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Numerical Study of the Time Fractional Burgers' Equation By Using Explicit and Implicit Schemes

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Abstract

The study discusses the numerical solution for a time fractional Burgers' equation using explicit (scheme 1) and implicit scheme (scheme 2), respectively. The approximation of the differential equation is discretized using the finite difference method (FDM). A non-linear term present in the Burgers' equation is approximated using the time-averaged values. The Von-Neumann analysis shows that the Scheme 1 is conditionally stable and Scheme 2 is unconditionally stable. The numerical solutions are compared with the exact solutions and are good in agreement. Also, the error is estimated between exact and numerical solutions.

Keywords: Time fractional derivatives; Burgers' equation; Caputo fractional derivative; finite difference method; Explicit and implicit schemes

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1. Introduction

Burgers' equations serve as a mathematical model for a wide variety of physical phenomena including turbulence, shock wave, and gas dynamics (Burgers (1948), Cole (1951), Hopf (1950)). Fractional calculus is a field of applied mathematics and engineering that deals with the integration and derivation of arbitrary orders. The application of fractional calculus to explain a real system better as compared to the use of integer order operators (Podlubny (1998)). Fractional calculus problems have gained importance and popularity mainly due to their applications in science and engineering. Thus, the research on fractional derivative based equations are increasing and this study takes the opportunity to derive the numerical solutions to time fractional Burgers' equation. Such a equation helpful in further understanding the flow phenomenon during fluid flow, advection diffusion, oscillation, etc. (Hilfer (2000), Machado et al. (2010), Valerio et al. (2014)).

There are number of numerical techniques available to solve the partial differential equations, and these can be directly or indirectly used to solve the fractional-based differential equations. For instance, the explicit and implicit based FDM is used for achieving the solutions to fractional advection diffusion equation and it is found that the scheme is convergent and stable. In addition, it is noted that the explicit FDM is conditionally stable and the implicit FDM is unconditionally stable (Rehim (2015), Liu et al. (2007)). Jiwari et al. (2012, 2013) implemented the quadrature method for cracking the numerical solutions for a transient Burgers' equation. Mukundan and Awasthi (2016) proposed a numerical scheme based on Method of Line-Implicit FDM (MOL-IFDM) to solve the classical Burgers' equation. The proposed scheme is linear with unconditionally stable. Jiwari (2015) introduced a hybrid scheme based on uniform Haar wavelet approximation to solve the non-linear Burgers' equation and which is found to be efficient in overall costs. Li et al. (2016) presented the linear FDM based method for solving the TFBE and found it to be computationally fast. Yokus and Kaya (2017) and Yokus (2018) discussed the extended FDM solutions for the TFBE (Caputo) and Space-TFBE (shifted Caputo) and compared them against the exact solution. Mohebbi (2018) presented the implicit Spectral-FDM scheme to solve the TFBE. Jiwari et al. (2019) developed a meshless method based on the quadrature method to solve the Burgers' and coupled Burgers' equations. These meshless method are used to capture the behavior of shock and generate a smooth solution. Onal and Esen (2020) worked on Crank-Nicolson-based (C-N) FDM to approximate the solutions for TFBE. The study discusses the efficiency of a C-N FDM and is examined with the exact solution. Akram et al. (2020) extended the FDM with the cubic B-spline to enrich the efficiency of an FDM solver for the TFBE. Abdi et al. (2021) examined the explicit decoupled scheme (C-N based scheme) for solving TFBE. It is noticed that the explicitly decoupled scheme is efficient than the classical C-N scheme. Recently, Doley et al. (2022) obtained a numerical solutions to the space fractional Burgers' equation using the Lax-Friedrichs-implicit scheme. The solution is good comparable with the exact solution and the scheme is found to be unconditionally stable.

The various numerical schemes available to solve the TFBE and classical Burgers' equation are discussed in the above literature. The present study takes a gap to discuss the IFDM (implict-FDM) and EFDM (explicit-FDM) based on time averaged discretization of a non-linear term. Since the number of calculations involved in C-N based methods is more, authors consider the implicit based approximations. The paper is structured as follows. Section 2 explores the fractional order Burgers' equation and describes the explicit and implicit schemes applied to the time fractional Burgers' equations. Section 3 discusses the numerical results, and Section 4 gives the conclusion.

2. Discretization of Time Fractional Burgers' Equation

The fractional order Burgers' equation is achieved when we replace the order of differential term of the equation with fractional order. With respect to that, we can generate the Burgers' equation with time fractional order as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u\frac{\partial u(x,t)}{\partial x} = \mu\frac{\partial^2 u(x,t)}{\partial x^2}, (x,t) \in [a,b] \times (0, T_{max}], \tag{1}$$

with initial values,

$$u(x,0) = u_0(x), \tag{2}$$

and boundary values

$$u\,(0,t) = \mathrm{B}_1, \;\; u(1,t) = \mathrm{B}_2. \tag{3}$$

Here, $u_0(x)$, B_1 and B_2 are known functions. $u(x)$ with respect to time is unknown functional. Equation (1) is known as the time-fractional viscous Burgers' equation (TFBE). When we drop the viscous term from Equation (1), then we will get the inviscid Burgers' equation (Towers (2020)).

The fractional calculus is mostly defined as the expansion of differential and integral with non-integer orders. It can be written in the conservative form as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial f(u)}{\partial x} = \mu\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{4}$$

where $f(u)$ is called flux function with $f(u) = \frac{u^2}{2}$ and μ is a viscous term, which is related to the Reynold number $R(= \frac{1}{\mu})$. R is the Reynolds number, which reflects the intensity of the viscosity. We can rewrite the above equation as given below:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + A\frac{\partial u(x,t)}{\partial x} = \mu\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{5}$$

where $A = \dfrac{df}{du} = u$.

Also, the term $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the α^{th} order Caputo fractional derivative of a function, which is defined as,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{d}{ds} f(s)(t-s)^{-\alpha} ds, \tag{6}$$

where $\Gamma(.)$ is a Gamma function. The time-fractional derivatives can be approximated using finite difference technique/method (FDM) and is written as,

$$\frac{\partial^\alpha u(x_i, t_{n+1})}{\partial t^\alpha} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \Big[\sum_{k=0}^{n} \Big(u_i^{n+1-k} - u_i^{n-k} \Big) \delta_{n,k}^\alpha \Big] + O(\Delta t)^{2-\alpha}. \tag{7}$$

The higher order factors are very small and thus $O(\Delta t)^{2-\alpha}$ is neglected,

$$\frac{\partial^\alpha u(x_i, t_{n+1})}{\partial t^\alpha} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \Big[\sum_{k=0}^{n} \Big(u_i^{n+1-k} - u_i^{n-k} \Big) \delta_{n,k}^\alpha \Big], \tag{8}$$

where $\delta_{n,k}^\alpha = (k+1)^{1-\alpha} - (k)^{1-\alpha}$.

Here the parameter $\delta_{n,k}^\alpha$ satisfy,
1. $\delta_{n,k}^\alpha > 0$, $k =, 1, 2, ...;$
2. $\delta_{n,k}^\alpha > \delta_{n,k+1}^\alpha$,$k = 0, 1, 2, 3, ...$

The numerical solution to the proposed TFBE are derived using the implicit and explicit scheme based on FDM. Applying the Caputo fractional derivatives in a time term of the TFBE. A non-linear term as time averaged FDM and central differences for the spatial derivatives. For the numerical solution of the time fractional Burgers' equation (1), we introduce a uniform grid mesh of points of coordinates (x_i, t_n) with $x_i = i\Delta x$, $i = 0, 1, 2, 3, ..m$ and $t_n = n\Delta t$, $n = 0, 1, 2, 3, \ldots N$.

2.1. Formulation of implicit time fractional Burgers' equation

By applying implicit approximation to the TFBE and using the average values n and $n+1$ time steps represents the current and next time steps. Then, we can get as follows:

$$A = u_i^n = \frac{u_i^n + u_i^{n+1}}{2}. \tag{9}$$

After applying Equation (9) in Equation (5), we get

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \Big(\frac{u_i^n + u_i^{n+1}}{2} \Big) \frac{\partial u(x,t)}{\partial x} = \mu \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{10}$$

Apply Caputo fractional derivative in the time direction (8) and central difference along space direction to the above equation (10). Eventually, the discretized form of TFBE is given by,

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\Big[\sum_{k=0}^{n}\Big(u_i^{n+1-k}-u_i^{n-k}\Big)\delta_{n,k}^{\alpha}\Big]+\Big(\frac{u_i^n+u_i^{n+1}}{2}\Big)\Big(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\Big) = \mu\frac{(u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1})}{\Delta x^2}. \tag{11}$$

Final discretization after rearranging Equation (11),

$$-\mu\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}u_{i-1}^{n+1}+\Big[1+\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{4\Delta x}(u_{i+1}^n-u_{i-1}^n)+2\mu\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\Big]u_i^{n+1} = u_i^n-\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{4\Delta x}(u_{i+1}^n-u_{i-1}^n)u_i^n-\Big[\sum_{k=1}^{n}\Big(u_i^{n+1-k}-u_i^{n-k}\Big)\delta_{n,k}^{\alpha}\Big], \tag{12}$$

where $R=\mu\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}$ and $S=\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{4\Delta x}$.

Then, we rewrite the above Equation (12) as,

$$-Ru_{i-1}^{n+1}+\Big[1+S(u_{i+1}^n-u_{i-1}^n)+2R\Big]u_i^{n+1} = u_i^n-S(u_{i+1}^n-u_{i-1}^n)u_i^n -\Big[\sum_{k=1}^{n}\Big(u_i^{n+1-k}-u_i^{n-k}\Big)\delta_{n,k}^{\alpha}\Big]. \tag{13}$$

2.2. Formulation of time fractional explicit scheme

A well-known method, namely the explicit scheme, will be employed to handle specific nonlinear time fractional Burgers' equation. By using Caputo fractional derivative in time and central difference in space direction in Equation (1), we arrive as follows:

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\Big[\sum_{k=0}^{n}\Big(u_i^{n+1-k}-u_i^{n-k}\Big)\delta_{n,k}^{\alpha}\Big]+\Big(\frac{u_i^n+u_i^{n+1}}{2}\Big)\Big(\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}\Big) = \mu\frac{(u_{i-1}^{n}-2u_i^{n}+u_{i+1}^{n})}{\Delta x^2} \tag{14}$$

$$\begin{aligned}\implies u_i^{n+1} = {} & u_i^n\Big(1-2\mu\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\Big)+\Big(\frac{\mu(\Delta t)^{\alpha}\Gamma(2-\alpha)}{\Delta x^2} +\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{2\Delta x}\Big(\frac{u_i^n+u_i^{n+1}}{2}\Big)\Big)u_{i-1}^n+\Big(\frac{\mu(\Delta t)^{\alpha}\Gamma(2-\alpha)}{\Delta x^2}\\ & -\frac{(\Delta t)^{\alpha}\Gamma(2-\alpha)}{2\Delta x}\Big(\frac{u_i^n+u_i^{n+1}}{2}\Big)\Big)u_{i+1}^n-\Big[\sum_{k=1}^{n}\Big(u_i^{n+1-k}-u_i^{n-k}\Big)\delta_{n,k}^{\alpha}\Big].\end{aligned} \tag{15}$$

After rearranging the equation (15), we obtain,

$$u_i^{n+1} = u_i^n(1-2R) + (R + S(u_i^n + u_i^{n+1}))u_{i-1}^n + (R - S(u_i^n + u_i^{n+1}))u_{i+1}^n - \Big[\sum_{k=1}^{n}\Big(u_i^{n+1-k} - u_i^{n-k}\Big)\delta_{n,k}^{\alpha}\Big]. \tag{16}$$

3. Stability Analysis

3.1. Stability of implicit time fractional Burgers' equation

This section investigate the stability of implicit-FDM based numerical scheme to the time fractional Burgers' equation. We introduce $u_j^n = \xi^n e^{ikj\Delta x}$ where k is the wave number and the corresponding amplification factor is $\xi = \xi(k)$. We investigate the above proposed scheme in Equation (13) and make $u_j^n = \xi^n e^{ikj\Delta x}$.

From above Equation (13), we get

$$\frac{\xi^{n+1}}{\xi^n} = \frac{1 - S(2i\sin\theta)\xi^n - \sum_{k=1}^{n}(\xi^{1-k} - \xi^{-k})\delta_{n,k}^{\alpha}}{1 + S(2i\sin\theta)\xi^n + 2R - 2R\cos\theta}. \tag{17}$$

When the system will stable then it must be $\left|\frac{\xi^{n+1}}{\xi^n}\right| \leq 1$,

$$\Big|\frac{\xi^{n+1}}{\xi^n}\Big| = \Big|\frac{1 - S(2i\sin\theta)\xi^n - \sum_{k=1}^{n}(\xi^{1-k} - \xi^{-k})\delta_{n,k}^{\alpha}}{1 + S(2i\sin\theta)\xi^n + 2R - 2R\cos\theta}\Big| \leq 1, \tag{18}$$

where $\Big|1 - S(2i\sin\theta)\xi^n - \sum_{k=1}^{n}(\xi^{1-k} - \xi^{-k})\delta_{n,k}^{\alpha}\Big| \leq \Big|1 + S(2i\sin\theta)\xi^n + 2R - 2R\cos\theta\Big|$.

It is obviously,

$$\Big|1 - S(2i\sin\theta)\xi^n - \sum_{k=1}^{n}(\xi^{1-k} - \xi^{-k})\delta_{n,k}^{\alpha}|\Big| \leq 1, \tag{19}$$

and

$$\Big|1 + S(2i\sin\theta)\xi^n + 2R - 2R\cos\theta\Big| \geq 1, \tag{20}$$

for all α, n and θ.

Because the denominator of the given equation (18) is always bigger than the numerator. That means, the absolute value of $\left|\frac{\xi^{n+1}}{\xi^n}\right|$ is much smaller than 1, i.e., thus the method (13) is unconditionally stable.

3.2. Stability analysis of explicit time fractional Burgers' equation

In this section, the stability of a proposed fractional numerical scheme Equation (16) is investigated by Von-Neumann method as defined by $u_j^n = \xi^n e^{ikj\Delta x}$. Thus, the obtained equation with respect to explicit scheme for TFBE is,

$$\frac{\xi^{n+1}}{\xi^n} = \Big[1 - 2R + 2Rcos(\theta) - 2i\Big(S(\xi^n + \xi^{n+1}\Big)sin(\theta) - \sum_{k=1}^{n}\Big(\xi^{1-k} - \xi^{-k}\Big)\delta_{n,k}^{\alpha}\Big]. \tag{21}$$

From Equation (21), while $\theta = 0$,

$$\left|\frac{\xi^{n+1}}{\xi^n}\right| = \left|\Big[1 - \sum_{k=1}^{n}\Big(\xi^{1-k} - \xi^{-k}\Big)\delta_{n,k}^{\alpha}\Big]\right| \leq 1. \tag{22}$$

It follows that, when $n = 0$, $\left|\frac{\xi^1}{\xi^0}\right| \leq 1$

$$\implies \left|\xi^1\right| \leq \left|\xi^0\right|. \tag{23}$$

And, for $n = 1$,

$$\left|\xi^2\right| \leq \left|1 - (\xi^0 - \xi^{-0})\delta_{n,1}^{\alpha}\right|\xi^1. \tag{24}$$

Thus,

$$\left|\xi^{(n+1)}\right| \leq \left|[1 - \sum_{k=1}^{n}(\xi^{1-k} - \xi^{-k})\delta_{n,k}^{\alpha}]\right|\xi^n, \tag{25}$$

while $\theta = \pi$. It will

$$\left|\frac{\xi^{n+1}}{\xi^n}\right| = \left|\Big[(1 - 4R) - \sum_{k=1}^{n}\Big(\xi^{1-k} - \xi^{-k}\Big)\delta_{n,k}^{\alpha}\Big]\right| \leq 1. \tag{26}$$

Hence, we can observe that the above Equation (22)-(26) that the proposed scheme is conditionally stable.

4. Numerical Results

This section illustrates the implementation of the proposed IFDM to solve the TFBE because of its unconditionally stable solutions. However, the explicit based FDM is conditionally stable.

Example 4.1.

Initially, the numerical implicit scheme is compared with the classical Burgers' equation (i.e., $\alpha = 1$). The results are compared with the MOL-IFDM scheme proposed by Mukundan and Awasthi (2016). The TFBE ($\alpha = 1$) at the homogeneous boundary condition,

$$u(0,t) = u(1,t) = 0, \tag{27}$$

and initial condition,

$$u(x,0) = \sin(\pi x), \tag{28}$$

are considered. The comparison of the results are given in Table 1.

Table 1. Comparison of the numerical results at $t = 0.01$, $\Delta t = 0.001$ and $\mu = 0.02$.

	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$
MOL-IFDM (Mukundan and Awasthi 2016)	0.57208	0.93970	0.95801	0.60175
Proposed IFDM	0.57201	0.93801	0.95782	0.60169

Example 4.2.

This example is to demonstrate the accuracy and stability of the proposed implicit method. We use absolute and maximum errors (difference between exact and numerical solution) as follows:

$$L_\infty = \parallel U(x,t) - u(x,t) \parallel_\infty = Max\{|U(x_i,t_n) - u(x_i,t_n)|\}, i = 0,1,2,3,...,m. \tag{29}$$

Consider the fractional problem (1) with source term the following exact solution which is given by Onal and Esen (2020),

$$u(x,t) = e^x t^2, \tag{30}$$

with initial condition,

$$u(x,0) = 0, 0 \leq x \leq 1, \tag{31}$$

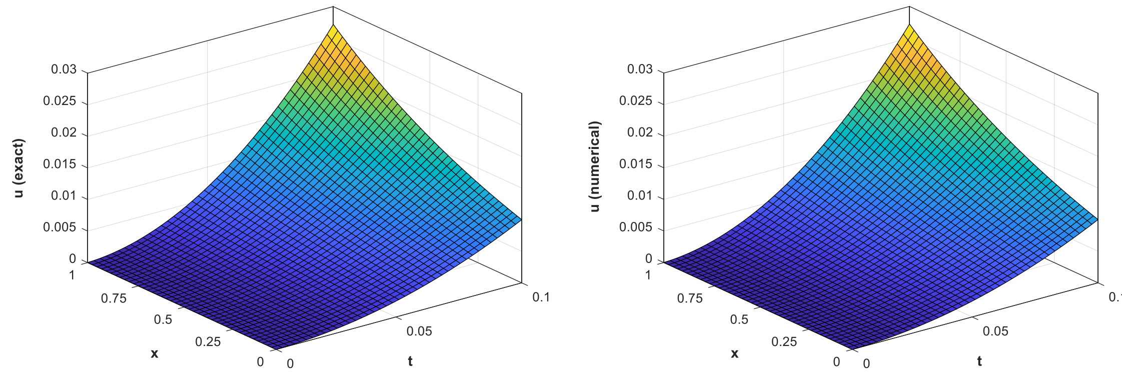
and boundary condition,

$$u(0,t) = t^2; u(1,t) = et^2, t \geq 0. \tag{32}$$

Moreover, the source term is given as,

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} e^x + t^4 e^{2x} - t^2 e^x, \tag{33}$$

Figure 1 represents the exact solution (left side) and the numerical solution (Proposed IFDM at the right side) for the TFBE for $\alpha = 0.5$. Both the solutions are well comparable to each other. Also, Table 2 describes the numerical error between both the solutions (numerical and exact solution). It is noted that the L_∞ decreases by reducing the space grid size (Δx) at $\Delta t = 0.025$.



(a) (b)

Figure 1. Comparison (a). Exact and (b) Numerical solution

Table 2. Maximum error between the exact and numerical solution for $t = 0.1$

Δx	L_∞
$1/20$	$1.78306000E-04$
$1/40$	$1.76959031E-04$
$1/60$	$1.76541798E-04$
$1/80$	$1.76320449E-04$
$1/100$	$1.76252412E-04$
$1/120$	$1.76071713E-04$

5. Conclusion

The study takes the discussion of numerical solution to the time fractional Burgers' equation. To achieve the simple and accurate solution, the explicit and implicit schemes in conjunction with the finite difference method has been proposed. In addition, the non-linear term in the TFBE is time averaged for the better accuracy.

Initially, both the schemes are checked for the stability and convergence. It is found that the explicit-based scheme is stable under certain condition. However, the implicit-based scheme is stable under all the conditions. Further, it is noted that both the schemes are convergent.

To check the accuracy of the scheme, the numerical solution which is achieved using implicit solution compared with the exact solution. The comparison shows the better agreement and the maximum error illustrates the same. In addition, it is observed that the decrease in space grid size reduces the numerical error.

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