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Integer Cordial Labeling of Alternate Snake Graph And Irregular Snake Graph

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Abstract

If a graph G admits integer cordial labeling, it is called an integer cordial graph. In this paper we prove that Alternate m-triangular Snake graph, Quadrilateral Snake graph, Alternate m-quadrilateral Snake graph, Pentagonal Snake graph, Alternate m-pentagonal Snake graph, Irregular triangular Snake graph, Irregular quadrilateral Snake graph, and Irregular pentagonal Snake graphs are integer cordial graphs.

Keywords: Integer cordial labelling; Alternate m-triangular snake graph; Irregular triangular snake graph

MSC 2010 No.: 05C76, 05C78

1. Introduction

In the present work, we contemplate a finite graph which is connected and undirected. We refer to a dynamic survey of graph labeling by Gallian (2020) for detailed survey on graph labeling. Cahit (1987) introduced the concept of cordial labeling and he proved that any graceful tree, any harmonius tree, ladder graph, or fan graph are cordial graphs. There are so many different labelings like product cordial labeling, graceful labeling, harmonious labeling, integer cordial labeling, etc.

Nicholas and Maya (2016) introduced the concept of integer cordial labeling. They have investigated that Cycle graph, Wheel graph, Star graph, Helm graph, and Closed Helm graphs are integer cordial graphs; Complete bipartite graph with *n* vertices if and only if *n* is even; Complete graph with *n* vertices is not integer cordial graph (Nicholas and Maya (2016)). Shah and Parmar (2019, 2020) have reported that Triangular snake graph, Double Triangular snake graph, Alternate triangular snake graph, *m*-Triangular snake graph, Quadrilateral snake graph, Double quadrilateral snake graph, *m*-Quadrilateral snake graph, Pentagonal snake graph, Double pentagonal snake graph, *m*-Pentagonal snake graph are integer cordial graph.

Sundaram et al. (2004) introduced product cordial labeling. A cycle graph with *n* vertices is product cordial if and only if *n* is odd; a complete graph with *n* vertices is not product cordial if *n* is greater than or equal to 4. All trees are product cordial graphs (Sundaram et al. (2004)). Sahaya Rani et al. (2018) introduced the concept of product integer cordial labeling. Star graph, and path graph are product integer cordial; Cycle graph with *n* vertices is product integer cordial if and only if *n* is odd (Sahaya Rani et al. (2018)).

2. Main Results

Theorem 2.1.

The m-Alternate triangular snake graph mAT_n is integer cordial graph for $m \geq 2$ & $n \geq 3$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are n vertices of the path P_n and joining u_{2i-1} and u_{2i} via new vertex $v_i^{(k)}$, for $1 \leq i \leq \lceil \frac{n-1}{2} \rceil$ & $1 \leq k \leq m$. So, the total number of vertices in $mAT_n = p = m\lceil \frac{n-1}{2} \rceil + n$.

There are six different types related to the values of m and n.

Type 1: If m is even and n is odd, p is odd.
We introduce the function $f : V \to \big[- \lfloor \frac{p}{2} \rfloor, ..., \lfloor \frac{p}{2} \rfloor \big]$ as follows:

$$f\big(u_i\big) = i - \tfrac{n+1}{2} \quad ; \quad 1 \leq i \leq n,$$

$$f\big(v_i^{(k)}\big) = i + \tfrac{(n-1)(k+1)}{4} ; \quad 1 \leq i \leq \tfrac{n-1}{2} \ \& \ 1 \leq k \leq m, \text{ where } k \text{ is odd},$$

$$f\big(v_i^{(k+1)}\big) = -\big[i + \tfrac{(n-1)(k+1)}{4}\big] ; \quad 1 \leq i \leq \tfrac{n-1}{2} \ \& \ 1 \leq k \leq m, \text{ where } k \text{ is odd}.$$

Type 2: If m and n both are even, p is even.
We introduce the function $f : V \to \big[-\frac{p}{2}, ..., \frac{p}{2}\big]^*$ as follows:

$$f\big(u_i\big) = \begin{cases} i - \frac{n+2}{2} ; & 1 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2} ; & \frac{n}{2} < i \leq n, \end{cases}$$

$$f\big(v_i^{(k)}\big) = i + \tfrac{n(k+1)}{4}\,; \quad 1 \le i \le \tfrac{n}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is odd,}$$

$$f\big(v_i^{(k+1)}\big) = -\big[i + \tfrac{n(k+1)}{4}\big]\,; \quad 1 \le i \le \tfrac{n}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is odd.}$$

Type 3: If m is odd and $n = 4t-1$, $t \in \mathbb{N}$ then p is even.
We introduce the function $f : V \to \big[-\frac{p}{2}, ..., \frac{p}{2}\big]^*$ as follows:

$$f\big(u_i\big) = \begin{cases} i - \frac{n+1}{2}\,; & 1 \le i \le \frac{n-1}{2}, \\ i - \frac{n-1}{2}\,; & \frac{n-1}{2} < i \le n, \end{cases}$$

$$f\big(v_i^{(1)}\big) = \begin{cases} i - \frac{3(n+1)}{4}\,; & 1 \le i \le \frac{n+1}{4}, \\ i + \frac{n+1}{4}\,; & \frac{n+1}{4} < i \le \frac{n-1}{2}, \end{cases}$$

$$f\big(v_i^{(k)}\big) = -\big[i + \tfrac{(n-1)(k+1)+2}{4}\big]\,; \quad 1 \le i \le \tfrac{n-1}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is even,}$$

$$f\big(v_i^{(k+1)}\big) = i + \tfrac{(n-1)(k+1)+2}{4}\,; \quad 1 \le i \le \tfrac{n-1}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is even.}$$

Type 4: If m is odd and $n = 4t$, $t \in \mathbb{N}$ then p is even.
We introduce the function $f : V \to \big[-\frac{p}{2}, ..., \frac{p}{2}\big]^*$ as follows:

$$f\big(u_i\big) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{n}{2}\,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f\big(v_i^{(1)}\big) = \begin{cases} i - \frac{3n+4}{4}\,; & 1 \le i \le \frac{n}{4}, \\ i + \frac{n}{4}\,; & \frac{n}{4} < i \le \frac{n}{2}, \end{cases}$$

$$f\big(v_i^{(k)}\big) = -\big[i + \tfrac{n(k+1)}{4}\big]\,; \quad 1 \le i \le \tfrac{n}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is even,}$$

$$f\big(v_i^{(k+1)}\big) = i + \tfrac{n(k+1)}{4}\,; \quad 1 \le i \le \tfrac{n}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is even.}$$

Type 5: If m is odd and $n = 4t+1$, $t \in \mathbb{N}$ then p is odd.
We introduce the function $f : V \to \big[-\lfloor\frac{p}{2}\rfloor, ..., \lfloor\frac{p}{2}\rfloor\big]$ as follows:

$$f\big(u_i\big) = i - \tfrac{n+1}{2}\,; \quad 1 \le i \le n,$$

$$f\big(v_i^{(1)}\big) = \begin{cases} i - \frac{3n+1}{4}\,; & 1 \le i \le \frac{n-1}{4}, \\ i + \frac{n-1}{4}\,; & \frac{n-1}{4} < i \le \frac{n-1}{2}, \end{cases}$$

$$f\big(v_i^{(k)}\big) = -\big[i + \tfrac{(n-1)(k+1)}{4}\big]\,; \quad 1 \le i \le \tfrac{n-1}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is even,}$$

$$f\big(v_i^{(k+1)}\big) = i + \tfrac{(n-1)(k+1)}{4}\,; \quad 1 \le i \le \tfrac{n-1}{2} \text{ \& } 1 \le k \le m, \text{ where } k \text{ is even.}$$

Type 6: If m is odd and $n = 4t+2$, $t \in \mathbb{N}$ then p is odd.

We introduce the function $f : V \rightarrow \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f\left(u_i\right) = \begin{cases} i - \frac{n+2}{2}\ ; & 1 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2}\ ; & \frac{n}{2} < i \leq n, \end{cases}$$

$$f\left(v_i^{(1)}\right) = \begin{cases} i - \frac{3n+2}{4}\ ; & 1 \leq i < \frac{n+2}{4}, \\ 0\ ; & i = \frac{n+2}{4}, \\ i + \frac{n-2}{4}\ ; & \frac{n+2}{4} < i \leq \frac{n}{2}, \end{cases}$$

$$f\left(v_i^{(k)}\right) = -\left[i + \frac{n(k+1)-2}{4}\right]; \quad 1 \leq i \leq \frac{n}{2}\ \&\ 1 \leq k \leq m, \text{ where } k \text{ is even,}$$

$$f\left(v_i^{(k+1)}\right) = i + \frac{n(k+1)-2}{4}\ ; \quad 1 \leq i \leq \frac{n}{2}\ \&\ 1 \leq k \leq m, \text{ where } k \text{ is even.}$$

Table 1. Edge condition for mAT_n

Sr. No.	m	n	p	Edge condition
1	m is even	n is odd	p is odd	$e_f\left(0\right) = \frac{(m+1)(n-1)}{2}$; $e_f\left(1\right) = \frac{(m+1)(n-1)}{2}$
2		n is even	p is even	$e_f\left(0\right) = \left\lfloor\frac{n(m+1)-1}{2}\right\rfloor$; $e_f\left(1\right) = \left\lceil\frac{n(m+1)-1}{2}\right\rceil$
3	m is odd	$n = 4t-1, t \in \mathbb{N}$	p is even	$e_f\left(0\right)=\frac{(m+1)(n-1)}{2}$; $e_f\left(1\right)=\frac{(m+1)(n-1)}{2}$
4		$n = 4t+1, t \in \mathbb{N}$	p is odd	
5		$n = 4t, t \in \mathbb{N}$	p is even	$e_f\left(0\right)=\left\lfloor\frac{n(m+1)-1}{2}\right\rfloor$; $e_f\left(1\right)=\left\lceil\frac{n(m+1)-1}{2}\right\rceil$
6		$n = 4t+2, t \in \mathbb{N}$	p is odd	

So, we get $\left|e_f(0) - e_f(1)\right| \leq 1$ in each case.

Hence, m-Alternate triangular snake graph mAT_n is integer cordial. ■

Theorem 2.2.

An alternate quadrilateral snake graph AQ_n is integer cordial graph, for $n \geq 3$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are n vertices of the path P_n and joining u_{2i-1} and u_{2i} via new vertices $v_{2i-1}^{(1)}$ and $v_{2i}^{(1)}$, for $1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor$. So, the total number of vertices in $AQ_n = p = 2n-1$ or $2n$, if n is odd or n is even, respectively.

There are two different types related to the values of n.

Type 1: If n is odd, p is also odd.
We introduce the function $f : V \rightarrow \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f\left(u_i\right) = i - \frac{n+1}{2}\ ; \quad 1 \leq i \leq n,$$

$$f\big(v_i^{(1)}\big) = \begin{cases} i-n\,; & 1 \le i \le \frac{n-1}{2}, \\ \quad i \quad ; & \frac{n-1}{2} < i \le (n-1). \end{cases}$$

Type 2: If n is even, p is also even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f\big(u_i\big) = \begin{cases} \quad i \quad ; & 1 \le i \le \frac{n}{2}, \\ i-(n+1)\,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f\big(v_i^{(1)}\big) = \begin{cases} -i+(n+1)\,; & 1 \le i \le \frac{n}{2}, \\ i-\frac{3n+2}{2} \quad ; & \frac{n}{2} < i \le n. \end{cases}$$

Table 2. Edge condition for AQ_n

Sr. No.	n	p	Edge condition
1	n is odd	p is odd	$e_f\big(0\big) = \left\lfloor \frac{5(n-1)}{4} \right\rfloor; e_f\big(1\big) = \left\lceil \frac{5(n-1)}{4} \right\rceil$
2	n is even	p is even	$e_f\big(0\big) = \left\lfloor \frac{5n-2}{4} \right\rfloor; e_f\big(1\big) = \left\lceil \frac{5n-2}{4} \right\rceil$

So, we get $\big|e_f(0) - e_f(1)\big| \le 1$ in each case.

Hence, Alternate quadrilateral snake graph AQ_n is integer cordial. ■

Theorem 2.3.

The m-Alternate quadrilateral snake graph mAQ_n is integer cordial graph, for $m \ge 2$ & $n \ge 3$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are n vertices of the path P_n and joining u_{2i-1} and u_{2i} via new vertices $v_{2i-1}^{(k)}$ and $v_{2i}^{(k)}$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$ & $1 \le k \le m$. So, the total number of vertices in $mAQ_n = p = n(m+1) - m$ or $n(m+1)$, if n is odd or n is even, respectively.

There are four different types related to the values of m and n.

Type 1: If m is even and n is odd, p is odd.
We introduce the function $f : V \to \left[-\left\lfloor \frac{p}{2} \right\rfloor, ..., \left\lfloor \frac{p}{2} \right\rfloor\right]$ as follows:

$$f\big(u_i\big) = i - \frac{n+1}{2} \quad ; \quad 1 \le i \le n,$$

$$f\big(v_i^{(k)}\big) = i + \frac{k(n-1)}{2}\,; \qquad 1 \le i \le (n-1) \text{ \& } 1 \le k \le m, \text{ where } k \text{ is odd},$$

$$f\big(v_i^{(k+1)}\big) = -\left[i + \frac{k(n-1)}{2}\right]; \quad 1 \le i \le (n-1) \text{ \& } 1 \le k \le m, \text{ where } k \text{ is odd}.$$

Type 2: If m and n both are even, p is even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f(u_i) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{n}{2} \,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f(v_i^{(k)}) = i + \frac{kn}{2}\,; \quad 1 \le i \le n \ \&\ 1 \le k \le m, \text{ where } k \text{ is odd,}$$

$$f(v_i^{(k+1)}) = -\left[i + \frac{kn}{2}\right]; \quad 1 \le i \le n \ \&\ 1 \le k \le m, \text{ where } k \text{ is odd.}$$

Type 3: If m and n are odd, p is odd.
We introduce the function $f : V \to \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f(u_i) = i - \frac{n+1}{2}\,; \quad 1 \le i \le n,$$

$$f(v_i^{(1)}) = \begin{cases} i - n\,; & 1 \le i \le \frac{n-1}{2}, \\ i \,; & \frac{n-1}{2} < i \le (n-1), \end{cases}$$

$$f(v_i^{(k)}) = -\left[i + \frac{k(n-1)}{2}\right]; \quad 1 \le i \le (n-1) \ \&\ 1 \le k \le m, \text{ where } k \text{ is even,}$$

$$f(v_i^{(k+1)}) = i + \frac{k(n-1)}{2}\,; \quad 1 \le i \le (n-1) \ \&\ 1 \le k \le m, \text{ where } k \text{ is even.}$$

Type 4: If m is odd and n is even, p is even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f(u_i) = \begin{cases} i \,; & 1 \le i \le \frac{n}{2}, \\ i - (n+1)\,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f(v_i^{(1)}) = \begin{cases} -i + (n+1)\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{3n+2}{2} \,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f(v_i^{(k)}) = -\left[i + \frac{kn}{2}\right]; \quad 1 \le i \le n \ \&\ 1 \le k \le m, \text{ where } k \text{ is even,}$$

$$f(v_i^{(k+1)}) = i + \frac{kn}{2}\,; \quad 1 \le i \le n \ \&\ 1 \le k \le m, \text{ where } k \text{ is even.}$$

Table 3. Edge condition for mAQ_n

Sr. No.	m	n	p	Edge condition
1	$m \in N$ & $m \ge 2$	n is odd	p is odd	$e_f(0) = \left\lfloor\frac{(n-1)(3m+2)}{4}\right\rfloor$; $e_f(1) = \left\lceil\frac{(n-1)(3m+2)}{4}\right\rceil$
2		n is even	p is even	$e_f(0) = \left\lfloor\frac{n(3m+2)-2}{4}\right\rfloor$; $e_f(1) = \left\lceil\frac{n(3m+2)-2}{4}\right\rceil$

So, we get $\left|e_f(0) - e_f(1)\right| \le 1$ in each case.

Hence, m-Alternate quadrilateral snake graph mAQ_n is integer cordial. ■

Theorem 2.4.

An alternate pentagonal snake graph APS_n is integer cordial graph, for $n \geq 3$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are n vertices of the path P_n and joining u_{2i-1} and u_{2i} to new vertices $v^{(1)}_{3i-2}, v^{(1)}_{3i-1}$ and $v^{(1)}_{3i}$, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. So, the total number of vertices in $APS_n = p = \frac{5n-3}{2}$ or $\frac{5n}{2}$, if n is odd or n is even, respectively.

There are four different types related to the values of n.

Type 1: If $n = 4t - 1$, $t \in \mathbb{N}$ then p is even.
We introduce the function $f : V \rightarrow \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f\big(u_i\big) = \begin{cases} i - \frac{n+3}{2}\,; & 1 \leq i \leq \frac{n+1}{2}, \\ i - \frac{n+1}{2}\,; & \frac{n+1}{2} < i \leq n, \end{cases}$$

$$f\big(v^{(1)}_i\big) = \begin{cases} i - \frac{5n+1}{4}\,; & 1 \leq i < \frac{3n-1}{4}, \\ i - \frac{n-3}{4}\,; & \frac{3n-1}{4} \leq i \leq \frac{3(n-1)}{2}. \end{cases}$$

Type 2: If $n = 4t + 1$, $t \in \mathbb{N}$ then p is odd.
We introduce the function $f : V \rightarrow \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f\big(u_i\big) = i - \frac{n+1}{2} \quad ; \quad 1 \leq i \leq n,$$

$$f\big(v^{(1)}_i\big) = \begin{cases} i - \frac{5n-1}{4}\,; & 1 \leq i \leq \frac{3(n-1)}{4}, \\ i - \frac{n-1}{4}\,; & \frac{3(n-1)}{4} < i \leq \frac{3(n-1)}{2}. \end{cases}$$

Type 3: If $n = 4t$, $t \in \mathbb{N}$ then p is even.
We introduce the function $f : V \rightarrow \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f\big(u_i\big) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2}\,; & \frac{n}{2} < i \leq n, \end{cases}$$

$$f\big(v^{(1)}_i\big) = \begin{cases} i - \frac{5n+4}{4}\,; & 1 \leq i \leq \frac{3n}{4}, \\ i - \frac{n}{4}\,; & \frac{3n}{4} < i \leq \frac{3n}{2}. \end{cases}$$

Type 4: If $n = 4t + 2$, $t \in \mathbb{N}$ then p is odd.
We introduce the function $f : V \rightarrow \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f\big(u_i\big) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2}\,; & \frac{n}{2} < i \leq n, \end{cases}$$

$$f\left(v_i^{(1)}\right) = \begin{cases} i - \frac{5n+2}{4}\ ; & 1 \leq i < \frac{3n+2}{4}, \\ 0\ ; & i = \frac{3n+2}{4}, \\ i - \frac{n+2}{4}\ ; & \frac{3n+2}{4} < i \leq \frac{3n}{2}. \end{cases}$$

Table 4. Edge condition for APS_n

Sr. No.	n	p	Edge condition
1	$n = 4t-1, t \in \mathbb{N}$	p is even	$e_f(0) = \frac{3(n-1)}{2}\ ; e_f(1) = \frac{3(n-1)}{2}$
2	$n = 4t+1, t \in \mathbb{N}$	p is odd	
3	$n = 4t, t \in \mathbb{N}$	p is even	$e_f(0) = \left\lfloor \frac{3n-1}{2} \right\rfloor\ ; e_f(1) = \left\lceil \frac{3n-1}{2} \right\rceil$
4	$n = 4t+2, t \in \mathbb{N}$	p is odd	

So, we get $\left|e_f(0) - e_f(1)\right| \leq 1$ in each case.

Hence, Alternate pentagonal snake graph APS_n is integer cordial. ■

Theorem 2.5.

The m-Alternate pentagonal snake graph $mAPS_n$ is integer cordial graph, for $m \geq 2$ & $n \geq 3$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are n vertices of the path P_n and joining u_{2i-1} and u_{2i} to new vertices $v_{3i-2}^{(k)}, v_{3i-1}^{(k)}$ and $v_{3i}^{(k)}$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ & $1 \leq k \leq m$. So, the total number of vertices in $mAPS_n = p = \frac{(n-1)(3m+2)+2}{2}$ or $\frac{n(3m+2)}{2}$, if n is odd or n is even, respectively.

There are six different types related to the values of m & n.

Type 1: If m is even and n is odd, p is odd.
We introduce the function $f : V \to \left[-\left\lfloor \frac{p}{2} \right\rfloor, ..., \left\lfloor \frac{p}{2} \right\rfloor \right]$ as follows:

$$f\left(u_i\right) = i - \frac{n+1}{2}\ ; \quad 1 \leq i \leq n,$$

$$f\left(v_i^{(k)}\right) = i + \frac{(n-1)(3k-1)}{4}\ ; \quad 1 \leq i \leq \frac{3(n-1)}{2} \text{ \& } 1 \leq k \leq m, \text{ where } k \text{ is odd},$$

$$f\left(v_i^{(k+1)}\right) = -\left[i + \frac{(n-1)(3k-1)}{4}\right]\ ; \quad 1 \leq i \leq \frac{3(n-1)}{2} \text{ \& } 1 \leq k \leq m, \text{ where } k \text{ is odd}.$$

Type 2: If m and n both are even, p is also even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2} \right]^*$ as follows:

$$f\left(u_i\right) = \begin{cases} i - \frac{n+2}{2}\ ; & 1 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2}\ ; & \frac{n}{2} < i \leq n, \end{cases}$$

$$f\left(v_i^{(k)}\right) = i + \frac{n(3k-1)}{4}\ ; \quad 1 \leq i \leq \frac{3n}{2} \text{ \& } 1 \leq k \leq m, \text{ where } k \text{ is odd},$$

$$f\left(v_i^{(k+1)}\right) = -\left[i + \frac{n(3k-1)}{4}\right]\ ; \quad 1 \leq i \leq \frac{3n}{2} \text{ \& } 1 \leq k \leq m, \text{ where } k \text{ is odd}.$$

Type 3: If m is odd and $n = 4t-1$, $t \in \mathbb{N}$, p is even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f\left(u_i\right) = \begin{cases} i - \frac{n+3}{2}\,; & 1 \le i \le \frac{n+1}{2}, \\ i - \frac{n+1}{2}\,; & \frac{n+1}{2} < i \le n, \end{cases}$$

$$f\left(v_i^{(1)}\right) = \begin{cases} i - \frac{5n+1}{4}\,; & 1 \le i \le \frac{3n-5}{4}, \\ i - \frac{n-3}{4}\,; & \frac{3n-5}{4} < i \le \frac{3(n-1)}{2}, \end{cases}$$

$$f\left(v_i^{(k)}\right) = -\left[i + \frac{(n-1)(3k-1)+2}{4}\right]; \quad 1 \le i \le \frac{3(n-1)}{2} \ \& \ 1 \le k \le m, \text{ where } k \text{ is even},$$

$$f\left(v_i^{(k+1)}\right) = i + \frac{(n-1)(3k-1)+2}{4}\,; \quad 1 \le i \le \frac{3(n-1)}{2} \ \& \ 1 \le k \le m, \text{ where } k \text{ is even}.$$

Type 4: If m is odd and $n = 4t$, $t \in \mathbb{N}$, p is even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f\left(u_i\right) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{n}{2}\,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f\left(v_i^{(1)}\right) = \begin{cases} i - \frac{5n+4}{4}\,; & 1 \le i \le \frac{3n}{4}, \\ i - \frac{n}{4}\,; & \frac{3n}{4} < i \le \frac{3n}{2}, \end{cases}$$

$$f\left(v_i^{(k)}\right) = -\left[i + \frac{n(3k-1)}{4}\right]; \quad 1 \le i \le \frac{3n}{2} \ \& \ 1 \le k \le m, \text{ where } k \text{ is even},$$

$$f\left(v_i^{(k+1)}\right) = i + \frac{n(3k-1)}{4}\,; \quad 1 \le i \le \frac{3n}{2} \ \& \ 1 \le k \le m, \text{ where } k \text{ is even}.$$

Type 5: If m is odd and $n = 4t+1$, $t \in \mathbb{N}$, p is odd.
We introduce the function $f : V \to \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f\left(u_i\right) = i - \frac{n+1}{2}\,; \quad 1 \le i \le n,$$

$$f\left(v_i^{(1)}\right) = \begin{cases} i - \frac{5n-1}{4}\,; & 1 \le i \le \frac{3(n-1)}{4}, \\ i - \frac{n-1}{4}\,; & \frac{3(n-1)}{4} < i \le \frac{3(n-1)}{2}, \end{cases}$$

$$f\left(v_i^{(k)}\right) = -\left[i + \frac{(n-1)(3k-1)}{4}\right]; \quad 1 \le i \le \frac{3(n-1)}{2} \ \& \ 1 \le k \le m, \text{ where } k \text{ is even},$$

$$f\left(v_i^{(k+1)}\right) = i + \frac{(n-1)(3k-1)}{4}\,; \quad 1 \le i \le \frac{3(n-1)}{2} \ \& \ 1 \le k \le m, \text{ where } k \text{ is even}.$$

Type 6: If m is odd and $n = 4t+2$, $t \in \mathbb{N}$, p is odd.
We introduce the function $f : V \to \left[-\left\lfloor\frac{p}{2}\right\rfloor, ..., \left\lfloor\frac{p}{2}\right\rfloor\right]$ as follows:

$$f\left(u_i\right) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{n}{2}\,; & \frac{n}{2} < i \le n, \end{cases}$$

$$f\left(v_i^{(1)}\right) = \begin{cases} i - \frac{5n+2}{4}\ ; & 1 \leq i < \frac{3n+2}{4}, \\ 0\ ; & i = \frac{3n+2}{4}, \\ i - \frac{n+2}{4}\ ; & \frac{3n+2}{4} < i \leq \frac{3n}{2}, \end{cases}$$

$$f\left(v_i^{(k)}\right) = -\left[i + \frac{n(3k-1)-2}{4}\right]\ ; \quad 1 \leq i \leq \frac{3n}{2}\ \&\ 1 \leq k \leq m, \text{ where } k \text{ is even},$$

$$f\left(v_i^{(k+1)}\right) = i + \frac{n(3k-1)-2}{4}\ ; \quad 1 \leq i \leq \frac{3n}{2}\ \&\ 1 \leq k \leq m, \text{ where } k \text{ is even}.$$

Table 5. Edge condition for $mAPS_n$

Sr. No.	m	n	p	Edge condition
1	m is even	n is odd	p is odd	$e_f(0) = \frac{(2m+1)(n-1)}{2}$; $e_f(1) = \frac{(2m+1)(n-1)}{2}$
2		n is even	p is even	$e_f(0) = \lfloor\frac{n(2m+1)-1}{2}\rfloor$; $e_f(1) = \lceil\frac{n(2m+1)-1}{2}\rceil$
3	m is odd	$n = 4t-1$, t $\in \mathbb{N}$	p is even	$e_f(0) = \frac{(2m+1)(n-1)}{2}$; $e_f(1) = \frac{(2m+1)(n-1)}{2}$
4		$n = 4t+1$, t $\in \mathbb{N}$	p is odd	
5		$n = 4t$, t $\in \mathbb{N}$	p is even	$e_f(0) = \lfloor\frac{(2m+1)(n-1)}{2}\rfloor$; $e_f(1) = \lceil\frac{(2m+1)(n-1)}{2}\rceil$
6		$n = 4t+2$, t $\in \mathbb{N}$	p is odd	

So, we get $\left|e_f(0) - e_f(1)\right| \leq 1$ in each case.

Hence, m-Alternate pentagonal snake graph $mAPS_n$ is integer cordial. ∎

Theorem 2.6.

Irregular triangular snake graph IT_n is integer cordial graph, for $n \geq 4$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are the n vertices of the path P_n and joining u_i and u_{i+2} to a new vertex $v_i^{(1)}$, for $1 \leq i \leq (n-2)$. So, the total number of vertices in $IT_n = p = 2(n-1)$.

There are two different types for the values of n.

Type 1: If n is odd, p is even.
We introduce the function $f : V \rightarrow \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f\left(u_i\right) = \begin{cases} -i\ ; & 1 \leq i \leq \frac{n+1}{2}, \\ i - \frac{n+1}{2}\ ; & \frac{n+1}{2} < i \leq n, \end{cases}$$

$$f\left(v_i^{(1)}\right) = \begin{cases} -[i + \frac{n+1}{2}]\ ; & 1 \leq i \leq \frac{n-3}{2}, \\ i+1\ ; & \frac{n-3}{2} < i \leq (n-2). \end{cases}$$

Type 2: If n is even, p is also even.
We introduce the function $f : V \rightarrow \left[-\frac{p}{2}, ..., \frac{p}{2}\right]^*$ as follows:

$$f(u_i) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{n}{2} & ; \quad \frac{n}{2} < i \le n, \end{cases}$$

$$f(v_i^{(1)}) = \begin{cases} i - n\,; & 1 \le i \le \frac{n-2}{2}, \\ i + 1\,; & \frac{n-2}{2} < i \le (n-2). \end{cases}$$

Table 6. Edge condition for IT_n

Sr. No.	n	p	Edge condition
1	n is odd	p is even	$e_f(0) = \lfloor \frac{3n-5}{2} \rfloor; e_f(1) = \lceil \frac{3n-5}{2} \rceil$
2	n is even		

So, we get $|e_f(0) - e_f(1)| \le 1$ in each case.

Hence, Irregular triangular snake graph IT_n is integer cordial. ■

Theorem 2.7.

Irregular quadrilateral snake graph IQ_n is integer cordial graph, for $n \ge 4$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are the n vertices of the path P_n and joining u_i and u_{i+2} to a new vertices $v_{2i-1}^{(1)}$ and $v_{2i}^{(1)}$, for $1 \le i \le (n-2)$. So, the total number of vertices in $IQ_n = p = 3n - 4$.

There are two different types for the values of n.

Type 1: If n is odd, p is also odd.
We introduce the function $f : V \to \left[-\lfloor \frac{p}{2} \rfloor, \ldots, \lfloor \frac{p}{2} \rfloor \right]$ as follows:

$$f(u_i) = i - \frac{n+1}{2}\,; \quad 1 \le i \le n,$$

$$f(v_i^{(1)}) = \begin{cases} i - \frac{3(n-1)}{2}\,; & 1 \le i \le (n-2), \\ i - \frac{n-3}{2}\,; & (n-2) < i \le 2(n-2). \end{cases}$$

Type 2: If n is even, p is also even.
We introduce the function $f : V \to \left[-\frac{p}{2}, \ldots, \frac{p}{2} \right]^*$ as follows:

$$f(u_i) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \le i \le \frac{n}{2}, \\ i - \frac{n}{2} & ; \quad \frac{n}{2} < i \le n, \end{cases}$$

$$f(v_i^{(1)}) = \begin{cases} i - \frac{3n-2}{2}\,; & 1 \le i \le (n-2), \\ i - \frac{n-4}{2}\,; & (n-2) < i \le 2(n-2). \end{cases}$$

So, we get $|e_f(0) - e_f(1)| \le 1$ in each case.

Table 7. Edge condition for IQ_n

Sr. No.	n	p	Edge condition
1	n is odd	p is odd	$e_f(0) = \lfloor \frac{4n-7}{2} \rfloor; e_f(1) = \lceil \frac{4n-7}{2} \rceil$
2	n is even	p is even	

Hence, Irregular quadrilateral snake graph IQ_n is integer cordial. ∎

Theorem 2.8.

Irregular pentagonal snake graph IPS_n is integer cordial graph, for $n \geq 4$.

Proof:

Consider $u_1, u_2, \ldots, u_n$ are the n vertices of the path P_n and joining u_i and u_{i+2} to the new vertices $v^{(1)}_{3i-2}, v^{(1)}_{3i-1}$ and $v^{(1)}_{3i}$, for $1 \leq i \leq (n-2)$. So, the total number of vertices in $IPS_n = p = 2(2n-3)$.

There are two different types for the values of n.

Type 1: If n is odd, p is even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2} \right]^*$ as follows:

$$f(u_i) = \begin{cases} i - \frac{n+3}{2}\,; & 1 \leq i \leq \frac{n+1}{2}, \\ i - \frac{n+1}{2}\,; & \frac{n+1}{2} < i \leq n, \end{cases}$$

$$f(v^{(1)}_i) = \begin{cases} -\left[i + \frac{n+1}{2}\right]; & 1 \leq i \leq \frac{3n-7}{2}, \\ i - (n-3)\,; & \frac{3n-7}{2} < i \leq 3(n-2). \end{cases}$$

Type 2: If n is even, p is also even.
We introduce the function $f : V \to \left[-\frac{p}{2}, ..., \frac{p}{2} \right]^*$ as follows:

$$f(u_i) = \begin{cases} i - \frac{n+2}{2}\,; & 1 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2}\,; & \frac{n}{2} < i \leq n, \end{cases}$$

$$f(v^{(1)}_i) = \begin{cases} -[i + \frac{n}{2}]\,; & 1 \leq i \leq \frac{3(n-2)}{2}, \\ i - (n-3)\,; & \frac{3(n-2)}{2} < i \leq 3(n-2). \end{cases}$$

Table 8. Edge condition for IPS_n

Sr. No.	n	p	Edge condition
1	n is odd	p is even	$e_f(0) = \left\lfloor \frac{5n-9}{2} \right\rfloor; e_f(1) = \left\lceil \frac{5n-9}{2} \right\rceil$
2	n is even		

So, we get $\left|e_f(0) - e_f(1)\right| \leq 1$ in each case.

Hence, Irregular pentagonal snake graph IPS_n is integer cordial. ∎

3. Conclusion

This paper concluded that *m*-Alternate triangular snake graph, Alternate quadrilateral snake graph, *m*-Alternate quadrilateral snake graph, Alternate pentagonal snake graph, *m*-Alternate pentagonal snake graph, Irregular triangular snake graph, Irregular quadrilateral snake graph, and Irregular pentagonal snake graph are integer cordial graphs.

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