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Approximate Controllability of Infinite-delayed Second-order Stochastic Differential Inclusions Involving Non-instantaneous Impulses

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Abstract

This manuscript investigates a broad class of second-order stochastic differential inclusions consisting of infinite delay and non-instantaneous impulses in a Hilbert space setting. We first formulate a new collection of sufficient conditions that ensure the approximate controllability of the considered system. Next, to investigate our main findings, we utilize stochastic analysis, the fundamental solution, resolvent condition, and Dhage's fixed point theorem for multi-valued maps. Finally, an application is presented to demonstrate the effectiveness of the obtained results.

Keywords: Second-order stochastic system; Approximate controllability; Fundamental solution; Delayed differential inclusion; Non-instantaneous impulses

MSC 2020 No.: 34K45, 60H15, 93B05, 93C10

1. Introduction

Numerous evolution processes in ecology, physics and population dynamics, among others, experience unexpected changes in their states at precise moments, and these abrupt changes are characterized as impulses (Gao et al. (2006)). Recently, the theory of non-instantaneous impulses (NIIs) has become a significant area of investigation (see Agarwal et al. (2017), Kumar and Abdal (2021), Kumar et al. (2017), and the references therein).

Controllability is a qualitative attribute of dynamical systems and its systematic study was initiated by Kalman et al. (1963). Post this seminal work various controllability results have been established extensively for both finite and infinite-dimensional spaces. However, Triggiani (1977) established that a differential equation (DE) is not exactly controllable if the semigroup or the associated control operator is compact in an infinite-dimensional space. Therefore, the approximate controllability of DEs has become an attractive area of research (see Kavitha et al. (2020), Mahmudov (2003), and Yan and Lu (2017)). Further, differential systems based on pragmatic principles in various fields of control theory, biology, finance, and many more, have either discontinuous or multi-valued maps in their expression. These systems are termed as differential inclusions (DIs).

It is well comprehended that noises or stochastic distresses are omnipresent and cannot be avoided while modelling real-life phenomena. To capture the dynamics of such systems, stochastic differential equations (SDEs) are appropriated. In the last few decades, SDEs involving the Wiener process have been studied by several researchers (Da Prato and Zabczyk (1992), Dineshkumar et al. (2021), Vijayakumar et al. (2020), Yan and Lu (2017)). On the other hand, the basic technique to handle the abstract deterministic second-order DEs, is administered by the theory of strongly continuous cosine family (Fattorini (1985), Travis and Webb (1978)). The approximate controllability of second-order delayed systems has been investigated by several researchers using various approaches, one can refer to Chalishajar (2012), Arora and Sukavanam (2016), Su and Fu (2018), and the references cited there. However, very little research has been done on delayed second order SDIs.

Various techniques are employed by the researchers to handle the approximate controllability of semilinear DEs, such as range condition, and resolvent condition, among others. The range condition approach was proposed by Naito (1987) and it was operated by numerous authors to analyze the approximate controllability results (see Muthukumar and Balasubramaniam (2009), and Palanisamy and Chinnathambi (2015), among others). However, it is strenuous to verify such a range condition for infinite-delayed DEs. Further, the resolvent approach was proposed by Bashirov and Mahmudov (1999) and extensively utilized by many other authors (see Arora and Sukavanam (2016), Su and Fu (2018) and the references cited there). No doubt the resolvent technique is easier to apply in concrete systems but the prerequisite condition to use this approach is that the nonlinear terms must be uniformly bounded.

However, in our considered system, the nonlinear functions are partly uniformly bounded and such systems can be found in various applied areas like heat conduction models with fading memory. Therefore, the theory of the cosine family together with the resolvent condition is not enough to prove the results. Thus, to tackle the obstacle, we use the theory of the fundamental solution associated with a second-order linear system. Moreover, the theory of fundamental solution has been used by many researchers (see Kumar and Yadav (2021), Liu (2009), and Su and Fu (2018), among others). To the best of our knowledge, the approximate controllability for second-order infinite-delayed SDIs with non-instantaneous impulses is an untouched topic so far. The primary aim of this article is to refill this existing crack and we believe it will further open some research questions for investigation. This fact is the novelty of our work. Also, our findings in this manuscript extend and generalize existing works of Arora and Sukavanam (2016) and Su and Fu (2018).

2. Main results

The purpose of this manuscript is to establish the solvability and approximate controllability of the following infinite-delayed second-order SDIs:

$$\begin{cases} d\xi'(t) \in [A\xi(t) + L(\xi_t) + f(t,\xi_t) + B\nu(t)]\,dt + G(t,\xi_t)dW(t), \\ \qquad\qquad t \in \bigcup_{j=0}^{k} (p_j, q_{j+1}] \subset \mathcal{J} = [0,a], \\ \xi(t) = \mathcal{I}_j^1(t, \xi_{q_j^-}),\ \xi'(t) = \mathcal{I}_j^2(t, \xi_{q_j^-}),\ t \in \bigcup_{j=1}^{k} (q_j, p_j], \\ \xi_0 = \Theta \in \mathcal{G},\ \xi'(0) = z^0 \in \mathscr{E}, \end{cases} \tag{1}$$

where $\xi(\cdot)$ is an $\mathscr{E}$-valued stochastic process and $\mathscr{E}$ is a separable Hilbert space; the control $\nu(\cdot) \in L_2^{\mathfrak{F}}(\mathcal{J}, \mathscr{V})$, where $\mathscr{V}$ is another Hilbert space; the $\mathscr{K}$-valued Wiener process $W(t)$ is defined on a probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}; \mathbf{P})$, where $\mathscr{K}$ is a separable Hilbert space with norm $\|\cdot\|_{\mathscr{K}}$. Let $0 = p_0 = q_0 < q_1 < p_1 < q_2 < \cdots < q_k < p_k < q_{k+1} = a < \infty$ and $\mathcal{L}(\mathscr{K}, \mathscr{E})$ stand for the space of all bounded linear operators from $\mathscr{K}$ into $\mathscr{E}$. The histories $\xi_t : (-\infty, 0] \to \mathscr{E}$, given by $\xi_t(\kappa) = \xi(t+\kappa)$ for $\kappa \leq 0$, belong to the phase space $\mathcal{G}$, and $L : \mathcal{G} \to \mathscr{E}$ and $B : \mathscr{V} \to \mathscr{E}$ are bounded linear operators. Suppose $A : D(A) \subset \mathscr{E} \to \mathscr{E}$ generates a strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$. The suitable conditions for the mappings f, G, $\mathcal{I}_j^1$ and $\mathcal{I}_j^2$ are to be described later. The $\mathfrak{F}_0$-adapted process Θ, having finite second moment, is independent of W.

Let the Wiener process W be defined on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}; \mathbf{P})$, having increasing, right continuous filtration $\{\mathfrak{F}_t\}_{t\geq 0}$, $\mathfrak{F}_a = \mathfrak{F}$ and $\mathfrak{F}_0$ incorporates all $\mathbf{P}$-null sets. Also, $W(t)$ has self-adjoint covariance operator $Q \in \mathcal{L}(\mathscr{K})$, with $Tr(Q) < \infty$. Further, suppose that there exists a bounded sequence $\{\lambda_i \geq 0\}_{i\in\mathbb{N}}$ and a complete orthonormal basis $\{z_i\}_{i\in\mathbb{N}} \subseteq \mathscr{K}$ such that $Qz_i = \lambda_i z_i$ for all $i \in \mathbb{N}$ and $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i$. Then, set $W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\alpha_i(t) z_i,\ t \geq 0$, for a collection of mutually independent Wiener processes $\{\alpha_i(\cdot)\}_{i\in\mathbb{N}}$.

Let $\mathcal{L}_Q \equiv \mathcal{L}_Q(Q^{1/2}\mathscr{K}, \mathscr{E})$ stand for the space all Q-Hilbert–Schmidt operators from $Q^{1/2}\mathscr{K}$ to $\mathscr{E}$ and be a separable Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{L}_Q} = \sum_{i=1}^{\infty} \|\sqrt{\lambda_i}\zeta z_i\|^2 < \infty$. Further, let $\mathcal{L}_2(\Omega, \mathscr{E}) \equiv \mathcal{L}_2(\Omega, \mathfrak{F}, \mathbf{P}, \mathscr{E})$ symbolize the Banach space of all strongly measurable, square-integrable, $\mathscr{E}$-valued stochastic processes.

In what follows, $C(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{E}))$ represents the Banach space of all continuous functions from $\mathcal{J}$ into $\mathcal{L}_2(\Omega, \mathscr{E})$ with the property that $\sup_{t\in\mathcal{J}} \mathbb{E}\|\xi(t)\|^2 < \infty$. Further, let $\mathcal{L}_2^{\mathfrak{F}}(\mathcal{J}, \mathscr{E})$ denote the closed subspace of $\mathcal{L}_2(\mathcal{J} \times \Omega, \mathscr{E})$ having $\mathfrak{F}_t$-adapted processes and $\mathcal{L}_2^0(\Omega, \mathcal{G})$ be a subspace of $\mathcal{L}_2(\Omega, \mathcal{G})$ comprising $\mathfrak{F}_0$-measurable functions. Define the set

$$\mathcal{PC}(\mathcal{J}, \mathscr{E}) = \{\xi : \mathcal{J} \to \mathscr{E} : \xi \in C((q_j, q_{j+1}]), \xi(q_j^-) = \xi(q_j) \text{ and } \xi(q_j^+) \text{ exist for } 0 \leq j \leq k\}.$$

Clearly, $(\mathcal{PC}(\mathcal{J},\mathscr{E}),\|\cdot\|_{\mathcal{PC}})$ is a Banach space with $\|\xi\|_{\mathcal{PC}}=\sup_{\eta\in\mathcal{J}}(\mathbb{E}\|\xi(\eta)\|^2)^{1/2}$.

Let $\mathcal{G}$ be the linear space of all $\mathscr{E}$-valued $\mathfrak{F}_0$-measurable functions on $(-\infty,0]$ with seminorm $\|\cdot\|_{\mathcal{G}}$ and satisfy the below axioms (Hale and Kato (1978), Hino et al. (1991)):

(**A**1) If $\xi:(-\infty,\gamma+\lambda]\to\mathscr{E}(\gamma\geq 0$ and $\lambda>0)$ is such that $\xi|_{[\gamma,\gamma+\lambda]}\in\mathcal{PC}([\gamma,\gamma+\lambda],\mathscr{E})$ and $\xi_\gamma\in\mathcal{G}$, then for every $t\in[\gamma,\gamma+\lambda]$, we have

- (i) $\xi_t\in\mathcal{G}$;
- (ii) for constant $K_0\geq 0$, $\|\xi(t)\|\leq K_0\|\xi_t\|_{\mathcal{G}}$;
- (iii) $\|\xi_t\|_{\mathcal{G}}\leq\Gamma(t-\gamma)\sup\{\|\xi(\eta)\|:\gamma\leq\eta\leq t\}+M(t-\gamma)\|\xi_\gamma\|_{\mathcal{G}}$, where Γ and M map $[0,\infty)$ into $[0,\infty)$, and all K_0, $\Gamma(\cdot)$, $M(\cdot)$ are independent of $\xi(\cdot)$. Also, $\Gamma(\cdot)$ is continuous and $M(\cdot)$ is locally bounded.

(**A**2) The space $\mathcal{G}$ is complete.

Define $\mathcal{PC}_a=\{\xi:(-\infty,a]\to\mathscr{E}$ such that $\xi_0=\Theta\in\mathcal{G}$ and $\xi|_{\mathcal{J}}\in\mathcal{PC}(\mathcal{J},\mathscr{E})\}$, and the map $\|\cdot\|_{\mathcal{PC}_a}$ defined by $\|\xi\|_{\mathcal{PC}_a}=\|\Theta\|_{\mathcal{G}}+\sup_{0\leq\eta\leq a}(\mathbb{E}\|\xi(\eta)\|^2)^{\frac{1}{2}}$ is a seminorm on $\mathcal{PC}_a$.

The following notations are utilized for further development: $\mathcal{P}(\mathscr{E})=\{\mathcal{U}\in 2^{\mathscr{E}}:\mathcal{U}\neq\emptyset\}$, $\mathcal{P}_{cl}(\mathscr{E})=\{\mathcal{U}\in\mathcal{P}(\mathscr{E}):\mathcal{U}$ is closed$\}$, $\mathcal{P}_{cv}(\mathscr{E})=\{\mathcal{U}\in\mathcal{P}(\mathscr{E}):\mathcal{U}$ is convex$\}$, $\mathcal{P}_{bd}(\mathscr{E})=\{\mathcal{U}\in\mathcal{P}(\mathscr{E}):\mathcal{U}$ is bounded$\}$, $\mathcal{P}_{cp}(\mathscr{E})=\{\mathcal{U}\in\mathcal{P}(\mathscr{E}):\mathcal{U}$ is compact$\}$, $M_*=\sup_{\eta\in\mathcal{J}}M(\eta)$ and $\Gamma_*=\sup_{\eta\in\mathcal{J}}\Gamma(\eta)$.

To acquire the desired results, we need the following assumptions:

(**R**1) Let $\{\mathcal{S}(t)\}_{t\in\mathbb{R}}\subset\mathcal{L}(\mathscr{E})$ be the sine family associated with $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$, and for all $t\in\mathcal{J}$, there exist $N_1,N_2>0$ such that $\|\mathcal{C}(t)\|^2\leq N_1$ and $\|\mathcal{S}(t)\|^2\leq N_2$. Also, $\mathcal{S}(t)$ is compact for $t>0$.

(**R**2) For some $l_0>0$, $\|L\|^2\leq l_0$ and let $N_B=\|B\|^2$.

(**R**3) The map $f:\tilde{H}_0\times\mathcal{G}\to\mathscr{E}$, $\tilde{H}_0=\bigcup_{j=0}^{k}[p_j,q_{j+1}]$ is measurable and $f(t,\cdot):\mathcal{G}\to\mathscr{E}$ is continuous for all $t\in\tilde{H}_0$. Also, there exists $M_1>0$ such that for $t\in\tilde{H}_0$ and $\varphi\in\mathcal{G}$,

$$\mathbb{E}\|f(t,\varphi)\|^2\leq M_1(1+\|\varphi\|_{\mathcal{G}}^2).$$

(**R**4) The multi-valued function $G:\tilde{H}_0\times\mathcal{G}\to\mathcal{P}_{bd,cl,cv}(\mathcal{L}(\mathscr{K},\mathscr{E}))$ is L^2-Carathéodory and it satisfies the following:

- (i) for $\varphi\in\mathcal{G}$, $\mathcal{N}_{G,\varphi}=\{g\in\mathcal{L}_2(\tilde{H}_0,\mathcal{L}(\mathscr{K},\mathscr{E})):g(t)\in G(t,\varphi)$ for $a.e.\ t\in\tilde{H}_0\}\neq\emptyset$;
- (ii) there exist a nondecreasing continuous positive valued map Ξ on $[0,\infty)$ and a function $\rho_g\in\mathcal{L}_2(\mathcal{J},\mathbb{R}^+)$ such that for $t\in\tilde{H}_0$ and $\varphi\in\mathcal{G}$,

$$\mathbb{E}\|G(t,\varphi)\|^2=\sup\{\mathbb{E}\|g_1\|^2:g_1\in G(t,\varphi)\}\leq\rho_g(t)\Xi(\|\varphi\|_{\mathcal{G}}^2);$$

- (iii) the following inequality holds:

$$\int_{K_{13}^*}^{\infty}\frac{1}{\eta+\Xi(\eta)}d\eta=\infty,\ \text{where } K_{13}^* \text{ is specified later.}$$

(**R**5) The maps $\mathcal{I}_j^i : H_j \times \mathcal{G} \to \mathscr{E},\ H_j = [q_j, p_j]$ are continuous and there exist $R_j^i > 0$ such that for all $t \in H_j,\ 1 \leq j \leq k$, and $\varphi_1,\ \varphi_2 \in \mathcal{G}$, the following hold:

$$\mathbb{E}\|\mathcal{I}_j^i(t,\varphi_1) - \mathcal{I}_j^i(t,\varphi_2)\|^2 \leq R_j^i\|\varphi_1 - \varphi_2\|_{\mathcal{G}}^2,\ \mathbb{E}\|\mathcal{I}_j^i(t,\varphi_1)\|^2 \leq R_j^i(1 + \|\varphi_1\|_{\mathcal{G}}^2),\ i = 1, 2.$$

(**R**6) For $1 \leq j \leq k$, the functions $\mathcal{I}_j^1 : H_j \times \mathcal{G} \to \mathscr{E}$ are completely continuous and there exist constants $\overline{R}_j^i$ such that for all $\varphi \in \mathcal{G}$ and $t \in H_j$, $\mathbb{E}\|\mathcal{I}_j^i(t,\varphi)\|^2 \leq \overline{R}_j^i,\ i = 1, 2.$

The fundamental solution $\mathscr{Q}(\cdot) \in \mathcal{L}(\mathscr{E})$ of the following linear system

$$\begin{cases} \frac{d^2}{dt^2}\xi(t) = A\xi(t) + L(\xi_t), t > 0, \\ \xi_0 = \Theta,\ \xi'(0) = z^0, \qquad t \leq 0, \end{cases}$$

is an operator valued function and is defined by

$$\mathscr{Q}(t) = \begin{cases} \mathcal{S}(t) + \int_0^t \mathcal{S}(t-\eta)L(\mathscr{Q}_\eta)d\eta, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $\mathscr{Q}_t(\kappa) = \mathscr{Q}(t+\kappa),\ \kappa \leq 0$ (see Su and Fu (2020)). Also, for $\mathscr{Q}(t),\ t \in \mathbb{R}$, the following hold (Su and Fu (2020)):

(i) $\{\mathscr{Q}(t)\}_{t\in\mathbb{R}} \in \mathcal{L}(\mathscr{E})$ is strongly continuous and satisfies that $\|\mathscr{Q}(t)\|^2 \leq N^* e^{\theta t},\ t \geq 0$, where $N^* > 1$ and $\theta \in \mathbb{R}$. Also, there exists $\overline{N} \geq 1$ such that for all $t \in \mathcal{J}$, $\|\mathscr{Q}(t)\|^2 \leq \overline{N}$.
(ii) If $\{\mathcal{S}(t)\}_{t>0}$ is compact, then $\{\mathscr{Q}(t)\}_{t>0}$ is compact.
(iii) For all $z \in \mathscr{E}$ and $t \geq 0$, the map $\mathscr{Q}(\cdot)z$ is continuously differentiable. Moreover, for $\overline{N}_1 \geq 1$, $\|\mathscr{Q}'(t)\|^2 \leq \overline{N}_1$ for all $t \in \mathcal{J}$.
(iv) $\mathscr{Q}(t)$ is uniformly continuous on $\mathcal{J}$.

Definition 2.1.

A stochastic process $\xi : (-\infty, a] \times \Omega \to \mathscr{E}$ is referred to be a mild solution of the system (1) if

(i) $\xi(t,\varpi)$ is measurable on $\mathcal{J} \times \Omega$ and $\xi(t)$ is $\mathfrak{F}_t$-adapted with $\mathbb{E}\|\xi(t)\|^2 < \infty$;
(ii) for $t \in \mathcal{J}$, ξ_t is $\mathcal{G}$-valued stochastic variable, and $\xi|_{\mathcal{J}} \in \mathcal{PC}(\mathcal{J}, \mathscr{E})$;
(iii) for $\nu \in \mathcal{L}_2^{\mathfrak{F}}(\mathcal{J}, \mathscr{V})$, the following equation:

$$\xi(t) = \begin{cases} \Theta(t),\ t \in (-\infty, 0], \\ \mathscr{Q}'(t)\Theta(0) + \mathscr{Q}(t)z^0 + \int_0^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta,\xi_\eta) + B\nu(\eta)\big]d\eta \\ \quad + \int_0^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in [0, q_1], \\ \mathcal{I}_j^1(t, \xi_{q_j^-}),\ t \in (q_j, p_{j+1}],\ 1 \leq j \leq k, \\ \mathscr{Q}'(t-p_j)\mathcal{I}_j^1(p_j, \xi_{q_j^-}) + \mathscr{Q}(t-p_j)\mathcal{I}_j^2(p_j, \xi_{q_j^-}) \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta,\xi_\eta)\big]d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)B\nu(\eta)d\eta \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in (p_j, q_{j+1}],\ 1 \leq j \leq k, \end{cases} \tag{2}$$

is satisfied, where $g \in \mathcal{N}_{G,\xi}$, and the map $\tilde{\Theta}(\cdot)$ is given by

$$\tilde{\Theta}(\tau) = \begin{cases} \Theta(\tau), & \tau \leq 0, \\ 0, & \tau > 0. \end{cases}$$

Now, for $\beta > 0$ and $s \in [p_j, q_{j+1})$, $0 \leq j \leq k$, define

$$\Upsilon_s^{q_{j+1}} = \int_s^{q_{j+1}} \mathscr{Q}(q_{j+1} - \eta) BB^* \mathscr{Q}^*(q_{j+1} - \eta) d\eta, \text{ and } \mathcal{R}(\beta, \Upsilon_s^{q_{j+1}}) = (\beta I + \Upsilon_s^{q_{j+1}})^{-1},$$

where $\mathscr{Q}^*$ and B^* are adjoint operators of $\mathscr{Q}$ and B, respectively. Evidently, $\Upsilon_s^{q_{j+1}}$ is a positive operator. Hence, $\mathcal{R}(\beta, \Upsilon_s^{q_{j+1}})$ is well defined.

(**R**7) The operator $\beta \mathcal{R}(\beta, \Upsilon_{p_j}^{q_{j+1}}) \to 0$ as $\beta \to 0^+$ in the strong operator topology.

Now, for any $\xi^{q_{j+1}} \in \mathscr{E}$, $\beta > 0$, and $t \in (p_j, q_{j+1}]$, define the control function

$$\begin{aligned}
\nu_\xi^\beta(t) =& B^* \mathscr{Q}^*(q_{j+1} - t) \mathcal{R}(\beta, \Upsilon_{p_j}^{q_{j+1}}) \\
& \times \left[\mathbb{E}\xi^{q_{j+1}} - \mathscr{Q}'(q_{j+1} - p_j) \mathcal{I}_j^1(p_j, \xi_{q_j^-}) - \mathscr{Q}(q_{j+1} - p_j) \mathcal{I}_j^2(p_j, \xi_{q_j^-}) \right] \\
& - B^* \mathscr{Q}^*(q_{j+1} - t) \bigg[\int_{p_j}^t \mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}}) \mathscr{Q}(q_{j+1} - \eta) \big[L(\tilde{\Theta}_\eta) + f(\eta, \xi_\eta) \big] d\eta \\
& + \int_{p_j}^t \mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}}) \big[\mathscr{Q}(q_{j+1} - \eta) g(\eta) - \psi_j(\eta) \big] dW(\eta) \bigg], \ 0 \leq j \leq k,
\end{aligned} \tag{3}$$

where $\mathcal{I}_0^1(0, \cdot) = \Theta(0)$, $\mathcal{I}_0^2(0, \cdot) = z^0$, $g \in \mathcal{N}_{G,\xi}$ and $\psi_j \in \mathcal{L}_2^{\mathfrak{F}}(\Omega, \mathcal{L}_2(p_j, q_{j+1}, \mathcal{L}_Q))$ such that $\xi^{q_{j+1}} = \mathbb{E}\xi^{q_{j+1}} + \int_{p_j}^{q_{j+1}} \psi_j(\eta) dW(\eta)$, $0 \leq j \leq k$ (Dauer and Mahmudov (2004)).

Theorem 2.1.

Suppose that (**R**1)-(**R**5) hold. If $\Theta \in \mathcal{G}$ and $z^0 \in \mathcal{L}_2(\Omega, \mathscr{E})$, then there is a mild solution for the system (1), provided $1 - 4\Gamma_*^2 K_{10}^* > 0$ and $\max_{1 \leq j \leq k}\{R_j^1 \Gamma_*^2, 2(\overline{N}_1 R_j^1 + \overline{N} R_j^2)\Gamma_*^2\} < 1$.

Proof:

Define the operator $\Phi : \mathcal{PC}_a \to 2^{\mathcal{PC}_a}$ by $\Phi\xi$, the collection of all $\sigma \in \mathcal{PC}_a$ with the property that

$$\sigma(t) = \begin{cases}
\Theta(t), \ t \in (-\infty, 0], \\
\mathscr{Q}'(t)\Theta(0) + \mathscr{Q}(t) z^0 + \int_0^t \mathscr{Q}(t - \eta) \big[L(\tilde{\Theta}_\eta) + f(\eta, \xi_\eta) + B\nu_\xi^\beta(\eta) \big] d\eta \\
\quad + \int_0^t \mathscr{Q}(t - \eta) g(\eta) dW(\eta), \ t \in [0, q_1], \\
\mathcal{I}_j^1(t, \xi_{q_j^-}), \ t \in (q_j, p_j], \ 1 \leq j \leq k, \\
\mathscr{Q}'(t - p_j) \mathcal{I}_j^1(p_j, \xi_{q_j^-}) + \mathscr{Q}(t - p_j) \mathcal{I}_j^2(p_j, \xi_{q_j^-}) + \int_{p_j}^t \mathscr{Q}(t - \eta) \big[L(\tilde{\Theta}_\eta) + f(\eta, \xi_\eta) \big] d\eta \\
\quad + \int_{p_j}^t \mathscr{Q}(t - \eta) B\nu_\xi^\beta(\eta) d\eta + \int_{p_j}^t \mathscr{Q}(t - \eta) g(\eta) dW(\eta), \ t \in (p_j, q_{j+1}], \ 1 \leq j \leq k,
\end{cases}$$

where $g \in \mathcal{N}_{G,\xi}$ and $\beta > 0$. Clearly, a fixed point of Φ will be a solution of (1). Further, let $u(\cdot) : (-\infty, a] \to \mathscr{E}$ be the map given as

$$u(t) = \begin{cases} \mathscr{Q}'(t)\Theta(0), & t \in \mathcal{J}, \\ \Theta(t), & t \in (-\infty, 0], \end{cases}$$

then $u \in \mathcal{PC}_a$. If $\xi(\cdot)$ is a solution of (1), break $\xi(\cdot)$ as $\xi(t) = y(t) + u(t), \ t \in (-\infty, a]$, which yields that $\xi_t = y_t + u_t$ for $t \in \mathcal{J}$ and

$$y(t) = \begin{cases} 0,\ t \in (-\infty, 0], \\ \mathscr{Q}(t)z^0 + \int_0^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y_\eta + u_\eta) + B\nu^\beta_{y+u}(\eta)\big]d\eta \\ \quad + \int_0^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in [0, q_1], \\ \mathcal{I}^1_j(t, y_{q_j^-} + u_{q_j^-}) - \mathscr{Q}'(t)\Theta(0),\ t \in (q_j, p_j],\ 1 \le j \le k, \\ \mathscr{Q}'(t-p_j)\mathcal{I}^1_j(p_j, y_{q_j^-} + u_{q_j^-}) + \mathscr{Q}(t-p_j)\mathcal{I}^2_j(p_j, y_{q_j^-} + u_{q_j^-}) - \mathscr{Q}'(t)\Theta(0) \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)L(\tilde{\Theta}_\eta)d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)\big[f(\eta, y_\eta + u_\eta) + B\nu^\beta_{y+u}(\eta)\big]d\eta \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in (p_j, q_{j+1}],\ 1 \le j \le k. \end{cases}$$

Consider the space $\mathcal{PC}^0_a = \{y \in \mathcal{PC}_a : y_0 = 0 \in \mathcal{G}\}$ endowed with the norm

$$\|y\|_{\mathcal{PC}^0_a} = \|y_0\|_{\mathcal{G}} + \sup_{0 \le \eta \le a} (\mathbb{E}\|y(\eta)\|^2)^{\frac{1}{2}} = \sup_{0 \le \eta \le a} (\mathbb{E}\|y(\eta)\|^2)^{\frac{1}{2}},\ y \in \mathcal{PC}^0_a.$$

Thus, $(\mathcal{PC}^0_a, \|\cdot\|_{\mathcal{PC}^0_a})$ is a Banach space. For $d > 0$, let $\mathcal{PCB}^0_d = \{y \in \mathcal{PC}^0_a : \|y\|_{\mathcal{PC}^0_a} \le d\}$.

For any $y \in \mathcal{PCB}^0_d$ and $t \in \mathcal{J}$, we extract

$$\|y_t + u_t\|^2_{\mathcal{G}} \le 2\Big(\|y_t\|^2_{\mathcal{G}} + \|u_t\|^2_{\mathcal{G}}\Big) \le 4\Gamma^2_*(d^2 + \overline{N}_1 K_0^2\|\Theta\|^2_{\mathcal{G}}) + 4M^2_*\|\Theta\|^2_{\mathcal{G}} = d_1.$$

Further, define the multi-valued map $\Pi : \mathcal{PC}^0_a \to 2^{\mathcal{PC}^0_a}$ given by Πy, the collection of all $\sigma \in \mathcal{PC}^0_a$ with the condition that for $1 \le j \le k$,

$$\sigma(t) = \begin{cases} 0,\ t \in (-\infty, 0], \\ \mathscr{Q}(t)z^0 + \int_0^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y_\eta + u_\eta) + B\nu^\beta_{y+u}(\eta)\big]d\eta \\ \quad + \int_0^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in [0, q_1], \\ \mathcal{I}^1_j(t, y_{q_j^-} + u_{q_j^-}) - \mathscr{Q}'(t)\Theta(0),\ t \in (q_j, p_j], \\ \mathscr{Q}'(t-p_j)\mathcal{I}^1_j(p_j, y_{q_j^-} + u_{q_j^-}) + \mathscr{Q}(t-p_j)\mathcal{I}^2_j(p_j, y_{q_j^-} + u_{q_j^-}) - \mathscr{Q}'(t)\Theta(0) \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)L(\tilde{\Theta}_\eta)d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)\big[f(\eta, y_\eta + u_\eta) + B\nu^\beta_{y+u}(\eta)\big]d\eta \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in (p_j, q_{j+1}]. \end{cases}$$

Let $\Pi = \Pi_1 + \Pi_2$, where the operators Π_1 and Π_2 are given by

$$(\Pi_1 y)(t) = \begin{cases} 0,\ t \in (-\infty, 0], \\ \mathscr{Q}(t)z^0,\ t \in [0, q_1], \\ \mathcal{I}^1_j(t, y_{q_j^-} + u_{q_j^-}) - \mathscr{Q}'(t)\Theta(0),\ t \in (q_j, p_j],\ 1 \le j \le k, \\ \mathscr{Q}'(t-p_j)\mathcal{I}^1_j(p_j, y_{q_j^-} + u_{q_j^-}) + \mathscr{Q}(t-p_j)\mathcal{I}^2_j(p_j, y_{q_j^-} + u_{q_j^-}) \\ \quad -\mathscr{Q}'(t)\Theta(0),\ t \in (p_j, q_{j+1}],\ 1 \le j \le k, \end{cases}$$

and $\Pi_2 y$ is the collection of all $\sigma \in \mathcal{PC}^0_a$ with the property that for $1 \le j \le k$,

$$\sigma(t) = \begin{cases} 0,\ t \in (-\infty, 0], \\ \int_0^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y_\eta + u_\eta) + B\nu^\beta_{y+u}(\eta)\big]d\eta \\ \quad + \int_0^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in [0, q_1], \\ 0,\ t \in (q_j, p_j], \\ \int_{p_j}^t \mathscr{Q}(t-\eta)L(\tilde{\Theta}_\eta)d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)\big[f(\eta, y_\eta + u_\eta) + B\nu^\beta_{y+u}(\eta)\big]d\eta \\ \quad + \int_{p_j}^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta),\ t \in (p_j, q_{j+1}]. \end{cases}$$

Therefore, the problem of solvability of (1) is reduced to show that Dhage's fixed point theorem for multi-valued can be applied for the operators Π_1 and Π_2.

Step 1. Π_1 is a contraction map on $\mathcal{PC}_a^0$.
Let $y^1,\ y^2 \in \mathcal{PC}_a^0$. Then, using ($\mathbf{R}5$), it follows that

$$\|\Pi_1 y^1 - \Pi_1 y^2\|^2_{\mathcal{PC}_a^0} \le R_0 \|y^1 - y^2\|^2_{\mathcal{PC}_a^0},$$

where $R_0 = \max_{1\le j\le k}\{R_j^1\Gamma_*^2, 2(\overline{N}_1 R_j^1 + \overline{N} R_j^2)\Gamma_*^2\} < 1$. Hence, Π_1 is a contraction.

Step 2. For $\sigma_1, \sigma_2 \in \Pi_2 y$, $\varrho \in [0,1]$ and $t \in (p_j, q_{j+1}],\ 0 \le j \le k$, we extract

$$\begin{aligned}
&\varrho\sigma_1(t) + (1-\varrho)\sigma_2(t)\\
&= \int_{p_j}^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y_\eta + u_\eta)\big]d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)(\varrho g_1(\eta) + (1-\varrho)g_2(\eta))dW(\eta)\\
&\quad + \int_{p_j}^t \mathscr{Q}(t-\eta)BB^*\mathscr{Q}^*(q_{j+1}-\eta)\Bigg[\mathcal{R}(\beta, \Upsilon_{p_j}^{q_{j+1}})\big[\mathbb{E}\xi^{q_{j+1}} - \mathscr{Q}'(q_{j+1}-p_j)\mathcal{I}_j^1(p_j, y_{q_j^-} + u_{q_j^-})\\
&\quad - \mathscr{Q}(q_{j+1}-p_j)\mathcal{I}_j^2(p_j, y_{q_j^-} + u_{q_j^-})\big]\\
&\quad + \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\psi_j(\tau)dW(\tau) - \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau)\big[L(\tilde{\Theta}_\tau) + f(\tau, y_\tau + u_\tau)\big]d\tau\\
&\quad - \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau)(\varrho g_1(\tau) + (1-\varrho)g_2(\tau))dW(\tau)\Bigg]d\eta,
\end{aligned}$$

where $\mathcal{I}_0^1(0,\cdot) = \Theta(0),\ \mathcal{I}_0^2(0,\cdot) = z^0$ and $g_1, g_2 \in \mathcal{N}_{G,y+u}$. Evidently, convexity of the set $\mathcal{N}_{G,y+u}$ entails that the set $\Pi_2 y$ is convex for each $y \in \mathcal{PC}_a^0$.

Step 3. For every $\mathcal{PCB}_d^0 \subset \mathcal{PC}_a^0$, the set $\Pi_2\mathcal{PCB}_d^0$ is bounded.
Indeed, it is ample to prove that for each $\sigma \in \Pi_2 y$ and $y \in \mathcal{PCB}_d^0$, $\|\sigma\|^2_{\mathcal{PC}_a^0} \le \overline{K}$ for some $\overline{K} > 0$. First, for $t \in [0, q_1]$, we compute

$$\begin{aligned}
\mathbb{E}\|\nu_{y+u}^\beta(t)\|^2 \le &\frac{7N_B\overline{N}}{\beta^2}\Bigg\{\|\mathbb{E}\xi^{q_1}\|^2 + K_0^2\overline{N}_1\|\Theta\|^2_{\mathcal{G}} + \overline{N}\mathbb{E}\|z^0\|^2 + \int_0^{q_1}\mathbb{E}\|\psi_0(\eta)\|^2_{\mathcal{L}_Q}d\eta\\
&+ l_0 q_1^2\overline{N}M_*^2\|\Theta\|^2_{\mathcal{G}} + q_1^2\overline{N}M_1(1+d_1) + \overline{N}\sup_{\tau\in[0,d_1]}\Xi(\tau^2)\int_0^{q_1}\rho_g(\eta)d\eta\Bigg\} = \overline{M}_{\nu^1}.
\end{aligned}$$

If $\sigma \in \Pi_2 y$, then for $t \in [0, q_1]$, we have

$$\begin{aligned}
\mathbb{E}\|\sigma(t)\|^2 \le &4l_0 q_1^2\overline{N}M_*^2\|\Theta\|^2_{\mathcal{G}} + 4q_1^2\overline{N}M_1(1+d_1) + 4q_1^2\overline{N}N_B\overline{M}_{\nu_1}\\
&+ 4\overline{N}\sup_{\tau\in[0,d_1]}\Xi(\tau^2)\int_0^{q_1}\rho_g(\eta)d\eta = K_1.
\end{aligned}$$

Similarly, for $t \in (p_j, q_{j+1}],\ 1 \le j \le k$, we extract

$$\begin{aligned}
\mathbb{E}\|\nu_{y+u}^\beta(t)\|^2 \le &\frac{7N_B\overline{N}}{\beta^2}\Bigg\{\|\mathbb{E}\xi^{q_{j+1}}\|^2 + \overline{N}_1 R_j^1(1+d_1) + \overline{N}R_j^2(1+d_1) + \int_{p_j}^{q_{j+1}}\mathbb{E}\|\psi_j(\eta)\|^2_{\mathcal{L}_Q}d\eta\\
&+ l_0 q_{j+1}^2\overline{N}M_*^2\|\Theta\|^2_{\mathcal{G}} + q_{j+1}^2\overline{N}M_1(1+d_1) + \overline{N}\sup_{\tau\in[0,d_1]}\Xi(\tau^2)\int_{p_j}^{q_{j+1}}\rho_g(\eta)d\eta\Bigg\}
\end{aligned}$$

$$=\overline{M}_{\nu^{j+1}},$$

and

$$\begin{aligned}\mathbb{E}\|\sigma(t)\|^2 \leq & 4l_0 q_{j+1}^2 \overline{N} M_*^2 \|\Theta\|_{\mathcal{G}}^2 + 4q_{j+1}^2 \overline{N} M_1(1+d_1) + 4q_{j+1}^2 \overline{N} N_B \overline{M}_{\nu^{j+1}} \\ & + 4\overline{N} \sup_{\tau\in[0,d_1]} \Xi(\tau^2) \int_{p_j}^{q_{j+1}} \rho_g(\eta) d\eta = K_{j+1}.\end{aligned}$$

Take $\overline{K} = \max\limits_{0\leq j\leq k} K_{j+1}$. Thus, $\|\sigma\|_{\mathcal{PC}_a^0}^2 \leq \overline{K}$ for all $\sigma \in \Pi_2 y$ and $y \in \mathcal{PCB}_d^0$.

Step 4. The set $\{\Pi_2 y : y \in \mathcal{PCB}_d^0\}$ is equicontinuous in $\mathcal{PC}_a^0$.
For $y \in \mathcal{PCB}_d^0$ and $\sigma \in \Pi_2 y$, there is $g \in \mathcal{N}_{G,y+u}$ such that for $t \in (p_j, q_{j+1}],\ 0 \leq j \leq k$,

$$\sigma(t) = \int_{p_j}^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y_\eta + u_\eta) + B\nu_{y+u}^\beta(\eta)\big] d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta) g(\eta) dW(\eta).$$

Thus, for $\epsilon > 0$ and $p_j < t_1 < t_2 \leq q_{j+1}$, we have

$$\begin{aligned}&\mathbb{E}\|\sigma(t_2) - \sigma(t_1)\|^2 \\ \leq & 12 l_0 q_{j+1} M_*^2 \|\Theta\|_{\mathcal{G}}^2 \int_{p_j}^{t_1-\epsilon} \mathbb{E}\|\mathscr{Q}(t_2-\eta) - \mathscr{Q}(t_1-\eta)\|^2 d\eta + 12\epsilon l_0 M_*^2 \|\Theta\|_{\mathcal{G}}^2 \int_{t_1-\epsilon}^{t_1} \mathbb{E}\|\mathscr{Q}(t_2-\eta) \\ & - \mathscr{Q}(t_1-\eta)\|^2 d\eta + 12 l_0 (t_2-t_1)^2 \overline{N} M_*^2 \|\Theta\|_{\mathcal{G}}^2 + 12 q_{j+1} M_1(1+d_1) \int_{p_j}^{t_1-\epsilon} \mathbb{E}\|\mathscr{Q}(t_2-\eta) \\ & - \mathscr{Q}(t_1-\eta)\|^2 d\eta + 12\epsilon M_1(1+d_1) \int_{t_1-\epsilon}^{t_1} \mathbb{E}\|\mathscr{Q}(t_2-\eta) - \mathscr{Q}(t_1-\eta)\|^2 d\eta \\ & + 12(t_2-t_1)^2 \overline{N} M_1(1+d_1) + 12 q_{j+1} N_B \overline{M}_{\nu^{j+1}} \int_{p_j}^{t_1-\epsilon} \mathbb{E}\|\mathscr{Q}(t_2-\eta) - \mathscr{Q}(t_1-\eta)\|^2 d\eta \\ & + 12\epsilon N_B \overline{M}_{\nu^{j+1}} \int_{t_1-\epsilon}^{t_1} \mathbb{E}\|\mathscr{Q}(t_2-\eta) - \mathscr{Q}(t_1-\eta)\|^2 d\eta + 12(t_2-t_1)^2 \overline{N} N_B \overline{M}_{\nu^{j+1}} \\ & + 12 \int_{p_j}^{t_1-\epsilon} \mathbb{E}\|\mathscr{Q}(t_2-\eta) - \mathscr{Q}(t_1-\eta)\|^2 \|g(\eta)\|_{\mathcal{L}_Q}^2 d\eta + 12 \int_{t_1-\epsilon}^{t_1} \mathbb{E}\|\mathscr{Q}(t_2-\eta) \\ & - \mathscr{Q}(t_1-\eta)\|^2 \|g(\eta)\|_{\mathcal{L}_Q}^2 d\eta + 12\overline{N}\Big(\sup_{\tau\in[0,d_1]} \Xi(\tau^2)\Big)\bigg(\int_{t_1}^{t_2} \rho_g(\eta) d\eta\bigg). \end{aligned} \tag{4}$$

The uniform continuity of $\mathscr{Q}(\cdot)$ implies that the right hand side of (4) tends to zero as $t_2 \to t_1$ for sufficiently small $\epsilon > 0$. Consequently, the family $\{\Pi_2 y : y \in \mathcal{PCB}_d^0\}$ is equicontinuous.

Step 5. Π_2 is a compact map.
From Steps 3 and 4, and by the Arzela–Ascoli theorem, it is sufficient to establish that for each $t \in \mathcal{J}$, $\mathcal{Z}(t) = \{(\Pi_2 y)(t) : y \in \mathcal{PCB}_d^0\}$ is relatively compact. Evidently for $t = 0$, the result is true. Let for $j \geq 0$, $t \in (p_j, q_{j+1}]$ be fixed. If $\sigma \in \Pi_2 y$ and $y \in \mathcal{PCB}_d^0$, then there exists $g \in \mathcal{N}_{G,y+u}$ such that

$$\sigma(t) = \int_{p_j}^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y_\eta + u_\eta) + B\nu_{y+u}^\beta(\eta)\big] d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta) g(\eta) dW(\eta).$$

By using the compactness of $\{\mathscr{Q}(t)\}_{t>0}$ and given assumptions, it yields that the set $\mathcal{Z}(t)$ is relatively compact for $t \in (p_j, q_{j+1}]$. Therefore, the map Π_2 is completely continuous.

Step 6. Π_2 has the closed graph.
Let $y^m \to y^0$ and $\sigma^m \to \sigma^0$ as $m \to \infty$, where $\sigma^m \in \Pi_2 y^m$. To achieve the desired outcome, it is sufficient to prove that $\sigma^0 \in \Pi_2 y^0$. Now, $\sigma^m \in \Pi_2 y^m$ means that there is a $g^m \in \mathcal{N}_{G,y^m+u}$ with the property that for any $t \in (p_j, q_{j+1}]$,

$$\begin{aligned}
\sigma^m(t) =& \int_{p_j}^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y^m_\eta + u_\eta)\big]d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)g^m(\eta)dW(\eta) \\
&+ \int_{p_j}^t \mathscr{Q}(t-\eta)BB^*\mathscr{Q}^*(q_{j+1}-\eta)\Bigg[\mathcal{R}(\beta, \Upsilon_{p_j}^{q_{j+1}})\big[\mathbb{E}\xi^{q_{j+1}} \\
&- \mathscr{Q}'(q_{j+1}-p_j)\mathcal{I}_j^1(p_j, y^m_{q_j^-} + u_{q_j^-}) - \mathscr{Q}(q_{j+1}-p_j)\mathcal{I}_j^2(p_j, y^m_{q_j^-} + u_{q_j^-})\big] \\
&+ \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\psi_j(\tau)dW(\tau) - \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau) \\
&\times \big[L(\tilde{\Theta}_\tau) + f(\tau, y^m_\tau + u_\tau)\big]d\tau - \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau)g^m(\tau)dW(\tau)\Bigg]d\eta.
\end{aligned}$$

Further, we must show that there is a $g^0 \in \mathcal{N}_{G,y^0+u}$ with the condition that for $t \in (p_j, q_{j+1}]$,

$$\begin{aligned}
\sigma^0(t) =& \int_{p_j}^t \mathscr{Q}(t-\eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, y^0_\eta + u_\eta)\big]d\eta + \int_{p_j}^t \mathscr{Q}(t-\eta)g^0(\eta)dW(\eta) \\
&+ \int_{p_j}^t \mathscr{Q}(t-\eta)BB^*\mathscr{Q}^*(q_{j+1}-\eta)\Bigg[\mathcal{R}(\beta, \Upsilon_{p_j}^{q_{j+1}})\big[\mathbb{E}\xi^{q_{j+1}} \\
&- \mathscr{Q}'(q_{j+1}-p_j)\mathcal{I}_j^1(p_j, y^0_{q_j^-} + u_{q_j^-}) - \mathscr{Q}(q_{j+1}-p_j)\mathcal{I}_j^2(p_j, y^0_{q_j^-} + u_{q_j^-})\big] \\
&+ \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\psi_j(\tau)dW(\tau) - \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau) \\
&\times \big[L(\tilde{\Theta}_\tau) + f(\tau, y^0_\tau + u_\tau)\big]d\tau - \int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau)g^0(\tau)dW(\tau)\Bigg]d\eta.
\end{aligned}$$

For $1 \leq j \leq k$, define the linear continuous operator $\overline{\Gamma} : \mathcal{L}_2((p_j, q_{j+1}], \mathscr{E}) \to C((p_j, q_{j+1}], \mathscr{E})$ by

$$\begin{aligned}
\overline{\Gamma}(g)(t) =& -\int_{p_j}^t \mathscr{Q}(t-\eta)BB^*\mathscr{Q}^*(q_{j+1}-\eta)\bigg(\int_{p_j}^\eta \mathcal{R}(\beta, \Upsilon_\tau^{q_{j+1}})\mathscr{Q}(q_{j+1}-\tau)g(\tau)dW(\tau)\bigg)d\eta \\
&+ \int_{p_j}^t \mathscr{Q}(t-\eta)g(\eta)dW(\eta).
\end{aligned}$$

It is deduced that $\overline{\Gamma} \circ \mathcal{N}_G$ is a closed graph function (Lasota and Opial (1965)). Also, using definition of $\overline{\Gamma}$ and the fact that $y^m \to y^0$, we conclude that $\sigma^0 \in \Pi_2 y^0$.

Step 7. We shall demonstrate that $\nabla = \{y \in \mathcal{PC}_a^0 : y \in \gamma\Pi_1 y + \gamma\Pi_2 y, \text{ for some } \gamma \in (0,1)\}$ is bounded. Let $y \in \nabla$ and if we specified

$$K_1^* = \frac{7N_B\overline{N}}{\beta^2}\Big\{\|\mathbb{E}\xi^{q_1}\|^2 + K_0^2\overline{N}_1\|\Theta\|^2_{\mathcal{G}} + \overline{N}\mathbb{E}\|z^0\|^2 + \int_0^{q_1}\mathbb{E}\|\psi_0(\eta)\|^2_{\mathcal{L}_Q}d\eta + l_0 q_1^2\overline{N}M_*^2\|\Theta\|^2_{\mathcal{G}}$$

$$
\begin{aligned}
&\quad + q_1^2\overline{N}M_1\Big\},\\
K_2^* =&5\overline{N}\mathbb{E}\|z^0\|^2 + 5l_0q_1^2\overline{N}M_*^2\|\Theta\|_{\mathcal{G}}^2 + 5q_1^2\overline{N}M_1 + 5q_1^2\overline{N}N_BK_1^*,\\
K_3^* =&5q_1\overline{N}M_1 + \frac{35q_1^3\overline{N}^3N_B^2M_1}{\beta^2},\ K_4^* = 5\overline{N} + \frac{35q_1^2\overline{N}^3N_B^2}{\beta^2},\\
K_5^* =& \max_{1\le j\le k}\frac{7N_B\overline{N}^2}{\beta^2}\Big\{\|\mathbb{E}\xi^{q_{j+1}}\|^2 + (\overline{N}_1R_j^1 + \overline{N}R_j^2)(1 + 2\Gamma_*^2\mathbb{E}\|y(q_j)\|^2 + 4M_*^2\|\Theta\|_{\mathcal{G}}^2\\
&\quad + 4\Gamma_*^2K_0^2\overline{N}_1\|\Theta\|_{\mathcal{G}}^2) + \int_{p_j}^{q_{j+1}}\mathbb{E}\|\psi_j(\eta)\|_{\mathcal{L}_Q}^2d\eta + l_0q_{j+1}^2\overline{N}M_*^2\|\Theta\|_{\mathcal{G}}^2 + q_{j+1}^2\overline{N}M_1\Big\},\\
K_6^* =& \max_{1\le j\le k}\{7(\overline{N}_1R_j^1 + \overline{N}R_j^2)(1 + 2\Gamma_*^2\mathbb{E}\|y(q_j)\|^2 + 4M_*^2\|\Theta\|_{\mathcal{G}}^2 + 4\Gamma_*^2K_0^2\overline{N}_1\|\Theta\|_{\mathcal{G}}^2)\\
&\quad + 7K_0^2\overline{N}_1\|\Theta\|_{\mathcal{G}}^2 + 7l_0q_{j+1}^2\overline{N}M_*^2\|\Theta\|_{\mathcal{G}}^2 + 7q_{j+1}^2\overline{N}M_1 + 7q_{j+1}^2\overline{N}N_BK_5^*\},\\
K_7^* =&7a\overline{N}M_1 + \frac{49a^3\overline{N}^3N_B^2M_1}{\beta^2},\ K_8^* = 7\overline{N} + \frac{49a^2\overline{N}^3N_B^2}{\beta^2},\\
R_*^1 =& \max\{2R_j^1 : 1\le j\le k\} + 2\overline{N}_1K_0^2\|\Theta\|_{\mathcal{G}}^2,
\end{aligned}
$$

and follow the calculations of **Step 3**, we can deduce that for any $t\in\mathcal{J}$,

$$
\mathbb{E}\|y(t)\|^2 \le K_9^* + K_{10}^*\|y_t+u_t\|_{\mathcal{G}}^2 + K_{11}^*\int_0^t\|y_\eta+u_\eta\|_{\mathcal{G}}^2d\eta + K_{12}^*\int_0^t\rho_g(\eta)\Xi(\|y_\eta+u_\eta\|_{\mathcal{G}}^2)d\eta,
$$

where $K_9^* = \max\{K_2^*, K_6^*, R_*^1\}$, $K_{10}^* = \max_{1\le j\le k}\{2R_j^1\}$, $K_{11}^* = \max\{K_3^*, K_7^*\}$, and $K_{12}^* = \max\{K_4^*, K_8^*\}$. Further, for any $t\in\mathcal{J}$, it follows that

$$
\|y_t+u_t\|_{\mathcal{G}}^2 \le 2\big(\|y_t\|_{\mathcal{G}}^2 + \|u_t\|_{\mathcal{G}}^2\big) \le 4\big(\Gamma_*^2\sup_{0\le\eta\le t}\mathbb{E}\|y(\eta)\|^2 + M_*^2\|\Theta\|_{\mathcal{G}}^2 + \Gamma_*^2K_0^2\overline{N}_1\|\Theta\|_{\mathcal{G}}^2\big).
$$

Consider the function given by

$$
\zeta(t) = 4\big(\Gamma_*^2\sup_{0\le\eta\le t}\mathbb{E}\|y(\eta)\|^2 + M_*^2\|\Theta\|_{\mathcal{G}}^2 + \Gamma_*^2K_0^2\overline{N}_1\|\Theta\|_{\mathcal{G}}^2\big),\ t\in\mathcal{J}. \tag{5}
$$

Then $\zeta(\cdot)$ is a nondecreasing function on $\mathcal{J}$ and for all $t\in\mathcal{J}$, we have

$$
\mathbb{E}\|y(t)\|^2 \le K_9^* + K_{10}^*\zeta(t) + K_{11}^*\int_0^t\zeta(\eta)d\eta + K_{12}^*\int_0^t\rho_g(\eta)\Xi(\zeta(\eta))d\eta. \tag{6}
$$

From (5) and (6), we deduce

$$
\zeta(t) \le K_{13}^* + K_{14}^*\int_0^t\zeta(\eta)d\eta + K_{15}^*\int_0^t\rho_g(\eta)\Xi(\zeta(\eta))d\eta,
$$

where $K_{13}^* = \frac{4\Gamma_*^2K_9^*+4M_*^2\|\Theta\|_{\mathcal{G}}^2+4\Gamma_*^2K_0^2\overline{N}_1\|\Theta\|_{\mathcal{G}}^2}{1-4\Gamma_*^2K_{10}^*}$, $K_{14}^* = \frac{4\Gamma_*^2K_{11}^*}{1-4\Gamma_*^2K_{10}^*}$ and $K_{15}^* = \frac{4\Gamma_*^2K_{12}^*}{1-4\Gamma_*^2K_{10}^*}$.

Let $\vartheta(t) = K_{13}^* + K_{14}^*\int_0^t\zeta(\eta)d\eta + K_{15}^*\int_0^t\rho_g(\eta)\Xi(\zeta(\eta))d\eta$, for $t\in\mathcal{J}$. Then $\zeta(t)\le\vartheta(t)$ and

$$
\vartheta'(t) = K_{14}^*\zeta(t) + K_{15}^*\rho_g(t)\Xi(\zeta(t)) \le \max\{K_{14}^*, K_{15}^*\rho_g(t)\}(\vartheta(t)+\Xi(\vartheta(t))),\ \text{for all } t\in\mathcal{J}.
$$

Further,

$$
\int_{K_{13}^*}^{\vartheta(t)}\frac{d\eta}{\eta+\Xi(\eta)} \le \int_0^a\max\{K_{14}^*, K_{15}^*\rho_g(\eta)\}d\eta < \infty.
$$

From the above inequality and assumption ($\mathbf{R}4$)(iii), we infer that $\vartheta(t) \leq \tilde{C}$, for some $\tilde{C} > 0$. Consequently, the set ∇ is bounded and by the fixed point theorem of Dhage (2006), it follows that $\Pi_1 + \Pi_2$ has a fixed point $y^* \in \mathcal{PC}_a^0$. Then $\xi^*(t) = y^*(t) + u(t)$, $t \in (-\infty, a]$ is a mild solution of the system (1). ■

Theorem 2.2.

Suppose that all hypotheses in Theorem 2.1, and ($\mathbf{R}6$)-($\mathbf{R}7$) hold. If f and G are uniformly bounded, then the system (1) is approximately controllable on $\mathcal{J}$.

Proof:

For $\beta > 0$, let $\xi^\beta(\cdot)$ be a mild solution of the system (1) on $(-\infty, a]$ under the control function $\nu_\xi^\beta(\cdot)$ given by (3). By substituting $t = q_{j+1}$ in (2) and using the fact that $I - \Upsilon_\eta^{q_{j+1}}\mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}}) = \beta\mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}})$, we obtain

$$
\begin{aligned}
\xi^\beta(q_{j+1}) =& \xi^{q_{j+1}} \\
&- \beta\mathcal{R}(\beta, \Upsilon_{p_j}^{q_{j+1}})\big[\mathbb{E}\xi^{q_{j+1}} - \mathscr{Q}'(q_{j+1} - p_j)\mathcal{I}_j^1(p_j, \xi_{q_j^-}^\beta) - \mathscr{Q}(q_{j+1} - p_j)\mathcal{I}_j^2(p_j, \xi_{q_j^-}^\beta)\big] \\
&+ \int_{p_j}^{q_{j+1}} \beta\mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}})\mathscr{Q}(q_{j+1} - \eta)\big[L(\tilde{\Theta}_\eta) + f(\eta, \xi_\eta^\beta)\big]d\eta \\
&+ \int_{p_j}^{q_{j+1}} \beta\mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}})\mathscr{Q}(q_{j+1} - \eta)g^\beta(\eta)dW(\eta) \\
&- \int_{p_j}^{q_{j+1}} \beta\mathcal{R}(\beta, \Upsilon_\eta^{q_{j+1}})\psi_j(\eta)dW(\eta).
\end{aligned}
$$

Now, the uniform boundedness of f and G yield that there is a subsequence, still named by $\{f(\cdot, \xi_\eta^\beta(\cdot)), g^\beta(\cdot)\}$, converging weakly to, say, $\{f(\cdot), g(\cdot)\}$ in $\mathscr{E} \times \mathcal{L}_Q$. Also, the compactness property of $\mathscr{Q}(\cdot)$ implies that as $\beta \to 0^+$, we have

$$\|\mathscr{Q}(q_{j+1} - \eta)\big(f(\eta, \xi_\eta^\beta) - f(\eta)\big)\|^2 \to 0,$$

$$\|\mathscr{Q}(q_{j+1} - \eta)\big(g^\beta(\eta) - g(\eta)\big)\|_{\mathcal{L}_Q}^2 \to 0.$$

Further, ($\mathbf{R}6$) yields that for each $1 \leq j \leq k$, there is a subsequence, represented by $\{\mathcal{I}_j^1(\cdot, \xi_{q_j^-}^\beta), \mathcal{I}_j^2(\cdot, \xi_{q_j^-}^\beta)\}$, converging weakly to say $\{\mathcal{I}_j^1(\cdot), \mathcal{I}_j^2(\cdot)\}$ in $\mathscr{E} \times \mathscr{E}$. Therefore, $\mathbb{E}\|\xi^\beta(q_{j+1}) - \xi^{q_{j+1}}\|^2 \to 0$ as $\beta \to 0^+$ for $0 \leq j \leq k$. Hence, the system (1) is approximately controllable on $\mathcal{J}$. ■

3. An Application

Consider the following stochastic differential inclusions having non-instantaneous impulses:

$$\begin{cases} \partial\left(\frac{\partial u(t,z)}{\partial t}\right) \in \Bigg(\frac{\partial^2 u(t,z)}{\partial z^2} + \int_{-\infty}^{t-1}\int_0^{\pi} \tilde{e}_1(\eta - t, z, x)u(\eta, x)dxd\eta + \int_{-\infty}^{t} w_1(t,\eta)\tilde{f}(\eta, u(\eta, z))d\eta \\ \qquad +B\nu(t,z)\Bigg)dt + \int_{-\infty}^{t} w_2(t,\eta)\mathcal{D}(\eta, u(\eta,z))d\eta\, d\alpha(t), \;\; (t,z) \in \bigcup_{j=0}^{k}(p_j, q_{j+1}] \times [0,\pi], \\ u(t,z) = \int_{-\infty}^{t} c_j^1(\eta - t, z)u(q_j^-, z)d\eta, \; (t,z) \in \bigcup_{j=1}^{k}(q_j, p_j] \times [0,\pi], \\ \frac{\partial u(t,z)}{\partial t} = \int_{-\infty}^{t} c_j^2(\eta - t, z)u(q_j^-, z)d\eta, \; (t,z) \in \bigcup_{j=1}^{k}(q_j, p_j] \times [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \; t \in \mathcal{J}, \; a > 1, \\ u(\kappa, z) = \Theta(\kappa, z), \; \kappa \leq 0, \; 0 \leq z \leq \pi, \\ \frac{\partial u(0,z)}{\partial t} = z^0 \in \mathscr{E}, \end{cases} \tag{7}$$

where $\Theta(\cdot,\cdot)$ is $\mathfrak{F}_0$-measurable, $\alpha(t)$ denotes standard Brownian motion and functions $c_j^i(\cdot,\cdot)$, $\mathcal{D}(\cdot,\cdot)$, $\tilde{e}_1(\cdot,\cdot,\cdot)$, $\tilde{f}(\cdot,\cdot)$, $w_1(\cdot,\cdot)$ and $w_2(\cdot,\cdot)$ are to be described later.

Let $\mathscr{E} = \mathcal{L}_2([0,\pi])$, $\mathscr{K} = \mathcal{L}_2([0,\pi])$, $U(\cdot)(\cdot) = u(\cdot,\cdot)$ and $\Theta(\cdot)(\cdot) = \Theta(\cdot,\cdot)$. Let $A : D(A) \subset \mathscr{E} \to \mathscr{E}$ be given by $Au = u''$ with domain $D(A) = \big\{u(\cdot) \in \mathscr{E} :$ u and u' are absolutely continuous, $u'' \in \mathscr{E}$ and $u(0) = u(\pi) = 0\big\}$. Then, A generates a compact semigroup $\{\mathcal{T}(t)\}_{t>0}$, which is strongly continuous and self-adjoint. Further, the eigenvalues of A are $-m^2, \; m \in \mathbb{N}$ with the corresponding normalised eigenvectors $\zeta_m(z) = \sqrt{\frac{2}{\pi}}\sin(mz)$. Additionally, the cosine family $\{\mathcal{C}(t)\}_{t\geq 0}$ generated by A is defined by $\mathcal{C}(t)u = \sum_{m=1}^{\infty}\cos(mt)\langle u, \zeta_m\rangle\zeta_m$ and the corresponding sine family $\{\mathcal{S}(t)\}_{t\geq 0}$ is given as $\mathcal{S}(t)u = \sum_{m=1}^{\infty}\frac{\sin(mt)}{m}\langle u, \zeta_m\rangle\zeta_m,$ for all $t \in \mathbb{R}$. Moreover, the family $\{\mathcal{S}(t)\}_{t>0}$ is compact and self-adjoint.

Consider the phase space $\mathcal{G} = \mathcal{PC}_0 \times \mathcal{L}_2(\hbar, \mathscr{E})$ (where $\hbar : (-\infty, 0] \to \mathbb{R}^+$) with the semi-norm

$$\|\varphi\|_{\mathcal{G}} = \|\varphi(0)\| + \left(\int_{-\infty}^{0}\hbar(\eta)\|\varphi(\eta)\|^2 d\eta\right)^{\frac{1}{2}},$$

where $\hbar$ and $\hbar\|\varphi\|^2$ are Lebesgue integrable on $(-\infty, 0)$, and $\varphi(\cdot)$ is continuous at 0 (see Hino et al. (1991)).

We now consider the following hypotheses for the system (7):

(i) For $\eta \leq 0$, the map $\tilde{e}_1(\eta,\cdot,\cdot) \in C([0,\pi]\times[0,\pi])$ and $\tilde{e}_1(\eta, 0, \cdot) = \tilde{e}_1(\eta, \pi, \cdot) = 0$. Let

$$l_0 = \int_0^{\pi}\int_{-\infty}^{-1}\frac{1}{\hbar(\eta)}\int_0^{\pi}|\tilde{e}_1(\eta, z, x)|^2 dxd\eta dz < \infty.$$

(ii) The function $\tilde{f} : \mathbb{R} \times \mathscr{E} \to \mathbb{R}$ is continuous, and Lipschitz continuous in the second variable with Lipschitz constant $\tilde{M}_1 > 0$. Also, $\tilde{f}$ is uniformly bounded.

(iii) The function $w_j : \tilde{H}_0 \times \mathbb{R} \to \mathbb{R},\ j = 1, 2$, is continuous with $|w_j(t, t+\eta)| < \lambda(\eta)$ and

$$\int_{-\infty}^{0} \frac{|\lambda(\eta)|^2}{\hbar(\eta)} d\eta < \infty.$$

(iv) There exists a continuous map $\tilde{c} : \mathbb{R} \to [0, \infty)$ such that the continuous function $\mathcal{D} : \mathbb{R} \times \mathscr{E} \to \mathbb{R}$ satisfies $\|\mathcal{D}(t, z)\| \le \tilde{c}(t)\|z\|,$ for $(t, z) \in \mathbb{R} \times \mathscr{E}$. Also, $\mathcal{D}$ is uniformly bounded.

(v) There are continuous functions $\overline{d}_j : \mathbb{R} \to [0, \infty)$ with the condition that the continuous maps $c_j^i : \mathbb{R} \times [0, \pi] \to \mathbb{R}$, satisfy $\|c_j^i(t, \eta)\| \le \overline{d}_j(t)$, for $(t, \eta) \in \mathbb{R} \times [0, \pi]$, $j \ge 1$ and $i = 1, 2$ with

$$R_{c_j^i} = \int_{-\infty}^{0} \frac{(\overline{d}_j(\eta))^2}{\hbar(\eta)} d\eta < \infty.$$

(vi) The function $\Theta(\cdot, \cdot)$ belongs to $\mathcal{L}_2^0(\Omega, \mathcal{G})$.

Define $L : \mathcal{G} \to \mathscr{E}$, $\tilde{f} : \tilde{H}_0 \times \mathcal{G} \to \mathscr{E}$, $G : \tilde{H}_0 \times \mathcal{G} \to \mathbb{R}$ and $\mathcal{I}_j^i : H_j \times \mathcal{G} \to \mathscr{E}$, respectively, as

$$L(\varphi)(z) = \int_{-\infty}^{-1} \int_0^{\pi} \tilde{e}_1(\eta, z, x)\varphi(\eta, x) dx d\eta,\ f(t, \varphi)(z) = \int_{-\infty}^{0} w_1(t, t+\eta)\tilde{f}(t+\eta, \varphi(\eta, z)) d\eta,$$

$$G(t, \varphi)(z) = \int_{-\infty}^{0} w_2(t, t+\eta)\mathcal{D}(t+\eta, \varphi(\eta, z)) d\eta,\ \mathcal{I}_j^i(t, \varphi)(z) = \int_{-\infty}^{0} c_j^i(\eta, z)\varphi(\eta, z) d\eta,$$

for $\varphi \in \mathcal{G}$, $j \ge 1$ and $i = 1, 2$.
Under the above assumptions, the system (7) can be transformed to the system (1). Further, assumptions (**R**1), (**R**3), (**R**4)(i), (**R**4)(ii) and (**R**5) are satisfied, and $\|L\|^2 \le l_0$.
Consider the real Hilbert space $\mathscr{V} = \left\{ \nu = \sum_{m=2}^{\infty} \nu_m \zeta_m : \sum_{m=2}^{\infty} \nu_m^2 < \infty \right\},$ with $\|\nu\| = \left(\sum_{m=2}^{\infty} \nu_m^2 \right)^{\frac{1}{2}}$.

Define $B\nu = 2\nu_2\zeta_1(z) + \sum_{m=2}^{\infty} \nu_m \zeta_m(z),$ for $\nu = \sum_{m=2}^{\infty} \nu_m \zeta_m \in \mathscr{V}$. Clearly, $B \in \mathcal{L}(\mathscr{V}, \mathscr{E})$ and adjoint of B is given by $B^*\mu = (2\mu_1 + \mu_2)\zeta_2(z) + \sum_{m=3}^{\infty} \mu_m \zeta_m(z),$ for $\mu = \sum_{m=1}^{\infty} \mu_m \zeta_m(z) \in \mathscr{E}$. Evidently, if for all $t \in \mathcal{J}$, $B^*\mathscr{Q}^*(t)\mu = 0$ then $\mu = 0$ and which further implies that (**R**7) is satisfied (see Su and Fu (2018)). Hence, if assumptions of Theorem 2.1 and (**R**6) hold, then by virtue of Theorem 2.2, the system (7) is approximately controllable on $\mathcal{J}$.

4. Conclusion

In this manuscript, approximate controllability for a class of second-order semilinear SDIs having unbounded delay and non-instantaneous impulses has been investigated. We utilized Dhage's fixed point theorem for the multi-valued maps to establish the solvability of the system. Finally, the obtained results are verified through an example. In the future, we will extend the results of this paper for the systems with state-dependent delay and random impulses as well as for the systems with delay in control and Poisson jumps.

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