Applications and Applied Mathematics: An International Journal (AAM)

Volume 17
Issue 3 *Special Issue No. 10 (October 2022)*

Article 12

10-2022

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Special Issue No. 10 (October 2022), pp. 141 – 156

Fear Effect in a Three Species Prey-predator Food-web System With Harvesting

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Received: November 20, 2021; Accepted: August 13, 2022

Abstract

Some recent studies and field experiments show that predators affect their prey not only by direct capture; they also induce fear in prey species, which reduces their reproduction rate. Considering this fact, we propose a mathematical model to study the fear effect of a middle predator on its prey in a three-species food web system with harvesting. The ecological feasibility of solutions to the proposed system is guaranteed in terms of positivity and boundedness. The local stability of stationary points in the proposed system is derived. Multiple co-existing stationary points for the proposed system are observed, which makes the problem more interesting compared to the similar models studied previously. The local existence of periodic solutions through Hopf bifurcations is additionally secured numerically in the case of both unique and multiple coexisting stationary points. It is also observed that the system can exhibit strange attractors in the form of chaos. A detailed numerical simulation is performed to ensure the existence of periodic solutions and period-doubling routes to chaos. Combined effects of fear and harvesting are also discussed numerically.

Keywords: Prey-predator system; Food-web model; Fear effect; Harvesting; Linearization; Routh-Hurwitz criterion; Stability; Bifurcation; Chaos; Numerical simulation

MSC 2020 No.: 34D20, 34C23, 65P20

1. Introduction

A food chain shows the relationship among organisms by the food they eat, and each level of a food chain represents a different trophic level. A food chain follows a direct and linear pathway of one species at a time. The food chain models of three or more species have been extensively studied by researchers. One of the most pioneer work for such models with Holling type-II functional response has been proposed and studies by Hastings and Powell (1991) where the authors showed the occurrence of chaotic dynamics. Later, Naji and Balasim (2007) explored a food chain model for three species with Beddington-DeAngelis functional response. The authors showed that the model could demonstrate chaotic dynamics for biologically feasible parametric values. Lv and Zhao (2008) proposed a three-species food chain model and observed that the system exhibits rich chaotic dynamics. By supplying allochthonous inputs to the top predator, Sahoo and Poria (2015) proposed a three-species prey-predator model with top predator harvesting.

A food web is an important ecological phenomenon that represents feeding relationships within a community. It also includes a variety of food chains that connect with one another. McCann and Hastings (1997) analyze the role of omnivory in food webs using a non-equilibrium perspective. The authors observe that the addition of omnivory to a simple food-chain model locally stabilizes the food web. Tanabe and Namba (2005) have shown that even if functional responses are linear, intraguild predation sometimes destabilizes food webs and induces chaos. A food web model involving two independent preys and a predator incorporating modified Holling type-II functional response is discussed by Gakkhar and Naji (2005). Namba et al. (2008) have numerically investigated bifurcation diagrams with all the parameters which are used in the food web model proposed by Tanabe and Namba (2005), including self-limitation of the intermediate consumer and predator. Visser et al. (2012) proposed a model to analyze the effect of adaptive foraging behavior within a three-species food web with intra-guild predation. The food web models with various ecological phenomena have been studied by several researchers (Hsu et al. (2015); Hntsa and Mengesha (2016); Namba et al. (2018); Gupta and Yadav (2020)).

Lima and Dill (1990) recognized various components of predation risk by evolving an abstraction of the predation procedure. According to the literature review, the ability of an individual to examine and control one or more of such components has an important role in decision-making in feeding animals. This also helps the animals to decide when and how to (a) avoid predators, (b) be social, and (c) breathe air for fish. Most species, including human beings, are prone to being influenced by fear. Usually, fear is produced by predators towards their prey in the process of capturing them. Fear effects are observed in the form of several behavioral changes and adaptations that are responsible for affecting the complete ecosystem (Brown et al. (1999); Trussell et al. (2006)). However, ecologists often ignore the role of predators to their prey because fear is mainly treated as a psychological effect (Clinchy et al. (2013)). The fear produced by predators among their prey may play an important role in ecology. A three-species food chain or food web model in the presence of a fear effect has been studied by several researchers (Hossain et al. (2020); Cong et al. (2021); Debnath et al. (2021); Sahoo and Samanta (2021)). Panday et al. (2018) recently proposed a food chain model demonstrating how fear can control chaos via the period-halving phenomenon.

2. Model Formulation

This section aims to propose a mathematical model for a simple food web model that incorporates both fear effects and harvesting. For this purpose, we first describe a simple food web model proposed by Tanabe and Namba (2005). The authors proposed the following mathematical model for a simple food web, where $P_1(t)$, $P_2(t)$ and $P_3(t)$, respectively, denote the densities of the basal resources, consumer populations, and top predators at time t:

$$\frac{dP_1}{dt} = P_1\,(r - a_{11}\,P_1) - a_{12}\,P_1\,P_2 - a_{13}\,P_1\,P_3, \tag{1a}$$

$$\frac{dP_2}{dt} = a_{21}\,P_1\,P_2 - a_{23}\,P_2\,P_3 - d_1\,P_2, \tag{1b}$$

$$\frac{dP_3}{dt} = a_{31}\,P_1\,P_3 + a_{32}\,P_2\,P_3 - d_2\,P_3, \tag{1c}$$

with initial condition

$$P_1(0) > 0, P_2(0) > 0, P_3(0) > 0. \tag{2}$$

Here, r represents the growth rate of the resource population. The parameters d_1 and d_2, respectively, denote the mortality rate of the intermediate-prey and top-predator. a_{11} is the intra-specific competition coefficient of basal resource. The parameters a_{12}, a_{13} and a_{23} are the rates of consumption, and a_{21}, a_{31} a_{32} measure the contribution of the resource to the growth of the consumer.

Motivated by the literature's works, we extend the food-web model (1) for three species by considering the fear of middle-predator on prey species with quadratic harvesting in both the predator species. The model is developed under the following two considerations:

(1) Many researchers who have worked with three species of prey-predator models have considered that the predator affects prey only by direct killing. But in the literature survey, we have observed that the fear of predator species also affects the growth rate of prey species. So, this fear mitigates the reproduction rate of the prey population. For this reason, it is relevant to include the fear term in our model to make it more effective. As a result, the growth rate of prey population can be modified as $\frac{dP_1}{dt} = \frac{P_1\,(r - a_{11}\,P_1)}{1 + k\,P_2}$. The authors of the articles (Debnath et al. (2021); Sasmal (2018); Wang et al. (2019)) have given a detailed description of the modified growth rate of the prey population, which indicates that such a modeling process is also valid.

(2) Several researchers (Clark (1976); Gupta and Chandra (2017); Kaur et al. (2021); Lenzini and Rebaza (2010); Satar and Naji (2019)) have studied ecological models using various types of harvesting. In particular, quadratic harvesting becomes relevant when the size of the population to be exploited becomes large (Gupta and Chandra (2017); Kaur et al. (2021)). A number of prey-predator models for three species with different types of predation rates have been studied by incorporating the fear effect. As far as our knowledge goes, no researcher has studied the combined effect of fear and quadratic harvesting.

The model (1) in presence of fear effect and quadratic harvesting of both predators is modified as follows:

$$\frac{dP_1}{dt} = \frac{P_1\,(r - a_{11}\,P_1)}{1 + k\,P_2} - a_{12}\,P_1\,P_2 - a_{13}\,P_1\,P_3 \equiv P_1\,\psi_1(P_1, P_2, P_3), \tag{3a}$$

$$\frac{dP_2}{dt} = a_{21}\,P_1\,P_2 - a_{23}\,P_2\,P_3 - d_1\,P_2 - h_1\,{P_2}^2 \equiv P_2\,\psi_2(P_1, P_2, P_3), \tag{3b}$$

$$\frac{dP_3}{dt} = a_{31}\,P_1\,P_3 + a_{32}\,P_2\,P_3 - d_2\,P_3 - h_2\,{P_3}^2 \equiv P_3\,\psi_3(P_1, P_2, P_3), \tag{3c}$$

with initial condition

$$P_1(0) > 0, P_2(0) > 0, P_3(0) > 0. \tag{4}$$

Here, k is the fear parameter of middle-predator on prey population, h_1 and h_2 are the rates of harvesting of middle and top-predator, respectively.

For simplification of forthcoming sections, we define $\psi_1(P_1, P_2, P_3) = \frac{(r - a_{11}\,P_1)}{1 + k\,P_2} - a_{12}\,P_2 - a_{13}\,P_3$, $\psi_2(P_1, P_2, P_3) = a_{21}\,P_1 - a_{23}\,P_3 - d_1 - h_1\,P_2$ and $\psi_3(P_1, P_2, P_3) = a_{31}\,P_1 + a_{32}\,P_2 - d_2 - h_2\,P_3$.

The proposed system (3a)-(3c) is defined in the region $\Gamma = \{(P_1, P_2, P_3) \in \mathbb{R}^3 |\ \ P_1 \geq 0, P_2 \geq 0, P_3 \geq 0\}$.

3. Feasibility of Solutions

In this section, we provide the results for positivity and establish the condition for boundedness of solutions in the system (3a)-(3c), which ensures the meaningfulness of the proposed system from an ecological point of view.

Theorem 3.1.

The solutions $(P_1(t), P_2(t), P_3(t))$ of system (3a)-(3c) with (4) remain positive for all $t \geq 0$.

Proof:

The system (3a)-(3c) along with initial condition (4) gives

$$P_1(t) = P_1(0)e^{\int_0^t \psi_1(P_1(T), P_2(T), P_3(T))dT} > 0,$$
$$P_2(t) = P_2(0)e^{\int_0^t \psi_2(P_1(T), P_2(T), P_3(T))dT} > 0,$$
$$P_3(t) = P_3(0)e^{\int_0^t \psi_3(P_1(T), P_2(T), P_3(T))dT} > 0.$$

This result ensures that solutions of the system (3a)-(3c) originating from the region Γ remain inside it for all $t \geq 0$. ∎

Theorem 3.2.

The solutions of the system (3a)-(3c) with initial condition (4) are uniformly bounded in the region $\Omega = \left\{(P_1, P_2, P_3) \in \mathbb{R}^3_+ : 0 < \chi(t) = P_1 + \frac{a_{12}}{a_{21}} P_2 + \frac{a_{13}}{a_{31}} P_3 < \frac{(r+M)^2}{4\,a_{11} M} + \phi, \text{ for any } \phi > 0\right\}$, if $2\,a_{12}a_{13}a_{21}a_{31}\,(a_{32}a_{23} + 2\,h_1\,h_2) > a_{13}^2 a_{21}^2 a_{32}^2 + a_{12}^2 a_{23}^2 a_{31}^2$.

Proof:

Let us define a function $\chi(t) = P_1 + \frac{a_{12}}{a_{21}} P_2 + \frac{a_{13}}{a_{31}} P_3$. Then the derivative of $\chi(t)$ with respect to time t given as

$$\frac{d\chi(t)}{dt} = \frac{dP_1}{dt} + \frac{a_{12}}{a_{21}} \frac{dP_2}{dt} + \frac{a_{13}}{a_{31}} \frac{dP_3}{dt}.$$

For a positive constant M, we have

$$\frac{d\chi(t)}{dt} + M\chi(t) = \frac{P_1 \left(r - a_{11} P_1\right)}{1 + k\, P_2} + M P_1 + \frac{a_{12} \left(M - d_1\right) P_2}{a_{21}} + \frac{a_{13} \left(M - d_2\right) P_3}{a_{31}} - Q(P_2, P_3),$$

where, $Q(P_2, P_3) = \frac{a_{12} h_1}{a_{21}}\, {P_2}^2 - \left(\frac{a_{13} a_{32}}{a_{31}} - \frac{a_{23} a_{12}}{a_{21}}\right) P_2 P_3 + \frac{a_{13} h_2}{a_{31}}\, {P_3}^2$.

The expression $Q(P_2, P_3)$ is a quadratic form and will be positive definite if it satisfies the condition $2\, a_{12}\, a_{13}\, a_{21}\, a_{31} \left(a_{32}\, a_{23} + 2\, h_1\, h_2\right) > a_{13}^2 a_{21}^2 a_{32}^2 + a_{12}^2\, a_{23}^2 a_{31}^2$.

If we choose $M = \min(d_1, d_2)$, then under the above condition, we can write

$$\frac{d\chi(t)}{dt} + M\chi(t) \leq -a_{11} \left(P_1 - \frac{r + M}{2\, a_{11}}\right)^2 + \frac{\left(r + M\right)^2}{4\, a_{11}}.$$

This gives $\frac{d\chi(t)}{dt} + M\chi(t) \leq \frac{(r+M)^2}{4\, a_{11}}$. Now, using the theory of differential inequality (Rota and Birkhoff (1962)), we obtain

$$0 < \chi(t) \leq \frac{\left(r + M\right)^2}{4\, a_{11} M} \left(1 - e^{-Mt}\right) + \chi(0)\, e^{-Mt}.$$

Therefore, $0 < \lim_{t\to\infty} \chi(t) \leq \frac{(r+M)^2}{4\, a_{11} M}$.

Hence, all solutions of the proposed system (3a)-(3c) with initial condition (4) which starts from Γ are bounded in the region Ω. ■

4. Existence and Stability Analysis of Stationary Points

In this section, we have investigated stationary points of the system (3a)-(3c) and obtained suitable conditions for their existence. We have also analyzed the local stability of all stationary points.

4.1. Existence of various stationary points

The stationary points of the system (3a)-(3c) are given as follows:

(I) $S_0 = (0, 0, 0)$ and $S_1 = \left(\frac{r}{a_{11}}, 0, 0\right)$ are, respectively, trivial and axial stationary points of the system (3a)-(3c), which always exist.

(II) The stationary points in $P_1\, P_2$-plane are $\left(\tilde{P}_1^{(1)}, \tilde{P}_2^{(1)}, 0\right)$ and $\left(\tilde{P}_1^{(2)}, \tilde{P}_2^{(2)}, 0\right)$ that are free of top-predator species. The P_2-components of these stationary points are the roots of the following quadratic equation

$$ka_{12}a_{21}{P_2}^2 + (h_1a_{11} + a_{12}a_{21})\, P_2 - ra_{21} + a_{11}d_1 = 0. \tag{5}$$

The two roots of Equation (5), are given by $\tilde{P}_2^{(1)} = \frac{-A_1+\sqrt{A_2}}{2\,k\,a_{12}a_{21}}$ and $\tilde{P}_2^{(2)} = \frac{-A_1-\sqrt{A_2}}{2\,k\,a_{12}a_{21}}$, where, $A_1 = (h_1a_{11} + a_{12}a_{21})$ and $A_2 = (h_1a_{11} + a_{12}a_{21})^2 - 4\,k\,a_{12}a_{21}\,(-ra_{21} + a_{11}d_1)$. The only possibility of positive root is $P_2 = \tilde{P}_2^{(1)}$ if $a_{11}d_1 < ra_{21}$. Hence, $\tilde{S} = \left(\frac{d_1+h_1\tilde{P}_2^{(1)}}{a_{21}}, \tilde{P}_2^{(1)}, 0\right)$ is only a feasible $P_1\, P_2$-planar stationary point of the system (3a)-(3c).

(III) In the $P_1\, P_3$-plane, the stationary point is given by $\hat{S} = \left(\frac{rh_2+a_{13}d_2}{a_{31}a_{13}+h_2a_{11}}, 0, \frac{a_{31}r-d_2a_{11}}{a_{31}a_{13}+h_2a_{11}}\right)$. This becomes positive only if $a_{31}r > d_2a_{11}$.

(IV) To discuss the existence of $S_{1*} = \left(P_{1*}^{(1)}, P_{2*}^{(1)}, P_{3*}^{(1)}\right)$ and $S_{2*} = \left(P_{1*}^{(2)}, P_{2*}^{(2)}, P_{3*}^{(2)}\right)$ stationary points of the system (3a)-(3c), we focus on the following quadratic equation in the P_2-component:

$$B_1{P_2}^2 + B_2P_2 + B_3 = 0, \tag{6}$$

where, $B_1 = a_{13}ka_{21}a_{32} + a_{13}ka_{31}h_1 - a_{12}ka_{31}a_{23} + a_{12}ka_{21}h_2, B_2 = a_{23}a_{11}a_{32} + a_{13}a_{31}h_1 + a_{13}a_{21}a_{32} - a_{13}ka_{21}d_2 + a_{13}ka_{31}d_1 + a_{12}a_{21}h_2 + a_{11}h_1h_2 - a_{12}a_{31}a_{23}$ and $B_3 = a_{11}d_1h_2 - a_{23}a_{11}d_2 + a_{13}a_{31}d_1 - ra_{21}h_2 - a_{13}a_{21}d_2 + ra_{31}a_{23}$.

The two roots of Equation (6) are $P_{2*}^{(1)} = \frac{-B_2+\sqrt{{B_2}^2-4\,B_1B_3}}{2\,B_1}$ and $P_{2*}^{(2)} = \frac{-B_2-\sqrt{{B_2}^2-4\,B_1B_3}}{2\,B_1}$.

There are the following possible cases.

Case 1: ${B_2}^2 < 4B_1B_3$.
In this case, the system (3a)-(3c) has no feasible stationary point because Equation (6) has no real root.

Case 2: ${B_2}^2 > 4B_1B_3$.

Subcase 1: If (i) $B_1 > 0$, $B_2 > 0$, $B_3 < 0$ or (ii) $B_1 > 0$, $B_2 < 0$, $B_3 < 0$, then $P_{2*} = P_{2*}^{(1)}$ is only a positive real root of Equation (6), and hence, the system (3a)-(3c) has only one feasible co-existing stationary point as $S_{1*} = \left(P_{1*}^{(1)}, P_{2*}^{(1)}, P_{3*}^{(1)}\right)$, where $P_{1*}^{(1)} = \frac{a_{23}a_{32}P_{2*}^{(1)}-a_{23}d_2+d_1h_2+h_1h_2P_{2*}^{(1)}}{h_2a_{21}-a_{31}a_{23}}$ and $P_{3*}^{(1)} = \frac{a_{31}d_1+a_{31}h_1P_{2*}^{(1)}+a_{32}a_{21}P_{2*}^{(1)}-d_2a_{21}}{h_2a_{21}-a_{31}a_{23}}$ if $h_2a_{21} > (<)a_{31}a_{23}$, $a_{23}a_{32}P_{2*}^{(1)} + d_1h_2 + h_1h_2P_{2*}^{(1)} > (<)a_{23}d_2$, $a_{31}d_1 + a_{31}h_1P_{2*}^{(1)} + a_{32}a_{21}P_{2*}^{(1)} > (<)d_2a_{21}$.

Secondly, if (i) $B_1 < 0$, $B_2 < 0$, $B_3 > 0$ or (ii) $B_1 < 0$, $B_2 > 0$, $B_3 > 0$, then $P_{2*} = P_{2*}^{(2)}$ is only positive real root of Equation (6) and hence, $S_{2*} = \left(P_{1*}^{(2)}, P_{2*}^{(2)}, P_{3*}^{(2)}\right)$, where $P_{1*}^{(2)} = \frac{a_{23}a_{32}P_{2*}^{(2)}-a_{23}d_2+d_1h_2+h_1h_2P_{2*}^{(2)}}{h_2a_{21}-a_{31}a_{23}}$ and $P_{3*}^{(2)} = \frac{a_{31}d_1+a_{31}h_1P_{2*}^{(2)}+a_{32}a_{21}P_{2*}^{(2)}-d_2a_{21}}{h_2a_{21}-a_{31}a_{23}}$ is only feasible co-existing stationary point of the system (3a)-(3c), which exists if the conditions $h_2a_{21} > (<)a_{31}a_{23}$,

$a_{23}a_{32}P_{2*}^{(2)} + d_1h_2 + h_1h_2P_{2*}^{(2)} > (<)a_{23}d_2$ and $a_{31}d_1 + a_{31}h_1P_{2*}^{(2)} + a_{32}a_{21}P_{2*}^{(2)} > (<)d_2a_{21}$ are satisfied.

Subcase 2: If (i) $B_1 > 0$, $B_2 < 0$, $B_3 > 0$ or (ii) $B_1 < 0$, $B_2 > 0$, $B_3 < 0$, then Equation (6) has two positive roots $P_{2*} = P_{2*}^{(1)}$ and $P_{2*} = P_{2*}^{(2)}$. Hence, the system (3a)-(3c) has two feasible co-existing stationary points $S_{i*} = \left(P_{1*}^{(i)}, P_{2*}^{(i)}, P_{3*}^{(i)}\right)$, where $P_{1*}^{(i)} = \frac{a_{23}a_{32}P_{2*}^{(i)} - a_{23}d_2 + d_1h_2 + h_1h_2P_{2*}^{(i)}}{h_2a_{21} - a_{31}a_{23}}$ and $P_{3*}^{(i)} = \frac{a_{31}d_1 + a_{31}h_1P_{2*}^{(i)} + a_{32}a_{21}P_{2*}^{(i)} - d_2a_{21}}{h_2a_{21} - a_{31}a_{23}}$, $i = 1, 2$. These stationary points exist only when the conditions $h_2a_{21} > (<)a_{31}a_{23}$, $a_{23}a_{32}P_{2*}^{(i)} + d_1h_2 + h_1h_2P_{2*} > (<)a_{23}d_2$, and $a_{31}d_1 + a_{31}h_1P_{2*}^{(i)} + a_{32}a_{21}P_{2*} > (<)d_2a_{21}$ are satisfied.

Subcase 3: If (i) $B_1 > 0$, $B_2 > 0$, $B_3 > 0$ or (ii) $B_1 < 0$, $B_2 < 0$, $B_3 < 0$, then Equation (6) has no positive real root. Hence, the system (3a)-(3c) has no co-existing stationary point.

Case 3: If ${B_2}^2 = 4B_1B_3$, then Equation (6) has only one real root, $\bar{P}_2 = \frac{-B_2}{2B_1}$, with following subcases:

Subcase 1: If (i) $B_1 < 0$, $B_2 < 0$ or (ii) $B_1 > 0$, $B_2 > 0$, then Equation (6) has no positive real root. Hence, the system (3a)-(3c) has no positive stationary point.

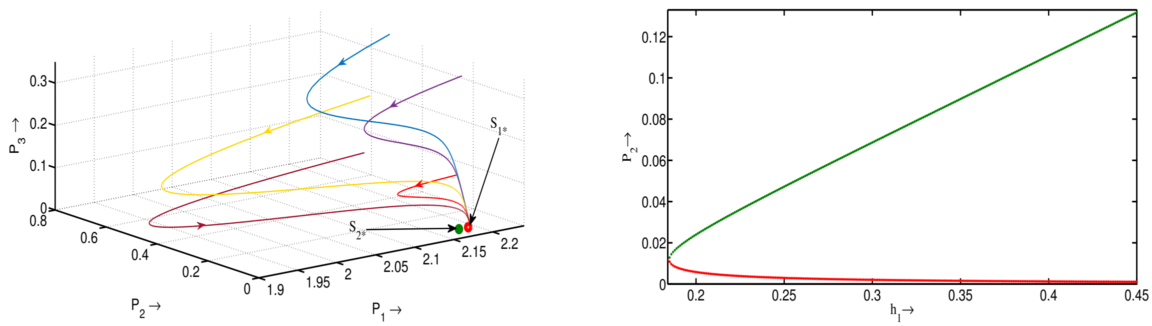
Subcase 2: If (i) $B_1 > 0$, $B_2 < 0$ or (ii) $B_1 < 0$, $B_2 > 0$, then the two roots of Equation (6) will coincide. Hence, the two stationary points S_{1*} and S_{2*} will coincide and the colliding stationary point is given by

$$\bar{S} = \left(\frac{a_{23}a_{32}\bar{P}_2 - a_{23}d_2 + d_1h_2 + h_1h_2\bar{P}_2}{h_2a_{21} - a_{31}a_{23}}, \bar{P}_2, \frac{a_{31}d_1 + a_{31}h_1\bar{P}_2 + a_{32}a_{21}\bar{P}_2 - d_2a_{21}}{h_2a_{21} - a_{31}a_{23}}\right).$$

The coinciding co-existing stationary point exists only when the conditions $h_2a_{21} > (<)a_{31}a_{23}$, $a_{23}a_{32}\bar{P}_2 + d_1h_2 + h_1h_2\bar{P}_2 > (<)a_{23}d_2$, $a_{31}d_1 + a_{31}h_1\bar{P}_2 + a_{32}a_{21}\bar{P}_2 > (<)d_2a_{21}$, are satisfied.

Example 4.1.

For a set of parameters' values $k = 10, r = 5, a_{11} = 2.3, a_{12} = 1, a_{13} = 0.28, a_{21} = 0.1, a_{23} = 6.8, d_1 = 0.1, a_{31} = 0.1, a_{32} = 0.02, d_2 = 0.2, h_2 = 1$ and taking h_1 as bifurcation parameter, we observe that the proposed system (3a)-(3c) possesses two interior stationary points $S_{1*} = (P_{1*}^{(1)}, P_{2*}^{(1)}, P_{3*}^{(1)})$ and $S_{2*} = (P_{1*}^{(2)}, P_{2*}^{(2)}, P_{3*}^{(2)})$ for $h_1 > h_1^*$, which collide with $\bar{S}\left(\bar{P}_1, \bar{P}_2, \bar{P}_3\right) = (2.16588, 0.01180, 0.01682)$ for critical threshold $h_1 = h_1^* = 0.18479$ and are mutually annihilated for $h_1 < h_1^*$. The corresponding phase portrait and bifurcation diagram for change in the number of interior stationary points are given in Figures 1(a) and 1(b), respectively.



(a) (b)

Figure 1. Figure 1(a) shows the phase portrait of solution trajectories of the system (3a)-(3c) corresponding to different initial values in which $S_{1*} = (2.16805, 0.00775, 0.01696)$ is stable stationary point and $S_{2*} = (2.16231, 0.01796, 0.01659)$ is unstable stationary point for $h_1 = 0.19$. P_2-component of interior stationary points are plotted in Figure 1(b), which represents change in number of interior stationary points with respect to bifurcation parameter h_1.

4.2. Local stability analysis of various stationary points

The main aim of this section is to analyze the local behavior of the system (3a)-(3c) in the neighborhood of the existing stationary points listed earlier.

Theorem 4.1.

(i) The trivial stationary point S_0 is always a saddle point.
(ii) The stationary point S_1 is stable if the conditions $r\, a_{21} < a_{11}\, d_1$ and $r\, a_{31} < a_{11}\, d_2$ are satisfied together; otherwise, it becomes unstable.
(iii) If $a_{31}\, \tilde{P}_1 + a_{32}\, \tilde{P}_2 < d_2$, then stationary point $\tilde{S}$ is stable; otherwise, it is unstable.
(iv) The stationary point $\hat{S}$ is stable if $a_{21}\hat{P}_1 < a_{23}\hat{P}_3 + d_1$; otherwise, it becomes unstable.

Proof:

(i) The variational matrix $V(S_0)$ has the following characteristic values: $\lambda_1 = r > 0$, $\lambda_2 = -d_1 < 0$ and $\lambda_3 = -d_2 < 0$. Here, one characteristic value is always positive. As a result, the stationary point S_0 is always a saddle point.

(ii) To analyze the stability of the stationary point $S_1 = (\frac{r}{a_{11}}, 0, 0)$, we evaluate the following variational matrix of the system (3a)-(3c) at the given stationary point S_1

$$V(S_1) = \begin{pmatrix} -r & -\frac{ra_{12}}{a_{11}} & -\frac{ra_{13}}{a_{11}} \\ 0 & \frac{ra_{21}}{a_{11}} - d_1 & 0 \\ 0 & 0 & \frac{ra_{31}}{a_{11}} - d_2 \end{pmatrix}.$$

The variational matrix $V(S_1)$ has the characteristic values as $\lambda_1 = -r$, $\lambda_2 = \frac{ra_{21}}{a_{11}} - d_1$ and $\lambda_3 = \frac{ra_{31}}{a_{11}} - d_2$. Thus, the result for the stability of stationary point $S_1 = (\frac{r}{a_{11}}, 0, 0)$ follows.

(iii) To know the behavior of the solutions of the system (3a)-(3c) near stationary point $\tilde{S} = (\tilde{P}_1, \tilde{P}_2, 0)$, we calculate the signs of the characteristic values of the following variational matrix:

$$V(\tilde{S}) = \begin{pmatrix} -\frac{a_{11}\tilde{P}_1}{1+k\tilde{P}_2} & -\frac{\tilde{P}_1\left(r-a_{11}\tilde{P}_1\right)k}{\left(1+k\tilde{P}_2\right)^2} - a_{12}\tilde{P}_1 & -a_{13}\tilde{P}_1 \\ a_{21}\tilde{P}_2 & -h_1\tilde{P}_2 & -a_{23}\tilde{P}_2 \\ 0 & 0 & a_{31}\tilde{P}_1 + a_{32}\tilde{P}_2 - d_2 \end{pmatrix}.$$

The characteristic polynomial of matrix $V(\tilde{S})$ is given by

$$Q_1(\lambda) = (\lambda - \lambda_1)\left(\lambda^2 + \frac{A}{1+k\tilde{P}_2}\lambda + \frac{\left(2\,k\,a_{12}\,a_{21}\tilde{P}_2 + a_{11}h_1 + a_{12}a_{21}\right)\tilde{P}_1\,\tilde{P}_2}{1+k\tilde{P}_2}\right), \tag{7}$$

where $\lambda_1 = a_{31}\,\tilde{P}_1 + a_{32}\,\tilde{P}_2 - d_2$ and $A = h_1\tilde{P}_2 + h_1\,k\,\tilde{P}_2^2 + a_{11}\,\tilde{P}_1$. This concludes the result for the local stability of stationary point $\tilde{S}$.

(iv) The variational matrix of the system (3a)-(3c) evaluated at $\hat{S} = (\hat{P}_1, 0, \hat{P}_3)$ is given by:

$$V(\hat{S}) = \begin{pmatrix} -a_{11}\hat{P}_1 & -ka_{13}\hat{P}_1\hat{P}_3 - a_{12}\hat{P}_1 & -a_{13}\hat{P}_1 \\ 0 & a_{21}\hat{P}_1 - a_{23}\hat{P}_3 - d_1 & 0 \\ a_{31}\hat{P}_3 & a_{32}\hat{P}_3 & -h_2\hat{P}_3 \end{pmatrix}.$$

The characteristic polynomial of matrix $V(\hat{S})$ is given as following

$$Q_2(\lambda) = (\lambda - \lambda_1)\left(\lambda^2 + \left(a_{11}\hat{P}_1 + h_2\hat{P}_3\right)\lambda + (a_{11}h_2 + a_{13}a_{31})\,\hat{P}_1\hat{P}_3\right), \tag{8}$$

where $\lambda_1 = a_{21}\hat{P}_1 - a_{23}\hat{P}_3 - d_1$. This is enough for the local stability result of $\hat{S}$. ∎

Theorem 4.2.

The co-existing stationary point $S_* = (P_{1*}, P_{2*}, P_{3*})$, is stable if the conditions $\xi_3 > 0$ and $\xi_1\,\xi_2 > \xi_3$ are satisfied.

Proof:

The variational matrix of the system (3a)-(3c) evaluated at $S_* = (P_{1*}, P_{2*}, P_{3*})$ is given as:

$$V(S_*) = \begin{pmatrix} -\frac{a_{11}P_{1*}}{1+kP_{2*}} & -\frac{(r-a_{11}P_{1*})kP_{1*}}{(1+kP_{2*})^2} - a_{12}P_{1*} & -a_{13}P_{1*} \\ a_{21}P_{2*} & -h_1P_{2*} & -a_{23}P_{2*} \\ a_{31}P_{3*} & a_{32}P_{3*} & -h_2P_{3*} \end{pmatrix}.$$

The characteristic equation of the matrix $V(S_*)$ is given as

$$\lambda^3 + \xi_1\,\lambda^2 + \xi_2\,\lambda + \xi_3 = 0, \tag{9}$$

where

$$\xi_1 = \frac{a_{11}P_{1*}}{1+kP_{2*}} + h_1P_{2*} + h_2P_{3*} > 0,$$

$$\xi_2 = \frac{P_{3*}P_{1*}\left(a_{11}h_2 + a_{13}a_{31} + a_{31}a_{13}kP_{2*}\right)}{1+kP_{2*}} + \left(h_1h_2 + a_{32}a_{23}\right)P_{2*}P_{3*} + \frac{P_{1*}P_{2*}\left(a_{11}h_1 + 2\,a_{21}a_{12}kP_{2*} + a_{21}ka_{13}P_{3*} + a_{12}a_{21}\right)}{1+kP_{2*}},$$

$$\xi_3 = \left(\frac{a_{11}\left(h_1h_2 + a_{32}a_{23}\right)}{1+kP_{2*}} - \frac{\left(\left(2\,a_{12}P_{2*} + a_{13}P_{3*}\right)k + a_{12}\right)\left(-h_2a_{21} + a_{31}a_{23}\right)}{1+kP_{2*}}\right)P_{1*}P_{2*}P_{3*} + a_{13}\left(a_{21}a_{32} + a_{31}h_1\right)P_{1*}P_{2*}P_{3*}.$$

Hence, from Routh-Hurwitz criterion the result for local stability of the co-existing stationary point $S_* = (P_{1*}, P_{2*}, P_{3*})$ follows. ∎

5. Hopf Bifurcation

It can be verified numerically that for the choice of a set of parameters' values taken in Example 4.1, the co-existing stationary point S_{2*} is always saddle whenever it exists, whereas S_{1*} can be stable 4.1 or unstable depending on the parametric restrictions. Therefore, we study the existence of periodic solution through Hopf bifurcation of the system (3a)-(3c) around the co-existing stationary point S_{1*}. We make use of the criterion derived by Liu (1994) to show the existence of periodic solutions through Hopf bifurcation. We choose fear parameter k as a bifurcation parameter so that S_{1*} is locally stable when $k > k_{crit}$ and unstable when $k < k_{crit}$. Co-existing stationary point S_{1*} loses its stability and periodic solutions appear at $k = k_{crit}$. We know that a simple Hopf bifurcation occurs when the following conditions are satisfied for the matrix $V(S_* = S_{1*}, k = k_{crit})$:
(1) $\xi_1 > 0, \xi_3 > 0$, and $\xi_1\xi_2 = \xi_3$ at $k = k_{crit}$.
(2) The transversality condition $\frac{d(Re(\lambda))}{dk}|_{S_*=S_{1*},k=k_{crit}} \neq 0$.

Since the analytical finding of Liu's criterion of the existence of periodic solution through Hopf bifurcation for the system (3a)-(3c) is much more complicated, we verify these conditions numerically. For this purpose, we consider a set of parameters: $r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_1 = 0.1$ and $h_2 = 0.27$ and we choose fear parameter k as bifurcation parameter. The system has a unique Hopf bifurcation threshold $k_{crit} = 1.59581$. For this choice of parameters' values the system (3a)-(3c) has unique co-existing stationary point S_{1*} as $k \in (0, 2.92316)$. The co-existing stationary point S_{1*} is stable for $k \in (1.59581, 2.92316)$ and unstable for $k < k_{crit} = 1.59581$. The characteristic equation corresponding to the matrix $V(S = S_{1*}, k_{crit})$ evaluated at $k = k_{crit} = 1.59581$ given as:

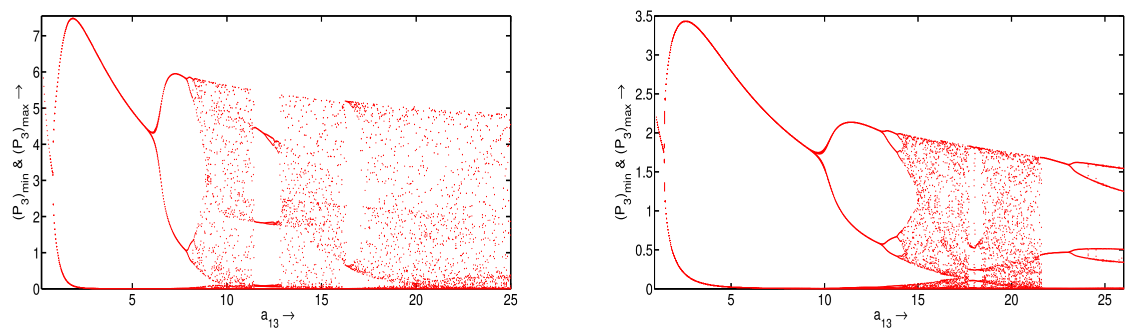
$$\lambda^3 + 0.28268\lambda^2 + 2.55614\lambda + 0.72257 = 0. \tag{10}$$

Here, we get $L_1 = 0.28268 > 0, L_3 = 0.72257 > 0, L_1L_2 = L_3 = 0.72257$ and $\frac{d(Re(\lambda))}{dk}|_{k=k_{crit}} = -0.10259 \neq 0$, with the characteristic values $-0.28268, 1.59879I$ and $-1.59879I$. As a result, all of Liu's criterion conditions for the existence of periodic solutions via Hopf bifurcation are satisfied, resulting in the appearance of periodic oscillations around S_{1*}.

6. Dynamics of the System Beyond Hopf Bifurcation

In this section, we discuss the different complex dynamical behaviors of the system (3a)-(3c) numerically for the parameters k, h_1 and h_2. First of all we examine the dynamics of the system (3a)-(3c) in the absence of fear effect k and harvesting h_1 and h_2. We consider a set of parameters' values as: $k = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0$ and choosing a_{13} as bifurcation parameter from the research article Tanabe and Namba (2005). For these parameters' values system (3a)-(3c) has unique Hopf bifurcation threshold $(a_{13})_{crit} = 0.79634$ and corresponding bifurcation diagram is shown in Figure 2(a), which agrees with the diagram given in Tanabe and Namba (2005). From Figure 2(a), it is clear that increase in value of bifurcation parameter a_{13} results in chaotic system behavior.

Now, we want to see the effect of a_{13} on the dynamics of the system (3a)-(3c) in the presence of harvesting (h_1 and h_2). To check the effect of a_{13}, we choose a set of parameters' values as: $k = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$. If we choose a_{13} as bifurcation parameter, then we get a unique Hopf bifurcation threshold $(a_{13})_{crit} = 1.39875$. The corresponding bifurcation diagram is shown in Figure 2(b), which confirms that if we increase the values of a_{13}, then the system (3a)-(3c) remains chaotic for a smaller range of values of a_{13}. It gives us a hint of control the chaos of the proposed system.

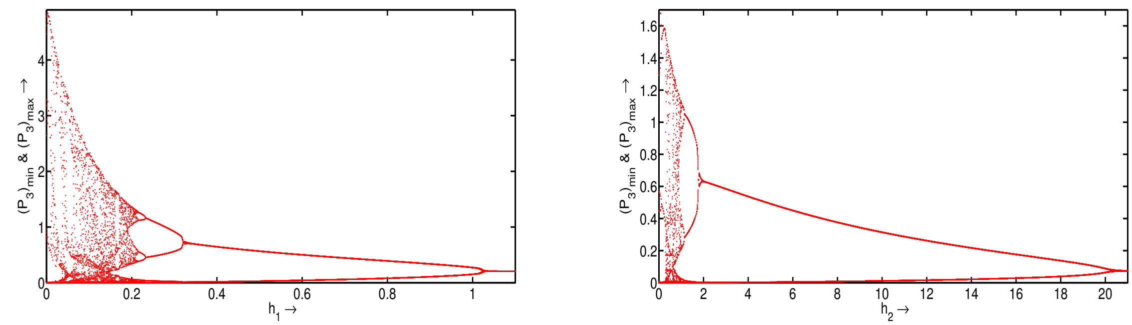


(a) (b)

Figure 2. (a) and (b) show variation in the top-predator species (P_3) with respect to (a_{13}). Left panel shows the dynamics in absence of both fear effect and harvesting whereas right panel shows the dynamics in absence of fear effect but in presence of harvesting.

To investigate the effects of harvesting on the dynamics of the system (3a)-(3c) in the absence of fear effect, we fix a set of parameters' values as follows $k = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 15, a_{21} = 1, a_{23} = 1, d_1 = 1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.027$. We choose h_1 as the bifurcation parameter. For these parameters' values the system (3a)-(3c) has Hopf bifurcation threshold as $(h_1)_{crit} = 1.04111$. In similar manner, if we choose a set of parameters' values as: $k = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 25, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2$ and taking h_2 as bifurcation parameter. The corresponding chaotic bifurcation diagrams are shown in Figures 3(a) and 3(b), respectively, which show that chaos can be controlled by harvesting an appropriate number of middle and top predator species.

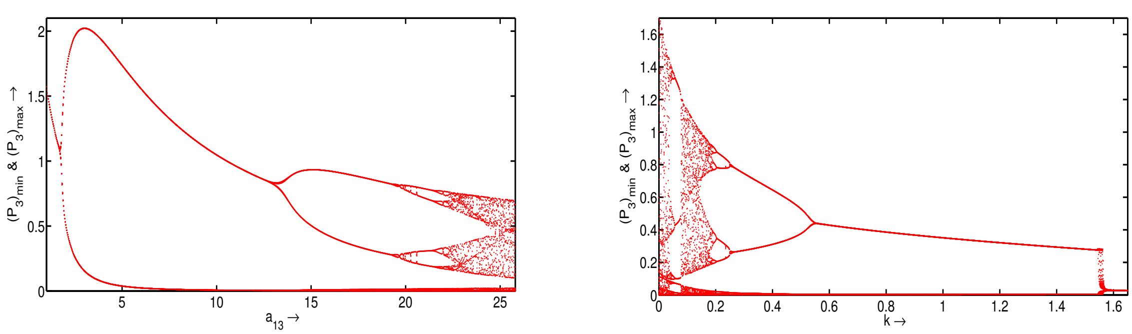
To know the effect of a_{13} on the dynamics of system (3a)-(3c) in the presence of fear effect k



(a) (b)

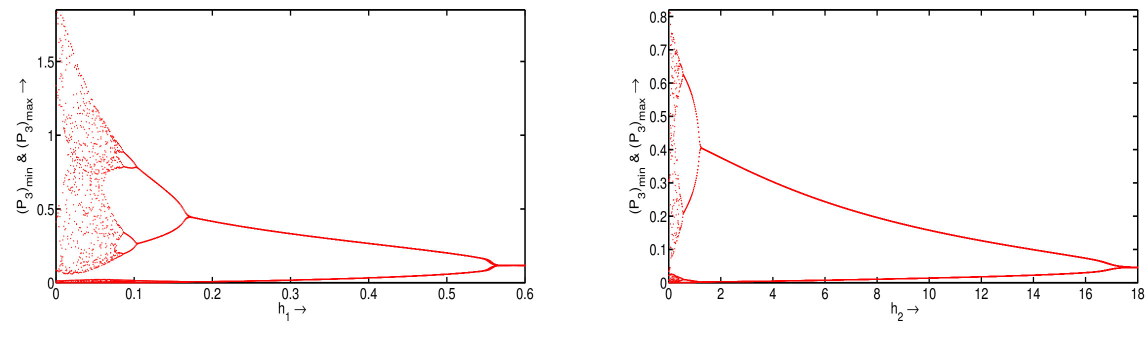
Figure 3. (a) and (b) represent change in the top-predator species (P_3) in absence of fear effect with respect to bifurcation parameters h_1 and h_2, respectively.

and harvesting (h_1 and h_2), we choose parameters' values as: $k = 0.25, r = 5, a_{11} = 0.4, a_{12} = 1, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$, by varying the values of a_{13}, we check the effect of consumption rate of prey by top-predator on the system dynamics. We notice that the increase in a_{13} makes system dynamics chaotic through period doubling routes, which can be seen from Figure 4(a). Next, we study the impact of fear effects on the dynamics of the model system (3a)-(3c) by choosing $r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$, and varying the value of k. We observe that the increase in fear effect k stabilizes the system (3a)-(3c) from chaotic instability, which can be seen by a bifurcation diagram plotted in Figure 4(b). The system remains chaotic because of the low cost of fear in prey growth. It shows that prey species become more aware and save themselves from their predators.



(a) (b)

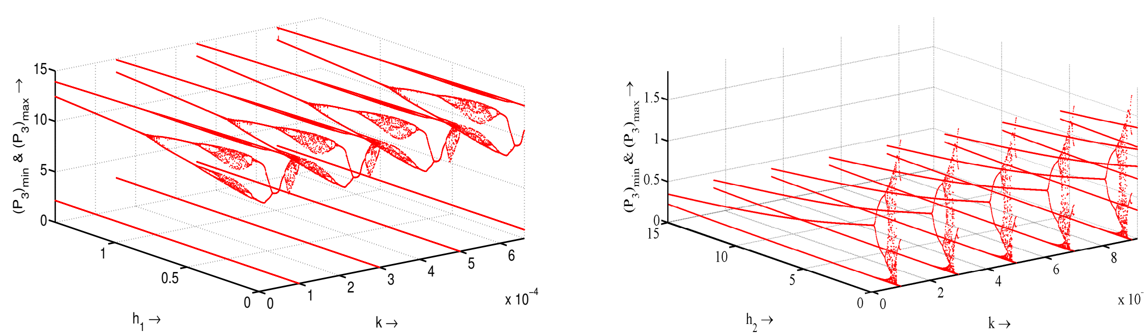
Figure 4. (a) and (b) describe change in the top-predator species (P_3) with respect to bifurcation parameters a_{13} and k in presence of both fear effect and harvesting respectively.



(a) (b)

Figure 5. (a) and (b) represent change in the top-predator species (P_3) with respect to bifurcation parameters h_1 and h_2 in presence of fear effect respectively.

We are also interested in understanding the effect of harvesting (h_1 and h_2) on the dynamics of the system (3a)-(3c) in the presence of fear effect. For this purpose, we first check the effect of harvesting h_1 on the system dynamics. We fixed a set of parameters' values as: $k = 0.25, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$, by varying h_1, we investigate the effect of harvesting of middle predator on dynamics of the system. The corresponding bifurcation diagram is shown in Figure 5(a). Similarly to check the behavior of proposed system with respect to harvesting h_2 of top predator, we choose parameters' values as: $k = 0.25, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 25, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2$ and vary h_2. The corresponding bifurcation diagram is shown in figure 5(b). From Figure 5(b), it is clear that the system becomes stable for increasing values of h_2.



(a) (b)

Figure 6. (a) represents change in the top-predator species (P_3) with respect to two bifurcation parameters k and h_1. (b) represents change in the top-predator species (P_3) with respect to two bifurcation parameters k and h_2.

Finally, to observe the combined effect of both fear and harvesting, bifurcation diagrams 6(a) and 6(b) are plotted for the pairs (k, h_1) and (k, h_2). These bifurcation diagrams clearly indicate that there are several chaotic bands for various values of k with respect to h_1 and h_2, respectively.

7. Conclusion

In this manuscript, we analyzed a food web system for three species by incorporating fear effects on prey species in which middle and top-predators are being harvested. The positivity as well as the boundedness of solutions of the proposed system are analyzed by using suitable differential inequalities. We have shown analytically the existence of ecological stationary points and observed that the proposed system has multiple co-existing stationary points. Furthermore, the local stability analysis of each stationary point of the system is performed with the help of characteristic values of the variational matrix.

It is numerically observed that in the case of two co-existing stationary points, one of them is always a saddle, whereas the second one may change its stability through Hopf bifurcation. We have obtained a periodic solution through Hopf bifurcation for the proposed model in the case of a unique co-existing stationary point. A similar result can be produced in the case of multiple co-existing stationary points.

The effects of fear and quadratic harvesting of middle and top-predators in a food-web model are analyzed in detail. We observe chaotic behavior of the proposed system in absence of middle predators' fear, middle predators' harvesting and top-predator harvesting for increasing values of prey consumption by top-predators. To study the effects of fear on prey species and the harvesting of middle and top-predators in the system, we have plotted the bifurcation diagrams with respect to each of these parameters. From the bifurcation diagrams, we observe that increase in either of the parameters middle predators' fear, middle predators' harvesting and top-predator harvesting can control chaos. The combined effect of both fear and harvesting is also observed numerically, in which more chaotic dynamics are obtained. In the case of multiple stationary points, the dynamics becomes more complex in the form of saddle-node bifurcation and bifurcation of periodic solutions in the future.

Acknowledgment:

The work of first author (R.P. Gupta) is supported by UGC-BSR Research Start-Up-Grant-2016-17 (No.R/Dev/M-14-33/0219). The second author (Dinesh K. Yadav) is thankful to UGC and BHU for the financial support under the Non-NET UGC fellowship (No.: R/Dev./Sch. (UGC Research Fellow)/2020-21/28397).

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