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# Ensemble energy average and energy flow relationships for nonstationary vibrating systems

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**Abstract**

This paper attempts to introduce a new point of view on energy analysis in structural dynamics with particular emphasis to its link with uncertainty and complexity. A linear, elastic system undergoing free vibrations, is considered. The system is subdivided into two subsystems and their respective energies together with the shared energy flow are analysed.

First, the ensemble energy average of the two subsystems, assuming uncertain the natural frequencies, is investigated. It is shown how the energy averages follow a simple law when observing the long-term response of the system, obtained by a suitable asymptotic expansion. The second part of the analysis shows how the ensemble energy average of a set of random samples is representative even of the single case if the system is complex enough.

The two previous points, combined, produce a result that applies to the energy sharing between two subsystems even independently of uncertainty: for complex systems, a simple energy sharing law is indeed stated. Moreover, in the case of absence of damping, a nonlinear relation between the energy flow and the energy (weighted) difference between the two subsystems is derived; on the other hand, when damping is present, this relationship becomes linear, including two terms: one is proportional to the energy (weighted) difference between the two subsystems, the other being proportional to its time derivative. Therefore, the approach suggests a way for deriving a general approach to energy sharing in vibration with results that, in some cases, are reminiscent of those met in Statistical Energy Analysis.

Finally, computational experiments, performed on systems of increasing complexity, validate the theoretical results.


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## 1. Uncertainty, energy and complexity: A point of view

Cases of uncertain systems are met frequently in engineering. Important examples are related to automotive components, aircraft fuselages or ship hulls. In general, these systems are very complicated because they are made of a large number of components, connected by a large number of joints. Since each connection must satisfy allowances on geometry, clearances, etc., they present always a certain degree of variability. Although, this does not necessarily affect the correct working of the whole system, different samples of the same production lot, may exhibit very different vibration and acoustic performances.

Uncertainty on the material and dynamic properties of the components also play a relevant role in this context [1]. Generally, this is not the case for the main structure, e.g. an aircraft or a ship, that, for strength and safety reasons, consists frequently of a relatively homogeneous structure (still or aluminium), carefully designed in any detail. However, for many subcomponents connected to the principal structure, such as seats, dashboards, panels, electronic equipments, etc., the material properties are often roughly known.

Other significant examples are related to structures coupled with a moving fluid, as in aerospace and naval engineering: turbulence generates random pressure fields exciting elastic structures; waves on the sea surface produce a random flow exciting the motion of floating and submerged bodies. In the simpler case, these problems can be modelled as random loads acting on a deterministic dynamic system; however, the problem discloses often a high complexity, because uncertainties involve the applied forces as well the system itself. In such cases, a more accurate model is represented by integral-differential equations with random coefficients, i.e. an uncertain operator is intrinsically associated to the investigated system [2–4].

It is interesting to note that, although the concept of uncertainty in dynamics does not imply necessarily the complexity of the system, a correlation between the two attributes exists. A system is complex when its description needs a model having a large number of degrees of freedom. This case is usually met when the characteristic wavelength produced by the exciting forces (or characterizing the initial conditions) is small with respect to the characteristic size of the considered system, implying the use of a fine mesh to produce a correct discretized model of the structure. Complexity arises also when the system consists of a very large number of different jointed subcomponents (beams, plates, shells, membranes etc.): a correct discretization results again in a fine mesh, i.e. in a large number of the model's degrees of freedom.

Complexity and uncertainty have a direct correlation: for systems with a large number of degrees of freedom, even a small inherent uncertainty of some parameters produces a considerable effect on its response [1]—at least in terms of space–state variables—especially at high frequency. This implies that uncertainty must be suitably taken into account in the analysis of complex systems.

At present, the analysis of uncertain systems can be approached in very different ways. Only some of them are discussed ahead, with the aim of collocating in this frame the present work.

The Monte Carlo simulation is a direct approach to the problem: it consists of solving a set of equations of motion, where each sample equation is solved with different parameters, following a given statistics. As a result, a random set of solutions is obtained, and a related statistics can be extracted [5,6]. Note that this approach is computationally very demanding when significant statistical results are requested and, at least for complex systems, this way of attacking the problem becomes computationally prohibitive.

However, the analysis of uncertainty in structural dynamic belongs to a class of well-established problems in the theory of stochastic differential equations [5–9]. Nevertheless, so far only the part of this theory related to random excitation is included in standard engineering analyses [10]; on the contrary, the part related to stochastic operators is not yet a familiar engineering tool and is still the subject of recent investigations [5].

Let us summarize the main goal of this theory: given the statistics of some parameters related to the governing equations of motion—initial and boundary conditions, material properties, etc.—provide the statistics of the corresponding solution. This approach is very general but does not change substantially the variable describing the problem: if $x$ is the unknown (in general a vector of functions depending on space and time) describing the deterministic system, it is asked to provide the probability density function of $x$, once the statistics of the problem's data are known. This statement of the problem is solved, in principle, by the Fokker–Plank–Kolmogorov (FPK) equation that is a central result of the theory of stochastic differential equations [7,8]. However, the great complexity of the FPK equation is reduced remarkably by less-demanding procedures, based on averaging techniques, that limit the solution to the knowledge of only few statistical moments of $x$ [7,11,12]. This approach inspires a whole class of solution procedures for stochastic differential equations including the method of weighted residuals and the polynomial chaos technique [3,5,13]. On the other hand, also the stochastic perturbation method, a perturbation technique revisited in a stochastic light, produces the statistical moments of the solution up to a desired order, provided that the statistical moments of the data are known [4,5,7].

Finally, another way to approach uncertainty exists, especially when joined with complexity. This is the way of statistical mechanics. It must be remarked that the theory of stochastic differential equations is not historically independent of statistical mechanics. It is a matter of fact that many techniques to deal with stochastic equations were initially developed in the context of statistical mechanics, before a systematic theory of stochastic differential equations was born. For example, the FPK equation has apparently its roots in the kinetic equation of Boltzmann [14], and was born in the frame of the description of diffusion processes in physics [15]. Another example is represented by the technique of averaging that is, as mentioned above, one of the basic ideas when analysing stochastic equations. An early attempt in this direction is provided, again, in statistical physics by the equation of Langevin in the analysis of the Brownian movement [14,16]: its solution represents the embryonic stage of the averaging technique.

Although the stochastic theory of differential equations received the first input and a heritage from statistical mechanics, at present, an essential element separates the two approaches. In the first, very general forms of differential equations are considered and the unknown $x$ is replaced by its probability density function, or its statistical moments [5,7]. In statistical mechanics, the analysis is indeed limited to special differential equations: those of Hamilton, but associated to a system with a very large number of degrees of freedom [14,17]. In this case the complexity of the system, namely related to the large number of molecules or atoms in a lattice, compels to introduce simpler descriptors of the system response in statistical terms. These are average energies associated to sub-sets of degrees of freedom of the original system. This point of view finds a strict formulation in the theory of canonical ensemble of Gibbs and Boltzmann [14].

In engineering, Statistical Energy Analysis (SEA) [18–21] has taken several elements from this fruitful way of approaching the problem. However, the nature of the systems investigated in structural dynamics is, in some sense, more complicated than that met in physics. Paradoxically,

the extreme complication of an atomic lattice, and its large number of degrees of freedom, makes simpler to deal with it by an effective statistical approach. Heat energy has disorder characteristics that allow a large simplification of the problem, not always allowed when considering the vibration motion of engineering structures: for example, the use of a linear proportionality between energy flow and energy differences [22,23].

Although the analogy with thermodynamic problems brought important results in the energy analysis of vibrating structures, this analogy still presents limits not yet completely understood. For example, while the analysis of the second-order moment in statistical mechanics is an obvious output of the theory [14], this is not the case of SEA and other energy approaches to structural dynamics. This point is still an argument of investigation in the context of energy methods for structural dynamics [24–27].

Finally, the absence of a trace of the second principle of thermodynamics in SEA and related energy methods, confirms the doubt about the incompleteness of the discussed analogy [28].

In the context of the analysis of uncertain systems and its link with complexity, a particular mention deserves the theory of fuzzy structures developed in Ref. [29]. There the attention is focused on a strict probabilistic analysis of a system resulting from the coupling between a deterministic main structure and a structural fuzzy, consisting of a complex system described in probabilistic terms. This is a case of practical interest and frequently applicable to engineering structural problems. However, in this approach, the energy analysis does not play a central role and, although some points of contacts with SEA (and energy methods in general) exist [30], it develops on a different ground, providing interesting results.

This paper, partly inspired by statistical mechanics, attempts to introduce a new point of view in energy modelling of structures with a special emphasis to the link with uncertainty and complexity. A linear, elastic system undergoing free vibrations, is considered. It is subdivided into two subsystems and their respective energies together with the shared energy flow are analysed.

The reasoning here developed can be schematically resumed as follows:

The ensemble energy average of the two subsystems, assuming uncertain natural frequencies, is investigated. It is shown that the energy averages follow a simple law when observing the long-term response of the system: to this aim a suitable asymptotic expansion of the energy average is developed in Section 2.

The second step of the analysis, developed in Section 3, shows how the ensemble energy average of a set of random samples is representative even of the single case if the system is complex enough.

The two previous points, combined, produce a result that applies to the energy sharing between two subsystems independently of uncertainty: for complex systems, a simple energy sharing law is stated. Moreover, considering conservative systems, a nonlinear relation between the energy flow and the energy (weighted) difference between the two subsystems is derived (Section 4). A connection of this analysis with the second principle of thermodynamics is considered in Section 5.

The presence of damping in the system (Section 6) modifies the results leading to linear relationships between energy difference and energy flow, including a two-fold contribution: the first is proportional to the energy difference between the two subsystems, the second is proportional to its time derivative. In this way the paper also suggests a new way to approach some results found in SEA.

Finally, the computational experiments of Section 7, performed on systems of increasing complexity, validate the theoretical results.

## 2. Energy and uncertainty: Ensemble energy average of nonstationary vibrations

In this section a freely vibrating linear elastic system, conservative, and subjected to given initial conditions, is considered. The system is partitioned into two subsystems and their energies are explicitely calculated as functions of time. Assuming the natural frequencies of the complete system to be random, in Section 2.1 the ensemble average of the two corresponding energies is provided. In Section 2.2 these are expanded using an asymptotic series in time domain: only few terms of the series are retained, providing the long-term response of the ensemble average. It consists, in general, of two main contributions: the stationary term, constant, that provides the asymptotic limit of the energy of the subsystem, detailed in Section 2.3; the nonstationary contribution, vanishing for large time and responsible of the energy sharing process, considered in Section 2.4.

### 2.1. Formulation of the problem

Consider a freely vibrating system $S$, isolated and conservative, satisfying given initial conditions. Two parts of $S$, $S_1$ and $S_2$, are considered such that $S \equiv S_1 \cup S_2$.

Let $\mathbf{\Phi}_i(\mathbf{x})$ and $q_i(t)$ be the orthonormal modes (vectors) and the Lagrangean coordinates of $S$, respectively, $\mathbf{x} \in S$ the vector of space coordinates and $t$ the time. The vibration field of $S_1$ and $S_2$ is described by

$$\mathbf{w}^{(1)}(\mathbf{x},t) = \sum_{i=1}^{N} \mathbf{\Phi}_i(\mathbf{x}) q_i(t), \quad \mathbf{x} \in S_1; \quad \mathbf{w}^{(2)}(\mathbf{x},t) = \sum_{i=1}^{N} \mathbf{\Phi}_i(\mathbf{x}) q_i(t), \quad \mathbf{x} \in S_2,$$

respectively, where $N$ modes of the system $S$ are considered in the response. The energies of $S_1$ and $S_2$ are $E^{(1)}(t), E^{(2)}(t)$, respectively, obtained as the sum of both the kinetic and potential contributions. For a continuous, linear, elastic system they are

$$E^{(r)}(t) = \frac{1}{2}\int_{S_r} \rho |\dot{\mathbf{w}}|^2 \,\mathrm{d}V + \frac{1}{2}\int_{S_r} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \,\mathrm{d}V,$$

$$E^{(r)}(t) = \frac{1}{2}\sum_{i,j=1}^{N} \dot{q}_i \dot{q}_j \int_{S_r} \rho \mathbf{\Phi}_i \mathbf{\Phi}_j \,\mathrm{d}V + \frac{1}{2}\sum_{i,j=1}^{N} q_i q_j \frac{1}{4}\int_{S_r} \left[\mathbf{D}\left(\nabla\mathbf{\Phi}_i + \nabla\mathbf{\Phi}_i^T\right)\right] : \left(\nabla\mathbf{\Phi}_i + \nabla\mathbf{\Phi}_i^{\mathrm{T}}\right) \mathrm{d}V,$$

where $r = 1, 2$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{D}$ are the stress, the deformation and the elastic tensor, respectively, and : denotes scalar product between tensors. With the positions

$$\alpha_{ij}^{(r)}(t) = \int_{S_r} \rho \mathbf{\Phi}_i \mathbf{\Phi}_j \,\mathrm{d}V, \quad \beta_{ij}^{(r)} = \frac{1}{4}\int_{S_r} \left[\mathbf{D}\left(\nabla\mathbf{\Phi}_i + \nabla\mathbf{\Phi}_i^{\mathrm{T}}\right)\right] : \left(\nabla\mathbf{\Phi}_i + \nabla\mathbf{\Phi}_i^T\right) \mathrm{d}V,$$

the energies can be written concisely as

$$E^{(r)}(t) = \frac{1}{2}\sum_{i,j=1}^{N} \alpha_{ij}^{(r)} \dot{q}_i(t)\dot{q}_j(t) + \frac{1}{2}\sum_{i,j=1}^{N} \beta_{ij}^{(r)} q_i(t) q_j(t). \tag{1}$$

The orthonormality conditions imply the equalities

$$\alpha_{ij}^{(1)} + \alpha_{ij}^{(2)} = \delta_{ij}, \quad \beta_{ij}^{(1)} + \beta_{ij}^{(2)} = \omega_i^2 \delta_{ij}. \tag{2}$$

If a discrete system is indeed considered, the whole set of masses can be partitioned into two sets $S_1$ and $S_2$. The vector of displacements $\mathbf{w} = \left[\mathbf{w}^{(1)}, \mathbf{w}^{(2)}\right]^{\mathrm{T}}$ is associated to them, where the two subvectors refer to the displacement of the masses belonging to $S_1$ and $S_2$, respectively. Correspondingly, the mass and the stiffness matrices and the $i$th eigenvector, are partitioned as follows:

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}, \quad \mathbf{\Phi}_i = \begin{Bmatrix} \mathbf{\Phi}_i^{(1)} \\ \mathbf{\Phi}_i^{(2)} \end{Bmatrix}.$$

In this case, letting $\alpha_{ij}^{(r)} = \mathbf{\Phi}_i^{(r)\mathrm{T}} \mathbf{M}_{rr} \mathbf{\Phi}_j^{(r)}$, $\beta_{ij}^{(r)} = \mathbf{\Phi}_i^{(r)\mathrm{T}} \mathbf{K}_{rr} \mathbf{\Phi}_j^{(r)}$, expressions (2) hold if the mixed energy terms $\frac{1}{2}\mathbf{\Phi}_i^{(r)\mathrm{T}} \mathbf{M}_{rs} \mathbf{\Phi}_j^{(s)}$, $\frac{1}{2}\mathbf{\Phi}_i^{(r)\mathrm{T}} \mathbf{K}_{rs} \mathbf{\Phi}_j^{(s)}$, $r \neq s$ are negligible. This implies that the physical coupling between the two subsystems stores a negligibly small amount of energy. Under this assumption, the sum of the energies $E^{(1)}(t) + E^{(2)}(t)$, determined by using Eq. (1), provides the total energy stored into the whole system. With this warning, the analysis ahead is valid whatever the nature of the considered system, discrete or continuous.

Assume that the system $S$ satisfies the initial conditions $\mathbf{w}(\mathbf{x}, 0) = 0$, $\dot{\mathbf{w}}(\mathbf{x}, 0) = \dot{\mathbf{w}}_0(\mathbf{x})$. The Lagrangean coordinates are:

$$q_i(t) = A_i \sin \omega_i t, \quad A_i = \frac{1}{\omega_i} \int_S \rho \dot{\mathbf{w}}_0 \cdot \mathbf{\Phi}_j \, \mathrm{d}V. \tag{3}$$

The explicit time dependency of the energy with respect to time is obtained as

$$E^{(r)}(t) = \frac{1}{4}\sum_{i,j=1}^{N} \left[ A_i A_j (\alpha_{ij}^{(r)} \omega_i \omega_j - \beta_{ij}^{(r)}) \cos(\omega_i + \omega_j)t + A_i A_j (\alpha_{ij}^{(r)} \omega_i \omega_j + \beta_{ij}^{(r)}) \cos(\omega_i - \omega_j)t \right],$$

$$E^{(r)}(t) = \sum_{i,j=1}^{N} \left[ a_{ij}^{(r)} \cos(\omega_i + \omega_j)t + b_{ij}^{(r)} \cos(\omega_i - \omega_j)t \right], \tag{4}$$

where

$$a_{ij}^{(r)} = \tfrac{1}{4} A_i A_j (\alpha_{ij}^{(r)} \omega_i \omega_j - \beta_{ij}^{(r)}), \quad b_{ij}^{(r)} = \tfrac{1}{4} A_i A_j (\alpha_{ij}^{(r)} \omega_i \omega_j + \beta_{ij}^{(r)}). \tag{5}$$

Suppose that inherent uncertainties affect the system $S$. As a consequence, expression (4) is not deterministic anymore, representing indeed a stochastic process. The attention is addressed to the population of energies $E^{(1)}(t), E^{(2)}(t)$ and more precisely to the time history of the ensemble average of this population.

The way the uncertain system $S$ is here described is through its natural frequencies $\omega_i$, regarded as a set of random variables, characterized by a joint probability density function $p(\omega_1, \omega_2, \ldots, \omega_N) = p(\Omega)$. Thus, the ensemble energy average of the subsystem $S_1$ is:

$$\bar{E}^{(1)}(t) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty E^{(1)} p(\omega_1, \omega_2, \ldots, \omega_N)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2 \ldots \mathrm{d}\omega_N = \int_{R^N} E^{(1)} p(\Omega)\, \mathrm{d}\Omega, \tag{6}$$

where $\mathrm{d}\Omega = \mathrm{d}\omega_1\, \mathrm{d}\omega_2 \ldots \mathrm{d}\omega_n$; an analogous expression is valid for the second subsystem $S_2$.

The expected energy provided by Eqs. (4) and (6) collapses into an expression leaded by simple time-dependent terms controlling the energy sharing between the two considered subsystems $S_1$ and $S_2$. The main simplification concerns the time scale over which the energy sharing between $S_1$ and $S_2$ is observed, as explained in the following section.

### 2.2. Asymptotic probabilistic analysis

Eqs. (4) and (6) must be combined to obtain the expression of the expected energy $\bar{E}^{(1)}(t)$ (and analogously for $\bar{E}^{(2)}(t)$)

$$\bar{E}^{(1)}(t) = \sum_{i,j=1}^{N} \int_{R^N} p(\Omega) \left[ a_{ij}^{(1)} \cos\left(\omega_i + \omega_j\right) t + b_{ij}^{(1)} \cos\left(\omega_i - \omega_j\right) t \right] \mathrm{d}\Omega, \tag{7}$$

where $R \equiv [0, \infty)$.

It is convenient to divide the terms of the summation (7) into different parts: the terms for which $i = j$, and those for which $i \neq j$. Letting

$$\begin{aligned} &\mathrm{d}\Omega_{ij} = \frac{\mathrm{d}\Omega}{\mathrm{d}\omega_i \mathrm{d}\omega_j}, \quad \mathrm{d}\Omega_i = \frac{\mathrm{d}\Omega}{\mathrm{d}\omega_i}, \\ &\Pi_{ij}\left(\omega_i, \omega_j\right) = \int_{R^{N-2}} p(\Omega)\, \mathrm{d}\Omega_{ij}, \quad \Pi_i(\omega_i) = \int_{R^{N-1}} p(\Omega)\, \mathrm{d}\Omega_i, \end{aligned} \tag{8}$$

where $\Pi_{ij}(\omega_i, \omega_j)$, $\Pi_i(\omega_i)$ are frequently called marginal probabilities, the following expression is produced:

$$\begin{aligned} \bar{E}^{(1)}(t) = &\sum_{i=1}^{N} \int_R \Pi_i(\omega_i) b^{(1)}(\omega_i)\, \mathrm{d}\omega_i + \sum_{i=1}^{N} \int_R \Pi_i(\omega_i) a^{(1)}(\omega_i) \cos\, 2\omega_i t\, \mathrm{d}\omega_i \\ &+ \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \int_{R^2} \Pi_{ik}\left(\omega_i, \omega_j\right) \left[ a^{(1)}\left(\omega_i, \omega_j\right) \cos\left(\omega_i + \omega_j\right) t + b^{(1)}\left(\omega_i, \omega_j\right) \cos\left(\omega_i - \omega_j\right) t \right] \mathrm{d}\omega_i\, \mathrm{d}\omega_j, \end{aligned} \tag{9}$$

where the dependency of the $a$'s and $b$'s on the natural frequencies is now explicit, and the simpler notation $a^{(1)}(\omega_i) = a^{(1)}(\omega_i, \omega_i)$, $b^{(1)}(\omega_i) = b^{(1)}(\omega_i, \omega_i)$ is used.

The actual limits of integration for each integral, precisely 0 and $\infty$, are replaced by $\omega_i^- = \bar{\omega}_i - \Delta\omega_i$, $\omega_i^+ = \bar{\omega}_i + \Delta\omega_i$, considering that, in practice, each natural frequency $\omega_i$ falls into the bandwidth $[\bar{\omega}_i - \Delta\omega_i,\ \bar{\omega}_i + \Delta\omega_i]$ with a probability close to one (the bandwidth size depends on the dispersion associated to each natural frequency). The first summation does not present any time dependency: its value is constant. The second and the third summations in the previous equation contain indeed the time variable: they can be dealt with integration by parts

(see Appendix A for a general derivation), leading to the so-called asymptotic expansion of the integrals with respect to $t$ [31]. In the second integral, integration by parts with respect to the $\omega_i$'s produces a summation whose first term is of order $t^{-1}$, while the other terms are of order $t^{-m}$ with $m>1$; on the other hand, the third integral produces terms of order $t^{-2}$ and, again, higher-order terms. Accordingly, the resulting asymptotic representation of the energy is (Appendix A)

$$\bar{E}^{(1)}(t) = \sum_{i=1}^{N} \int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i) b^{(1)}(\omega_i)\, \mathrm{d}\omega_i + \frac{1}{2t} \sum_{i=1}^{N} \left[\Pi_i(\omega_i) a^{(1)}(\omega_i) \sin 2\omega_i t\right]\Big|_{\omega_i^-}^{\omega_i^+} + o\left(t^{-1}\right), \tag{10}$$

where only the first-order terms are explicitly retained. For time $t$ large (a sharp meaning of this condition is detailed later and also specified in Appendix A), only the constant contribution and the dominant term $t^{-1}$ of the energy expression is kept, being those of higher order neglected. Note that, for some particular systems, the coefficients $a^{(1)}(\omega_i)$ could vanish. In this case, being zero the terms of order $t^{-1}$, those of order $t^{-2}$ must be kept and a second-order approximation is needed (see Appendix B).

Expression (10), neglecting the terms $o(t^{-1})$, is the asymptotic expansion of the ensemble energy average. Note that the second term in Eq. (10) tends to zero as time increases, while the first is constant. This means that the expected energy of the subsystem $S_1$ has a typical trend: an initial transient controlled by the $t^{-1}$-vanishing terms is responsible of the energy sharing between the two subsystems; as the time increases, the energy flow tends to zero approaching a steady-state condition, i.e. $\mathrm{d}\bar{E}^{(1)}/\mathrm{d}t = 0$, where each subsystem stores stably a certain amount of energy. In a thermodynamic sense, this is the equilibrium condition and, as it is shown ahead, it is approached through an entropy increasing process.

The two energy terms are now individually considered.

### 2.3. Stationary energy term

It must be remarked preliminarily that the total energy $\bar{E}_0$ of the system $S$, that is time invariant being it conservative, can be expressed as

$$\bar{E}_0 = \bar{E}^{(1)}(t) + \bar{E}^{(2)}(t) = \lim_{t\to\infty} \bar{E}^{(1)}(t) + \bar{E}^{(2)}(t),$$

i.e. using expression (9) or (10):

$$\bar{E}_0 = \sum_{i=1}^{N} \int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i) \left[b^{(1)}(\omega_i) + b^{(2)}(\omega_i)\right] \mathrm{d}\omega_i.$$

Using Eqs. (5) and (2) the previous energy expression becomes:

$$\bar{E}_0 = \frac{1}{2} \sum_{i=1}^{N} \int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i) (A_i\omega_i)^2\, \mathrm{d}\omega_i. \tag{11}$$

Note that for a small stochastic perturbation of the system's eigenfrequencies, the term $(A_i\omega_i)^2$ can be considered almost independent of the natural frequency $\omega_i$. In fact, Eq. (3) shows that $(A_i\omega_i)^2$ depends only on the modeshape $\mathbf{\Phi}_i$, that is known to be not sensitive to small perturbations of the natural frequencies [1]. If $\varepsilon$ is the order of magnitude of a given system's perturbation, the

perturbation of the natural frequency is of order $\varepsilon$, while the modeshape's perturbation is of order $\varepsilon^2$ [1]. In this case it is easily seen that the total energy $\bar{E}_0$, associated with the initial condition $\dot{\mathbf{w}}_0(\mathbf{x})$, becomes a deterministic quantity $E_0$

$$\bar{E}_0 = E_0 = \frac{1}{2}\sum_{i=1}^{N}\left[\int_S \rho\dot{\mathbf{w}}_0 \cdot \mathbf{\Phi}_j \,\mathrm{d}V\right]^2.$$

Consider now the steady-state energy, or equilibrium energy $\bar{E}^{(1)}_{\mathrm{eq}}$, of the system $S_1$ (or $S_2$)

$$\bar{E}^{(1)}_{\mathrm{eq}} = \lim_{t\to\infty}\bar{E}^{(1)} = \sum_{i=1}^{N}\int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i) b^{(1)}(\omega_i)\,\mathrm{d}\omega_i.$$

In Appendix C an approximated explicit expression for $b^{(1)}$ is found

$$\alpha^{(1)}_{ii} \approx \frac{m^{(1)}}{m^{(1)}+m^{(2)}}, \quad \beta^{(1)}_{ii} \approx \frac{m^{(1)}}{m^{(1)}+m^{(2)}}\,\omega_i^2,$$
$$b^{(1)}(\omega_i) \approx \tfrac{1}{2}\alpha^{(1)}_{ii}(A_i\omega_i)^2, \tag{12}$$

where $m^{(1)}$ and $m^{(2)}$ are the masses associated to $S_1$ and $S_2$, respectively. This result is demonstrated in Appendix C under the hypothesis of two coupled subsystems of the same type, e.g. two beams, two acoustic cavities, two plates, having homogenous properties. With these expressions, the steady equilibrium energy becomes

$$\bar{E}^{(1)}_{\mathrm{eq}} = \frac{m^{(1)}}{m^{(1)}+m^{(2)}}\frac{1}{2}\sum_{i=1}^{N}\int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i)(A_i\omega_i)^2\,\mathrm{d}\omega_i$$

that, compared with Eq. (11), provides

$$\bar{E}^{(1)}_{\mathrm{eq}} = \frac{m^{(1)}}{m_0}\bar{E}_0,$$

where $m_0$ is the total mass of $S$. An analogous expression holds for the energy of subsystem $S_2$.

The obtained equality takes also the forms

$$\frac{\bar{E}^{(1)}_{\mathrm{eq}}}{m^{(1)}} = \frac{\bar{E}^{(2)}_{\mathrm{eq}}}{m^{(2)}} = \frac{\bar{E}_0}{m_0}.$$

Considering that this equilibrium condition is determined under the hypothesis that two similar subsystems are coupled, it can be written in an alternative form. Suppose that the modal response of the whole system includes the natural frequencies up to $\omega_{\max}$. In general, the mass of a system is related to the number $N$ of modes contained in the frequency bandwidth $[0, \omega_{\max}]$. For example, for an acoustic cavity, the mode count leads to

$$N = \left(\frac{\omega_{\max}}{c}\right)^3 \frac{m}{6\pi^2\rho},$$

where $c, m, \rho$ are the speed of sound, the mass of the trapped gas and its mass density, respectively. For a bending plate it is

$$N = \frac{\omega_{\max} m}{3.6\rho c_L h^2},$$

where $c_L$, $m$, $\rho$, $h$ are the speed of longitudinal waves, the plate mass, the material mass density and the thickness, respectively. As a general rule $m = Nf(\omega_{\max})$, where the form of the function $f$ depends on the kind of system considered and on its properties. If the two coupled subsystems have the same form of the function $f$, then

$$\frac{m^{(1)}}{m_0} = \frac{N^{(1)}}{N}, \quad \frac{m^{(2)}}{m_0} = \frac{N^{(2)}}{N}, \quad \frac{\bar{E}_{\text{eq}}^{(1)}}{N^{(1)}} = \frac{\bar{E}_{\text{eq}}^{(2)}}{N^{(2)}} = \frac{\bar{E}_0}{N}, \tag{13}$$

where $N^{(1)}$ and $N^{(2)}$ ($N^{(1)} + N^{(2)} = N$) are the number of modes of $S_1$ and $S_2$, respectively, contained into the frequency range $[0, \omega_{\max}]$. This expression states that, in steady conditions, the energy per mode of each subsystem is equal to the initial energy per mode of the whole system. This expresses a condition partly reminiscent of the equipartion principle in statistical mechanics and it is in agreement with the general result obtained by the entropy approach in Ref. [28].

### 2.4. Nonstationary energy term and binomial expression of the energy average

The second term of Eq. (10) produces

$$\frac{1}{t}\sum_{i=1}^{N}\left[\Pi_i(\omega_i^+)a^{(1)}(\omega_i^+)\sin 2\omega_i^+ t - \Pi_i(\omega_i^-)a^{(1)}(\omega_i^-)\sin 2\omega_i^+ t\right]. \tag{14}$$

The following considerations clarify the kind of solution obtained by the ensemble energy average and the limit of the asymptotic probabilistic expansion given by Eq. (14).

Note that the coefficients $a^{(1)}(\omega_i)$ tend to zero as $i$ increases (Appendix C). Thus, only few terms in the previous summation provide a significant contribution to the ensemble energy average. To simplify the reasoning, consider only the first term of the energy series

$$\bar{E}^{(1)}(t) \approx \bar{E}_0\frac{N^{(1)}}{N} + \frac{1}{t}\left[\Pi_1(\omega_1^+)a^{(1)}(\omega_1^+)\sin 2\omega_1^+ t - \Pi_1(\omega_1^-)a^{(1)}(\omega_1^-)\sin 2\omega_1^+ t\right]. \tag{15}$$

It appears that this analysis needs only the knowledge of a reasonable probability density function of the natural frequencies in order to evaluate $\Pi_1(\omega_1)$ and only the modelling of the first mode of the dynamical system $S$. The model can be made more accurate including some other modes appearing in the sum (14), but the main result remains valid: the ensemble energy average of a system with $N$-degrees of freedom, can be described by considering only few modes of the system, even when $N$ is very large. This means that, e.g. a finite element model, can be used to identify few modes and the associated natural frequencies of a given system. This analysis is computationally inexpensive, because the discrete model needs only few nodes. This information is sufficient to estimate the coefficients $a^{(1)}(\bar{\omega}_1)$, and Eq. (10) is applied.

The time range over which the probabilistic asymptotic expansion holds, is characterized by the general condition given in Appendix A (see Eq. A.3), that assumes the more simple form $t \gg 1/\sigma_{\omega_1}$ for a Gaussian distribution of the natural frequencies, where $\sigma_{\omega_1}$ is the root mean square of the first natural frequency (see Appendix A). It is known that, in real structures, the larger the order of the natural frequency, the larger its dispersion [1]. This means that when $t \gg 1/\sigma_{\omega_1}$, then, for any frequency $\omega_i$ of the system $S$, $t \gg 1/\sigma_{\omega_1}$.

The considerations above about the expression of the nonstationary energy contribution (14), suggest that, at least over a suitable time range, it has an overall trend, or envelope, essentially controlled by $t^{-1}$. Then, considering that the coefficients $a^{(1)}(\omega_i)$ have the dimension of an energy, the envelope trend of the ensemble energy average, denoted by $\langle \bar{E}^{(1)} \rangle$ (accounting both for the stationary and nonstationary contributions), is kept by the following binomial expression:

$$\left\langle \bar{E}^{(1)} \right\rangle = \bar{E}_0 \left( \frac{N^{(1)}}{N} + \frac{T}{t} \right), \tag{16}$$

where $T$ is a suitable constant (positive or negative) having dimension of a time. Finally, considering that, because of Eqs. (2) and (5), $a^{(1)}(\omega_i) = -a^{(2)}(\omega_i)$, the expression below for subsystem $S_2$ holds

$$\left\langle \bar{E}^{(2)} \right\rangle = \bar{E}_0 \left( \frac{N^{(2)}}{N} - \frac{T}{t} \right). \tag{17}$$

Note that, from Eqs. (16) and (17), it follows that the sign of $T$ must be positive if $\langle \bar{E}^{(1)} \rangle / N^{(1)} > \langle \bar{E}^{(2)} \rangle / N^{(2)}$, negative in the opposite case.

Considering Eq. (13), the previous two relationships can be also written as:

$$\left\langle \bar{E}^{(1)} \right\rangle = \bar{E}_0 \left( \frac{m^{(1)}}{m_0} + \frac{T}{t} \right), \quad \left\langle \bar{E}^{(2)} \right\rangle = \bar{E}_0 \left( \frac{m^{(2)}}{m_0} - \frac{T}{t} \right).$$

As remarked in Section 2.2, for those systems for which the coefficients $a^{(1)}(\omega_i)$ vanish, a second-order probabilistic asymptotic expansion must be used, as shown in Appendix B. In this case in Eqs. (16) and (17) (and in the last two as well) the term $T/t$ is replaced by $\pm(T/t)^2$ (the sign to be used depending, again, on the same condition discussed above).

## 3. Complexity and uncertainty: Dispersion of the energy samples around the mean

The previous two sections considered the problem of the estimate of the ensemble average of a partition of an uncertain systems $S$. The results given by Eqs. (10), (15), (16) and (17) represent a rather simple behaviour of the average energy of the subsystems $S_1$ and $S_2$. It is important to estimate the second statistical moment of the energy. In this section, we use a simple but general argument to show how the dispersion of the energy samples around the mean value, as determined in the previous sections, is sensitive to the number of modes $N$, i.e. to the complexity of the system: for $N$ large, a small deviation with respect to the ensemble average is expected.

For the sake of simplicity, assume that a given subsystem $S$ (e.g. $S_1$ ) is an $N$-degrees of freedom system, having masses and stiffness distributed in a certain volume. Note that this schematization can be used for a continuous system as well, when only $N$ modes are considered in its response.

Consider the system reaching the steady condition, accordingly with Eq. (13). Subdivide the system $S$ in a number $M$ of subsystems $S_k$, $k = 1, 2 \ldots M$, such that each of them contains a large number of degrees of freedom (or nodes if $S$ originates from the discretization of a continuous system, such that any subsystems $S_k$ extends over a region much larger with respect to the dimension of a characteristic wavelength) implying $M \ll N$, or equivalently $M = cN$ with $c \ll 1$. In this case, since the energy stored into the physical couplings between the $M$ subsystems is

negligible with respect to that stored into them (in fact, the number of springs and masses included in each subsystem, is much larger than the number of springs connecting the considered subsystem with the others, being $M \ll N$), the energy of the system $S$ and its ensemble average are

$$E = \sum_{k=1}^{M} E^{(k)} \to \bar{E} = \sum_{k=1}^{M} \bar{E}^{(k)}, \tag{18}$$

$E^{(k)}$ denotes the energy of $S_k$. Because of Eq. (13), the equality $\bar{E}^{(k)}/m^{(k)} = \bar{E}/m$ holds, where $m = \sum_{k=1}^{M} m_k$. Since the considered system $S$ is homogeneous, then $m^{(k)} = m/M$. Thus

$$\bar{E}^{(k)} = \bar{E}/M = \bar{\varepsilon} \to \bar{E} = M\bar{\varepsilon}. \tag{19}$$

From Eq. (18)

$$E - \bar{E} = \sum_{k=1}^{M} (E^{(k)} - \bar{\varepsilon}) = \sum_{k=1}^{M} \Delta E^{(k)} \to \overline{\left(E - \bar{E}\right)^2} = \sum_{h=1}^{M} \sum_{k=1}^{M} \overline{\Delta E^{(k)} \Delta E^{(h)}}.$$

Reasonably, fluctuations with respect to the average in different subsystems are weakly correlated, implying $\overline{\Delta E^{(h)} \Delta E^{(k)}} \ll \overline{(\Delta E^{(h)})^2}$, $h \neq k$. Moreover, assume that $\overline{(\Delta E^{(k)})^2} = \sigma^2$, i.e. the standard deviations of the fluctuations in each subsystem have the same value. Thus

$$\overline{\left(E - \bar{E}\right)^2} = M\sigma^2. \tag{20}$$

From Eqs. (19) and (20) one obtains

$$\frac{\sqrt{\overline{\left(E - \bar{E}\right)^2}}}{\bar{E}} = \frac{\sigma}{\bar{\varepsilon}} \frac{1}{\sqrt{M}} = \frac{\sigma}{\sqrt{c}\bar{\varepsilon}} \frac{1}{\sqrt{N}}. \tag{21}$$

From this equation it follows that when $N$ is large, then the *rms* of the energy is small with respect to its average, i.e. the dispersion of the energy data around the mean is small. This result is referred to an equilibrium condition, but it is reasonable that this property, at least qualitatively, holds over the whole time axis, including the transient.

Therefore, it is shown that if the system has a large number of modes or degrees of freedom, i.e. it is complex, the ensemble average is a rather good representation of the energy behaviour of each sample of the population. This is an argument of practical value: generally, one is interested in predicting the behaviour of the single sample, rather than that of the average of the samples. Real structures are samples of a population, while the average structure, represented by the ensemble average, is a conceptual abstraction not existing in reality. If, for a certain class of systems, actually those for $N$ large, the dispersion of the energies is small, it means that the ensemble average, e.g. given by Eq. (15) or Eqs. (16) and (17), is a good approximation of the expected behaviour of each possible sample of the population, i.e. of the single case. On the contrary, for systems having a small number of degrees of freedom or modes, although the same equations produce the expected value of the energy, this value is a poor representation of the energy response of a random sample of the population. As a consequence, it is expected that for complex systems, the behaviour of the energy has the simpler trend described in the previous section, with respect to that observed in few-degrees of freedom systems, for which this simplification does not hold.

A final but important remarks concerns the number of modes and the initial condition of the problem. In order to develop the theory above, we need to assume that $N$ is large: more specifically this $N$ is the actual upper bound of the modal summation appearing in Eq. (1). This means that the number of modes actually activated by the initial condition $\mathbf{w}(\mathbf{x},0)=0$, $\dot{\mathbf{w}}(\mathbf{x},0)=\dot{\mathbf{w}}_0(\mathbf{x})$ must be large. For example, even if the system has a large number of degrees of freedom, but $\dot{\mathbf{w}}_0(\mathbf{x})$ is proportional to a given system's mode shape, it is clear that just a single mode participates to the system's response and it behaves like a single-degrees of freedom system, so that the developed analysis does not hold. Thus, the condition that the system is equipped with a large number of modes is a necessary condition to state that the system is complex, but it is not sufficient. The role of the initial condition is indeed crucial: it must activate a large number of modes. This is, for example, the case in which $\dot{\mathbf{w}}_0(\mathbf{x})$ is the Dirac impulse, corresponding to a typical shock response problem. In Section 6 the performed numerical experiments use indeed this initial condition.

## 4. Energy flow and weighted energy difference: Nonlinear correlation law

In the previous sections the energy behaviour of two coupled subsystems is investigated. On this basis, nonlinear relationships for the energy flow between the two subsystems and their respective energies can be determined.

The difference between Eqs. (16) and (17) produces indeed

$$\frac{N^{(1)}N^{(2)}}{N}\left(\frac{\left\langle \bar{E}^{(1)}\right\rangle}{N^{(1)}}-\frac{\left\langle \bar{E}^{(2)}\right\rangle}{N^{(2)}}\right)=\bar{E}_0\frac{T}{t}.$$

On the other hand, the energy flow $\varphi(t)$ between the two subsystems can be expressed as

$$\varphi=-\frac{\mathrm{d}\left\langle \bar{E}^{(1)}\right\rangle}{\mathrm{d}t}=\frac{\mathrm{d}\left\langle \bar{E}^{(2)}\right\rangle}{\mathrm{d}t}=\bar{E}_0\frac{T}{t^2},$$

where $\varphi(t)$ is assumed to be positive when the energy is transferred form $S_1$ to $S_2$. Finally, the combination of the last two written equations provides

$$\varphi=\frac{1}{\bar{E}_0T}\left(\frac{N_1N_2}{N_0}\right)^2\left[\frac{\left\langle \bar{E}^{(1)}\right\rangle}{N_1}-\frac{\left\langle \bar{E}^{(2)}\right\rangle}{N_2}\right]^2 \tag{22}$$

or, alternatively, using Eq. (13)

$$\varphi=\frac{1}{\bar{E}_0T}\left(\frac{m^{(1)}m^{(2)}}{m_0}\right)^2\left[\frac{\left\langle \bar{E}^{(1)}\right\rangle}{m^{(1)}}-\frac{\left\langle \bar{E}^{(2)}\right\rangle}{m^{(2)}}\right]^2. \tag{23}$$

Eqs. (22) and (23) provide a quadratic relationship between the energy flow and the weighted energy difference. Although they sound not conformal with the usual results presented in energy methods, e.g. in SEA, it must be remarked that the energy here considered refers to a

nonstationary, asymptotic, time domain response, i.e. to completely different conditions with respect to those usually investigated in SEA.

Finally, for those systems for which a second-order asymptotic expansion is needed, following an identical procedure, it follows:

$$\varphi = \frac{2\bar{E}_0}{T}\left(\frac{N^{(1)}N^{(2)}}{\bar{E}_0 N}\right)^{3/2}\left[\left|\frac{\left\langle \bar{E}^{(1)}\right\rangle}{N^{(1)}} - \frac{\left\langle \bar{E}^{(2)}\right\rangle}{N^{(2)}}\right|\right]^{3/2}.$$

## 5. Connection with the second principle of thermodynamics

The introduction of the second principle of thermodynamics in vibration is considered in Ref. [28]. Referring the reader to that work for an additional point of view on this argument, in the present context the concept of entropy and its connection with the previous analysis are considered in a more intuitive way.

In classical thermodynamics, the entropy variation $\mathrm{d}H$ of a system that exchanges the heat quantity $\mathrm{d}Q$ with the environment, is given by

$$\mathrm{d}H = \frac{\mathrm{d}Q}{\vartheta},$$

where $\vartheta$ is the temperature of the system.

If two coupled systems are considered, the total entropy variation with respect to time is:

$$\dot{H}_0 = \dot{H}^{(1)} + \dot{H}^{(2)} = \frac{\mathrm{d}Q^{(1)}/\mathrm{d}t}{\vartheta^{(1)}} + \frac{\mathrm{d}Q^{(2)}/\mathrm{d}t}{\vartheta^{(2)}},$$

where $^{(1)}$ and $^{(2)}$ denote quantities related to the subsystem $S_1$ and $S_2$, respectively.

In order to define, for a vibrating system, an analogous of the previous definition of the entropy, $\mathrm{d}Q$, i.e. the amount of thermal energy exchanged with the environment, can be associated to the energy amount $\mathrm{d}\langle\bar{E}\rangle$. (Remind that this is representative of the energy of a given sample of the structure only for complex systems). Consequently, the analogous of the temperature $\vartheta$ is determined by analysing Eqs. (22) and (23): the power flow $\mathrm{d}\langle\bar{E}\rangle/\mathrm{d}t$ is indeed controlled by the quantities $\langle\bar{E}^{(i)}\rangle/m^{(i)}$ or $\langle\bar{E}^{(i)}\rangle/N^{(i)}$, depending on which equation is considered. On the other hand, it is known from thermology that the heat flow $\mathrm{d}Q/\mathrm{d}t$ is controlled by the difference between the subsystems' temperatures. Thus, $\langle\bar{E}^{(i)}\rangle/m^{(i)}$ or $\langle\bar{E}^{(i)}\rangle/N^{(i)}$, are the analogous of $\vartheta$, i.e. they are the temperatures of the vibrating subystems. This suggests the following definition of entropy for a vibrating system:

$$\mathrm{d}H^{(1)} = N^{(1)}\frac{\mathrm{d}\left\langle \bar{E}^{(1)}\right\rangle}{\left\langle \bar{E}^{(1)}\right\rangle}, \quad \mathrm{d}H^{(2)} = N^{(2)}\frac{d\left\langle \bar{E}^{(2)}\right\rangle}{\left\langle \bar{E}^{(2)}\right\rangle},$$

$$H^{(1)} = N^{(1)}\ln\left\langle \bar{E}^{(1)}\right\rangle + \text{constant}, \quad H^{(2)} = N^{(2)}\ln\left\langle \bar{E}^{(2)}\right\rangle + \text{constant},$$

where with this definition, the entropy is a nondimensional quantity. The proportionality between the entropy and the logarithm of the energy is in agreement with a result found in Ref. [28]. In this context, this is a well-posed entropy definition: an entropy increasing property can be indeed demonstrated. In fact

$$H_0 = H^{(1)} + H^{(2)} = N^{(1)} \ln\left\langle \bar{E}^{(1)} \right\rangle + N^{(2)} \ln\left\langle \bar{E}^{(2)} \right\rangle + \text{constant}. \tag{24}$$

Using the expressions of $\langle \bar{E}^{(1)} \rangle$ and $\langle \bar{E}^{(2)} \rangle$ given by Eqs. (16) and (17), the entropy rate follows:

$$\dot{H}_0 = \frac{T}{t^2}\left[\frac{1}{(1/N_0) - (T/N_2 t)} - \frac{1}{(1/N_0) + (T/N_1 t)}\right].$$

If $T > 0$, then the expression in square brackets is certainly positive; on the contrary, if $T < 0$, then the expression in square brackets is negative. Since this expression is multiplied by $T$, it is concluded that the considered mechanical system satisfies, in any case, the condition

$$\frac{\mathrm{d}H_0}{\mathrm{d}t} > 0, \tag{25}$$

i.e. the Boltzmann's inequality, expressing the second principle of thermodynamics. If $t \to \infty$, the entropy rate tends to zero, i.e. the thermodynamic equilibrium is approached. Note that since the considered energies are $\langle \bar{E}^{(1)} \rangle$ and $\langle \bar{E}^{(2)} \rangle$, Eq. (25) holds for those systems whose subsystems' energy are well represented by Eqs. (16) and (17), i.e. by the trends of the ensemble energy average. This is true only if the system is complex in the sense discussed in the previous section. This result is in agreement with the findings shown in Ref. [28].

It is interesting to remark that the monotonically entropy increase towards the equilibrium condition (equal temperatures of the two vibrating systems) is a result that holds always for the systems usually considered in thermodynamics. For the mechanical engineering structures here considered this is not indeed an obvious property. In this context, an interesting example of a system that does not present conformance to this 'one way' energy exchange law is considered in Ref. [32], where the response of a complex elastic system coupled to an elemental oscillator (single-degree of freedom) is investigated.

## 6. Stochastic asymptotic expansion for nonconservative systems: The effect of damping

The previous analysis is valid for conservative systems. In this section, although following the same asymptotic expansion technique, the dissipation effect is included in the analysis.

The system response, in presence of damping, can be expressed in the form

$$q_i(t) = A_i \mathrm{e}^{-\delta_i \omega_i t} \sin \omega_i t$$

provided that the $\omega_i$'s are the damped natural frequencies and the modes are assumed to be real; the $\delta_i$'s are the modal damping coefficients and, again, the $A_i$'s are related to the prescribed initial

conditions. The expected value of the energy (in analogy with Eq. (9)) is

$$\bar{E}^{(1)}(t) = \sum_{i=1}^{N} \int_R \Pi_i(\omega_i) b^{(1)}(\omega_i) \mathrm{e}^{-\delta_i \omega_i t}\, \mathrm{d}\omega_i + \sum_{i=1}^{N} \int_R \Pi_i(\omega_i) a^{(1)}(\omega_i) \mathrm{e}^{-\delta_i \omega_i t} \cos 2\omega_i t\, \mathrm{d}\omega_i$$
$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \int_{R^2} \Pi_{ij}(\omega_i, \omega_j) \mathrm{e}^{-(\delta_i \omega_i + \delta_j \omega_j)t} \left[ a^{(1)}(\omega_i, \omega_j) \cos(\omega_i + \omega_j) t + b^{(1)}(\omega_i, \omega_j) \cos(\omega_i - \omega_j) t \right] \mathrm{d}\omega_i\, \mathrm{d}\omega_j.$$

Applying to the previous integrals the asymptotic expansion (i.e. integrating by parts as shown in Appendix A) and retaining only the terms up to the order $t^{-2}$, one obtains

$$\bar{E}^{(1)}(t) = \sum_{i=1}^{N} \left\{ \left[ \frac{\Pi_i a_i^{(1)} (\delta_i \cos 2\omega_i t - \sin 2\omega_i t) \mathrm{e}^{-2\delta_i \omega_i t}}{2t(1+\delta_i^2)} \right]_{\omega_i^-}^{\omega_i^+} + \left[ \frac{\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\Pi_i a_i^{(1)}\right) (\cos 2\omega_i t - 2\delta_i \sin 2\omega_i t) \mathrm{e}^{-2\delta_i \omega_i t}}{4t^2(1+4\delta_i^2)} \right]_{\omega_i^-}^{\omega_i^+} \right\}$$
$$- \sum_{i=1}^{N} \left\{ \left[ \frac{\Pi_i b_i^{(1)} \mathrm{e}^{-2\delta_i \omega_i t}}{2t\delta_i} \right]_{\omega_i^-}^{\omega_i^+} + \left[ \frac{\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\Pi_i b_i^{(1)}\right) \mathrm{e}^{-2\delta_i \omega_i t}}{(2t\delta_i)^2} \right]_{\omega_i^-}^{\omega_i^+} \right\}$$
$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left\{ \left[ \frac{\Pi_{ij} a_{ij}^{(1)}}{t^2} \mathrm{e}^{-(\delta_i \omega_i + \delta_{ji} \omega_{ji}) t} \frac{(\delta_i \delta_j - 1) \cos(\omega_i + \omega_j) t - (\delta_i + \delta_j) \sin(\omega_i + \omega_j) t}{(\delta_i \delta_j - 1)^2 + (\delta_i + \delta_j)^2} \right]_{\omega_i^-}^{\omega_i^+} \right\}_{\omega_j^-}^{\omega_j^+}$$
$$- \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left\{ \left[ \frac{\Pi_{ij} b_{ij}^{(1)}}{t^2} \mathrm{e}^{-(\delta_i \omega_i + \delta_{ji} \omega_{ji}) t} \frac{(\delta_i \delta_j + 1) \cos(\omega_i - \omega_j) t + (\delta_i - \delta_j) \sin(\omega_i - \omega_j) t}{(\delta_i \delta_j + 1)^2 + (\delta_i - \delta_j)^2} \right]_{\omega_i^-}^{\omega_i^+} \right\}_{\omega_j^-}^{\omega_j^+} + o(t^{-2}). \tag{26}$$

An analogous expression for $\langle \bar{E}^{(2)} \rangle$ holds. Namely, the first two summations have been expanded by using a double integration by parts in order to include both the terms of order $t^{-1}$ and $t^{-2}$, while the second and the third summations are integrated by parts only once, being their contribution of order $t^{-2}$ at the first step of integration. Thus, neglecting terms of order $t^{-3}$ and higher order, it is recognized that the trend of the energy is controlled by vanishing exponential terms divided by $t$ or $t^2$. This means that the main trend of the energy is controlled by a suitable combination of terms of the type $\mathrm{e}^{-\delta\omega t}/t, \mathrm{e}^{-\delta\omega t}/t^2$, i.e. reasonably, the energy trend can be fitted by an expression of the form

$$\left\langle \bar{E}^{(1)} \right\rangle = \mu^{(1)} \frac{\mathrm{e}^{-\delta\omega t}}{t} \left( 1 + \frac{T^{(1)}}{t} \right), \quad \left\langle \bar{E}^{(2)} \right\rangle = \mu^{(2)} \frac{\mathrm{e}^{-\delta\omega t}}{t} \left( 1 - \frac{T^{(2)}}{t} \right), \tag{27}$$

where $\mu^{(1)}, \mu^{(2)}, \delta, \omega, T^{(1)}, T^{(2)}$ are suitable coefficients. Analogies and differences with respect to the case of a conservative system are worked out by comparison with Eqs. (16) and (17).

Note that, in the conservative case, only the terms of order $t^{-1}$ were kept—besides those constant—, while in the present case those up to $t^{-2}$ have been included. In fact, in the analysis of conservative systems the expansion up to $t^{-1}$ is sufficient to describe the basic mechanism of energy sharing: the energy initially stored into one system is progressively transferred to the other

so that an equilibrium is reached when the modal energy of the two subsystems are equal. This process leads to energy trends monotonically varying towards the equilibrium condition. On the other hand for nonconservative systems, the physical mechanism of energy sharing is more complicated. The system initially at rest, undergoes indeed two opposite effects: an energy injection is received by the first system, that initially energized, but an energy loss is due to the inherent dissipation. The combination of the two effects leads in this case to an increasing of the energy at early times (energy transmitted by the first system) while, once some energy is stored into the system, it is decreased because of the dissipation. The resulting trend exhibits a relative maximum before the energy definitely vanishes (see Figs. 22 and 23 in Section 7). This trend can be only described including the terms of order $t^{-2}$ besides that of order $t^{-1}$. If only the term of order $t^{-1}$ is kept , i.e. Eq. (27) is simplified as

$$\left\langle \bar{E}^{(1)} \right\rangle = \mu^{(1)} \frac{\mathrm{e}^{-\delta\omega t}}{t} \quad \text{and} \quad \left\langle \bar{E}^{(2)} \right\rangle = \mu^{(2)} \frac{\mathrm{e}^{-\delta\omega t}}{t},$$

then both the energies monotonically vanish towards the equilibrium (see Figs. 22 and 23). Although this is the energy trend exhibited for late time, in this case an interesting part of the energy response should be lost.

Finally, starting from Eq. (27) and making some additional hypotheses, conclusions concerning the power-flow relationship can be drawn.

Let us consider the expression of the total energy of the system $\langle \bar{E}^{(1)} \rangle + \langle \bar{E}^{(2)} \rangle$. On the basis of Eq. (27) it follows:

$$\left\langle \bar{E}^{(1)} \right\rangle + \left\langle \bar{E}^{(2)} \right\rangle = (\mu^{(1)} + \mu^{(2)}) \frac{\mathrm{e}^{-\delta\omega t}}{t} + (\mu^{(1)} T^{(1)} - \mu^{(2)} T^{(2)}) \frac{\mathrm{e}^{-\delta\omega t}}{t^2}.$$

A simplification in the analysis of the power flow follows by the assumption of neglecting the contribution of the term $(\mu^{(1)} T^{(1)} - \mu^{(2)} T^{(2)})(\mathrm{e}^{-\delta\omega t}/t^2)$. This assumption can be justified considering that, on the basis of Eqs. (2), (5) and using expression (26), most of the contributions of type $\mathrm{e}^{-\delta\omega t}/t^2$ contained into the expression of $\langle \bar{E}^{(1)} \rangle + \langle \bar{E}^{(2)} \rangle$ are cancelled. Thus, it is assumed:

$$\mu^{(1)} T^{(1)} = \mu^{(2)} T^{(2)}, \quad \left\langle \bar{E}^{(1)} \right\rangle + \left\langle \bar{E}^{(2)} \right\rangle = (\mu^{(1)} + \mu^{(2)}) \frac{\mathrm{e}^{-\delta\omega t}}{t}. \tag{28}$$

The energy balance of the whole system implies

$$\frac{\mathrm{d}\left[ \left\langle \bar{E}^{(1)} \right\rangle + \left\langle \bar{E}^{(2)} \right\rangle \right]}{\mathrm{d}t} = \frac{\mathrm{d}\langle \bar{E} \rangle}{\mathrm{d}t} = P_{\mathrm{diss}}, \tag{29}$$

where $P_{\mathrm{diss}}$ is the power dissipated into the system. Combining Eqs. (28) and (29), the following expression of the dissipated power in terms of the energy is produced

$$P_{\mathrm{diss}} = -\delta\omega \langle \bar{E} \rangle - \frac{\langle \bar{E} \rangle}{t}. \tag{30}$$

It is interesting to notice that this expression sounds similar, except a higher-order term, to the expression generally used in SEA.

Once a relationship between dissipated power and stored energy is found as in Eq. (30), the power flow expression is determined simply writing for subsystems 1 and 2 their respective

energy balances

$$\frac{\mathrm{d}\left\langle \bar{E}^{(1)}\right\rangle}{\mathrm{d}t} = P^{(1)}_{\mathrm{diss}} - \varphi, \quad \frac{\mathrm{d}\left\langle \bar{E}^{(2)}\right\rangle}{\mathrm{d}t} = P^{(2)}_{\mathrm{diss}} + \varphi, \tag{31}$$

where $\varphi(t)$ is the energy flow between the two subsystems, assumed to be positive when the energy is transferred form $S_1$ to $S_2$. Using Eqs. (27), (30) and (31), the power flow takes the form (see Appendix D):

$$\varphi = -\frac{\mu^{(1)}T^{(1)}}{2\left(T^{(1)}+T^{(2)}\right)}\left\{\delta\omega\left[\frac{\langle E^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle E^{(2)}\rangle}{\mu^{(2)}}\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\langle E^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle E^{(2)}\rangle}{\mu^{(2)}}\right]\right\} \tag{32}$$

showing a linear dependence of the power flow on the weighted energy difference

$$\left[\langle E^{(1)}\rangle/\mu^{(1)} - \langle E^{(2)}\rangle/\mu^{(2)}\right].$$

An extrapolation of Eq. (32) for the steady case, leads to an expression qualitatively reminiscent of one of the basic result of SEA.

A conclusive remark concerns the time interval of validity of the asymptotic Eq. (27) and consequently of the energy flow expression (32).

From expression (26) it appears that four summations are included in the asymptotic energy. Considering the second summation, the condition under which the terms of type $\mathrm{e}^{-\delta\omega t}/t^3$ are negligible with respect to those $\mathrm{e}^{-\delta\omega t}/t$ and $\mathrm{e}^{-\delta\omega t}/t^2$ implies (following the same scheme shown in Appendix A) $t \gg 1/\delta_i\sigma_{\omega_i}$. For the remaining summations the condition for neglecting terms of type $\mathrm{e}^{-\delta\omega t}/t^3$ is still $t \gg 1/\sigma_{\omega_i}$ (for $\delta_i \ll 1$), similar to that discussed in Appendix A. The presence of the condition $t \gg 1/\delta_i\sigma_{\omega_i}$, that is more restrictive than $t \gg 1/\sigma_{\omega_i}$, shows that the asymptotic expansion for nonconservative systems is valid in a time range the lower bound of which depends on the damping. Thus, asymptotic Eqs. (16) and (17) for conservative systems and asymptotic Eq. (27) for dissipative systems are valid under different time scales. This explains why the expansion (27), as well as Eq. (32), do not produce for $\delta_i = 0$, as a particular case, the results obtained in the previous sections for conservative systems.

It can be concluded that, in presence of damping the asymptotic probabilistic analysis leads to Eqs. (27) and (32). If damping is not very small, the constraint $t \gg 1/\delta_i\sigma_{\omega_i}$ is not too restrictive. However, for light damping, Eqs. (27) and (32) are valid only for very late time (i.e. $t \gg 1/\delta_i\sigma_{\omega_i}$, where $\delta_i$ is very small); thus, for small damping, the early time energy response (i.e. that obtained under the less restrictive condition $t \gg 1/\sigma_{\omega_i}$) can be still represented by Eqs. (16), (17) and (23).

## 7. Numerical experiments

The simplest system that can be investigated is the two degrees of freedom oscillator. The system, described in Fig. 1, has the following parameters $m_1 = m_2 = 1\,\mathrm{kg}$, $k_1 = k_3 = 1\,\mathrm{N/m}$, $k_2 = 0.1\,\mathrm{N/m}$. The initial conditions are all set to zero, except for the initial velocity of the oscillator 1, for which $V_0 = 1\,\mathrm{m/s}$. A collection of 10 000 similar systems is generated allowing the coefficients $k_1$, $k_3$ to be uniformly distributed in the bandwidth [0.9 N/m;1.1 N/m], i.e. the samples present a maximum difference of 10% in their stiffness with respect to the average values

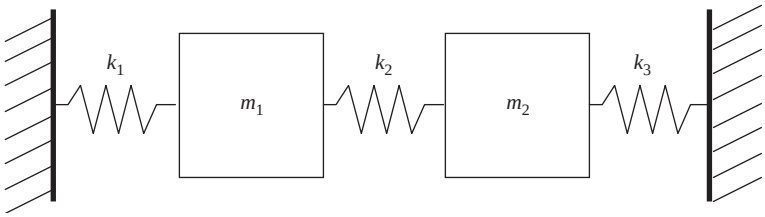


Fig. 1. Sketch of the 2 dof used in the numerical simulations.

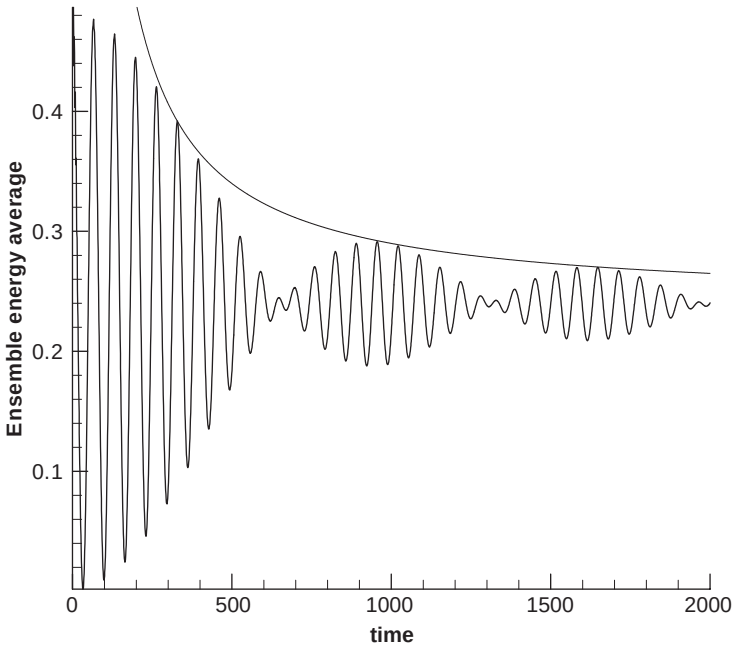


Fig. 2. Two dof system: ensemble energy average with random stiffness (10 000 samples) vs $t^{-1}$ vanishing trend (smooth curve).

$k_1 = k_3 = 1\,\mathrm{N/m}$. The ensemble energy average of the oscillator 1 is shown in Fig. 2 (all the energy plot in this paper use joule units). Its trend is rapidly decreasing and it tends to half of its initial energy, reaching the equilibrium condition, in agreement with the results of Eq. (16). Moreover, in Fig. 2, the envelope of the ensemble energy average is drawn, using a hyperbolic expression $T/t$ —where the constant $T$ is chosen to fit the desired envelope—confirming the trend predicted by Eq. (16). Finally in Fig. 3 the ensemble energy average is represented together with the energy time history of one of the sample of the set. It is apparent that, in this case (only two degrees of freedom), the ensemble average does not match at all the single sample energy trend, in agreement with the arguments provided in Section 3. In particular, note that the energy of the sample does not present any asymptotic tendency to the equilibrium condition, but the energy is just fluctuating between the two oscillators.

A second example is provided considering the simply supported beam sketched in Fig. 4. It is ideally divided into two subsystems of different lengths $L$ and $3L$, respectively. The characteristics

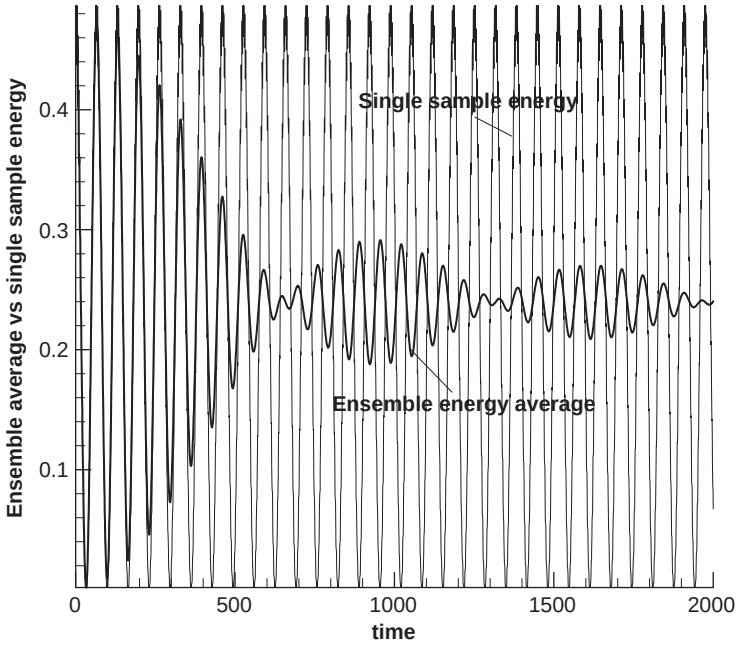


Fig 3. Two dof system: comparison between the ensemble energy average and the energy of a sample of the set.

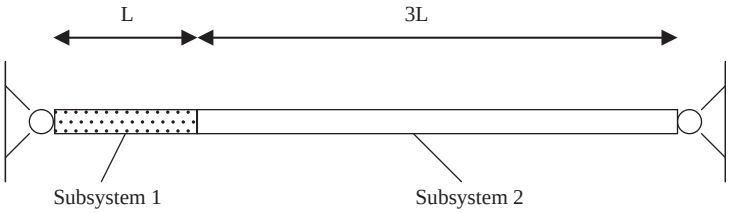


Fig 4. Sketch of the partition of the beam used in the numerical test.

of the beam are $\rho A = 1\,\mathrm{kg/m}$, $L_0 = 4L = 1\,\mathrm{m}$, $EI = 1\,\mathrm{N\,m^2}$, where $\rho A$, $L_0$, $EI$ are the mass per unit length, the total length and the bending stiffness, respectively. The initial conditions on the displacement are set to zero, while a velocity spike (Dirac's impulse) is localized on the subsystem 1. Therefore, the initial energy is stored only into subsystem 1, while that stored into subsystem 2 is zero. The simulation is performed by using 20 modes in the beam response.

Fig. 5 shows the time histories of the ensemble energy average of the two subsystems. It appears how an equilibrium condition is approached, where, accordingly with Eqs. (13), (16) and (17) and considering that the respective masses of $S_1$ and $S_2$ are $\rho AL$ and $3\rho AL$, the energy of $S_1$ is one third of that stored in $S_2$. Fig. 6 shows the trends of the two energies fitted by the expressions

$$\left\langle \bar{E}^{(1)} \right\rangle = \bar{E}_0 \left[ \frac{m^{(1)}}{m_0} + \left( \frac{T}{t} \right)^2 \right], \quad \left\langle \bar{E}^{(2)} \right\rangle = \bar{E}_0 \left[ \frac{m^{(2)}}{m_0} - \left( \frac{T}{t} \right)^2 \right],$$

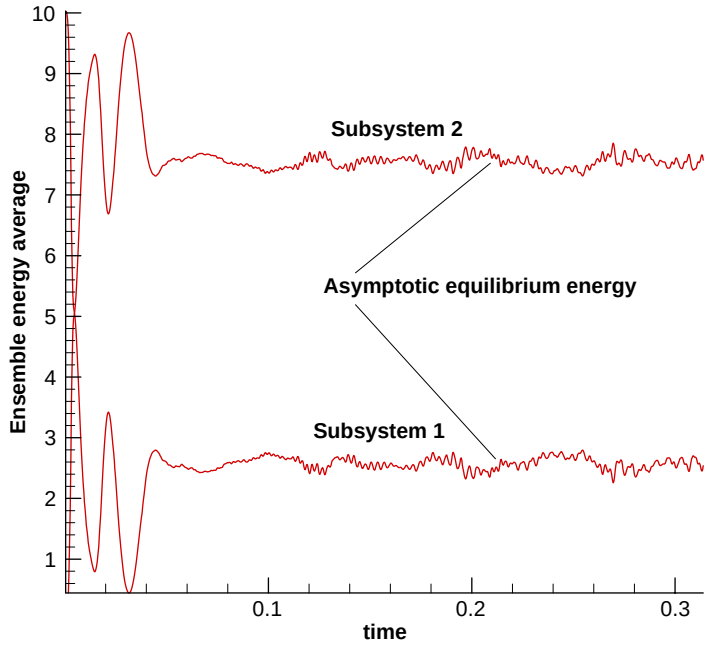


Fig 5. Simply supported beam: ensemble energy average of the two subsystems (20 modes).

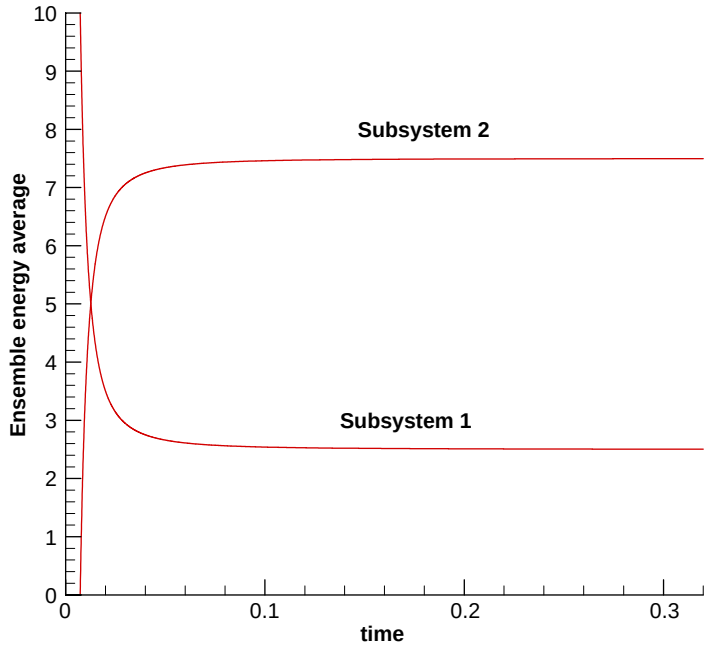


Fig. 6. Simply supported beam: envelopes of the ensemble energy of the two subsystems (20 modes).

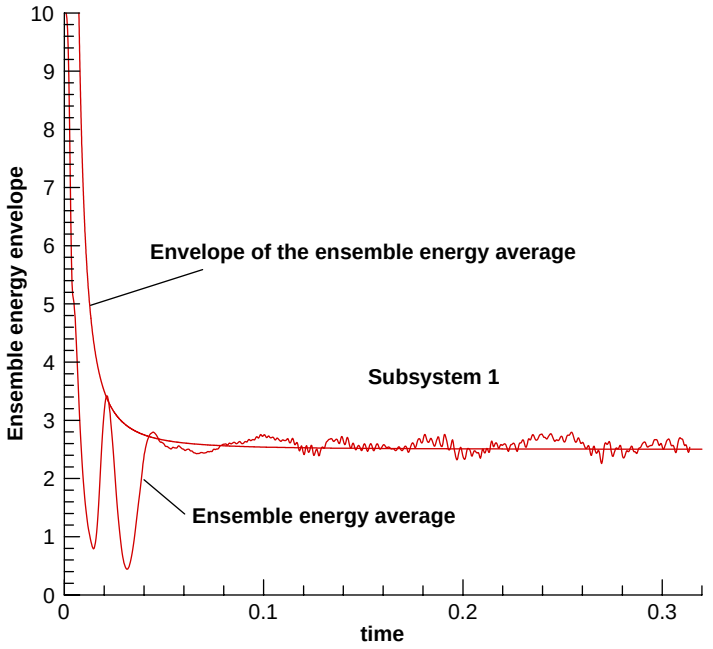


Fig. 7. Simply supported beam: theoretical envelope versus numerical ensemble energy average.

discussed in Section 2.4 (in the case of the beam, the coefficients $a^{(1)}(\omega_i)$ vanish and the second-order expansion presented in Appendix B must be used), where $\bar{E}_0$ is known and equal to the initial energy stored into $S_1$ (about 10 J); moreover $m^{(1)}/m_0 = 0.25$ and $m^{(2)}/m_0 = 0.75$ and, again, the constant $T$ is suitably chosen to fit the envelope. Fig. 7 shows the comparison between the determined envelope and the ensemble energy average for the subsystem 1. The results show that the ensemble energy trends predicted in Section 2.4 are in good agreement with the results of the numerical simulations.

A second set of computational experiments is performed on the supported beam. A population of 30 beam samples, with characteristics identical to those of the previous one, is generated by varying the length $L_0$, uniformly distributed in the range [0.9 m;1.1 m]. The initial energy distribution is the same of the previous case. The analysis consists of comparing, for $S_1$, the energy average with the energy response of a sample of the population. The analysis is made by varying the number of modes included into the beam's response.

Fig. 8 shows the comparison between the average and the energy sample when only 4 modes are considered. It appears how, except at very early times, the time history of the average energy does not match the energy of the sample. Moreover, the average energy clearly reaches an equilibrium condition with small fluctuations around it, while this is not the case for the energy of the sample. Figs. 9–11 show the same comparison when increasing the number of modes included into the beam's response up to 8, 16 and 32, respectively. As it appears form the figures, the more the number of modes is increased, the more the ensemble energy average matches the energy of the sample. In Fig. 11, it is apparent how, although the energy of the sample still exhibits larger

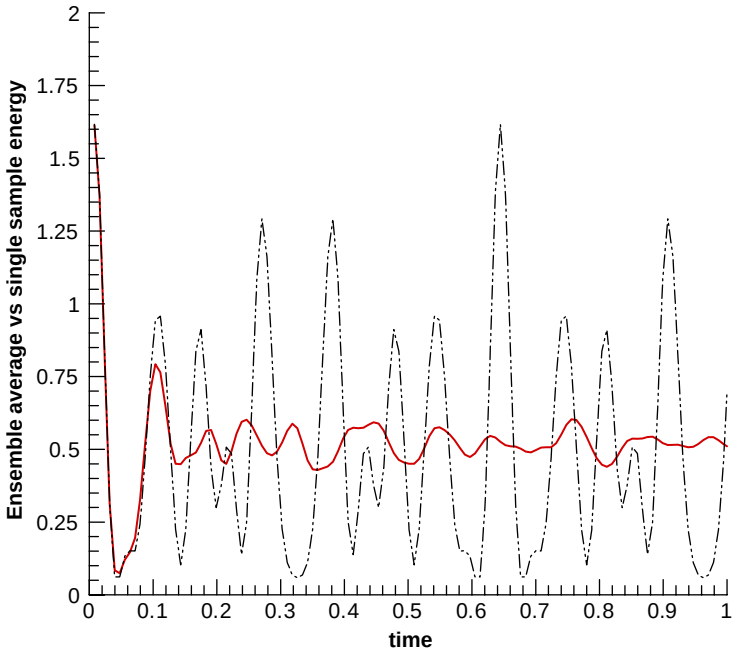


Fig. 8. Simply supported beam: ensemble energy average versus the single sample energy (4 modes).

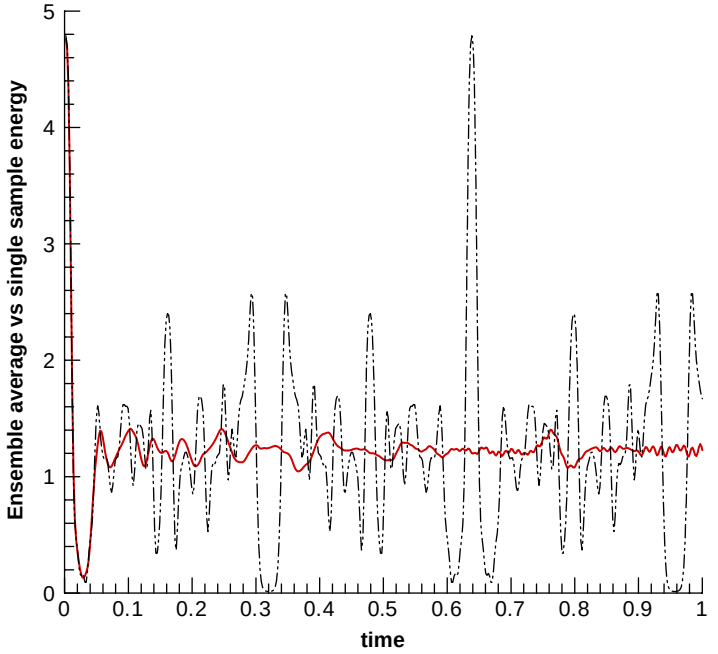


Fig. 9. Simply supported beam: ensemble energy average versus the single sample energy (8 modes).

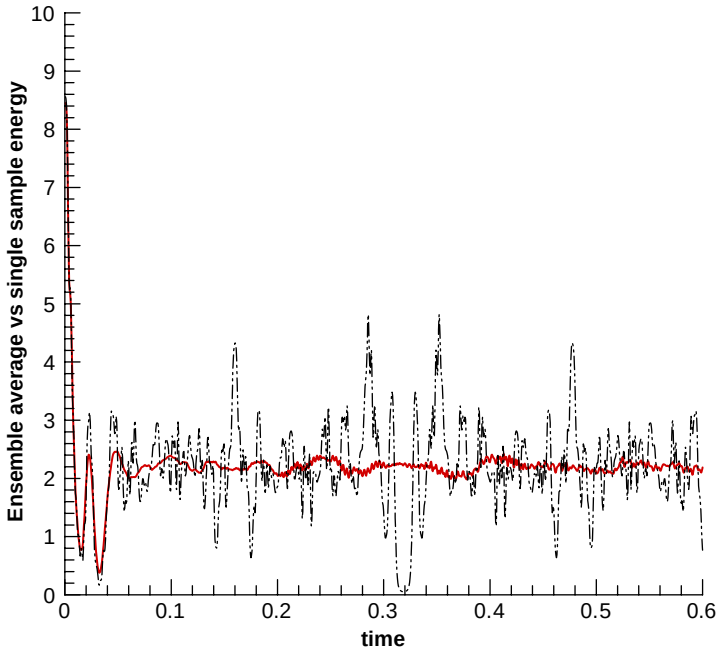


Fig. 10. Simply supported beam: ensemble energy average versus the single sample energy (16 modes).

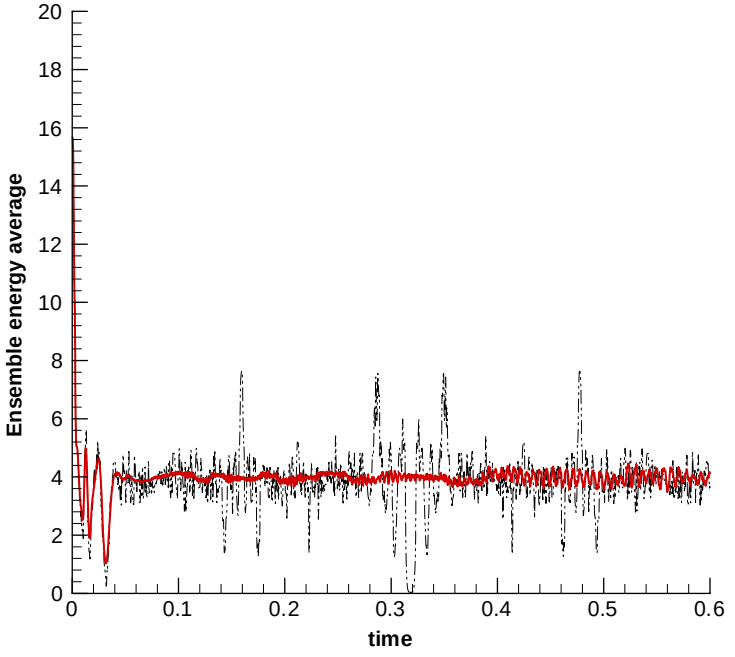


Fig. 11. Simply supported beam: ensemble energy average versus the single sample energy (32 modes).

fluctuations with respect to those of the ensemble average, the two trends are qualitatively similar and the energy of the sample also reaches an equilibrium condition.

To check the result provided by Eq. (21), for each of the analysed cases, differing for the number $N$ of the modes included in the model, the rms of the energy has been calculated for the subsystem 1, as

$$\overline{\left(E^{(1)}(t)-\bar{E}^{(1)}(t)\right)^2}=\frac{1}{N}\sum_{k=1}^{N}\left[E_k^{(1)}(t)-\bar{E}^{(1)}(t)\right]^2,$$

where $E_k^{(1)}(t)$ is the energy of $S_1$ for the $k$th sample of the population and $N=30$. For each case, the quantity

$$\frac{\sqrt{\overline{\left(E^{(1)}(t)-\bar{E}^{(1)}(t)\right)^2}}}{\bar{E}^{(1)}(t)}$$

has been determined. In order to eliminate the time dependency, a time average of this quantity is taken over the interval [0,1] (note from Eq. (21), that this operation does not alter the dependency of the rms on the number of degrees of freedom $N$). In Fig. 12 the rms values so obtained are plotted versus the number of modes used to represent the beam's response: 11 cases are analysed increasing the number of modes $N$ from 4 up to 32. A least-squares procedure, using the fitting function of the form $r/\sqrt{M}$ ($r$ is the fitting constant), produces the solid line shown in Fig. 12, confirming the exact trend predicted by Eq. (21).

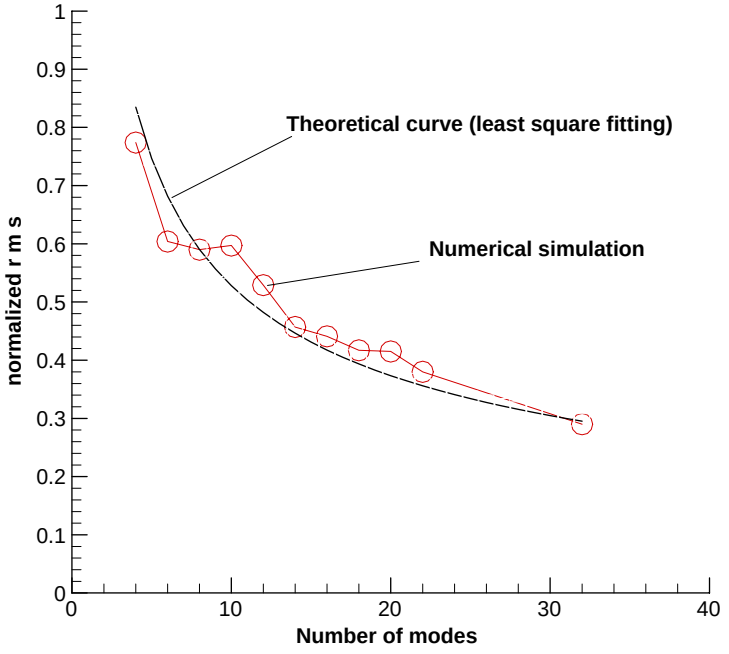


Fig 12. Simply supported beam: rms trend versus the number of modes (from 4 up to 32 modes).

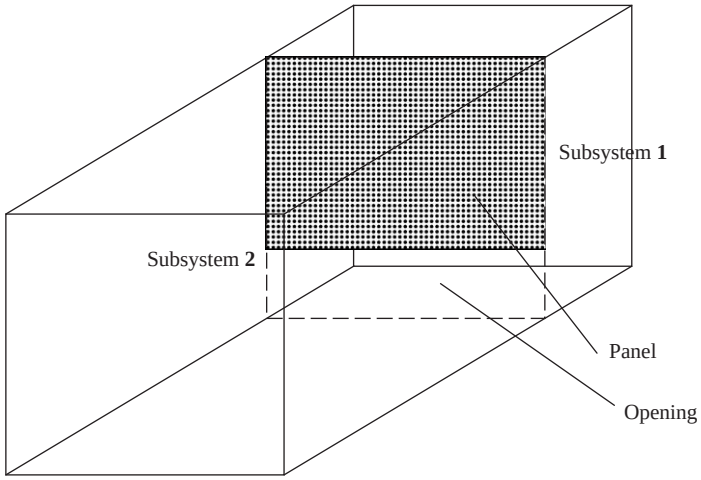


Fig. 13. Sketch of the acoustic box used for the numerical simulations.

The last set of computational experiments are performed on a three-dimensional acoustic cavity depicted in Fig. 13. It consists of a cubic box, side length 1 m, with a rectangular vertical panel placed at one-third of the side length, partitioning the inner volume into two subsystems as shown in Fig. 13. One side of the panel is shorter than the box side, leaving a rectangular opening of dimensions $1\,\mathrm{m} \times 0.25\,\mathrm{m}$ between the two subsystems. The speed of sound in the cavity is 1 m/s.

The numerical model uses finite differences with a grid $24 \times 24 \times 24$ corresponding to 13 824 degrees of freedom (or modes) included into the box response. Initial conditions consists of a pressure spike localized into the subsystem 1 (the smallest). Therefore, the initial energy is wholly stored into $S_1$.

Fig. 14 shows the energy response of the two subsystems. The energies reach the equilibrium condition approximately in the way predicted in Sections 2.2 and 2.3: in fact, the asymptotic energy of $S_1$ is almost half of the energy of $S_2$. Fig. 15 represents the trends of the two energies fitted with the expressions given by Eqs. (16) and (17). In Fig. 16 a direct comparison between the fitting curve and the numerical energy trend for the subsystem $S_1$ is plotted. The important fact that is demonstrated by the numerical experiments summarized in Figs. 14–16, is that now the energy trends for the single case obey the laws expressed by Eqs. (16) and (17), originally derived for the ensemble energy averages. This point definitely supports the demonstration and the arguments given in Section 2.4: when the number of degrees of freedom is large (13 824 in the present case) the dispersion of the energy samples around the ensemble average tends to zero. Thus, in these conditions, the ensemble energy average correctly represents the single case for which Eqs. (16) and (17) apply as well.

A direct comparison between the single sample and the ensemble average obtained by numerical experiments is also provided. A set of 10 acoustic cavity samples is generated by varying the opening dimension whose short side, with average value 0.42 m, is uniformly distributed in the range [0.21 m;0.63 m]. In Fig. 17 the ensemble energy average of the smallest subsystem is compared with the energy of a sample of the population. An

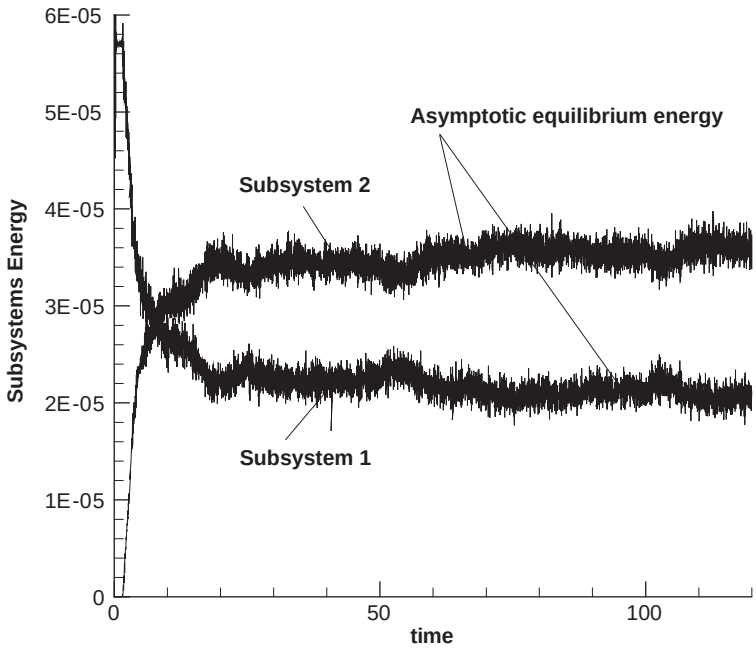


Fig. 14. Acoustic cavity: energy of the two subsystems.

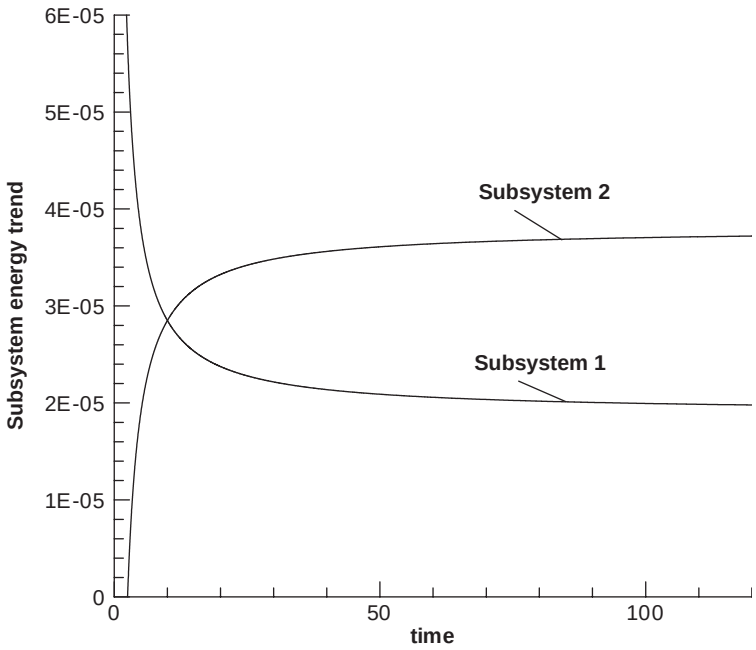


Fig. 15. Acoustic cavity: theoretical trends of the subsystems energies.

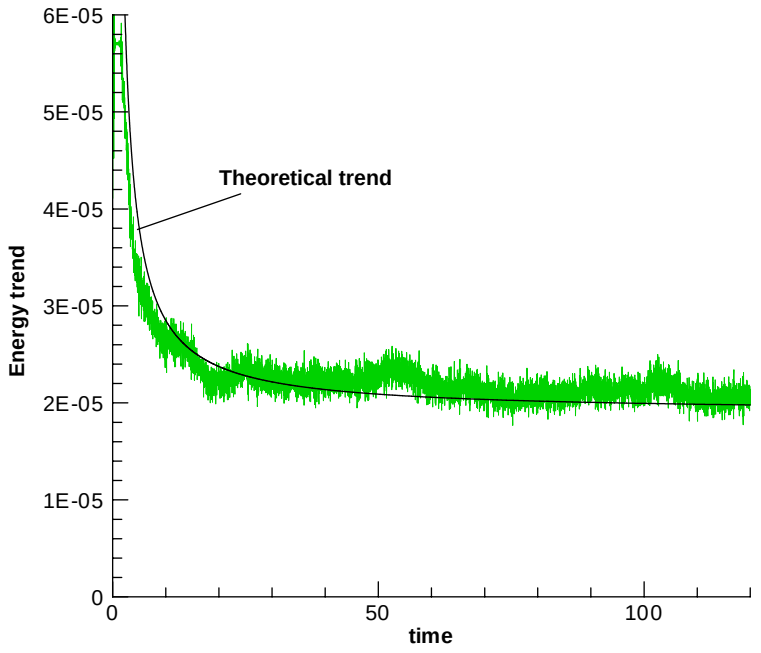


Fig. 16. Acoustic cavity: comparison between numerical trend versus theoretical trend of energy.

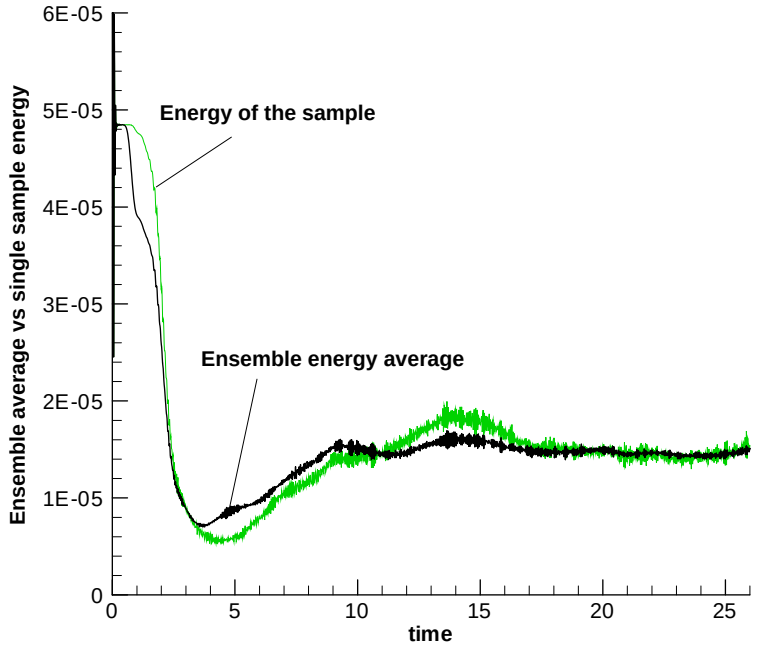


Fig. 17. Acoustic cavity: average energy versus single case energy—random length of the partitioning panel.

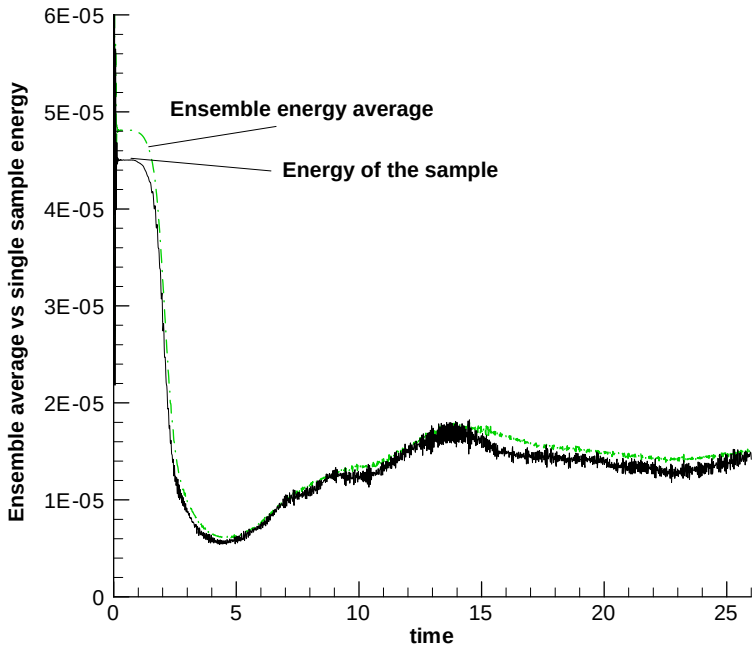


Fig. 18. Acoustic cavity: average energy versus single case energy—random speed of sound.

analogous comparison is shown in Fig. 18 when varying the speed of sound (average is 1 m/s) uniformly distributed in the range [0.9 m;1.1 m]. Again, these results confirm the validity of the results of Section 2.4.

In order to validate Eqs. (22) and (23) expressing a quadratic energy flow relationship, the direct correlation between the energy flow $\varphi = -\mathrm{d}\langle \bar{E}^{(1)} \rangle / \mathrm{d}t$ and the temperature difference $\langle \bar{E}^{(1)} \rangle / N_1 - \langle \bar{E}^{(2)} \rangle / N_2$ is studied for the acoustic cavity with opening of dimensions $1\,\mathrm{m} \times 0.25\,\mathrm{m}$. Note that, as it has been observed in many numerical experiments, when the opening is small, then the trend of the energy response is more smooth, as it is indicated also by comparing Figs. 16 (opening $1\,\mathrm{m} \times 0.25\,\mathrm{m}$) and 17 (opening $1\,\mathrm{m} \times 0.42\,\mathrm{m}$). This leads to conclude that when the energy stored into the physical coupling is smaller (in fact, the number of nodes at the boundary between the two subsystems is smaller in the case of the opening $1\,\mathrm{m} \times 0.25\,\mathrm{m}$), as specified in Sections 2 and 3, the energies provided by Eqs. (16) and (17) better match the energy trend (that is, in fact, monotonic, see Fig. 16).

The determination of $\mathrm{d}\langle \bar{E}^{(1)} \rangle / \mathrm{d}t$ would imply numerical derivation of the energy signal shown in Fig. 16. Obvious numerical pitfalls suggest a preliminary smoothing of the high frequency small fluctuations of the signal. This is made by using a moving time average producing the curves represented in Fig. 19. The resulting plot of the energy flow $\varphi$ versus the temperature difference between the two subsystems is represented in Fig. 20. The solid line is a quadratic function obtained by a least-squares fitting of the numerical data. The quadratic law represents well the numerical data, confirming the validity of Eqs. (22) and (23).

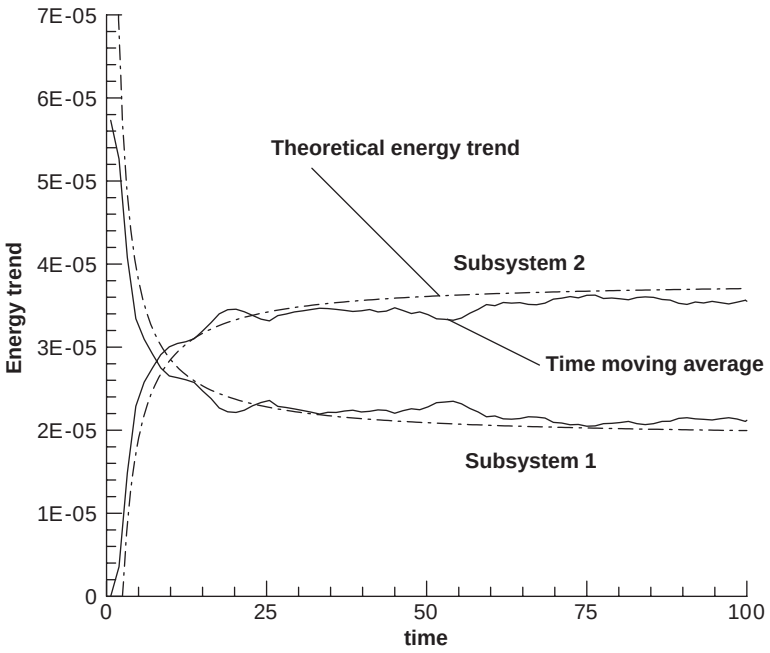


Fig. 19. Acoustic cavity: time moving average of the subsystems energies versus their theoretical trends.

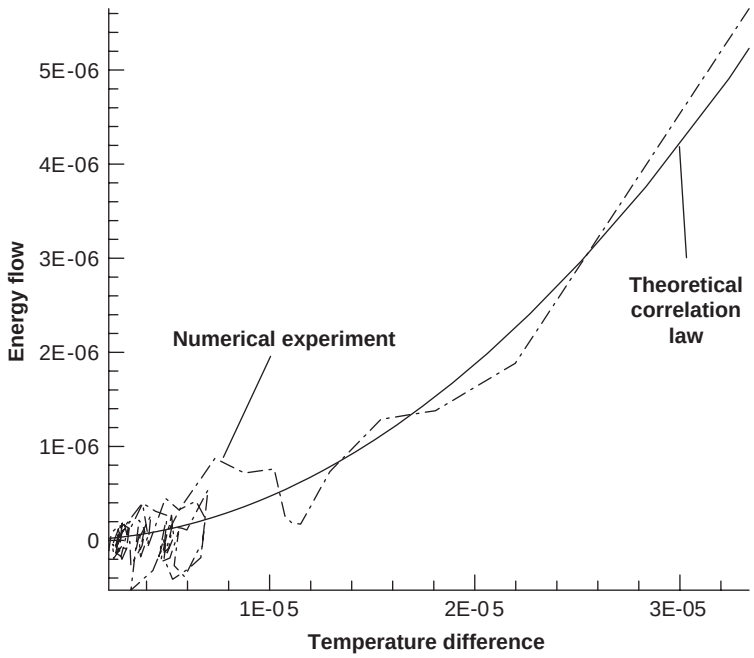


Fig. 20. Acoustic cavity: energy flow vs temperature difference between the two subsystems.

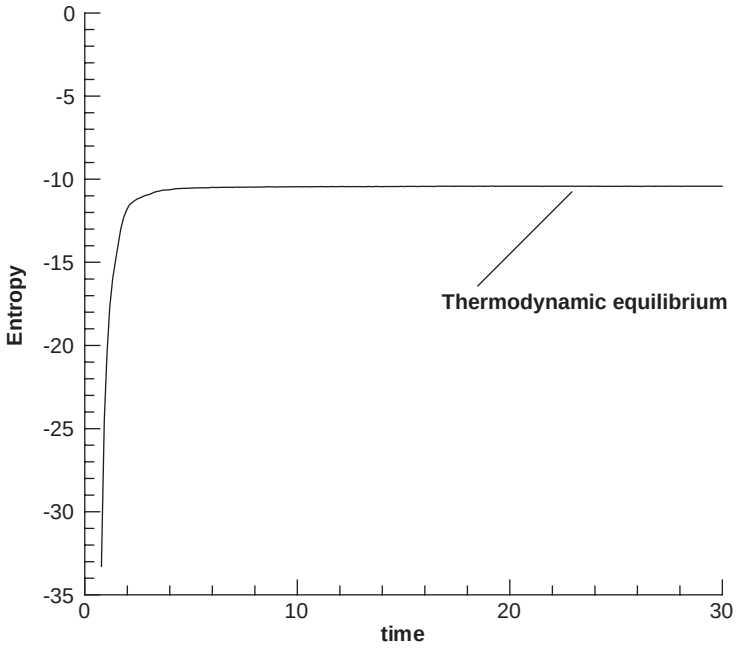


Fig. 21. Acoustic cavity: time history of the entropy of the acoustic cavity.

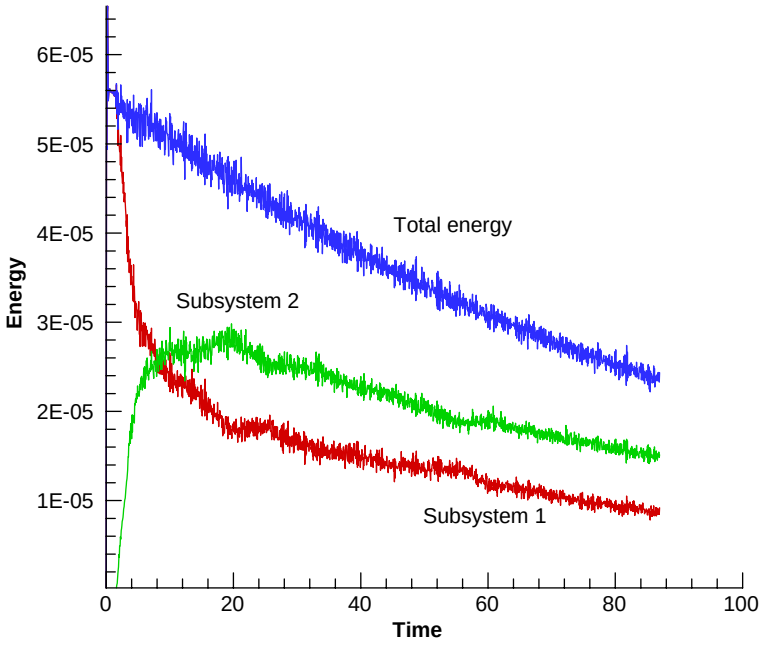


Fig. 22. Acoustic cavity with light dissipation: energy trend of the two subsystems.

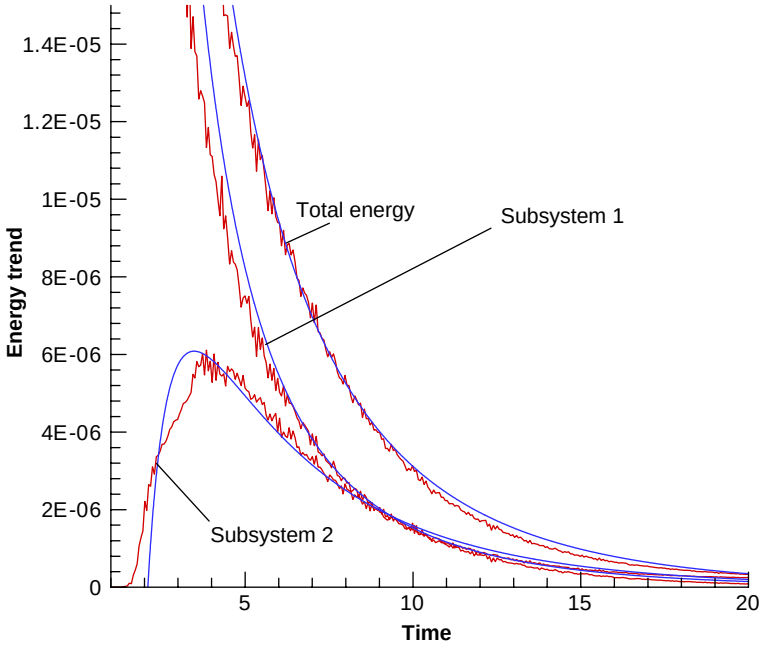


Fig. 23. Acoustic cavity with heavy dissipation: theoretical energy trend of the two subsystems (smooth curves) versus numerical simulations.

Finally, the entropy of the whole system, consisting of the two coupled cavities, is determined using definition (24) and it is represented in Fig. 21 (entropy is a nondimensional quantity). This confirm the increasing entropy trend demonstrated in Section 5. The entropy increases very rapidly till to the reaching of the equilibrium condition; after, its trend becomes rather flat.

A final verification deserves the analysis in presence of damping. In Fig. 22, the system of the coupled acoustic boxes is again considered (the same case analysed in Fig. 14) introducing a viscous-type dissipation effect. Fig. 23 shows the same simulation but with a damping effect increased by a factor six. In both cases it appears that the subsystem 2, that initially at rest, exhibits a two-fold energy trend: at early time, the effect of the energy transfer from subsystem 1 to subsystem 2 prevails with respect to the inherent damping, thus producing an energy increasing into the subsystem 2; at late time, the dissipation effect is dominant, since the energy rate transfer between the two subsystems is reduced being their energies comparable. This leads to a characteristic maximum in the energy stored into subsystem 2.

However, it is apparent that, for smaller damping at early time (see Fig. 22), the energy trend of the two subsystems is not very different with respect to that found in the case of absence of damping (see Fig. 14 for time smaller with respect to the intersection point). In fact, at this stage, the effect of the energy release from subsystem 1 towards subsystem 2 is the dominant effect, i.e. the same observed in absence of damping. The different behaviour appears only at late time, where the damping effect becomes more important being the energy sharing rate now very weak.

The case of higher damping, shown in Fig. 23, is indeed different. In fact, the large energy dissipation effect appears to be important even at early time, deeply modifying the response with respect to the case of absence of damping.

In Fig. 23 the comparison of the results of the numerical experiment with the trends given by Eqs. (27) and (28) (with coefficients determined by a best-fitting procedure and verifying the condition $\mu^{(1)}T^{(1)} = \mu^{(2)}T^{(2)}$) is shown and a satisfactory agreement is found.

## 8. Conclusions

The main results contained in this paper can be summarized as follows:

- the ensemble energy average of a population of structures is developed, when the natural frequencies are random;
- an asymptotic probabilistic expansion in time domain, shows the long-term response of the ensemble energy average of the two subsystems; they exhibit a simple law during the transient both in absence and in presence of damping;
- the complexity of the subsystems reduces the dispersion of the energy samples around the ensemble energy average: the more the complexity of the system, the less the energy deviation from the mean energy.

The first and second points relates energy and uncertainty; the third uncertainty and complexity. The combination of the three points provides a direct correlation between energy and complexity, independently of uncertainty: energy exhibits simple properties when the system is complex, i.e. in a sense, simplification in the energy analysis comes from the complexity of the system. In this case, the energy response of the single case can be predicted by using the simplified model valid for the ensemble average.

A third point concerns the implications of this analysis on the energy flow relationship: a distinction between undamped and damped systems is considered. For conservative systems, it is shown how the energy flow and the energy difference are related by a quadratic relationship, that is a rather unusual result when compared with those of the existing energy flow techniques (like, e.g. SEA).

The inclusion of damping in the analysis shows that the energy flow is indeed expressed by a linear combination of the energy difference between the two subsystems and of the time derivative of this difference.

Finally, it is shown how the found property of the average energy for complex systems leads to an analogy with the second principle of thermodynamics.

## Appendix A. Probabilistic asymptotic expansion

In this appendix, the asymptotic expansion of the time-dependent integrals appearing into Eq. (9) is derived.

Integration by parts of $\int_R \Pi_i(\omega_i)a^{(1)}(\omega_i)\cos 2\omega_i t\,\mathrm{d}\omega_i$ with respect to $\omega$ (for the sake of simplicity, index $i$ is omitted) produces the time dependence of order $t^{-1}$

$$\int_{\omega^-}^{\omega^+} \Pi(\omega)a^{(1)}(\omega)\cos\,2\omega t\,\mathrm{d}\omega = \frac{1}{2t}\left[\Pi(\omega)a^{(1)}(\omega)\sin\,2\omega t\right]\Big|_{\omega^-}^{\omega^+} - \frac{1}{2t}\int_{\omega^-}^{\omega^+}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\Pi(\omega)a^{(1)}(\omega)\right)\sin\,2\omega t\,\mathrm{d}\omega.$$

Integrating again by parts the last integral, an additional term of order $t^{-2}$ is produced

$$\begin{aligned}\int_{\omega^-}^{\omega^+} &\Pi(\omega)a^{(1)}(\omega)\cos\,2\omega t\,\mathrm{d}\omega \\ &= \frac{1}{2t}\left[\Pi(\omega)a^{(1)}(\omega)\sin\,2\omega t\right]\Big|_{\omega^-}^{\omega^+} + \frac{1}{(2t)^2}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\Pi(\omega)a^{(1)}(\omega)\right)\cos 2\omega t\right]\Big|_{\omega^-}^{\omega^+} \\ &\quad - \frac{1}{(2t)^2}\int_{\omega^-}^{\omega^+}\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\Pi(\omega)a^{(1)}(\omega)\right)\cos\,2\omega t\,\mathrm{d}\omega.\end{aligned}$$

Integration by parts can be recursively iterated, thus generating an infinite series of terms of order $t^{-1}, t^{-2}, t^{-3}, \ldots$ etc. Therefore:

$$\int_{\omega^-}^{\omega^+} \Pi(\omega)a^{(1)}(\omega)\cos\,2\omega t\,\mathrm{d}\omega = \frac{1}{2t}\left[\Pi(\omega)a^{(1)}(\omega)\sin\,2\omega t\right]\Big|_{\omega^-}^{\omega^+} + \frac{1}{(2t)^2}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\Pi(\omega)a^{(1)}(\omega)\right)\cos\,2\omega t\right]\Big|_{\omega^-}^{\omega^+} + o\left(t^{-2}\right). \qquad \text{(A.1)}$$

On the other hand, integration by parts of the second time-dependent integral in Eq. (9)

$$\int_{R^2}\Pi_{ij}\left(\omega_i,\omega_j\right)\left[a^{(1)}\left(\omega_i,\omega_j\right)\cos\left(\omega_i+\omega_j\right)t + b^{(1)}\left(\omega_i,\omega_j\right)\cos\left(\omega_i-\omega_j\right)t\right]\mathrm{d}\omega_i\,\mathrm{d}\omega_j,$$

where $i\neq j$, generates indeed only the term of order $t^{-2}$ and higher-order terms. In fact, letting $f\left(\omega_i,\omega_j\right) = \Pi_{ij}\left(\omega_i,\omega_j\right)a_{ij}^{(1)}\left(\omega_i,\omega_j\right)$ or, analogously, $f\left(\omega_i,\omega_j\right) = \Pi_{ij}\left(\omega_i,\omega_j\right)b_{ij}^{(1)}\left(\omega_i,\omega_j\right)$, the previous integral consists of terms of the form

$$\int_{R^2} f\left(\omega_i,\omega_j\right)\cos\left(\omega_i\pm\omega_j\right)t\,\mathrm{d}\omega_i\,\mathrm{d}\omega_j$$

or

$$\int_{R^2} f\left(\omega_i,\omega_j\right)\cos\left(\omega_i\pm\omega_j\right)t\,\mathrm{d}\omega_i\,\mathrm{d}\omega_j = \int_{\omega_i^-}^{\omega_i^+}\left[\int_{\omega_j^-}^{\omega_j^+} f\left(\omega_i,\omega_j\right)\cos\left(\omega_i\pm\omega_j\right)t\,\mathrm{d}\omega_j\right]\mathrm{d}\omega_i.$$

Integrating by parts (with the same technique previously seen) it is produced

$$\int_{\omega_i^-}^{\omega_i^+}\left[\int_{\omega_i^-}^{\omega_i^+} f(\omega_i,\omega_j)\cos(\omega_i\pm\omega_j)t\,\mathrm{d}\omega_j\right]\mathrm{d}\omega_i$$
$$=\int_{\omega_i^-}^{\omega_i^+}\left[\left(\pm\frac{1}{t}\right)\left[f(\omega_i,\omega_j)\sin(\omega_i\pm\omega_j)t\right]\right]\Bigg|_{\omega_j^-}^{\omega_j^+}+o(t^{-2})\,\mathrm{d}\omega_i.$$

Neglecting the second-order term in the integrand, and integrating, again by parts, the following expression is obtained:

$$\int_{R^2} f(\omega_i,\omega_j)\cos(\omega_i\pm\omega_j)t\,\mathrm{d}\omega_i\,\mathrm{d}\omega_j=\left(\pm\frac{1}{t^2}\right)\left[\left[f(\omega_i,\omega_j)\cos(\omega_i,\pm\omega_j)t\right]\Big|_{\omega_i^-}^{\omega_i^+}\right]_{\omega_j^-}^{\omega_j^+}+o(t^{-3})$$
$$=\left(\pm\frac{1}{t^2}\right)\left\{f\left(\omega_i^+,\omega_j^-\right)\cos\left(\omega_i^+\pm\omega_j^-\right)t-f\left(\omega_i^-,\omega_j^-\right)\cos\left(\omega_i^-\pm\omega_j^-\right)t\right.$$
$$\left.-f\left(\omega_i^+,\omega_j^+\right)\cos\left(\omega_i^+\pm\omega_j^+\right)t+f\left(\omega_i^-,\omega_j^+\right)\cos\left(\omega_i^-\pm\omega_j^+\right)t\right\}+o(t^{-2}). \tag{A.2}$$

Therefore, on the basis of Eqs. (A.1) and (A.2), when retaining only the terms up to the first order (namely those arising from Eq. (A.1)), i.e. for large $t$, the asymptotic expansion (10) is proven.

Finally, let provide an estimate of how large $t$ must be in order that the approximation given by Eq. (10) be valid. Considering Eq. (A.1), the condition that allows to neglect the terms of order $t^{-1}$ with respect to those of order $t^{-1}$ is

$$\frac{1}{2t}\left|\left[\Pi(\omega_i)a^{(1)}(\omega_i)\right]_{\bar\omega_i\pm\Delta\omega_i}\right|\gg\frac{1}{(2t)^2}\left|\left[\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\Pi(\omega_i)a^{(1)}(\omega_i)\right)\right]_{\bar\omega_i\pm\Delta\omega_i}\right|,$$

where $\omega_i^+=\bar\omega_i+\Delta\omega_i$, $\omega_i^-=\bar\omega_i-\Delta\omega_i$ being the expressions evaluated for $\omega_i=\bar\omega_i\pm\Delta\omega_i$. This implies

$$t\gg\left|\frac{\left[(\mathrm{d}/\mathrm{d}w)\left(\Pi(\omega_i)a^{(1)}(\omega_i)\right)\right]_{\bar\omega\pm\Delta\omega}}{\left[\Pi(\omega_i)a^{(1)}(\omega_i)\right]_{\bar\omega\pm\Delta\omega}}\right| \tag{A.3}$$

showing that the time limit depends on the probabilistic properties of the natural frequencies.

An example of application of the found relationship is given considering a Gaussian distribution for the natural frequencies of the system. In this case, any marginal probability $\Pi(\omega_i)$derived from the joint Gaussian probability $p(\Omega)$, is still Gaussian, thus

$$\Pi(\omega_i)=\frac{1}{\sqrt{2\pi}\sigma_{\omega_i}}\mathrm{e}^{-(\omega_i-\bar\omega_i)^2/\sigma_{\omega_i}^2}.$$

Using this expression and approximately neglecting the dependency of the $a_{\mathrm{i}}^{(1)}$'s on the natural frequency (this is justified by the second of Eq. (3), by Eq. (5) and considering that the coefficients $\alpha$ and $\beta$ depends on the modeshapes and not directly on the natural

frequencies), Eq. (A.3) takes the form

$$t \gg \left| \frac{\left[(\mathrm{d}/\mathrm{d}\omega)\Pi(\omega_i)\right]_{\bar{\omega}_i \pm \Delta\omega_i}}{\left[\Pi(\omega_i)\right]_{\bar{\omega}_i \pm \Delta\omega_i}} \right| \rightarrow t \gg \frac{\Delta\omega_i}{\sigma_{\omega_i}^2}.$$

This expression details, under the hypothesis of Gaussian distribution, the limits of validity of the asymptotic probabilistic expansion presented in Section 2.2. The term $\Delta\omega_i$, that is half the integration interval used to compute the energy average, must be large enough to cover the range where $\Pi(\omega_i)$ provides a significant contribution. It means that, in general, $\Delta\omega_i$ has the order of magnitude of $\sigma_\omega$, i.e. the asymptotic expansion is valid for $t \gg 1/\sigma_{\omega_i}$.

## Appendix B. Second-order expansion

If $a^{(1)}(\omega_i) \equiv 0$, Eq. (A.1) takes the form

$$\int_{\omega^-}^{\omega^+} \Pi(\omega) a^{(1)}(\omega) \cos\, 2\omega t \, \mathrm{d}\omega = \frac{1}{(2t)^2} \left[ \frac{\mathrm{d}}{\mathrm{d}\omega} \left( \Pi(\omega) a^{(1)}(\omega) \right) \cos\, 2\omega t \right] \Bigg|_{\omega^-}^{\omega^+} + o\left(t^{-2}\right),$$

where the series starts with terms of order $t^{-1}$. Therefore in this case, being the terms of order $t^{-1}$ absent, the asymptotic expansion must take into account those of order $t^{-2}$ and, besides the previous contribution, also that of Eq. (A.2) must be included. Thus, when $a^{(1)}(\omega_i) \equiv 0$, Eq. (10) modifies as:

$$\begin{aligned} \bar{E}^{(1)}(t) = {} & \sum_{i=1}^{N} \int_{\omega_i^-}^{\omega_i^+} \Pi_i(\omega_i) b^{(1)}(\omega_i) \, \mathrm{d}\omega_i + \frac{1}{(2t)^2} \sum_{i=1}^{N} \left[ \frac{\mathrm{d}}{\mathrm{d}\omega} \left( \Pi_i(\omega_i) a^{(1)}(\omega_i) \right) \cos\, 2\omega_i t \right] \Bigg|_{\omega^-}^{\omega^+} \\ & + \frac{1}{t^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left[ \left[ \Pi_{ij}(\omega_i, \omega_j) a^{(1)}(\omega_i, \omega_j) \cos(\omega_i + \omega_j) t + \Pi_{ij}(\omega_i, \omega_j) b^{(1)}(\omega_i, \omega_j) \cos(\omega_i - \omega_j) t \right] \Big|_{\omega_i^-}^{\omega_i^+} \right]_{\omega_j^-}^{\omega_j^+} + o(t^{-2}). \end{aligned}$$

An example of vanishing $a^{(1)}(\omega_i)$ is the case of a simply supported beam, or the parallelepiped acoustic cavity, when the two subsystems have identical shapes.

## Appendix C. $b^{(r)}(\omega_i)$ coefficients

Relationships (12) can be proven limiting the analysis to two coupled homogenous subsystems of the same type (cavity–cavity, beam–beam, plate–plate, etc.). As an example, let evaluate these coefficients initially for a parallelepiped acoustic cavity with rigid walls subdivided into two subsystems. (Note that using a substantially identical procedure, this demonstration applies to rods, beams, rectangular membranes and rectangular plates.)

The $i$th mode is

$$\Phi_{i(n,m,k)}(\mathbf{x}) = \sqrt{\frac{16\rho c^2}{V}} \cos\frac{\pi n x}{L_x} \cos\frac{\pi m y}{L_y} \cos\frac{\pi k z}{L_z}$$

with an associated natural frequency

$$\omega_{i(n,m,k)} = c \left[ \left( \frac{\pi n}{L_x} \right)^2 + \left( \frac{\pi m}{L_y} \right)^2 + \left( \frac{\pi k}{L_z} \right)^2 \right]^{1/2},$$

where $L_x$, $L_y$, $L_z$ and $c$ are the lengths along the coordinate axes and the speed of sound, respectively. The cavity consists of the two subsystems
$S_1 \equiv \left[x \in (0, L_x^{(1)}),\ y \in (0, L_y),\ z \in (0, L_z)\right]$ and $S_2 \equiv \left[x \in (L_x^{(1)},\ L_x - L_x^{(1)}),\ y \in (0, L_y),\ z \in (0, L_z)\right]$.

After some mathematics it is produced

$$\alpha_{ii}^{(1)} = \frac{1}{2\rho c^2}\int_{S_1} \Phi_{i(m,n,k)}^2 \,\mathrm{d}V, \quad \beta_{ii}^{(1)} = \frac{1}{2\rho}\int_{S_1} \left|\mathrm{grad}\ \Phi_{i(m,n,k)}\right|^2 \mathrm{d}V,$$

where in the acoustic case, the coefficients $\alpha_{ii}^{(1)}$ are related to the potential energy, while the coefficients $\beta_{ii}^{(1)}$ to the kinetic energy. The previous expressions lead to

$$\begin{aligned}
\alpha_{ii}^{(1)} &= \frac{1}{2\rho c^2}\int_{S_1} \Phi_{i(m,n,k)}^2 \,\mathrm{d}V = \frac{L_x^{(1)}}{L_x} + \frac{1}{\pi n}\cos\frac{\pi n L_x^{(1)}}{L_x}\sin\frac{\pi n L_x^{(1)}}{L_x},\\
\beta_{ii}^{(1)} &= \frac{1}{2\rho}\int_{S_1} \left|\mathrm{grad}\ \Phi_{i(m,n,k)}\right|^2 \mathrm{d}V = \frac{L_x^{(1)}}{L_x}c^2\left[\left(\frac{\pi n}{L_x}\right)^2 + \left(\frac{\pi m}{L_y}\right)^2 + \left(\frac{\pi k}{L_z}\right)^2\right]\\
&\quad - \frac{c^2}{\pi n}\left[\left(\frac{\pi n}{L_x}\right)^2 - \left(\frac{\pi m}{L_y}\right)^2 - \left(\frac{\pi k}{L_z}\right)^2\right].
\end{aligned}$$

If an increasing order of the mode $i$th is considered, then $n$ becomes larger and it follows

$$\begin{aligned}
\alpha_{ii}^{(1)} &\approx \frac{L_x^{(1)}}{L_x} = \frac{\rho L_x^{(1)} L_y L_z}{\rho L_x L_y L_z} = \frac{m^{(1)}}{m^{(1)} + m^{(2)}},\\
\beta_{ii}^{(1)} &\approx \frac{L_x^{(1)}}{L_x}c^2\left[\left(\frac{\pi n}{L_x}\right)^2 + \left(\frac{\pi m}{L_y}\right)^2 + \left(\frac{\pi k}{L_z}\right)^2\right] = \frac{L_x^{(1)}}{L_x}\omega_{i(n,m,k)}^2 = \frac{m^{(1)}}{m^{(1)} + m^{(2)}}\omega_{i(n,m,k)}^2,
\end{aligned} \tag{C.1}$$

$$b^{(1)}(\omega_i) = \tfrac{1}{2}\alpha_{ii}^{(1)}\left(A_i\omega_{i(n,m,k)}\right)^2,$$

that is the desired relationship.

An intuitive argument can be provided for generalize this result. In fact, it is possible to make weaker the assumption made before about the simple configuration of the cavity (parallelepiped with rigid walls), considering an acoustic cavity of different shape, with a distributed wall impedance, with scattering internal objects, providing an approximate expression for its modes $\Psi_{(n',m',k')}$ [33]. If the perturbations with respect to the original system are of order $\varepsilon$ (small), then the following expression holds at the first order

$$\Psi_i = \Phi_i + \sum_{\substack{j=1\\ j\neq i}}^{N} V_{ij}\Phi_j,$$

where $V_{ij}$ are suitable projection coefficients of order $\varepsilon$. Thus it follows:

$$\int_{S_1}\frac{\Psi_i^2}{2\rho c^2}\,\mathrm{d}V = \int_{S_1}\frac{\Phi_i^2}{2\rho c^2}\,\mathrm{d}V + 2\sum_{\substack{j=1\\ j\neq i}}^{N} V_{ij}\int_{S_1}\frac{\Phi_i\Phi_j}{2\rho c^2}\,\mathrm{d}V + \sum_{\substack{j=1,r=1\\ j\neq i, r\neq i}}^{N} V_{ij}V_{ir}\int_{S_1}\frac{\Phi_i\Phi_j}{2\rho c^2}\,\mathrm{d}V.$$

Neglecting second-order terms

$$\int_{S_1} \frac{\Psi_i^2}{2\rho c^2}\,\mathrm{d}V = \int_{S_1} \frac{\Phi_i^2}{2\rho c^2}\,\mathrm{d}V + 2\sum_{\substack{j=1\\ j\neq i}}^{N} V_{ij} \int_{S_1} \frac{\Phi_i \Phi_j}{2\rho c^2}\,\mathrm{d}V.$$

It is clear also that, increasing $i-j$, very rapidly

$$\int_{S_1} \frac{\Phi_i^2}{2\rho c^2}\,\mathrm{d}V \gg \int_{S_1} \frac{\Phi_i \Phi_j}{2\rho c^2}\,\mathrm{d}V, \quad i \neq j,$$

because the second integral and $V_{ij}$ drop off in value very fast [33]. Thus:

$$\alpha_{ii}^{(1)} = \int_{S_1} \frac{\Psi_i^2}{2\rho c^2}\,\mathrm{d}V \approx \int_{S_1} \frac{\Phi_i^2}{2\rho c^2}\,\mathrm{d}V \approx \frac{m^{(1)}}{m^{(1)}+m^{(2)}}.$$

Since

$$\alpha_{ii}^{(1)} = \tilde{\omega}_i^2 \int_{S_1} \frac{\Psi_i^2}{2\rho c^2}\,\mathrm{d}V,$$

where $\tilde{\omega}_i^2$ is the natural frequency associated to $\Psi_i$, is proportional to the space average potential energy, while

$$\beta_{ii}^{(1)} = \frac{1}{2\rho} \int_{S_1} \left|\mathrm{grad}\,\Phi_i\right|^2 \mathrm{d}V$$

is indeed proportional to the space average kinetic energy, it follows $\alpha_{ii}^{(1)}\tilde{\omega}_i^2 \approx \beta_{ii}^{(1)}$. Therefore, when increasing the order of the $i$th mode, very rapidly the following equality is approached:

$$b^{(1)}(\tilde{\omega}_i) \approx \tfrac{1}{2}\alpha_{ii}^{(1)}(A_i\tilde{\omega}_i)^2.$$

Correspondingly, for the previous arguments, the coefficients $a^{(1)}(\tilde{\omega}_i)$ vanish very rapidly as $i$ increases.

## Appendix D. Power flow expression in presence of damping

Let us produce an explicit form of the terms of the balance

$$\frac{\mathrm{d}\left\langle \bar{E}^{(1)}\right\rangle}{\mathrm{d}t} = P_{\mathrm{diss}}^{(1)} - \varphi, \quad \frac{\mathrm{d}\left\langle \bar{E}^{(2)}\right\rangle}{\mathrm{d}t} = P_{\mathrm{diss}}^{(2)} + \varphi. \tag{D.1}$$

From Eq. (27)

$$\begin{aligned}
\frac{\mathrm{d}\left\langle \bar{E}^{(1)}\right\rangle}{\mathrm{d}t} &= -\delta\omega\mu^{(1)}\frac{\mathrm{e}^{-\delta\omega t}}{t} - \mu^{(1)}\frac{\mathrm{e}^{-\delta\omega t}}{t^2} - \delta\omega\mu^{(1)}T^{(1)}\frac{\mathrm{e}^{-\delta\omega t}}{t^2} - 2\mu^{(1)}T^{(1)}\frac{\mathrm{e}^{-\delta\omega t}}{t^3},\\
\frac{\mathrm{d}\left\langle \bar{E}^{(2)}\right\rangle}{\mathrm{d}t} &= -\delta\omega\mu^{(2)}\frac{\mathrm{e}^{-\delta\omega t}}{t} - \mu^{(2)}\frac{\mathrm{e}^{-\delta\omega t}}{t^2} + \delta\omega\mu^{(2)}T^{(2)}\frac{\mathrm{e}^{-\delta\omega t}}{t^2} + 2\mu^{(2)}T^{(2)}\frac{\mathrm{e}^{-\delta\omega t}}{t^3}.
\end{aligned} \tag{D.2}$$

From Eqs. (30) and (27)

$$P_{\text{diss}}^{(1)} = -\delta\omega\langle E^{(1)}\rangle - \frac{\langle E^{(1)}\rangle}{t} = -\delta\omega\mu^{(1)}\frac{e^{-\delta\omega t}}{t} - \delta\omega\mu^{(1)}T^{(1)}\frac{e^{-\delta\omega t}}{t^2} - \mu^{(1)}\frac{e^{-\delta\omega t}}{t^2} - \mu^{(1)}T^{(1)}\frac{e^{-\delta\omega t}}{t^3},$$

$$P_{\text{diss}}^{(2)} = -\delta\omega\langle E^{(2)}\rangle - \frac{\langle E^{(2)}\rangle}{t} = -\delta\omega\mu^{(2)}\frac{e^{-\delta\omega t}}{t} + \delta\omega\mu^{(2)}T^{(2)}\frac{e^{-\delta\omega t}}{t^2} - \mu^{(2)}\frac{e^{-\delta\omega t}}{t^2} + \mu^{(2)}T^{(2)}\frac{e^{-\delta\omega t}}{t^3}. \quad \text{(D.3)}$$

Introducing Eqs. (D.2) and (D.3) into Eq. (D.1) the expression of the power flow is obtained:

$$\varphi = \mu^{(1)}T^{(1)}\frac{e^{-\delta\omega t}}{t^3} \quad \text{and} \quad \varphi = \mu^{(2)}T^{(2)}\frac{e^{-\delta\omega t}}{t^3}. \quad \text{(D.4)}$$

These expressions, because of the first of Eq. (28), provide the same result.

$$\frac{d}{dt}\left[\frac{\langle \bar{E}^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle \bar{E}^{(2)}\rangle}{\mu^{(2)}}\right] = -\delta\omega(T^{(1)} + T^{(2)})\frac{e^{-\delta\omega t}}{t^2} - 2(T^{(1)} + T^{(2)})\frac{e^{-\delta\omega t}}{t^3}.$$

From Eq. (27)

$$\frac{\langle \bar{E}^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle \bar{E}^{(2)}\rangle}{\mu^{(2)}} = (T^{(1)} + T^{(2)})\frac{e^{-\delta\omega t}}{t^2}. \quad \text{(D.5)}$$

Derivating:

$$\frac{d}{dt}\left[\frac{\langle \bar{E}^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle \bar{E}^{(2)}\rangle}{\mu^{(2)}}\right] = -\delta\omega(T^{(1)} + T^{(2)})\frac{e^{-\delta\omega t}}{t^2} - 2(T^{(1)} + T^{(2)})\frac{e^{-\delta\omega t}}{t^3}. \quad \text{(D.6)}$$

Substituting Eq. (D.5) into Eq. (D.6) it is produced

$$\delta\omega\left[\frac{\langle \bar{E}^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle \bar{E}^{(2)}\rangle}{\mu^{(2)}}\right] + \frac{d}{dt}\left[\frac{\langle \bar{E}^{(1)}\rangle}{\mu^{(1)}} - \frac{\langle \bar{E}^{(2)}\rangle}{\mu^{(2)}}\right] = -2(T^{(1)} + T^{(2)})\frac{e^{-\delta\omega t}}{t^3}$$

and considering Eq. (D.4), the result of Eq. (32) is proven.

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