Estimation of the Position and Time of Emission of a Source

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Abstract

We consider the problem of the estimation of the position and of the moment of beginning of emission of a source by observations from K detectors on the plane. We propose the conditions of identifiability and construct a linear estimator of unknown parameters. Then we verify the consistency and describe the limit distribution of this estimator.

MSC 2000 Classification: 62M02, 62G10, 62G20.

Key words: *Parameter estimation, inhomogeneous Poisson processes, localization of a source, asymptotic properties.*

1 Introduction

Consider the following model of observations: we have K detectors located on the plane at the points $\mathbb{D}_k$, $k = 1, \ldots, K$, with coordinates $D_k = (x_k, y_k)$, $k = 1, \ldots, K$, respectively. At some point $\mathbb{S}_0$ with coordinates $D_0 = (x_0, y_0)$ we have a source which starts emission at the moment τ_0.

We suppose that the values x_0, y_0 and τ_0 are unknown and we want to estimate the parameter $\vartheta_0 = (x_0, y_0, \tau_0)$ by the observations of K independent stochastic processes $X^T = (X_1^T, \ldots, X_K^T)$, where $X_k^T = \left(X_k(t),\ 0 \leq t \leq T\right)$, $k = 1, \ldots, K$, are the observations recorded by the k-th detector.

There are many real problems which can be described with the help of such models. An example of such configuration with 5 detectors and a source is given in Figure 1 below.

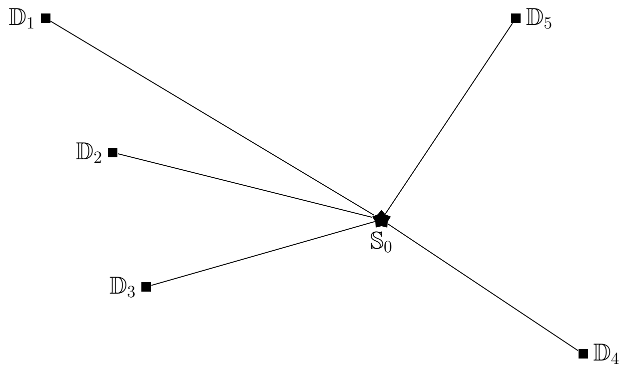

Figure 1: Observation model: $\mathbb{S}_0$ is the source, $\mathbb{D}_1, \ldots, \mathbb{D}_5$ are the detectors

For example, if we have a radioactive source at (x_0, y_0) which starts its emission at the moment τ_0, then the k-th detector receives an inhomogeneous Poisson process $X_k^T = \left(X_k(t),\ 0 \leq t \leq T\right)$ with intensity function

$$\lambda_k(\vartheta_0, t) = S_k(t - \tau_k) + \lambda_0, \qquad 0 \leq t \leq T. \tag{1}$$

Here $\lambda_0 > 0$ is the intensity of the noise, and for $k = 1, \ldots, K$, the "signals" $S_k(t) = 0$ for $t < 0$ and $S_k(t) > 0$ for $t > 0$, and the parameter τ_k is the moment of arrival of the signal at the k-th detector. We have

$$\tau_k = \tau_k(\vartheta_0) = \tau_0 + \nu^{-1}\|D_k - D_0\|_2, \tag{2}$$

where $\nu > 0$ is the rate of propagation of the signals and $\|\cdot\|_2$ is the Euclidean distance in $\mathcal{R}^2$. The functions $S_k(\cdot)$, $k = 1, \ldots, K$, positions D_k, $k = 1, \ldots, K$, the parameters ν and λ_0 are supposed to be known, and the main problem is to estimate $\vartheta_0 = (x_0, y_0, \tau_0)$ by the observations X^T.

A similar statistical problem can be considered in the case where the observations $X_k^T = \left(X_k(t),\ 0 \leq t \leq T\right)$ recorded by the k-th detector are Gaussian processes

$$\mathrm{d}X_k(t) = S_k(t - \tau_k)Y_k(t)\mathrm{d}t + \varepsilon\sigma_k(t)\mathrm{d}W_k(t), \quad X_k(0) = 0, \quad 0 \leq t \leq T, \tag{3}$$

where $\varepsilon \in (0, 1]$ is the level of noise, $S_k(t) = 0$ for $t < 0$, and $W_k(\cdot)$ are independent Wiener processes. The moment of arriving τ_k is still defined by the relation (2), and the (hidden Markov) stochastic processes $Y_k(\cdot)$ satisfy the linear equations

$$\mathrm{d}Y_k(t) = f_k(t)Y_k(t)\mathrm{d}t + \varepsilon b_k(t)\mathrm{d}V_k(t), \quad Y_k(0) = y_0 \neq 0, \quad 0 \leq t \leq T.$$

Of course, there are many other stochastic models (different time series, diffusion processes, *etc.*) for which similar problems can be considered.

We are interested in the situations where the estimation of ϑ_0 is possible with small errors (consistent estimation). Therefore we have to introduce one or another type of asymptotics. In the case of inhomogeneous Poisson processes, the intensities (1) can be replaced by the intensities

$$\lambda_{k,n}(\vartheta_0, t) = nS_k\big(t - \tau_k(\vartheta_0)\big) + n\lambda_0, \qquad 0 \le t \le T, \tag{4}$$

where $n \to \infty$, *i.e.*, we have *asymptotics of large signals.*

If the observations are given by (3), then the consistent estimation of ϑ_0 will be possible if we consider the *asymptotics of small noise* $\varepsilon \to 0$.

The same mathematical models can be used in the following, somewhat inverse, problem. Suppose that we have K emitters $\mathbb{D}_1, \dots, \mathbb{D}_K$ and a receiver $\mathbb{S}_0$ which records stochastic processes $X^T = (X_1^T, \dots, X_K^T)$. If the emitters send weak optical signals, then $X_k^T = \big(X_k(t),\ 0 \le t \le T\big)$ are inhomogeneous Poisson processes with intensity functions (1), and the delays τ_k are defined by (2). We have the same problem: how to estimate ϑ_0 by the observations X^T? For example, if τ_0 is known, then we are in a situation close to the GPS-localization problem.

Of course, if $\mathbb{S}_0$ receives the Gaussian signals (3) and τ_0 is known, then once more we are in the framework of GPS-localization problem.

The both statistical statements of the problems are evidently of essential importance for applications. The engineers are already working with such models during decades but the detailed mathematical and statistical study seems not to be well developed. We can mention here the works [2],[6],[10],[11],[12] and references therein.

This work is a continuation of the study initiated in [1, 4, 5, 7, 9]. In all these works except [1], it is supposed that τ_0 is known and we have to estimate $\vartheta_0 = D_0 = (x_0, y_0)$. Let us remind some of the results obtained in these papers.

In the work [5], the observed processes are inhomogeneous Poisson processes with the intensity functions (4). The functions $S_k(\cdot)$ are supposed to be sufficiently smooth and this allow to verify that the maximum likelihood estimator (MLE) $\hat\vartheta_n$ and the Bayesian estimators (BEs) $\tilde\vartheta_n$ are consistent, asymptotically normal:

$$\sqrt{n}\big(\hat\vartheta_n - \vartheta_0\big) \Longrightarrow \mathcal{N}\big(0, \mathrm{I}(\vartheta_0)^{-1}\big) \quad \text{and} \quad \sqrt{n}\big(\tilde\vartheta_n - \vartheta_0\big) \Longrightarrow \mathcal{N}\big(0, \mathrm{I}(\vartheta_0)^{-1}\big),$$

and we also have the convergence of polynomial moments and the asymptotic efficiency of both the estimators. Here $\mathrm{I}(\vartheta_0)$ is the Fisher information matrix.

In the work [4], it was supposed that the front of the signals has cusp-type singularity in 0, *i.e.*, $S_k(t) = a|t|^\kappa \mathbb{1}_{\{t\geq 0\}} + o(t)$, where $\kappa \in (0, 1/2)$. It was shown that the BEs are consistent, converge in distribution:

$$n^{\frac{1}{2\kappa+1}}\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \zeta_1,$$

we have the convergence of polynomial moments, and the BEs are asymptotically efficient. Here ζ_1 is some random vector defined with the help of the fBm.

The case of signals with change-point type singularity in 0 (discontinuous intensity), *i.e.*, $S_k(t) = a\mathbb{1}_{\{t\geq 0\}} + o(t)$, was studied in the work [7]. It was proved that the BEs are consistent, converge in distribution:

$$n\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \zeta_2,$$

we have the convergence of polynomial moments and the BEs are asymptotically efficient. Here ζ_2 is some random vector defined with the help of two homogeneous Poisson processes.

All three convergences can be joint in the following writing:

$$\tilde\vartheta_n = \vartheta_0 + \varphi_n\zeta_n, \qquad \varphi_n \to 0, \qquad \zeta_n \Longrightarrow \zeta,$$

with different rates $\varphi_n \to 0$ and limit distributions ζ.

The work [9] is also devoted to the problem of estimation of $\vartheta_0 = (x_0, y_0)$, but the observations are now given by (3). The properties of the MLE $\hat\vartheta_\varepsilon$ and of the BEs $\tilde\vartheta_\varepsilon$ are again studied in the three cases (similar to those above) in the asymptotic $\varepsilon \to 0$. These estimators have three different rates of convergence too and their limit distributions can be written in symbolic way as follows:

$$\tilde\vartheta_\varepsilon = \vartheta_0 + \varphi_\varepsilon\zeta_\varepsilon, \qquad \varphi_\varepsilon \to 0, \qquad \zeta_\varepsilon \Longrightarrow \zeta.$$

There are two approach for estimation $\vartheta_0 = (x_0, y_0, \tau_0)$. One is to collect all the data X^T and, using this data, calculate the likelihood ratio and then the MLE and the BEs of ϑ_0. This approach was used in the works [5, 4, 7, 9]. Another approach can be described as follows. Suppose that on the base of the observations recorded by each detector, we first estimate the moments $\tau_k(\vartheta_0)$, $k = 1, \ldots, K$, of arriving of the signals at the detectors, and then the obtained estimators, say $\tau_{1,n}, \ldots, \tau_{K,n}$, are transmitted to the center of data treatment, where using these values the estimator of ϑ_0 is constructed. Note that all the information needed for the consistent estimation of ϑ_0 is contained in the moments of arriving of the signals.

The study of estimators in the case of unknown τ_0 and unknown position $D_0 = (x_0, y_0)$ was initiated in the work [1], where the vector $\vartheta_0 = (x_0, y_0, \tau_0)$

was estimated with the help of the second approach and the least squares estimator (LSE). The equation obtained there for the estimator of ϑ_0 was nonlinear, and in order to reduce it to a linear system, the parameter space was extended introducing one more component as follows. Introduce an unknown vector $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, where

$$\gamma_1 = x_0, \qquad \gamma_2 = y_0, \qquad \gamma_3 = \tau_0, \qquad \gamma_4 = \frac{1}{2}\bigl(x_0^2 + y_0^2 - \nu^2\tau_0^2\bigr).$$

Now the equation for γ is linear and the LSE can be easily calculated and studied. The conditions proposed there allowed to prove the consistency of this estimator.

For simplicity of exposition we consider below the Gaussian model of observations.

2 Estimation of ϑ_0: first approach

Remind the model of observations. We have K detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$ located on the plane at the points $D_1, \ldots, D_K$, where $D_k = (x_k, y_k)$, and a source $\mathbb{S}_0$ located at the point $D_0 = (x_0, y_0)$. The source $\mathbb{S}_0$ starts emitting signals at some unknown moment τ_0, and the detectors receive signals $S_k\bigl(t - \tau_k(\vartheta_0)\bigr)$ at the moments $\tau_1(\vartheta_0), \ldots, \tau_K(\vartheta_0)$ in presence of white Gaussian noise. The observations are $X^T = (X_1^T, \ldots, X_K^T)$, where $X_k^T = \bigl(X_k(t),\ 0 \le t \le T\bigr)$ and

$$\mathrm{d}X_k(t) = S_k\bigl(t - \tau_k(\vartheta_0)\bigr)\mathrm{d}t + \varepsilon\sigma_k(t)\mathrm{d}W_k(t), \quad X_k(0) = 0, \quad 0 \le t \le T. \quad (5)$$

Here $\vartheta_0 = (x_0, y_0, \tau_0)$ and $\tau_k(\vartheta_0)$ is defined by the relation (2). In our statement of the problem we suppose that during the observation time $[0, T]$ all the detectors receive signals.

We suppose that the functions $S_k(\cdot)$ and $\sigma_k(\cdot)$, the positions $D_1, \ldots, D_K$ of the detectors and the signal propagation rate $\nu > 0$ are known. The region Θ of possible values of ϑ_0 is also known, but its definition needs a special study, which will be discussed later in this work. The set Θ is open, bounded and convex subset of $\mathcal{R}^3$. Here we note that for a given configuration of detectors $D_1, \ldots, D_K$, the region $\mathcal{D}_0$ of possible values of $D_0 = (x_0, y_0)$ and the interval $\mathcal{T}_0$ of possible values of τ_0 are related. Let us denote

$$r_m = \min_{k=1,\ldots,K} \inf_{D_0 \in \mathcal{D}_0} \|D_k - D_0\|_2, \qquad r_M = \max_{k=1,\ldots,K} \sup_{D_0 \in \mathcal{D}_0} \|D_k - D_0\|_2,$$

$\tau_m = \nu^{-1} r_m$ and $\tau_M = \nu^{-1} r_M$. Then we obviously have $\mathcal{T}_0 \subset (-\tau_m, T - \tau_M)$.

We have to estimate $\vartheta_0 \in \Theta$ by the observations X^T. The signals are supposed to be bounded and the functions $\sigma_k(\cdot)$ separated from 0. Therefore, the corresponding measures are equivalent and the likelihood ratio function is given by

$$L\big(\vartheta,X^T\big)=\exp\Bigg\{\sum_{k=1}^{K}\int_{\tau_k(\vartheta)\big)}^{T}\frac{S_k\big(t-\tau_k(\vartheta)\big)}{\varepsilon^2\sigma_k^2(t)}\,\mathrm{d}X_k(t)\\ -\sum_{k=1}^{K}\int_{\tau_k(\vartheta)\big)}^{T}\frac{S_k\big(t-\tau_k(\vartheta)\big)^2}{2\varepsilon^2\sigma_k(t)^2}\,\mathrm{d}t\Bigg\},\qquad \vartheta\in\Theta.$$

The MLE $\hat\vartheta_\varepsilon$ is defined by the relation

$$L\big(\hat\vartheta_\varepsilon,X^T\big)=\sup_{\vartheta\in\Theta}L\big(\vartheta,X^T\big).$$

If we suppose that ϑ_0 is a random vector with prior density $p(\vartheta)$, $\vartheta\in\Theta$, then the BE for quadratic loss function is the conditional expectation

$$\tilde\vartheta_\varepsilon=\int_\Theta\vartheta\,p\big(\vartheta\,\big|\,X^T\big)\,\mathrm{d}\vartheta,\qquad p\big(\vartheta\,\big|\,X^T\big)=\frac{p(\vartheta)L\big(\vartheta,X^T\big)}{\int_\Theta p(\vartheta)L\big(\vartheta,X^T\big)\,\mathrm{d}\vartheta}\,.$$

We suppose that the function $p(\cdot)$ is continuous and strictly positive on Θ.

Let us denote $\|\cdot\|_3$ the Euclidian norm in $\mathcal{R}^3$ and put

$$G(\vartheta,\vartheta_0)=\sum_{k=1}^{K}\int_{\tau_k(\vartheta)\wedge\tau_k(\vartheta_0)}^{T}\big[S_k\big(t-\tau_k(\vartheta)\big)-S_k\big(t-\tau_k(\vartheta_0)\big)\big]^2\sigma_k(t)^{-2}\,\mathrm{d}t,$$

$$g_{\mathbb{K}}(\mu)=\inf_{\vartheta_0\in\mathbb{K}}\ \inf_{\|\vartheta-\vartheta_0\|_3>\mu}G(\vartheta,\vartheta_0).$$

Introduce the following condition.

I. (***Identifiability***) *For any compact $\mathbb{K}\subset\Theta$ and any $\mu>0$ we have*

$$g_{\mathbb{K}}(\mu)>0. \tag{6}$$

This is, in some sense, the main condition in this work and we will discuss the sufficient conditions providing (6) later.

We suppose as well that the following condition is fulfilled too.

R. (***Regularity***) *The functions $S_k(\cdot)$, $k=1,\ldots,K$ have two continuous derivatives, $S_k(t)=0$ for $t\leq 0$ and $S_k(t)\neq 0$ for $t>0$, $k=1,\ldots,K$.*

Note that

$$\frac{\partial\tau_k(\vartheta_0)}{\partial x_0}=-\frac{x_k-x_0}{\nu\|D_k-D_0\|}=-\frac{m_{k,x}}{\nu},\quad \frac{\partial\tau_k(\vartheta_0)}{\partial y_0}=-\frac{m_{k,y}}{\nu}\quad\text{and}\quad\frac{\partial\tau_k(\vartheta_0)}{\partial\tau_0}=1$$

with obvious notation. Here $(m_{k,x},m_{k,y})$ is a unit vector which shows the direction from the source $\mathbb{S}_0$ to the k-th detector $\mathbb{D}_k$.

We introduce as well $I_k=I_k(\vartheta)$ by

$$I_k(\vartheta)=\int_{\tau_k(\vartheta)}^{T}\frac{S_k'\big(t-\tau_k(\vartheta)\big)^2}{\nu^2\sigma_k(t)^2}\,\mathrm{d}t,\qquad k=1,\ldots,K.$$

The Fisher information matrix in our problem is

$$\mathrm{I}(\vartheta)=\begin{pmatrix}\sum_{k=1}^K m_{k,x}^2 I_k & \sum_{k=1}^K m_{k,x}m_{k,y}I_k & \nu\sum_{k=1}^K m_{k,x}I_k\\ \sum_{k=1}^K m_{k,x}m_{k,y}I_k & \sum_{k=1}^K m_{k,y}^2 I_k & \nu\sum_{k=1}^K m_{k,y}I_k\\ \nu\sum_{k=1}^K m_{k,x}I_k & \nu\sum_{k=1}^K m_{k,x}I_k & \nu^2\sum_{k=1}^K I_k\end{pmatrix}.$$

This matrix can re-written as follows. Let us denote $\mathcal{R}_*^K$ the space of K dimensional vectors $\mathbf{a}=(a_1,\ldots,a_K)^\top$ with the scalar product and the corresponding norm defined by the relations

$$\langle\mathbf{a},\mathbf{b}\rangle_*=\sum_{k=1}^K a_k b_k I_k,\qquad \|\mathbf{a}\|_*=\sum_{k=1}^K a_k^2 I_k.$$

Then using the vectors $\mathbf{m}_x=(m_{1,x},\ldots,m_{K,x})^\top$, $\mathbf{m}_y=(m_{1,y},\ldots,m_{K,y})^\top$ and $\mathbf{n}=(\nu,\ldots,\nu)^\top$, we can write

$$\mathrm{I}(\vartheta)=\begin{pmatrix}\|\mathbf{m}_x\|_*^2 & \langle\mathbf{m}_x,\mathbf{m}_y\rangle_* & \langle\mathbf{m}_x,\mathbf{n}\rangle_*\\ \langle\mathbf{m}_x,\mathbf{m}_y\rangle_* & \|\mathbf{m}_y\|_*^2 & \langle\mathbf{m}_y,\mathbf{n}\rangle_*\\ \langle\mathbf{m}_x,\mathbf{n}\rangle_* & \langle\mathbf{m}_y,\mathbf{n}\rangle_* & \|\mathbf{n}\|_*^2\end{pmatrix}.$$

Hence the Fisher information matrix is the Gram matrix. Recall that here $\mathbf{m}_x=\mathbf{m}_x(\vartheta)$, $\mathbf{m}_y=\mathbf{m}_y(\vartheta)$ and $I_k=I_k(\vartheta)$.

The next is condition is the following.

N. (*Non degeneracy*) *The Fisher information matrix is uniformly non degenerate:*

$$\inf_{\vartheta\in\Theta}\ \inf_{e\,:\,\|e\|_3=1} e^\top\mathrm{I}(\vartheta)e>0. \tag{7}$$

As it is proved below, the family of measures which corresponds to our statistical experiment is locally asymptotically normal, and therefore we have the Hajek-Le Cam lower bound: for any estimator $\bar\vartheta_\varepsilon$, it holds

$$\lim_{\delta\to0}\ \varliminf_{\varepsilon\to0}\ \sup_{\|\vartheta-\vartheta_0\|_3\le\delta}\varepsilon^{-2}\,\mathbf{E}_\vartheta\|\bar\vartheta_\varepsilon-\vartheta\|_3^2\ge\mathbf{E}_{\vartheta_0}\|\zeta\|_3^2,$$

where $\zeta \sim \mathcal{N}\bigl(0, \mathrm{I}(\vartheta_0)^{-1}\bigr)$ (see, *e.g.*, [8]). We call an estimator ϑ^*_ε asymptotically efficient if for all $\vartheta_0 \in \Theta$ we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\sup_{\|\vartheta-\vartheta_0\|_3\le\delta}\varepsilon^{-2}\,\mathbf{E}_\vartheta\|\vartheta^*_\varepsilon-\vartheta\|_3^2=\mathbf{E}_{\vartheta_0}\|\zeta\|_3^2,$$

Theorem 1. *Let the conditions* **I**, **R** *and* **N** *be fulfilled. Then the MLE* $\hat\vartheta_\varepsilon$ *and the BEs* $\tilde\vartheta_\varepsilon$ *are uniformly consistent, uniformly on compacts asymptotically normal:*

$$\frac{\hat\vartheta_\varepsilon-\vartheta_0}{\varepsilon}\Longrightarrow\zeta\quad\text{and}\quad\frac{\tilde\vartheta_\varepsilon-\vartheta_0}{\varepsilon}\Longrightarrow\zeta,\quad\zeta\sim\mathcal{N}\bigl(0,\mathrm{I}(\vartheta_0)^{-1}\bigr),$$

we have the uniform on compacts convergence of polynomial moments: for any $p>0$*, it holds*

$$\varepsilon^{-p}\,\mathbf{E}_{\vartheta_0}\|\hat\vartheta_\varepsilon-\vartheta_0\|_3^p\longrightarrow\mathbf{E}_{\vartheta_0}\|\zeta\|_3^p\quad\text{and}\quad\varepsilon^{-p}\,\mathbf{E}_{\vartheta_0}\|\tilde\vartheta_\varepsilon-\vartheta_0\|_3^p\longrightarrow\mathbf{E}_{\vartheta_0}\|\zeta\|_3^p,$$

and both the estimators are asymptotically efficient.

Proof. The desired properties of the estimators follow from Theorem 3.5.1 of [8] if we check the conditions 1–3 of this theorem. The conditions 1 and 2 for our model are obviously fulfilled. In order to verify the condition 3, we change the variables $\vartheta=\vartheta_0+\varepsilon u$, $u\in\mathbb{U}_\varepsilon=\{u\,:\,\vartheta_0+\varepsilon u\in\Theta\}$. Remark that this statistical experiment is regular [8]. The normalized likelihood ratio

$$Z_\varepsilon(u)=\frac{L\bigl(\vartheta_0+\varepsilon u,X^T\bigr)}{L\bigl(\vartheta_0,X^T\bigr)},\qquad u\in\mathbb{U}_\varepsilon,$$

admits the representation

$$Z_\varepsilon(u)=\exp\biggl\{u^\top\Delta_\varepsilon\bigl(\vartheta_0,X^T\bigr)-\frac{1}{2}\,u^\top\mathrm{I}(\vartheta_0)u+r_\varepsilon\biggr\},$$

where $r_\varepsilon\xrightarrow{\mathbf{P}}0$ and

$$\begin{aligned}\Delta_\varepsilon\bigl(\vartheta_0,X^T\bigr)&=\sum_{k=1}^K\int_{\tau_k(\vartheta_0)}^T\frac{\dot S_k\bigl(t-\tau_k(\vartheta_0)\bigr)}{\varepsilon\sigma_k(t)^2}\,\bigl[\mathrm{d}X_k(t)-S_k\bigl(t-\tau_k(\vartheta_0)\bigr)\mathrm{d}t\bigr]\\&=\sum_{k=1}^K\int_{\tau_k(\vartheta_0)}^T\frac{\dot S_k\bigl(t-\tau_k(\vartheta_0)\bigr)}{\sigma_k(t)}\,\mathrm{d}W_k(t)\ \sim\ \mathcal{N}\bigl(0,\mathrm{I}(\vartheta_0)\bigr).\end{aligned}$$

Here we denoted

$$\dot S_k\bigl(t-\tau_k(\vartheta_0)\bigr)=\frac{\partial S_k\bigl(t-\tau_k(\vartheta_0)\bigr)}{\partial\vartheta_0}\,.$$

Further, using Taylor formula we can write

$$\frac{G(\vartheta_0+\varepsilon u,\vartheta_0)}{\varepsilon^2} = \sum_{k=1}^K \int_{\tau_k(\vartheta_0))}^T \left(u^\top \frac{\partial S_k\bigl(t-\tau_k(\vartheta_0)\bigr)}{\partial \vartheta_0}\right)^2 \sigma_k(t)^{-2}\,\mathrm{d}t\,\bigl(1+o(1)\bigr)$$
$$= u^\top \mathrm{I}(\vartheta_0) u\,\left(1+o(1)\right).$$

Hence we can find $\delta > 0$ such that, for $\|\vartheta - \vartheta_0\|_3 = \varepsilon \|u\|_3 < \delta$, it holds

$$\frac{G(\vartheta_0+\varepsilon u,\vartheta_0)}{\varepsilon^2} \geq \frac{1}{2}\, u^\top \mathrm{I}(\vartheta_0) u \geq \frac{\kappa_1}{2}\|u\|_3^2,$$

where we denoted

$$\kappa_1 = \inf_{\vartheta\in\Theta}\ \inf_{e\,:\,\|e\|_3=1} e^\top \mathrm{I}(\vartheta) e > 0$$

from condition (7). For $\varepsilon\|u\|_3 > \delta$, by condition (6) we can write

$$\frac{G(\vartheta_0+\varepsilon u,\vartheta_0)}{\varepsilon^2} \geq \frac{g_{\mathbb{K}}(\delta)}{\varepsilon^2} \geq \kappa_2 \|u\|_3^2.$$

The last inequality was obtained as follows. The set Θ being bounded, we can write

$$\sup_{\vartheta_1,\vartheta_2\in\Theta} \|\vartheta_1 - \vartheta_2\| \leq D$$

with some $D > 0$. Hence, for $\varepsilon\|u\|_3 > \delta$, we have $\varepsilon^2\|u\|_3^2 \leq D^2$ and

$$\frac{g_{\mathbb{K}}(\delta)}{\varepsilon^2} \geq \frac{g_{\mathbb{K}}(\delta)}{D^2}\|u\|_3^2 = \kappa_2\|u\|_3^2.$$

Now we put $\kappa = \kappa_1 \wedge \kappa_2$, and for $u \in \mathbb{U}_\varepsilon$, we obtain the estimate

$$\frac{G(\vartheta_0+\varepsilon u,\vartheta_0)}{\varepsilon^2} \geq \kappa\|u\|_3^2.$$

So, the conditions (3.5.13) and (3.5.14) of [8] are fulfilled with any $\lambda(\varepsilon) \to \infty$ such that $\varepsilon\lambda(\varepsilon) \to 0$ (for example, we can take $\lambda(\varepsilon) = \varepsilon^{-1/2}$). □

3 Identifiability conditions

In this section we discuss possible configurations $D_1, \ldots, D_K$ of the detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$, positions D_0 of the source $\mathbb{S}_0$ and moments τ_0 of the emission start. We denote $\rho(\cdot,\cdot)$ the Euclidian distance in the plane and, for simplicity, suppose that the signal propagation speed ν is equal to 1.

Recall that by the identifiability condition **I**, the function $g(\mu) > 0$ for any compact $\mathbb{K}$ and any (small) $\mu > 0$. If for some $\vartheta \in \Theta$ and some $\mu > 0$ satisfying $\|\vartheta - \vartheta_0\|_3 \geq \mu > 0$ we have

$$\sum_{k=1}^{K} \int_{\tau_k(\vartheta_0)\wedge\tau_k(\vartheta)}^{T} \frac{\left[S_k\left(t-\tau_k(\vartheta)\right) - S_k\left(t-\tau_k(\vartheta_0)\right)\right]^2}{\sigma_k(t)^2}\,\mathrm{d}t = 0,$$

then we obtain the equalities

$$\tau_1(\vartheta) = \tau_1(\vartheta_0), \ldots, \tau_K(\vartheta) = \tau_K(\vartheta_0),$$

because by the condition **R** we have $S_k(t) = 0$ for $t < 0$ and $S_k(t) \neq 0$ for $t > 0$. Surely, in such situation the consistent estimation is impossible, because for two different values of unknown parameter we obtain the same statistical model.

We see that the question of identifiability is reduced to the following one: *when having* $\tau_1(\vartheta), \ldots, \tau_K(\vartheta)$ *is it possible to find* ϑ*?* More precisely, what are the configurations of detectors $D_1, \ldots, D_K$, positions D_0 of source and the moments τ_0, which allow to identify ϑ_0 by $\tau_1(\vartheta), \ldots, \tau_K(\vartheta)$.

A necessary and sufficient (geometric) condition

Let us recall that a (non-degenerate) hyperbola branch is the locus of a point D for which $\rho(F_2, D) - \rho(F_1, D) = \delta$, where F_1 and F_2 are two given points (foci), and $\delta \in \left(0, \rho(F_1, F_2)\right)$ is a given constant. Recall also that an affine transformation of the plane preserve conics (and, in particular, hyperbola branches), lines and convexity property.

Theorem 2. *A system with K detectors located at points $D_1, \ldots, D_K \in \mathcal{R}^2$ will be identifiable (without further restrictions on D_0 and τ_0) if and only if the detectors are not located on a same (non-degenerate) hyperbola branch or line.*

Proof. If the detectors are on the same line, any two points symmetric with respect to this line will give the same arrival times at the detectors. Further, if there exists a hyperbola branch passing through all the detectors, denoting F_1 and F_2 the foci of the hyperbola, we have $\rho(F_2, D_k) = \rho(F_1, D_k) + \delta$, where δ does not depend on the choice of the detector $\mathbb{D}_k$ (*cf.* Figure 2). So, if the source can be located at F_2 (e.g., if the arrival time at D_k is equal to the distance between D_k and F_2, and the emission time is 0), it can also be located at F_k (with emission time δ).

Inversely, if there exist two possible combinations of sources and emission times, then the detectors are on a hyperbola branch having these sources as

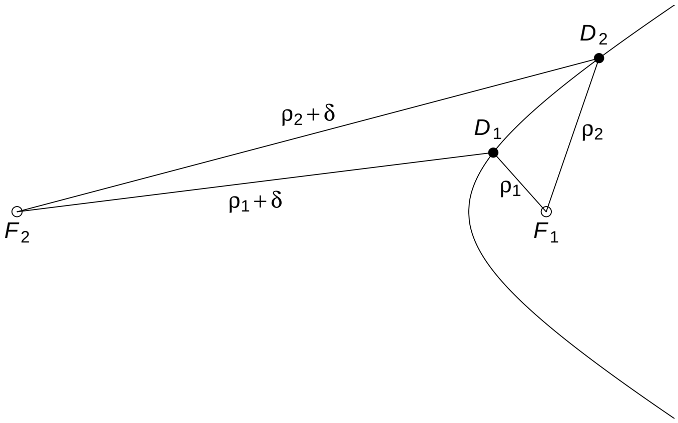

Figure 2: Detectors on a hyperbola branch

foci, and the difference of the distances to foci equal to the difference of the emission times (it degenerates to a line if the difference of the emission times is 0). □

Non-identifiability with three detectors

Proposition 1. *A system with* 3 *(or less) detectors located at arbitrary points* $D_1, D_2, D_3 \in \mathcal{R}^2$ *will not be identifiable (without further restrictions on* D_0 *and* τ_0*).*

Proof. If the detectors are aligned, then there is clearly no identifiability (*cf.* Theorem 2).

Otherwise, one can clearly find an affine transformation of the plane mapping the points D_1, D_2, D_3 to, say, points $(1, 1), (2, 1/2), (3, 1/3)$. The latters lay on the positive branch of the hyperbola $xy = 1$, and hence the points D_1, D_2, D_3 also lay on a hyperbola branch. So, there is no identifiability according to Theorem 2. □

Identifiability and non-identifiability with four detectors

Theorem 3. *We consider a system with* 4 *detectors located at the points* $D_1, D_2, D_3, D_4 \in \mathcal{R}^2$*. We distinguish the* 5 *following cases (cf. Figure 3):*

a) *the* 4 *detectors are aligned;*

b) *there exist* 3 *aligned detectors, while the fourth detector is not on the same line;*

c) *the detectors are in general linear position (i.e., any* 3 *of them are not aligned) and form a non-convex quadrilateral;*

d) *the detectors are in general linear position and form a parallelogram;*

e) *all the remaining cases, that is, the detectors are in general linear position and form a convex quadrilateral which is not a parallelogram (i.e., at least two opposite sides of the quadrilateral lay on intersecting lines).*

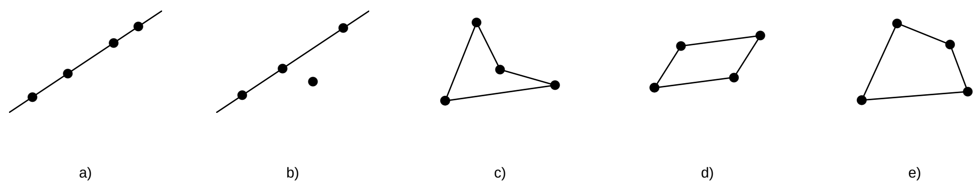

Figure 3: Cases a) – e)

The system is identifiable (without further restrictions on D_0 *and* τ_0*) in the cases b), c) and d), and is not identifiable in the cases a) and e).*

Proof. The case a) is immediate (*cf.* Theorem 2).

In the case b) the detectors are not aligned, so it is sufficient to notice that they can neither lay on a same hyperbola branch (since otherwise, this hyperbola would be intersected by a line in 3 different points, which is not possible) and apply Theorem 2.

In the case c) the detectors are also not aligned, so it is again sufficient to show that they can neither lay on a same hyperbola branch. Indeed, if they were on a same hyperbola branch, the quadrilateral would be convex (the "interior" of a hyperbola branch being convex itself).

Now we turn to the proof of the case d). In this case, since any parallelogram can be mapped to the unit square $[0, 1]^2$ by an affine transformation, it is sufficient to show that the points $(0, 0), (0, 1), (1, 0), (1, 1)$ cannot lay on a same hyperbola branch.

Recall that the general equation of a conic is

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0, \tag{8}$$

and that this conic will be a (possibly degenerate) hyperbola if and only if its *discriminant* $\Delta = B^2 - 4AC > 0$.

Requiring the points $(0, 0), (0, 1), (1, 0), (1, 1)$ satisfy the equation (8) implies $H = 0$, $G = -C$, $F = -A$ and $B = 0$. If $A = 0$, the equation (8) becomes $Cy^2 - Cy = 0$, and the conic degenerates to the pair of lines $y = 0$

and $y = 1$. So we can suppose that $A \neq 0$ and, dividing by A and denoting $M = -C/A$, the equation (8) becomes

$$x^2 - My^2 - x + My = 0,$$

and it will be a (possibly degenerate) hyperbola if and only if $M > 0$.

If $M = 1$, this hyperbola degenerates to a pair of lines $x = y$ and $x = -y$ (*cf.* Figure 4, solid plot).

If $M > 1$, the points $(0, 0)$ and $(1, 0)$ will lay on one of the branches of the hyperbola, and the points $(0, 1)$ and $(1, 1)$ on another. Indeed, it is not difficult to see that substituting, for example, $y = 1/2$ in the equation (8) yields the quadratic equation $x^2 - x + M/4 = 0$, which has no solution, since its discriminant equals $1 - M < 0$. So, the line $y = 1/2$ does not intersect the hyperbola, and hence separates its branches (*cf.* Figure 4, dashed plot).

Similarly, if $M < 1$, the points $(0, 0)$ and $(0, 1)$ will lay on one of the branches of the hyperbola, and the points $(1, 0)$ and $(1, 1)$ on another, since, for example, the line $x = 1/2$ does not intersect the hyperbola, and hence separates its branches (*cf.* Figure 4, dotted plot).

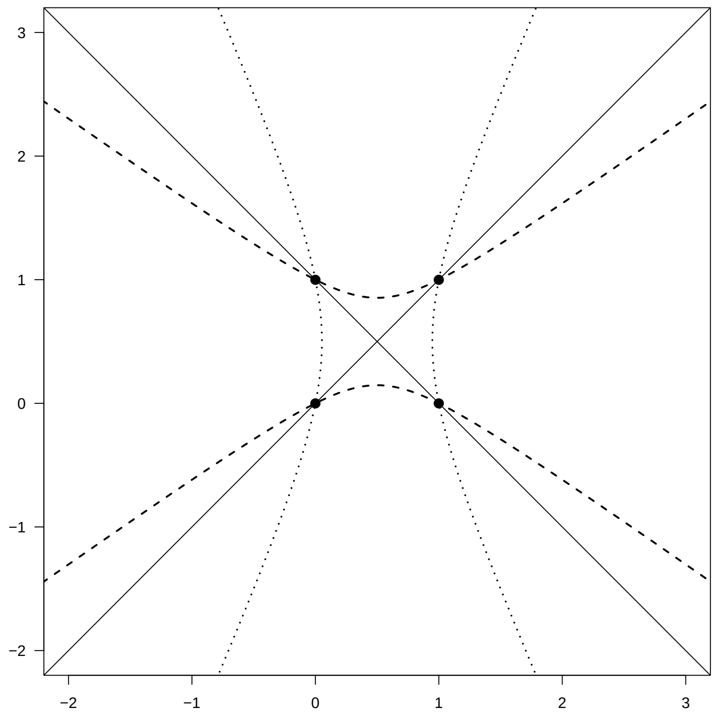

Figure 4: $M = 1$ (solid), $M > 1$ (dashed) and $M < 1$ (dotted) in the case d)

So, in all the three situations the detectors does not lay on a same hyperbola branch, which proves the case d).

It remains to prove the case e). First we give an heuristic (though non rigorous) proof. Somewhat similarly to the case d), the general equation of the conic passing through our four points will be driven by one parameter (say m). Recall that at least two opposite sides of the quadrilateral formed by our points lay on intersecting lines. This pair of lines (which is a degenerate hyperbola, *cf.* Figure 5, solid plot) corresponds to some value $m = m_0$ of the parameter. In the vicinity of m_0 the conic will remain a hyperbola, but the branches will be positioned differently on each side of m_0 (*cf.* Figure 5, dashed and dotted plots), and one of them will contain the four detectors.

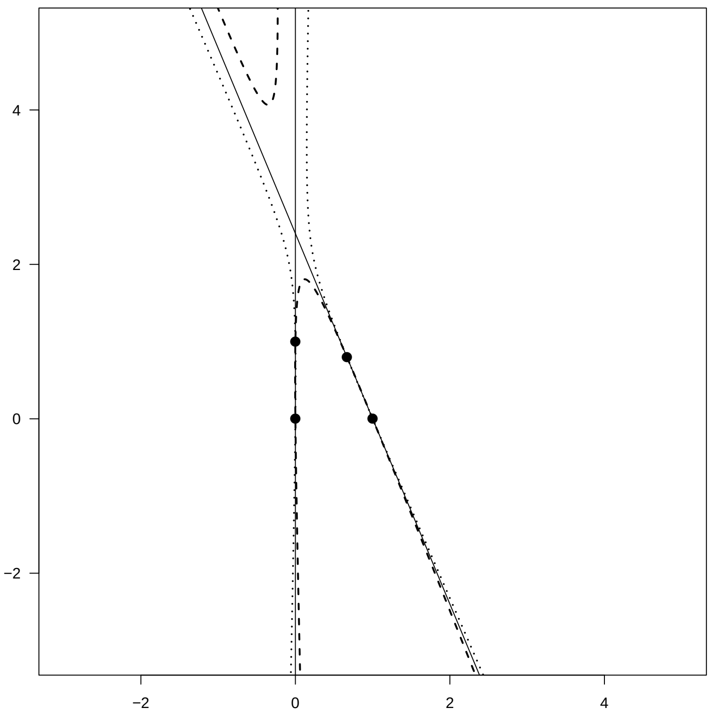

Figure 5: Case e)

To make this proof rigorous, first let us note that with an affine transformation we can map the detectors to the points $(0,0),(0,1),(1,0),(\alpha,\beta)$, where $\alpha \in (0,1)$, $\beta \in (0,1]$ and (since the quadrilateral must stay convex) $\beta > 1-\alpha$.

Requiring the points $(0,0),(0,1),(1,0),(1,1)$ satisfy the general equation (8) of a conic implies $H = 0$, $G = -C$, $F = -A$ and $B = A\,\frac{1-\alpha}{\beta} + C\,\frac{1-\beta}{\alpha}$. Dividing by A (we can suppose that $A \neq 0$) and denoting $m = C/A$, we obtain the equation

$$x^2 + \left(\frac{1-\alpha}{\beta} + m\,\frac{1-\beta}{\alpha}\right)xy + my^2 - x - my = 0. \tag{9}$$

Note that for $m = 0$, this conic degenerates to a pair of lines intersecting at $(0, \gamma)$ with $\gamma = \frac{\beta}{1-\alpha} > 1$ (*cf.* Figure 5, solid plot).

The discriminant of the conic (9) is

$$\Delta(m) = \left(\frac{1-\alpha}{\beta} + m\,\frac{1-\beta}{\alpha}\right)^2 - 4m.$$

This function Δ is continuous, and we have $\Delta(0) = \left(\frac{1-\alpha}{\beta}\right)^2 > 0$. Hence, for m sufficiently close to 0, this conic is a hyperbola. So, it is enough to show that for $m > 0$ (and sufficiently small), our four points lay on the same branch of this hyperbola (*cf.* Figure 5, dashed plot).

For this, let us verify that the line $y = \gamma$ does not intersects the conic (9). Substituting $y = \gamma = \frac{\beta}{1-\alpha}$ in the equation (9), we obtain the following quadratic equation (with respect to x):

$$x^2 + m\,\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\,x + m\left(\frac{\beta}{1-\alpha}\right)\left(\frac{\beta}{1-\alpha} - 1\right) = 0.$$

Its discriminant is given by

$$m^2\,\frac{\beta^2(1-\beta)^2}{\alpha^2(1-\alpha)^2} - 4m\left(\frac{\beta}{1-\alpha}\right)\left(\frac{\beta}{1-\alpha} - 1\right) = am^2 - bm = am\left(m - \frac{b}{a}\right)$$

with evident notations $a, b > 0$. So, for $m \in (0, b/a)$, this discriminant is negative, and hence the line $y = \gamma$ does not intersects the conic (9), which concludes the proof. $\square$

4 Estimation of ϑ_0: second approach

We have the same K detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$ located on the plane at the points $D_1, \ldots, D_K$, and a source $\mathbb{S}_0$ located at the point $D_0 = (x_0, y_0)$. At some moment τ_0 the source starts emission and the detectors receive the measurements $X^T = (X_1^T, \ldots, X_K^T)$, where $X_k^T = \big(X_k(t),\ 0 \leq t \leq T\big)$ satisfy the equations (5). As above, we have to estimate $\vartheta_0 = (x_0, y_0, \tau_0)$ by the observations X^T.

In this second approach the strategy of estimation is the following. First, on the base of observations X_k^T recorded by the k-th detector, we construct an estimator $\tau^*_{k,\varepsilon}$ of the moment $\tau_{0,k}=\tau_k(\vartheta_0)\in\mathcal{T}_k$ of the arrival of the signal at the detector. Here $\mathcal{T}_k$ is the set of admissible moments of arrival at the k-th detector. Then, having all the estimators $\tau^*_{k,\varepsilon}$, $k=1,\ldots,K$, we construct an estimator $\vartheta^*_\varepsilon=(x^*_\varepsilon,y^*_\varepsilon,\tau^*_\varepsilon)$ of $\vartheta_0=(x_0,y_0,\tau_0)$.

Therefore, we have K independent problems of estimation of $\tau_{0,k}$ by the observations

$$\mathrm{d}X_k(t)=S_k(t-\tau_{0,k})\mathrm{d}t+\varepsilon\sigma_k(t)\mathrm{d}W_k(t),\quad X_k(0)=0,\quad 0\le t\le T,$$

where $S_k(t)=0$ for $t<0$, and $W_k(\cdot)$ are independent Wiener processes. We suppose that $S_k(\cdot)$ and $\sigma_k(\cdot)$ are bounded known functions, and that the functions $\sigma_k(\cdot)$ are separated from 0. This model of observations and the properties of the MLE and of the BEs are well known. The identifiability conditions in these problems are: for any $\mu>0$, we have

$$g_k(\mu)=\inf_{|\tau_k-\tau_{0,k}|\ge\mu}\int_{\tau_{0,k}\wedge\tau_k}^{T}\frac{\left[S_k(t-\tau_k)-S_k(t-\tau_{0,k})\right]^2}{\sigma_k(t)^2}\,\mathrm{d}t>0. \tag{10}$$

Let us recall the asymptotic ($\varepsilon\to 0$) properties of the MLE $\hat\tau_{k,\varepsilon}$ and of the BEs $\tilde\tau_{k,\varepsilon}$ supposing that the condition (10) is fulfilled for all k.

If the functions $S_k(\cdot)$ have two continuous derivatives and the Fisher information is uniformly positive:

$$\inf_{\tau_k\in\mathcal{T}_k}\mathrm{I}_k(\tau_k)>0,\qquad \mathrm{I}_k(\tau_k)=\int_{\tau_k}^{T}\frac{S'_k(t-\tau_k)^2}{\sigma_k(t)^2}\,\mathrm{d}t,$$

then the MLE $\hat\tau_{k,\varepsilon}$ and the BEs $\tilde\tau_{k,\varepsilon}$ are consistent, asymptotically normal:

$$\frac{\hat\tau_{k,\varepsilon}-\tau_{0,k}}{\varepsilon}\Longrightarrow\xi_1\quad\text{and}\quad\frac{\tilde\tau_{k,\varepsilon}-\tau_{0,k}}{\varepsilon}\Longrightarrow\xi_1,\quad \xi_1\sim\mathcal{N}\big(0,\mathrm{I}_k(\tau_{0,k})^{-1}\big), \tag{11}$$

the polynomial moments converge, and both the estimators are asymptotically efficient (see Theorem 3.5.1 in [8]).

If the functions $S_k(\cdot)$ have cusp-type singularities, *i.e.*, for small values of t we have $S_k(t)=a_k|t|^\kappa\mathbb{1}_{\{t\ge 0\}}+o(t)$, where $\kappa\in(0,1/2)$, then the MLE and the BEs have different limit distributions:

$$\frac{\hat\tau_{k,\varepsilon}-\tau_{0,k}}{\varepsilon^{\frac{2}{2\kappa+1}}}\Longrightarrow\xi_2\quad\text{and}\quad\frac{\tilde\tau_{k,\varepsilon}-\tau_{0,k}}{\varepsilon^{\frac{2}{2\kappa+1}}}\Longrightarrow\xi_3,$$

the polynomial moments converge and the BEs are asymptotically efficient (see [3]). Note that the random variables ξ_2 and ξ_3 are defined with the help of the fBm.

If the functions $S_k(\cdot)$ have change-point type singularities, *i.e.*, for small values of t we have $S_k(t) = a_k \mathbb{1}_{\{t\geq 0\}} + o(t)$, then the MLE and the BEs once more have different limit distributions:

$$\frac{\hat{\tau}_{k,\varepsilon} - \tau_{0,k}}{\varepsilon^2} \Longrightarrow \xi_4 \quad \text{and} \quad \frac{\tilde{\tau}_{k,\varepsilon} - \tau_{0,k}}{\varepsilon^2} \Longrightarrow \xi_5, \tag{12}$$

the polynomial moments converge and the BEs are asymptotically efficient (see Theorems 6.2.2 and 6.2.3 in [8]). The random variables ξ_4 and ξ_5 are defined with the help of a two-sided Wiener process.

The convergences (11)–(12) can be unified as follows:

$$\frac{\hat{\tau}_{k,\varepsilon} - \tau_{0,k}}{\varphi_\varepsilon} = \hat{\xi}_{k,\varepsilon}, \quad \frac{\tilde{\tau}_{k,\varepsilon} - \tau_{0,k}}{\varphi_\varepsilon} = \tilde{\xi}_{k,\varepsilon}, \qquad \hat{\xi}_{k,\varepsilon} \Longrightarrow \hat{\xi}_k, \quad \tilde{\xi}_{k,\varepsilon} \Longrightarrow \tilde{\xi}_k,$$

where $\varphi_\varepsilon \to 0$ and $\hat{\xi}_k, \tilde{\xi}_k$ are corresponding rates and limits.

We do not consider these particular models with different regularity conditions, but suppose that we already have some estimators $\tau^*_{k,\varepsilon}$, $k = 1, \ldots, K$, of the parameters $\tau_{0,k}$, $k = 1, \ldots, K$, which have the above properties, that is,

$$\frac{\tau^*_{k,\varepsilon} - \tau_{0,k}}{\varphi_\varepsilon} \Longrightarrow \xi^*_k \quad \text{and} \quad \mathbf{E}_{\vartheta_0} \left| \frac{\tau^*_{k,\varepsilon} - \tau_{0,k}}{\varphi_\varepsilon} \right|^p \longrightarrow \mathbf{E}_{\vartheta_0} |\xi^*_k|^p \tag{13}$$

for any $p > 0$.

Our goal is to find conditions on the configuration of detectors $D_1, \ldots, D_K$ and the position of the source D_0 which allow to construct a computationally simple consistent estimator ϑ^*_ε of ϑ_0 by the "observations" $\tau^*_{k,\varepsilon}$, $k = 1, \ldots, K$.

Suppose that we can observe $\tau_{0,1}, \ldots, \tau_{0,K}$. Then the parameters x_0, y_0 and τ_0 satisfy the equations

$$(x_k - x_0)^2 + (y_k - y_0)^2 = \nu^2(\tau_{0,k} - \tau_0)^2, \qquad k = 1, \ldots, K.$$

Hence

$$x_k^2 + y_k^2 + x_0^2 + y_0^2 - 2x_k x_0 - 2y_k y_0 = \nu^2 \tau_{0,k}^2 + \nu^2 \tau_0^2 - 2\nu^2 \tau_{0,k} \tau_0.$$

This is a non linear equation with respect to ϑ_0. Introduce the notations

$$\gamma_1 = x_0, \qquad \gamma_2 = y_0, \qquad \gamma_3 = \tau_0, \qquad \gamma_4 = \frac{1}{2}(\nu^2 \tau_0^2 - x_0^2 - y_0^2),$$
$$z_k = \frac{1}{2}(x_k^2 + y_k^2 - \nu^2 \tau_{0,k}^2)$$
$$a_{1,k} = x_k, \qquad a_{2,k} = y_k, \qquad a^\circ_{3,k} = -\nu^2 \tau_{0,k}, \qquad a_{4,k} = 1.$$

We embed the initial problem with unknown three dimensional parameter $(\gamma_1,\gamma_2,\gamma_3)$ in another problem with unknown parameter $\gamma=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ satisfying the system of equation

$$a_{1,k}\gamma_1+a_{2,k}\gamma_2+a_{3,k}^{\circ}\gamma_3+a_{4,k}\gamma_4=z_k,\qquad k=1,\ldots,K.$$

Let us replace $a_{3,k}^{\circ}$ and z_k by the "observable" values $a_{3,k}=-\nu^2\tau_{k,\varepsilon}^{*}$ and $z_{k,\varepsilon}=\frac{1}{2}(x_k^2+y_k^2-\nu^2\tau_{k,\varepsilon}^{*2})$. Then we obtain the system of equations

$$a_{1,k}\gamma_1+a_{2,k}\gamma_2+a_{3,k}\gamma_3+a_{4,k}\gamma_4=z_{k,\varepsilon},\qquad k=1,\ldots,K.$$

We define an estimator $\gamma_\varepsilon^*=(\gamma_{1,\varepsilon},\gamma_{2,\varepsilon},\gamma_{3,\varepsilon},\gamma_{4,\varepsilon})$ using the least squares method:

$$\gamma_\varepsilon^*=\underset{\gamma}{\operatorname{argmin}}\sum_{k=1}^{K}\left[z_{k,\varepsilon}-\sum_{j=1}^{4}a_{j,k}\gamma_j\right]^2.$$

Introducing the vector $Z_\varepsilon=(Z_{1,\varepsilon},\ldots,Z_{4,\varepsilon})^\top$ and the matrix $\mathbb{A}_\varepsilon=(A_{i,j})_{4\times4}$ by

$$Z_{j,\varepsilon}=\sum_{k=1}^{K}a_{j,k}z_{k,\varepsilon}\quad\text{and}\quad A_{i,j}=\sum_{k=1}^{K}a_{i,k}a_{j,k},$$

we have

$$\gamma_\varepsilon^*=\mathbb{A}_\varepsilon^{-1}Z_\varepsilon.$$

We denote $\mathbb{A}_0$ the limit of the matrix $\mathbb{A}_\varepsilon$ with

$$a_{3,k}=-\nu^2\tau_{k,\varepsilon}=-\nu^2\tau_{0,k}-\varphi_\varepsilon\nu^2\xi_{k,\varepsilon}^*\longrightarrow a_{3,k}^{\circ}=-\nu^2\tau_{0,k}.$$

Remark that $\mathbb{A}_0=\mathbb{A}_0(\vartheta_0)$ because the term

$$a_{3,k}^{\circ}=-\nu^2\tau_0-\nu\left[(x_k-x_0)^2+(y_k-y_0)^2\right]^{1/2}.$$

We also have

$$\mathbb{A}_\varepsilon=\mathbb{A}_0(\vartheta_0)+\varphi_\varepsilon\mathbb{B}_\varepsilon,$$

where the matrix $\mathbb{B}_\varepsilon=(B_{i,j})_{4\times4}$ has zero elements except

$$B_{3,j}=-\nu^2\sum_{k=1}^{K}\xi_{k,\varepsilon}^*a_{j,k},\quad j\neq3,\qquad B_{i,3}=-\nu^2\sum_{k=1}^{K}\xi_{k,\varepsilon}^*a_{i,k},\quad i\neq3,$$

$$B_{3,3}=\nu^4\sum_{k=1}^{K}\left[2\tau_{0,k}\xi_{k,\varepsilon}^*+\varphi_\varepsilon(\xi_{k,\varepsilon}^*)^2\right]=2\nu^4\sum_{k=1}^{K}\tau_{0,k}\xi_{k,\varepsilon}^*+O(\varphi_\varepsilon).$$

Let us denote $\xi^* = (\xi_1^*, \ldots, \xi_K^*)^\top$, where ξ_k^*, $k = 1, \ldots, K$, are independent random variables from (13), and denote $\mathbb{B}_0(\xi^*, \vartheta_0)$ the matrix obtained from $\mathbb{B}_\varepsilon$ by replacing in its elements $\xi_{k,\varepsilon}^*$, $k = 1, \ldots, K$, by ξ_k^*, $k = 1, \ldots, K$, and also putting $O(\varphi_\varepsilon) = 0$ in the expression of $B_{3,3}$. Then we can write

$$\mathbb{B}_\varepsilon \Longrightarrow \mathbb{B}_0(\xi^*, \vartheta_0).$$

Introduce the random matrix

$$\mathbb{C}_0(\xi^*, \vartheta_0) = \mathbb{A}_0(\vartheta_0)^{-1}\, \mathbb{B}_0(\xi^*, \vartheta_0)\, \mathbb{A}_0(\vartheta_0)^{-1}$$

and random vector

$$\zeta^* = \mathbb{A}_0(\vartheta_0)^{-1}\, Y - \mathbb{C}_0(\xi^*, \vartheta_0)\, z,$$

where $z = (z_1, \ldots, z_K)^\top$ and $Y = (\tau_{0,1}\xi_1^*, \ldots, \tau_{0,K}\xi_K^*)^\top$.

We need the following condition:

A. *The configuration of detectors $D_1, \ldots, D_K$ and the set Θ are such that the matrix $\mathbb{A}_0 = \mathbb{A}_0(\vartheta)$ is uniformly non degenerate:*

$$\inf_{\vartheta_0 \in \Theta}\ \inf_{e\,:\,\|e\|_4 = 1} e^\top \mathbb{A}_0(\vartheta_0) e > 0.$$

Remark that under this condition we have the equality

$$\mathbb{A}_0(\vartheta_0)^{-1} z = \gamma. \tag{14}$$

Theorem 4. *Suppose that the conditions* (13) *and* **A** *are fulfilled. Then the estimator γ_ε^* is consistent and we have the convergence in distribution*

$$\frac{\gamma_\varepsilon^* - \gamma}{\varphi_\varepsilon} \Longrightarrow \zeta^*. \tag{15}$$

Proof. We have the representation

$$\gamma_\varepsilon^* = \mathbb{A}_0(\vartheta_0)^{-1} z + \big(\mathbb{A}_\varepsilon^{-1} - \mathbb{A}_0(\vartheta_0)^{-1}\big) z + \mathbb{A}_\varepsilon^{-1}(Z_\varepsilon - z).$$

Therefore, using (14) we get

$$\varphi_\varepsilon^{-1}(\gamma_\varepsilon^* - \gamma) = \varphi_\varepsilon^{-1}\big(\mathbb{A}_\varepsilon^{-1} - \mathbb{A}_0(\vartheta_0)^{-1}\big) z + \mathbb{A}_\varepsilon^{-1}\varphi_\varepsilon^{-1}(Z_\varepsilon - z).$$

We have expansion

$$z_{k,\varepsilon} = \frac{1}{2}\big[x_k^2 + y_k^2 - \nu^2(\tau_{0,k} + \varphi_\varepsilon \xi_{k,\varepsilon}^*)^2\big] = z_k - \nu^2 \tau_{0,k}\xi_{k,\varepsilon}^* \varphi_\varepsilon \big(1 + O(\varphi_\varepsilon)\big).$$

Hence

$$\varphi_\varepsilon^{-1}(Z_\varepsilon - z) = -\nu^2\, Y_\varepsilon\bigl(1 + O(\varphi_\varepsilon)\bigr),$$

where $Y_\varepsilon = (\tau_{0,1}\xi^*_{1,\varepsilon}, \ldots, \tau_{0,K}\xi^*_{K,\varepsilon})^\top \Longrightarrow Y$, and we obtain the convergence

$$\mathbb{A}_\varepsilon^{-1}\varphi_\varepsilon^{-1}(Z_\varepsilon - z) \Longrightarrow \mathbb{A}_0(\vartheta_0)^{-1}Y.$$

Further, we have

$$\begin{aligned}\bigl[\mathbb{A}_0(\vartheta_0) + \varphi_\varepsilon\mathbb{B}_\varepsilon\bigr]^{-1} &= \bigl[\mathbb{A}_0(\vartheta_0)\bigl(\mathbb{I} + \varphi_\varepsilon\mathbb{A}_0(\vartheta_0)^{-1}\mathbb{B}_\varepsilon\bigr)\bigr]^{-1}\\ &= \bigl[\mathbb{I} - \varphi_\varepsilon\mathbb{A}_0(\vartheta_0)^{-1}\mathbb{B}_\varepsilon\bigr]\mathbb{A}_0(\vartheta_0)^{-1} + O(\varphi_\varepsilon^2)\\ &= \mathbb{A}_0(\vartheta_0)^{-1} - \varphi_\varepsilon\mathbb{A}_0(\vartheta_0)^{-1}\mathbb{B}_\varepsilon\mathbb{A}_0(\vartheta_0)^{-1} + O(\varphi_\varepsilon^2).\end{aligned}$$

Hence

$$\begin{aligned}\varphi_\varepsilon^{-1}\Bigl(\bigl[\mathbb{A}_0(\vartheta_0) + \varphi_\varepsilon\mathbb{B}_\varepsilon\bigr]^{-1} - \mathbb{A}_0(\vartheta_0)^{-1}\Bigr)z &= -\mathbb{A}_0(\vartheta_0)^{-1}\mathbb{B}_\varepsilon\mathbb{A}_0(\vartheta_0)^{-1}z + O(\varphi_\varepsilon)\\ &\Longrightarrow -\mathbb{C}_0(\xi^*, \vartheta_0)z.\end{aligned}$$

So, the convergence (15) is proved. □

Remark that in the work [11] a similar approach of estimation was considered but the limit behavior of errors was not studied.

5 Example with four detectors

In this section we consider the problem of localizing the source and finding its emission time by combining measurements collected from four detectors arranged on a rectangle. This topology is common in localization applications (see, for instance, [11]).

In the work [1], it was shown that if the moment of emission τ_0 is unknown, at least four detectors are needed in order to localize the source (see also Section 3), and that in this situation it will be located at the point of intersection of three hyperbolas. Intersecting hyperbolas numerically is a strongly nonlinear problem with high computational costs, but we avoid such difficulties by using the geometric properties of a rectangle, and show that with the help of four detectors arranged in a rectangle we can obtain exact expressions for $\vartheta_0 = (x_0, y_0, \tau_0)$, where as before (x_0, y_0) is the position of the source and τ_0 is the moment of the beginning of the emission. We determine the location of the source and its time of emission by evaluating the difference in arrival time of signals at four spatially separated detectors, whose positions are $D_1 = \bigl(-\frac{a}{2}, -\frac{b}{2}\bigr)$, $D_2 = \bigl(\frac{a}{2}, -\frac{b}{2}\bigr)$, $D_3 = \bigl(-\frac{a}{2}, \frac{b}{2}\bigr)$ and $D_4 = \bigl(\frac{a}{2}, \frac{b}{2}\bigr)$

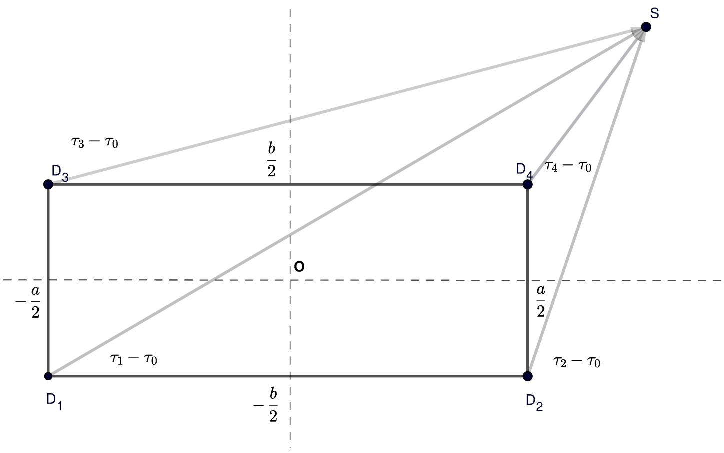

Figure 6: Rectangular grid of four detectors

(see Figure 6). Also, for simplicity, we suppose that the signal propagation speed ν is equal to 1.

As above we suppose that we already have the estimators $\tau^*_{1,\varepsilon}, \ldots, \tau^*_{4,\varepsilon}$ satisfying the relations (13). To introduce the estimator $\vartheta^*_\varepsilon = (x^*_{0,\varepsilon}, y^*_{0,\varepsilon}, \tau^*_{0,\varepsilon})$ of ϑ_0, we first suppose that these estimators are "without errors", that is, $\tau^*_{1,\varepsilon} = \tau_{0,1}, \ldots, \tau^*_{4,\varepsilon} = \tau_{0,4}$, and obtain explicit expressions for the "estimators" (true values)

$$x_0 = \Phi_x(\tau_{0,1}, \ldots, \tau_{0,4}), \quad y_0 = \Phi_y(\tau_{0,1}, \ldots, \tau_{0,4}) \quad \text{and} \quad \tau_0 = \Phi_\tau(\tau_{0,1}, \ldots, \tau_{0,4}).$$

Then we replace $\tau_{0,1}, \ldots, \tau_{0,4}$ by "observations" $\tau^*_{1,\varepsilon}, \ldots, \tau^*_{4,\varepsilon}$, and after slight modification of functions $\Phi_x(\cdot)$, $\Phi_y(\cdot)$ and $\Phi_\tau(\cdot)$, obtain the *estimator of substitution* $\vartheta^*_\varepsilon = (x^*_{0,\varepsilon}, y^*_{0,\varepsilon}, \tau^*_{0,\varepsilon})$ of ϑ_0. The asymptotic properties of the estimator ϑ^*_ε follow directly from these expressions.

In order to simplify the notations, we denote in what follows $\tau_k = \tau_{0,k}$, $k = 1, \ldots, 4$.

Proposition 2. *Denote* $\Delta = \tau_1 - \tau_2 - \tau_3 + \tau_4$.

1. *We have* $\Delta = 0$ *if and only if the source is located on one of the coordinate axes.*

2. *If* $\Delta \neq 0$, *we have*

$$\begin{cases} x_0 = \frac{(\tau_1-\tau_2)\,(\tau_3-\tau_4)\,(\tau_3+\tau_4-\tau_1-\tau_2)}{2a\Delta} \\ y_0 = \frac{(\tau_1-\tau_3)\,(\tau_2-\tau_4)\,(\tau_2+\tau_4-\tau_1-\tau_3)}{2a\Delta} \\ \tau_0 = \frac{\tau_1^2-\tau_2^2-\tau_3^2+\tau_4^2}{2\Delta}\,. \end{cases}$$

3. If $\Delta = 0$, *one of the following situations happens:*

- $\tau_1 = \tau_3$ *and* $\tau_2 = \tau_4 \neq \tau_1$, *then*

$$\begin{cases} x_0 = \frac{\tau_1-\tau_2}{2}\sqrt{\frac{a^2+b^2-(\tau_1-\tau_2)^2}{a^2-(\tau_1-\tau_2)^2}} \\ y_0 = 0 \\ \tau_0 = \tau_1 - \sqrt{\left(x_0+\frac{a}{2}\right)^2+\frac{b^2}{4}}\,, \end{cases}$$

- $\tau_1 = \tau_2$ *and* $\tau_3 = \tau_4 \neq \tau_1$, *then*

$$\begin{cases} x_0 = 0 \\ y_0 = \frac{\tau_1-\tau_3}{2}\sqrt{\frac{a^2+b^2-(\tau_1-\tau_3)^2}{a^2-(\tau_1-\tau_3)^2}} \\ \tau_0 = \tau_1 - \sqrt{\left(y_0+\frac{b}{2}\right)^2+\frac{a^2}{4}}\,, \end{cases}$$

- *all* τ_i *are equal, then*

$$(x_0, y_0) = (0, 0) \quad \text{and} \quad \tau_0 = \tau_1 - \frac{\sqrt{a^2+b^2}}{2}\,.$$

Proof. Combining the reception times with the unknown emission time τ_0, we obtain the following system of equations

$$(\mathcal{S}) \begin{cases} \left(x_0+\frac{a}{2}\right)^2+\left(y_0+\frac{b}{2}\right)^2 = (\tau_1-\tau_0)^2 & (16) \\ \left(x_0-\frac{a}{2}\right)^2+\left(y_0+\frac{b}{2}\right)^2 = (\tau_2-\tau_0)^2 & (17) \\ \left(x_0+\frac{a}{2}\right)^2+\left(y_0-\frac{b}{2}\right)^2 = (\tau_3-\tau_0)^2 & (18) \\ \left(x_0-\frac{a}{2}\right)^2+\left(y_0-\frac{b}{2}\right)^2 = (\tau_4-\tau_0)^2. & (19) \end{cases}$$

Notice that the sum of the left hand sides of the equations (16) and (19) is equal to the sum of the left hand sides of the equations (17) and (18). Hence, we have

$$(\tau_1-\tau_0)^2 - (\tau_2-\tau_0)^2 - (\tau_3-\tau_0)^2 + (\tau_4-\tau_0)^2 = 0, \tag{20}$$

or equivalently

$$2\Delta\tau_0 = \tau_1^2 - \tau_1^2 - \tau_3^2 + \tau_4^2. \tag{21}$$

We notice that if $\Delta \neq 0$, the equation (20) admits the solution

$$\tau_0 = \frac{\tau_1^2-\tau_2^2-\tau_3^2+\tau_4^2}{2\Delta}\,.$$

Plugging this value into the system $(\mathcal{S})$, we then obtain

$$x_0 = \frac{(\tau_1 - \tau_2)(\tau_3 - \tau_4)(\tau_3 + \tau_4 - \tau_1 - \tau_2)}{2a\Delta}$$

and

$$y_0 = \frac{(\tau_1 - \tau_3)(\tau_2 - \tau_4)(\tau_2 + \tau_4 - \tau_1 - \tau_3)}{2a\Delta}.$$

Let us now consider the case $\Delta = 0$. In this case, using (21), we get

$$\tau_1^2 - \tau_2^2 - \tau_3^2 + \tau_4^2 = 0,$$

or equivalently

$$(\tau_1 - \tau_2)(\tau_1 + \tau_2) + (\tau_4 - \tau_3)(\tau_4 + \tau_3) = 0.$$

Using $\Delta = 0$, we also have $\tau_4 - \tau_3 = \tau_2 - \tau_1$, and we can write

$$(\tau_1 - \tau_2)(\tau_1 - \tau_3 + \tau_2 - \tau_4) = 0.$$

Further, as $\tau_2 - \tau_4 = \tau_1 - \tau_3$, we finally obtain

$$2(\tau_1 - \tau_2)(\tau_1 - \tau_3) = 0.$$

It then follows that we must have $\tau_1 = \tau_2$ and/or $\tau_1 = \tau_3$, that is, $x_0 = 0$ and/or $y_0 = 0$. Consequently, the set of source points for which $\Delta = 0$ forms a cross centered at the origin (see Figure 6).

Let us now determine the time of emission of the source and its coordinates when it is located on this cross. We will consider the case $y_0 = 0$ only (the case $x_0 = 0$ can be treated in a similar way). Due to the configuration of detectors, if $y_0 = 0$, we have $\tau_1 = \tau_3$ and $\tau_2 = \tau_4$. Thus, the system of equations $(\mathcal{S})$ can be replaced by the system

$$(\mathcal{S}') \begin{cases} \left(x_0 + \frac{a}{2}\right)^2 + \frac{b^2}{4} = (\tau_1 - \tau_0)^2 \\ \left(x_0 - \frac{a}{2}\right)^2 + \frac{b^2}{4} = (\tau_2 - \tau_0)^2. \end{cases}$$

From the first equation, taking into account that $\tau_1 \geq \tau_0$, we obtain

$$\tau_0 = \tau_1 - \sqrt{\left(x_0 + \frac{a}{2}\right)^2 + \frac{b^2}{4}}. \tag{22}$$

Subtracting the second equation of the system $(\mathcal{S}')$ from the first yields

$$2ax_0 = (\tau_1 - \tau_0)^2 - (\tau_2 - \tau_0)^2.$$

Replacing the value of τ_0 found in (22) and denoting $\beta = \tau_1 - \tau_2$, we get

$$2ax_0 + \beta^2 = 2\beta\sqrt{\left(x_0 + \frac{a}{2}\right)^2 + \frac{b^2}{4}}\,.$$

Elevating both sides of the equation to the square, we obtain

$$4\,(a^2 - \beta^2)\,x_0^2 = \beta^2(a^2 + b^2 - \beta^2).$$

Remark that $\beta = \tau_1 - \tau_2 = \rho(D_1, D_0) - \rho(D_2, D_0)$, and hence

$$|\beta| < \rho(D_1, D_2) = a$$

thanks to the triangle inequality and the fact that the points D_0, D_1 and D_2 are not aligned. So, $a^2 + b^2 - \beta^2 > a^2 - \beta^2 > 0$, and thus

$$x_0 = \pm\frac{\beta}{2}\sqrt{\frac{a^2 + b^2 - \beta^2}{a^2 - \beta^2}}\,.$$

If $\beta = 0$, all τ_i are equal, and we have $x_0 = 0$. Otherwise, in order to choose the sign of x_0, let us first note that if $\beta > 0$, then $\tau_1 > \tau_2$, and hence $\rho(D_1, D_0) > \rho(D_2, D_0)$, which yields $x_0 > 0$. Similarly, if $\tau_1 < \tau_2$, we get $x_0 < 0$. So, β and x_0 have the same sign, and we finally get

$$x_0 = \frac{\tau_1 - \tau_2}{2}\sqrt{\frac{a^2 + b^2 - (\tau_1 - \tau_2)^2}{a^2 - (\tau_1 - \tau_2)^2}}\,,$$

which conclude the proof. □

Let us now consider the construction of the estimator of substitution. Denote $\Delta_\varepsilon = \tau^*_{1,\varepsilon} - \tau^*_{2,\varepsilon} - \tau^*_{3,\varepsilon} + \tau^*_{4,\varepsilon}$ and remark that

$$\varphi_\varepsilon^{-1}(\Delta_\varepsilon - \Delta) = \xi^*_{1,\varepsilon} - \xi^*_{2,\varepsilon} - \xi^*_{3,\varepsilon} + \xi^*_{4,\varepsilon} \Longrightarrow \xi^*_1 - \xi^*_2 - \xi^*_3 + \xi^*_4.$$

We have

$$\begin{aligned} x^*_{0,\varepsilon} &= \frac{(\tau^*_{1,\varepsilon} - \tau^*_{2,\varepsilon})\,(\tau^*_{3,\varepsilon} - \tau^*_{4,\varepsilon})\,(\tau^*_{3,\varepsilon} + \tau^*_{4,\varepsilon} - \tau^*_{1,\varepsilon} - \tau^*_{2,\varepsilon})}{2a\Delta_\varepsilon}\,\mathbb{1}_{\{\mathcal{M}^c\}} \\ &\qquad + \frac{\tau^*_{1,\varepsilon} - \tau^*_{2,\varepsilon}}{2}\sqrt{\frac{a^2 + b^2 - (\tau^*_{1,\varepsilon} - \tau^*_{2,\varepsilon})^2}{a^2 - (\tau^*_{1,\varepsilon} - \tau^*_{2,\varepsilon})^2}}\,\mathbb{1}_{\{\mathcal{M},\mathcal{N}_{1,3},\mathcal{N}_{2,4},\mathcal{N}^c_{1,4}\}}, \\ y^*_{0,\varepsilon} &= \frac{(\tau^*_{1,\varepsilon} - \tau^*_{3,\varepsilon})\,(\tau^*_{2,\varepsilon} - \tau^*_{4,\varepsilon})\,(\tau^*_{2,\varepsilon} + \tau^*_{4,\varepsilon} - \tau^*_{1,\varepsilon} - \tau^*_{3,\varepsilon})}{2a\Delta_\varepsilon}\,\mathbb{1}_{\{\mathcal{M}^c\}} \\ &\qquad + \frac{\tau^*_{1,\varepsilon} - \tau^*_{3,\varepsilon}}{2}\sqrt{\frac{a^2 + b^2 - (\tau^*_{1,\varepsilon} - \tau^*_{3,\varepsilon})^2}{a^2 - (\tau^*_{1,\varepsilon} - \tau^*_{3,\varepsilon})^2}}\,\mathbb{1}_{\{\mathcal{M},\mathcal{N}_{1,2},\mathcal{N}_{3,4},\mathcal{N}^c_{1,4}\}} \end{aligned}$$

and

$$\tau^*_{0,\varepsilon} = \frac{\tau^{*2}_{1,\varepsilon} - \tau^{*2}_{2,\varepsilon} - \tau^{*2}_{3,\varepsilon} + \tau^{*2}_{4,\varepsilon}}{2\Delta_\varepsilon}\, \mathbb{1}_{\{\mathcal{M}^c\}}
+ \left(\tau^*_{1,\varepsilon} - \sqrt{\left(x_0 + \frac{a}{2}\right)^2 + \frac{b^2}{4}}\right) \mathbb{1}_{\{\mathcal{M}, \mathcal{N}_{1,3}, \mathcal{N}_{2,4}, \mathcal{N}^c_{1,4}\}}
+ \left(\tau^*_{1,\varepsilon} - \sqrt{\left(y_0 + \frac{b}{2}\right)^2 + \frac{a^2}{4}}\right) \mathbb{1}_{\{\mathcal{M}, \mathcal{N}_{1,2}, \mathcal{N}_{3,4}, \mathcal{N}^c_{1,4}\}}.$$

Here, the sets $\mathcal{M} = \mathcal{M}_\varepsilon$ and $\mathcal{N}_{i,j} = \mathcal{N}_{i,j;\varepsilon}$, $i, j = 1, 2, 3, 4$, are defined by

$$\mathcal{M}_\varepsilon = \left\{|\Delta_\varepsilon| \le \varphi_\varepsilon^{1/2}\right\} \quad \text{and} \quad \mathcal{N}_{i,j;\varepsilon} = \left\{|\tau^*_{i,\varepsilon} - \tau^*_{j,\varepsilon}| \le \varphi_\varepsilon^{1/2}\right\}.$$

Proposition 3. *Suppose that the conditions* (13) *are fulfilled. Then, the estimator* $\vartheta^*_\varepsilon = (x^*_{0,\varepsilon}, y^*_{0,\varepsilon}, \tau^*_{0,\varepsilon})$ *is consistent:*

$$x^*_{0,\varepsilon} \xrightarrow{\mathbf{P}} x_0, \qquad y^*_{0,\varepsilon} \xrightarrow{\mathbf{P}} y_0, \qquad \tau^*_{0,\varepsilon} \xrightarrow{\mathbf{P}} \tau_0,$$

and we have the convergence in distribution

$$\varphi_\varepsilon^{-1}(\vartheta^*_\varepsilon - \vartheta_0) \Longrightarrow \zeta^*. \tag{23}$$

Here the vector $\zeta^* = (\zeta^*_1, \zeta^*_2, \zeta^*_3)$ *has the components*

$$\zeta^*_i = c_{i,1}\xi^*_1 + c_{i,2}\xi^*_2 + c_{i,3}\xi^*_3 + c_{i,4}\xi^*_4, \qquad i = 1, 2, 3, \tag{24}$$

with deterministic coefficients $c_{i,k}$.

Proof. Suppose that $\Delta \ne 0$. Then, for any $p > 0$, we have

$$\begin{aligned}
\mathbf{P}_{\vartheta_0}(\mathcal{M}_\varepsilon) &= \mathbf{P}_{\vartheta_0}\left(|\Delta_\varepsilon - \Delta + \Delta| \le \varphi_\varepsilon^{1/2}\right) \le \mathbf{P}_{\vartheta_0}\left(|\Delta| - |\Delta_\varepsilon - \Delta| \le \varphi_\varepsilon^{1/2}\right) \\
&= \mathbf{P}_{\vartheta_0}\left(|\Delta_\varepsilon - \Delta| \ge |\Delta| - \varphi_\varepsilon^{1/2}\right) \le \mathbf{P}_{\vartheta_0}\left(|\Delta_\varepsilon - \Delta| \ge \frac{1}{2}|\Delta|\right) \\
&\le \frac{2^p}{|\Delta|^p}\, \mathbf{E}_{\vartheta_0}|\Delta_\varepsilon - \Delta|^p \le \frac{C}{|\Delta|^p}\, \varphi_\varepsilon^p \longrightarrow 0.
\end{aligned} \tag{25}$$

Therefore, for $\Delta \ne 0$, we have

$$\mathbf{P}_{\vartheta_0}(\mathcal{M}^c_\varepsilon) \ge 1 - \frac{C}{|\Delta|^p}\, \varphi_\varepsilon^p \longrightarrow 1 \tag{26}$$

and

$$\mathbf{P}_{\vartheta_0}(\mathcal{M}_\varepsilon, \mathcal{N}_{1,3;\varepsilon}, \mathcal{N}_{2,4;\varepsilon}, \mathcal{N}^c_{1,4;\varepsilon}) \le \mathbf{P}_{\vartheta_0}(\mathcal{M}^c_\varepsilon) \le \frac{C}{|\Delta|^p}\, \varphi_\varepsilon^p \longrightarrow 0. \tag{27}$$

If $\Delta = 0$, we can write

$$\begin{aligned}\mathbf{P}_{\vartheta_0}(\mathcal{M}_\varepsilon^c) &= \mathbf{P}_{\vartheta_0}\left(|\tau_{1,\varepsilon}^* - \tau_1 - \tau_{2,\varepsilon}^* + \tau_2 - \tau_{3,\varepsilon}^* + \tau_3 + \tau_{4,\varepsilon}^* - \tau_4| \leq \varphi_\varepsilon^{1/2}\right)\\ &\geq 1 - \mathbf{P}_{\vartheta_0}\left(|\tau_{1,\varepsilon}^* - \tau_1| + |\tau_{2,\varepsilon}^* - \tau_2| + |\tau_{3,\varepsilon}^* - \tau_3| + |\tau_{4,\varepsilon}^* - \tau_4| > \varphi_\varepsilon^{1/2}\right)\\ &= 1 - \mathbf{P}_{\vartheta_0}\left(|\xi_{1,\varepsilon}^*| + |\xi_{2,\varepsilon}^*| + |\xi_{3,\varepsilon}^*| + |\xi_{4,\varepsilon}^*| > \varphi_\varepsilon^{-1/2}\right)\\ &\geq 1 - C\varphi_\varepsilon^{p/2}, \end{aligned} \tag{28}$$

where we used Tchebyshev's inequality and the boundedness of the expectations in (13).

Further, if $\tau_i \neq \tau_j$, we have similarly the estimate

$$\begin{aligned}\mathbf{P}_{\vartheta_0}(\mathcal{N}_{i,j;\varepsilon}) &= \mathbf{P}_{\vartheta_0}\left(|\tau_{i,\varepsilon}^* - \tau_i - \tau_{j,\varepsilon}^* + \tau_j + \tau_i - \tau_j| \leq \varphi_\varepsilon^{1/2}\right)\\ &\leq \mathbf{P}_{\vartheta_0}\left(|\tau_i - \tau_j| - |\tau_{i,\varepsilon}^* - \tau_i| - |\tau_{j,\varepsilon}^* - \tau_j| \leq \varphi_\varepsilon^{1/2}\right)\\ &\leq \mathbf{P}_{\vartheta_0}\left(|\tau_{i,\varepsilon}^* - \tau_i| + |\tau_{j,\varepsilon}^* - \tau_j| \geq |\tau_i - \tau_j| - \varphi_\varepsilon^{1/2}\right)\\ &\leq \frac{C}{|\tau_i - \tau_j|^p}\,\varphi_\varepsilon^p \longrightarrow 0. \end{aligned} \tag{29}$$

Finally, if $\tau_i = \tau_j$, we obtain the estimate

$$\mathbf{P}_{\vartheta_0}(\mathcal{N}_{i,j;\varepsilon}) \geq 1 - C\varphi_\varepsilon^{p/2}. \tag{30}$$

Now, the consistency of ϑ_ε^* follows from the consistency of the estimators $\tau_{k,\varepsilon}^*$, $k = 1, \ldots, 4$, and the obtained estimates (25)–(30).

The explicit expression (24) for the limit distribution of $\varphi_\varepsilon^{-1}(\vartheta_\varepsilon^* - \vartheta_0)$ in (23) can be obtained from the expressions of the estimators $x_{0,\varepsilon}^*, y_{0,\varepsilon}^*, \tau_{0,\varepsilon}^*$ given above, the representations $\tau_{k,\varepsilon}^* = \tau_k + \varphi_\varepsilon \xi_{k,\varepsilon}^*$, Taylor formula and the limits (13). We do not give it here because the calculations are elementary but rather cumbersome. □

Acknowledgments

This research was financially supported by the Ministry of Education and Science of the Russian Federation (research project No. FSWF–2020–0022) and by RFBR and CNRS (research project No. 20–51–15001).

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