + \mathcal{U}_6 + \dots \tag{9}$$

## 4. Applying HPM to Fuzzy Differential Equations

Let us consider the following $n^{th}$ order fuzzy differential equation,

$$\widetilde{u}^{(n)}(t) + f(t, \widetilde{u}(t), \widetilde{u}'(t), \widetilde{u}''(t), \widetilde{u}'''(t), \dots, \widetilde{u}^{(n)}(t)) = 0, \ \ t \in [0,1], \tag{10}$$

with initial conditions,

$$\widetilde{u}^{(i)} = (g_i(z), k_i(z)), \quad i = 0,1,2,3,\dots n-1.$$

By the HPM (He (1999a); He (2000a)), to establish a homotopy

$$(1-q)\widetilde{u}^{(n)} + q[\widetilde{u}^{(n)}(t) + f(t, \widetilde{u}(t), \widetilde{u}'(t), \widetilde{u}''(t), \widetilde{u}'''(t), \dots, \widetilde{u}^{(n)}(t)] = 0, \tag{11}$$

$$\widetilde{u}^{(n)}(t) + q[f(t, \widetilde{u}(t), \widetilde{u}'(t), \widetilde{u}''(t), \widetilde{u}'''(t), \dots, \widetilde{u}^{(n)}(t))] = 0, \tag{12}$$

where $q \in [0,1]$ is an embedding parameter.

Substituting $q = 0$ in Equation $(11)$, we obtain $\widetilde{u}^{(n)}(t) = 0$, and substitute $q = 1$ in equation $(11)$. Hence, we obtain

$$\widetilde{u}^{(n)}(t) + f(t, \widetilde{u}(t), \widetilde{u}'(t), \widetilde{u}''(t), \widetilde{u}'''(t), \dots, \widetilde{u}^{(n)}(t)) = 0.$$

This is known as deformation in topology; $\widetilde{u}^{(n)}(t)$ and $\widetilde{u}^{(n)}(t) + f(t, \widetilde{u}(t), \widetilde{u}'(t), \widetilde{u}''(t), \ldots, \widetilde{u}^{(n)}(t))$ are called homotopic. According to HPM, assume that the solution of Equation $(11)$ or $(12)$ can be expressed as a series in $q$,

$$\widetilde{u}(t) = \widetilde{u}_0(t) + q\widetilde{u}_1(t) + q^2\widetilde{u}_2(t) + q^3\widetilde{u}_3(t) + q^4\widetilde{u}_4(t) + q^5\widetilde{u}_5(t) + q^6\widetilde{u}_6(t) + \ldots \tag{13}$$

When $q \to 1$, Equation $(11)$ or $(12)$ corresponds to Equation $(10)$ and $(13)$ and becomes the approximate solution of Equation $(10)$,

$$\widetilde{u}(t) = \widetilde{u}_0(t) + \widetilde{u}_1(t) + \widetilde{u}_2(t) + \widetilde{u}_3(t) + \widetilde{u}_4(t) + \widetilde{u}_5(t) + \widetilde{u}_6(t) + \ldots \tag{14}$$

The approximate solution to the equation $(10)$ is as follows. The series equation $(14)$ is usually convergent, leading to the exact solution of equation $(10)$. For approximate solutions, one can use either the closed form or the truncated form.

## 5. Analysis of Variational Iteration Method

Consider the following general non-linear equation,

$$\mathcal{K}u(t) + \mathcal{M}u(t) = v(t), \tag{15}$$

where $\mathcal{K}$ and $\mathcal{M}$ are linear and non-linear operators respectively. $v(t)$ is the non-homogeneous term. He has modified the general Lagrange multiplier method to an iteration method known as correction functional.

The basic character of the method is to construct a correction functional for the above equation, which follows,

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\mathcal{K}u_n + \mathcal{M}u_n - v(s))ds, \tag{16}$$

where $\lambda$ is a general Lagrange multiplier that can be ideally determined by variational theory, $u_n$ is the $n^{th}$ approximate solution, and $\widetilde{u}_n$ is a restricted variation. $\gamma\widetilde{u}_n = 0$. It is to be noted that the Lagrange multiplier $\lambda$ can be constant or a function. It is required to determine optimally via integration by parts and employing a limited variation.

A general formula $\lambda$ for the $n^{th}$ order differential equation,

$$u^{(n)} + f(u(t), u'(t), u''(t), u'''(t), \ldots, u^{(n)}(t)) = 0, \tag{17}$$

$$\lambda = (-1)^n \frac{(s-t)^{n-1}}{(n-1)!}. \tag{18}$$

The following approximations $u_n + 1$ calculated by any initial function $u_0$ after the Lagrange multiplier has been obtained. As a consequence, by taking the limit the result is obtained,

$$u = \lim_{n\to\infty} u_n. \tag{19}$$

The corrections functional $(16)$ generates a series of approximations, with the precise solution determined at the limit of the approximations. Then, under an appropriate initial term $u_0(t)$, the solution of problem $(15)$ is considered a fixed point of the following functional,

$$u_{n+1} = u_n + \int_0^t \lambda(\mathcal{K}u_n + \mathcal{M}u_n - v(s))ds. \tag{20}$$

## 6. Applying VIM to Fuzzy Differential Equations

Now, consider the following $n^{th}$ order fuzzy differential equation,

$$\begin{gathered} u^{(n)} + f(t, u(t), u'(t), u''(t), u'''(t), \ldots, u^{(n)}(t)) = 0, t \in [0,1], \\ \widetilde{u}^{(i)}(0) = (g_i(z), k_i(z)) \qquad i = 0, 1, 2, \ldots, n-1. \end{gathered} \tag{21}$$

The correction functional for equation $(21)$ is

$$\underline{u}_{n+1}(t,z) = \underline{u}_n(t,z) + \int_0^t \underline{\lambda}\Big[\frac{d^n}{ds^n}\underline{u}_n + f(t, \underline{u}_n(s), \underline{u}'_n(s), \underline{u}''_n(s), \underline{u}'''_n(s), \ldots, \underline{u}^{(n)}_n(s))\Big]ds,$$

$$\overline{u}_{n+1}(t,z) = \overline{u}_n(t,z) + \int_0^t \overline{\lambda}\Big[\frac{d^n}{ds^n}\overline{u}_n + f(t, \overline{u}_n(s), \overline{u}'_n(s), \overline{u}''_n(s), \overline{u}'''_n(s), \ldots, \overline{u}^{(n)}_n(s))\Big]ds.$$

The Lagrange multiplier is defined in the following way,

$$\underline{\lambda}(s,t) = \overline{\lambda}(s,t) = (-1)^n \frac{(s-t)^{n-1}}{(n-1)!}. \tag{22}$$

The exact value of Lagrange multipliers is obtained if $f$ is a linear operator as defined by the Euler-Lagrange differential equations.

The Iteration formula is,

$$\underline{u}_{n+1}(t,z) = \underline{u}_n(t,z) + \int_0^t (-1)^n \frac{(s-t)^{n-1}}{(n-1)!} \times \Big[\frac{d^n}{ds^n}\underline{u}_n + f(t, \underline{u}_n(s), \underline{u}'_n(s), \underline{u}''_n(s), \ldots, \underline{u}^{(n)}_n(s)\Big]ds,$$
$$\overline{u}_{n+1}(t,z) = \overline{u}_n(t,z) + \int_0^t (-1)^n \frac{(s-t)^{n-1}}{(n-1)!} \times \Big[\frac{d^n}{ds^n}\overline{u}_n + f(t, \overline{u}_n(s), \overline{u}'_n(s), \overline{u}''_n(s), \ldots, \overline{u}^{(n)}_n(s)\Big]ds.$$

## 7. Numerical Applications

**Example 7.1.**

Consider the second-order linear fuzzy differential equation that follows:

$$u''(t) - 4u'(t) + 4u(t) = 0, t \in [0,1]. \tag{23}$$

Subject to the initial conditions,

$$\widetilde{u}(0) = (2+\eta, 4-\eta), \widetilde{u}'(0) = (5+\eta, 7-\eta).$$

The exact solution of $(23)$ is,

$$\begin{gathered} \underline{U}(t,\eta) = (2+\eta)e^{2t} + (1-\eta)te^{2t}, \\ \overline{U}(t,\eta) = (4-\eta)e^{2t} + (\eta-1)te^{2t}. \end{gathered}$$

Using HPM, we have series to find approximate solutions,

$$\begin{aligned}
\underline{u}_0(t;\eta) &= (2+\eta)+(5+\eta)t,\\
\underline{u}_1(t;\eta) &= 6t^2-\frac{2}{3}(5+\eta)t^3,\\
\underline{u}_2(t;\eta) &= 8t^3-\frac{2}{3}(8+\eta)t^4+\frac{2}{15}(5+\eta)t^5,\\
\underline{u}_3(t;\eta) &= 8t^4-\frac{8}{15}(11+\eta)t^5+\frac{8}{90}(13+2\eta)t^6-\frac{4}{315}(5+\eta)t^7,\\
&\vdots\\
\overline{u}_0(t;\eta) &= (4-\eta)+(7-\eta)t,\\
\overline{u}_1(t;\eta) &= 6t^2-\frac{2}{3}(7-\eta)t^3,\\
\overline{u}_2(t;\eta) &= 8t^3-\frac{2}{3}(10-\eta)t^4+\frac{2}{15}(7-\eta)t^5,\\
\overline{u}_3(t;\eta) &= 8t^4-\frac{8}{15}(13-\eta)t^5+\frac{4}{90}(34-4\eta)t^6-\frac{4}{315}(7-\eta)t^7,\\
&\vdots
\end{aligned}$$

Similarly, $\underline{u}_4,\underline{u}_5,\underline{u}_6\ldots$, and $\overline{u}_4,\overline{u}_5,\overline{u}_6\ldots$ can be estimated following in this manner, and the approximate series solutions are obtained as follows:

$$\begin{aligned}
\underline{u}(t;\eta) = (2+\eta)+(5+\eta)t+6t^2-\frac{2}{3}(5+\eta)t^3+8t^3-\frac{2}{3}(8+\eta)t^4+\frac{2}{15}(5+\eta)t^5&\\
+8t^4-\frac{8}{15}(11+\eta)t^5+\frac{8}{90}(13+2\eta)t^6-\frac{4}{315}(5+\eta)t^7+\cdots&,\\
\overline{u}(t;\eta) = (4-\eta)+(7-\eta)t+6t^2-\frac{2}{3}(7-\eta)t^3+8t^3-\frac{2}{3}(10-\eta)t^4+\frac{2}{15}(7-\eta)t^5&\\
+8t^4-\frac{8}{15}(13-\eta)t^5+\frac{4}{90}(34-4\eta)t^6-\frac{4}{315}(7-\eta)t^7+\cdots&.
\end{aligned}$$

Using the HPM, this is an approximate solution to a given problem.

In addition, when applying VIM to get an approximate solution, the equation $(23)$ is of the form,

$$\mathcal{K}\underline{u}+\mathcal{M}\underline{u}=0, \qquad \mathcal{K}\overline{u}+\mathcal{M}\overline{u}=0, \tag{24}$$

where,

$$\begin{aligned}
\mathcal{K}\underline{u} &= \frac{d^2\underline{u}}{dt^2}, & \mathcal{K}\overline{u} &= \frac{d^2\overline{u}}{dt^2}\\
\mathcal{M}\underline{u} &= -4\frac{d\underline{u}}{dt}+4\underline{u}, & \mathcal{M}\overline{u} &= -4\frac{d\overline{u}}{dt}+4\overline{u}
\end{aligned}$$

It is a linear and non-linear term, respectively. Then, the correction functional for equation $(24)$ is

$$\left.\begin{aligned}
\underline{u}_{n+1}(t,\eta) &= \underline{u}_n(t,\eta)+\int_0^t \underline{\lambda}\left\{\tfrac{d^2}{ds^2}\underline{u}_n+\mathcal{M}\underline{u}_n\right\}ds\\
\overline{u}_{n+1}(t,\eta) &= \overline{u}_n(t,\eta)+\int_0^t \overline{\lambda}\left\{\tfrac{d^2}{ds^2}\overline{u}_n+\mathcal{M}\overline{u}_n\right\}ds, \quad n\geq 0
\end{aligned}\right\}. \tag{25}$$

Considering the variation in the independent variables, $\underline{u}_n$ and $\overline{u}$,

$$\gamma\mathcal{M}(\underline{u}_n(0)) = 0, \qquad \gamma\mathcal{M}(\overline{u}_n(0)) = 0$$

$$\begin{aligned}
\gamma\underline{u}_{n+1}(t,\eta) &= \gamma\underline{u}_n(t,\eta) + \gamma\int_0^t \underline{\lambda}\left\{\frac{d^2}{ds^2}\underline{u}_n + \mathcal{M}(\underline{u}_n)\right\}ds,\\
\gamma\underline{u}_{n+1}(t,\eta) &= \gamma\underline{u}_n(t,\eta) - \underline{\lambda}'\gamma\underline{u}_n + \underline{\lambda}\gamma\underline{u}_n' + \int_0^t \left\{\frac{\partial^2\underline{\lambda}}{\partial s^2}\right\}\gamma\underline{u}_n(s)ds = 0,\\
\gamma\overline{u}_{n+1}(t,\eta) &= \gamma\overline{u}_n(t,\eta) + \gamma\int_0^t \overline{\lambda}\left\{\frac{d^2}{ds^2}\overline{u}_n + \mathcal{M}(\overline{u}_n)\right\}ds,\\
\gamma\overline{u}_{n+1}(t,\eta) &= \gamma\overline{u}_n(t,\eta) - \overline{\lambda}'\gamma\overline{u}_n + \overline{\lambda}\gamma\overline{u}_n' + \int_0^t \left\{\frac{\partial^2\overline{\lambda}}{\partial s^2}\right\}\gamma\overline{u}_n(s)ds = 0.
\end{aligned}$$

As a result, the Euler Lagrange equations are obtained,

$$\frac{\partial^2\underline{\lambda}(s,t)}{\partial s^2} = 0, \qquad \frac{\partial^2\overline{\lambda}(s,t)}{\partial s^2} = 0. \tag{26}$$

and the boundary conditions,

$$\begin{aligned}
1 - \underline{\lambda}'(s,t) &= 0, & 1 - \overline{\lambda}'(s,t) &= 0,\\
\underline{\lambda}(s,t) &= 0, & \overline{\lambda}(s,t) &= 0.
\end{aligned}$$

As a result, determine the Lagrange multiplier and substitute it into the functional, which gives

$$\underline{\lambda}(s,t) = (s-t), \qquad \overline{\lambda}(s,t) = (s-t). \tag{27}$$

Substituting $(27)$ into $(25)$, then obtain the results in the iteration formulation,

$$\left.\begin{aligned}
\underline{u}_{n+1}(t,\eta) &= \underline{u}_n(t,\eta) + \int_0^t (s-t)\Big[\frac{d^2}{ds^2}\underline{u}_n - 4\frac{d}{ds}\underline{u}_n + 4\underline{u}_n\Big]ds\\
\overline{u}_{n+1}(t,\eta) &= \overline{u}_n(t,\eta) + \int_0^t (s-t)\Big[\frac{d^2}{ds^2}\overline{u}_n - 4\frac{d}{ds}\overline{u}_n + 4\overline{u}_n\Big]ds, \quad n \geq 0
\end{aligned}\right\}. \tag{28}$$

Choose,

$$\begin{aligned}
\underline{u}_0(t,\eta) &= (2+\eta) + (5+\eta)t\\
\underline{u}_1(t,\eta) &= (2+\eta) + (5+\eta)t + 6t^2 - \left(\frac{10}{3} + \frac{2\eta}{3}\right)t^3\\
\underline{u}_2(t,\eta) &= (2+\eta) + (5+\eta)t + 6t^2 + \left(\frac{14}{3} - \frac{2\eta}{3}\right)t^3 - \left(\frac{16}{3} + \frac{2\eta}{3}\right)t^4 + \left(\frac{2}{3} + \frac{2\eta}{15}\right)t^5\\
\underline{u}_3(t,\eta) &= (2+\eta) + (5+\eta)t + 6t^2 + \left(\frac{14}{3} - \frac{2\eta}{3}\right)t^3 - \left(\frac{16}{3} + \frac{2\eta}{3}\right)t^4 + \left(\frac{2}{3} + \frac{2\eta}{15}\right)t^5\\
&\vdots\\
\overline{u}_0(t,\eta) &= (4-\eta) + (7-\eta)t
\end{aligned}$$

$$
\begin{aligned}
\overline{u}_1(t,\eta) &= (4-\eta)+(7-\eta)t+6t^2-\left(\frac{14}{3}-\frac{2\eta}{3}\right)t^3\\
\overline{u}_2(t,\eta) &= (4-\eta)+(7-\eta)t+6t^2+\left(\frac{10}{3}+\frac{2\eta}{3}\right)t^3-\left(\frac{20}{3}-\frac{2\eta}{3}\right)t^4+\left(\frac{14}{15}-\frac{2\eta}{15}\right)t^5\\
\overline{u}_3(t,\eta) &= (4-\eta)+(7-\eta)t+6t^2+\left(\frac{10}{3}+\frac{2\eta}{3}\right)t^3-\left(\frac{20}{3}-\frac{2\eta}{3}\right)t^4+\left(\frac{14}{15}-\frac{2\eta}{15}\right)t^5\\
&\vdots
\end{aligned}
$$

Similarly, $\underline{u}_4, \underline{u}_5, \underline{u}_6 \ldots$, and $\overline{u}_4, \overline{u}_5, \overline{u}_6 \ldots$ can be estimated following in this manner, and the approximate series solutions are obtained as follows:

$$
\underline{u}\,(t,\eta) = (2+\eta)+(5+\eta)t+6t^2+\left(\frac{14}{3}-\frac{2\eta}{3}\right)t^3-\left(\frac{16}{3}+\frac{2\eta}{3}\right)t^4+\left(\frac{2}{3}+\frac{2\eta}{15}\right)t^5+\cdots
$$

$$
\overline{u}\,(t,\eta) = (4-\eta)+(7-\eta)\,t+6t^2+\left(\frac{10}{3}+\frac{2\eta}{3}\right)t^3-\left(\frac{20}{3}-\frac{2\eta}{3}\right)t^4+\left(\frac{14}{15}-\frac{2\eta}{15}\right)t^5+\cdots
$$

**Table 1.** Comparison to the exact solutions and the approximate solutions were obtained by HPM & VIM

| | $\underline{U}$ | | | $\overline{U}$ | | |
|---|---|---|---|---|---|---|
| $\eta$ | EXACT | HPM | VIM | EXACT | HPM | VIM |
| 0 | 2.5649 | 2.5715 | 2.5641 | 4.37634 | 4.7727 | 4.7627 |
| 0.1 | 2.6748 | 2.6816 | 2.6741 | 4.6534 | 4.6627 | 4.6527 |
| 0.2 | 2.7847 | 2.7917 | 2.7840 | 4.5435 | 4.5526 | 4.5428 |
| 0.3 | 2.8947 | 2.9017 | 2.8939 | 4.4336 | 4.4426 | 4.4329 |
| 0.4 | 3.0046 | 3.0118 | 3.0038 | 4.3237 | 4.3325 | 4.3230 |
| 0.5 | 3.1145 | 3.1218 | 3.1138 | 4.2138 | 4.2224 | 4.2130 |
| 0.6 | 3.2244 | 3.2319 | 3.2237 | 4.1038 | 4.1124 | 4.1031 |
| 0.7 | 3.3344 | 3.3420 | 3.3336 | 4.1038 | 4.1124 | 4.1031 |
| 0.8 | 3.4443 | 3.4520 | 3.4436 | 3.8840 | 3.8923 | 3.8833 |
| 0.9 | 3.5542 | 3.5621 | 3.5535 | 3.7741 | 3.7822 | 3.7733 |
| 1.0 | 3.6641 | 3.6721 | 3.6634 | 3.6641 | 3.6721 | 3.6634 |

**Example 7.2.**

Consider the fourth-order linear fuzzy differential equation:

$$
u^{(4)}(t)-u(t)=0, \quad t\in[0,1], \tag{29}
$$

subject to the initial conditions,

$$
\widetilde{u}(0)=(\eta-1,1-\eta),\ \widetilde{u}'(0)=(\eta-1,1-\eta),\ \widetilde{u}''(0)=(\eta-1,1-\eta),\ \widetilde{u}'''(0)=(\eta-1,1-\eta)
$$

The exact solution of $(29)$ is given by,

$$
\begin{aligned}
\underline{U}(t;\eta) &= (\eta-1)e^t,\\
\overline{U}(t;\eta) &= (1-\eta)e^t.
\end{aligned}
$$

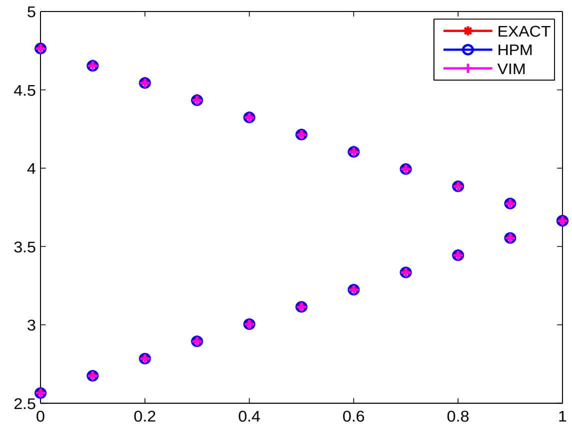


**Figure 1.** Comparison of Exact with HPM and VIM

Now, using HPM, we have a series to find the approximate solution,

$$\begin{aligned}
\underline{u}_0(t;\eta) &= (\eta-1)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right),\\
\underline{u}_1(t;\eta) &= (\eta-1)\left(\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}+\frac{t^7}{7!}\right),\\
\underline{u}_2(t;\eta) &= (\eta-1)\left(\frac{t^8}{8!}+\frac{t^9}{9!}+\frac{t^{10}}{10!}+\frac{t^{11}}{11!}\right),\\
\underline{u}_3(t;\eta) &= (\eta-1)\left(\frac{t^{12}}{12!}+\frac{t^{13}}{13!}+\frac{t^{14}}{14!}+\frac{t^{15}}{15!}\right),\\
&\vdots\\
\overline{u}_0(t;\eta) &= (1-\eta)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right),\\
\overline{u}_1(t;\eta) &= (1-\eta)\left(\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}+\frac{t^7}{7!}\right),\\
\overline{u}_2(t;\eta) &= (1-\eta)\left(\frac{t^8}{8!}+\frac{t^9}{9!}+\frac{t^{10}}{10!}+\frac{t^{11}}{11!}\right),\\
\overline{u}_3(t;\eta) &= (1-\eta)\left(\frac{t^{12}}{12!}+\frac{t^{13}}{13!}+\frac{t^{14}}{14!}+\frac{t^{15}}{15!}\right),\\
&\vdots
\end{aligned}$$

Similarly, $\underline{u}_4, \underline{u}_5, \underline{u}_6 \ldots$, and $\overline{u}_4, \overline{u}_5, \overline{u}_6 \ldots$ can be estimated following in this manner, and the approximate series solutions are obtained by HPM as follows:

$$\begin{aligned}
\underline{u}(t;\eta) &= (\eta-1)e^t,\\
\overline{u}(t;\eta) &= (1-\eta)e^t.
\end{aligned}$$

Now, applying VIM to get an approximate solution, the equation $(29)$ is of the form,

$$\mathcal{K}\underline{u} + \mathcal{M}\underline{u} = 0, \qquad \mathcal{K}\overline{u} + \mathcal{M}\overline{u} = 0, \tag{30}$$

where,

$$\mathcal{K}\underline{u} = \frac{d^4\underline{u}}{dt^4} - \underline{u}, \qquad \mathcal{K}\overline{u} = \frac{d^4\overline{u}}{dt^4} - \overline{u}$$

$$\mathcal{M}\underline{u} = 0, \qquad \mathcal{M}\overline{u} = 0.$$

The correction functional for the equation $(29)$ is

$$\left.\begin{aligned} \underline{u}_{n+1}(t;\eta) &= \underline{u}_n(t;\eta) + \int_0^t \underline{\lambda}\big[\frac{d^4}{ds^4}\underline{u}_n - \underline{u}\big]ds \\ \overline{u}_{n+1}(t;\eta) &= \overline{u}_n(t;\eta) + \int_0^t \overline{\lambda}\big[\frac{d^4}{ds^4}\overline{u}_n - \overline{u}\big]ds \end{aligned}\right\}. \tag{31}$$

Consider the variation in the independent variables, $\underline{u}_n$ and $\overline{u}$, and finding $\lambda$ in the VIM. We have,

$$\left.\begin{array}{ll} -\underline{\lambda}(s) + \underline{\lambda}^{(4)}(s) = 0, & -\overline{\lambda}(s) + \overline{\lambda}^{(4)}(s) = 0 \\ 1 - \underline{\lambda}'''(s) = 0, & 1 - \overline{\lambda}'''(s) = 0 \\ \underline{\lambda}''(s) = 0, & \overline{\lambda}''(s) = 0 \\ \underline{\lambda}'(s) = 0, & \overline{\lambda}'(s) = 0 \\ \underline{\lambda}(s) = 0, & \overline{\lambda}(s) = 0 \end{array}\right\}. \tag{32}$$

Solving the equation $(32)$ to find the Lagrange multipliers,

$$\overline{\lambda} = \underline{\lambda} = \frac{1}{2}\left[\sinh(s-t) - \sin(s-t)\right]. \tag{33}$$

Substitute $(33)$ into $(31)$ to obtain the iteration formulation,

$$\left.\begin{aligned} \underline{u}_{n+1}(t,\eta) &= \underline{u_n}(t,\eta) + \int_0^t [\sinh(s-t) - \sin(s-t)]\left[\frac{d^4\underline{u}_n}{ds^4} - \underline{u}\right]ds \\ \overline{u}_{n+1}(t,\eta) &= \overline{u}_n(t,\eta) + \int_0^t [\sinh(s-t) - \sin(s-t)]\left[\frac{d^4\overline{u}_n}{ds^4} - \overline{u}\right]ds, \quad n \geq 0 \end{aligned}\right\}. \tag{34}$$

Choose,

$$\underline{u}_0(t,\eta) = (\eta - 1)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)$$
$$\underline{u}_1(t,\eta) = (\eta - 1)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$\underline{u}_2(t,\eta) = (\eta - 1)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

$$\underline{u}_3(t,\eta) = (\eta - 1)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$\vdots$$
$$\overline{u}_0(t,\eta) = (1 - \eta)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)$$
$$\overline{u}_1(t,\eta) = (1 - \eta)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$\overline{u}_2(t,\eta) = (1 - \eta)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$\overline{u}_3(t,\eta) = (1 - \eta)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$\vdots$$

We started with initial approximation values and derived the approximate solution using the iteration formula $(34)$,

$$\underline{u}(t,\eta) = (\eta - 1)e^t,$$
$$\overline{u}(t,\eta) = (1 - \eta)e^t.$$

**Table 2.** Comparison to the exact solutions and the approximate solutions were obtained by HPM & VIM

| | $\underline{U}$ | | | $\overline{U}$ | | |
|---|---|---|---|---|---|---|
| $\eta$ | EXACT | HPM | VIM | EXACT | HPM | VIM |
| 0 | -1.1052 | -1.1052 | -1.1052 | 1.1052 | 1.1052 | 1.1052 |
| 0.1 | -0.9946 | -0.9946 | -0.9946 | 0.9946 | 0.9946 | 0.9946 |
| 0.2 | -0.8841 | -0.8841 | -0.8841 | 0.8841 | 0.8841 | 0.8841 |
| 0.3 | -0.7736 | -0.7736 | -0.7736 | 0.7736 | 0.7736 | 0.7736 |
| 0.4 | -0.6631 | -0.6631 | -0.6631 | 0.6631 | 0.6631 | 0.6631 |
| 0.5 | -0.5526 | -0.5526 | -0.5526 | 0.5526 | 0.5526 | 0.5526 |
| 0.6 | -0.4421 | -0.4421 | -0.4421 | 0.4421 | 0.4421 | 0.4421 |
| 0.7 | -0.3315 | -0.3315 | -0.3315 | 0.3315 | 0.3315 | 0.3315 |
| 0.8 | -0.2210 | -0.2210 | -0.2210 | 0.2210 | 0.2210 | 0.2210 |
| 0.9 | -0.1105 | -0.1105 | -0.1105 | 0.1105 | 0.1105 | 0.1105 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 |

As shown in Figures 1 and 2, the exact values are compared to those of HPM and VIM, respectively. As a result of the above findings, the proposed method HPM shows excellent agreement with other existing methods of VIM. According to the analysis, HPM can be a suitable mathematical tool for solving higher-order fuzzy differential equations.

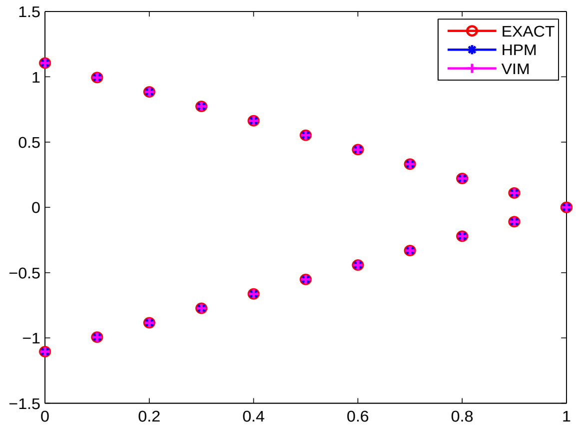


**Figure 2.** Comparison of Exact with HPM and VIM

## 8. Conclusion

In this paper, the authors described a linear fuzzy differential equations algorithm that was simulated using HPM and VIM, as well as a comparison between exact with HPM and VIM. For determining approximate solutions to fuzzy differential equations, the previously mentioned techniques can be employed as an alternative and equivalent method. HPM, like the perturbation method, does not require any parameters in the equation. This approach is powerful and efficient since it provides accurate approximations. HPM and VIM produce approximate solutions that are infinite power series with appropriate initial conditions that can be described in the closed-form of exact solutions. Finally, when compared to the exact solution, the HPM outperforms the VIM in solving higher-order fuzzy differential equations. Furthermore, the obtained solution shows that HPM and VIM results satisfy the properties of triangular shape fuzzy numbers.

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