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AN ENTROPY FORMULATION FOR THE ANALYSIS OF ENERGY FLOW BETWEEN MECHANICAL RESONATORS

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Several energy-based methods to approach noise and vibro-acoustic problems are actually under development. These techniques provide a chance of describing the vibro-acoustic behaviour of complex systems by the energies of a limited number of sub-components. This process is the base of one of the most acknowledged methods in this field, i.e. the statistical energy analysis (SEA). However, SEA invokes only the first law of thermodynamics, i.e. the energy conservation principle. On the contrary, it seems that the formulation of a complete theory of energy transmission among oscillators would claim also for the second principle of thermodynamics. Such direction of investigation, via the entropy concept, is developed in this paper leading to a theoretical energy flow analysis to predict the energy exchanged among complex systems. Some classical SEA results are obtained as a special case of a more general approach.

1. INTRODUCTION

The concept of entropy was originally developed by Clausius in the classical theory of Thermodynamics [1]. In that frame entropy is defined in terms of exchanged heat and system temperature along a thermodynamic transformation. The development of Statistical Thermodynamics provided further insights into this concept leading to the Boltzmann definition of entropy [2–4]. It involves the macro- and micro-states of a general system of particles and provides an important generalisation of the entropy concept that goes beyond the limits of the physical context in which it was originally developed.

This paper is focused on the implication of the use of the second principle of thermodynamics in the energy analysis of mechanical oscillators. The problem is of practical importance. It is a matter of fact that very popular methods in this field, such as statistical energy analysis (SEA) [5–7], try to state an analogy between energy flow in mechanical systems and heat flow in thermal problems. Moreover, the energy conservation principle, i.e. the first law of thermodynamics, is used to provide the balance equations of the system's components. It is natural to wonder whether the second law of thermodynamics expressed by the entropy concept, finds a place in this context. A chance in this sense is here provided by introducing the entropy of a general resonator. This leads to the definition of an associated *thermodynamic temperature* that would have an important role in mechanical energy transfer.

It is revealing that some conclusions which are part of the bases of SEA, are here obtained starting from a new point of view about the power flow, i.e. by using an entropy formulation, that, at least apparently, nothing shares with the known approach to SEA.

This fact could open unexpected perspectives in power flow analysis showing new potentialities for energy methods.

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2. FORMULATION OF AN ENTROPY-BASED POWER FLOW APPROACH

It is here considered useful to give a general feeling of the proposed approach, focusing on the methodological aspects and on its innovative contents, rather than providing a rigorous mathematical formulation that, at least initially, is not intuitive. Formal arguments, related to the definition of entropy and its properties, are provided in the next sections.

It is well known that a quantitative formulation of the second principle of thermodynamics can pass through the concept of entropy. One of the most important results of classical and statistical thermodynamics is the well-known Boltzmann's inequality:

$$\mathrm{d}H/\mathrm{d}t \geq 0 \tag{1}$$

where H is the entropy of an isolated system subjected to a thermodynamic transformation and t is the time (the symbol H is here used following the original notation of Boltzmann). Equation (1) simply states that the entropy of an isolated system never decreases.

At the beginning one could be discouraged in attempting this way of approaching the power flow analysis between mechanical resonators. In fact, entropy is a complex quantity, whose physical meaning seems sometimes shifty even when it is met in the frame of classical thermodynamics. Nevertheless, such an operation is possible and fruitful.

To be convinced that this point of view deserves to be investigated, let us show, momentarily intuitively, how equation (1) plays a crucial role in the problem of energy transmission.

Suppose that we are able to define an entropy H for a mechanical N-dof resonator, and assume that for such a system an entropy equation (1) holds. Moreover, let us admit that H exhibits an additive property, i.e. the entropy of a composite system is just the sum of the individual entropy of each component.

Consider two resonators $\mathbf{R_1}$ and $\mathbf{R_2}$ whose total energy and entropy are E_1, H_1 and E_2, H_2, respectively (these can be elemental oscillators as well as complex resonators). When the resonators, initially separated, are coupled together and the obtained global system is left to itself, an energy transfer takes place. On the basis of the previous considerations, the following information to study the power flow between $\mathbf{R_1}$ and $\mathbf{R_2}$ can be used:

(a) energy conservation: $\mathrm{d}E_1/\mathrm{d}t + \mathrm{d}E_2/\mathrm{d}t = 0$
(b) Boltzmann's inequality: $\mathrm{d}H/\mathrm{d}t \geq 0$
(c) additive entropy property: $H = H_1 + H_2$

By combining (b) and (c) one has

$$\frac{\mathrm{d}H_1}{\mathrm{d}t} + \frac{\mathrm{d}H_2}{\mathrm{d}t} \geq 0 \Rightarrow \frac{\mathrm{d}H_1}{\mathrm{d}E_1}\frac{\mathrm{d}E_1}{\mathrm{d}t} + \frac{\mathrm{d}H_2}{\mathrm{d}E_2}\frac{\mathrm{d}E_2}{\mathrm{d}t} \geq 0$$

and using equation (a):

$$\left[\frac{\mathrm{d}H_1}{\mathrm{d}E_1} - \frac{\mathrm{d}H_2}{\mathrm{d}E_2}\right]\dot{W} \geq 0, \quad \dot{W} = \frac{\mathrm{d}E_1}{\mathrm{d}t} = -\frac{\mathrm{d}E_2}{\mathrm{d}t} \tag{2}$$

where $\dot{W}$ denotes the energy flow.

Equation (2) is a direct consequence of the Boltzmann inequality and of the energy conservation principle and expresses a relevant property of the quantity in brackets. In fact, the previous inequality states that the power flow direction (i.e. the sign of $\dot{W}$) depends on the sign of the difference between the entropy rates of the two resonators.

More precisely:

$$\mathrm{d}H_1/\mathrm{d}E_1 > \mathrm{d}H_2/\mathrm{d}E_2 \Rightarrow \dot{W} > 0$$
$$\mathrm{d}H_1/\mathrm{d}E_1 < \mathrm{d}H_2/\mathrm{d}E_2 \Rightarrow \dot{W} < 0.$$

In the first case the resonator $\mathbf{R_1}$ increases its energy [see equation (2)] and a net energy flows from $\mathbf{R_2}$ to $\mathbf{R_1}$, the converse being valid in the second case. Thus, it is natural to assume the quantity $\mathrm{d}H/\mathrm{d}E$ as a measure of the tendency of the system to absorb external energy, and, consequently, it is also natural to assume the inverse of it as the system's tendency to release its energy. This last quantity is the thermodynamic temperature T of the system. Thus:

$$1/T = \mathrm{d}H/\mathrm{d}E. \tag{3}$$

With this definition of T, one can assert that energy flows from the resonator having the higher temperature to the one having the lower temperature.

Thus, a rational and strict definition of temperature is obtained by equation (3).

The previous relationship is a key point of the present approach. In this equation three fundamental quantities are involved: energy, temperature and entropy. This point of view has a direct implication in the study of the power flow between resonators. Since, as stated by equations (2) and (3), T controls the power flow, the simplest way to express the power flow between $\mathbf{R_1}$ and $\mathbf{R_2}$ is by the linear relationship:

$$\dot{W} \propto (T_2 - T_1) \Rightarrow \dot{W} \propto \left(\frac{1}{\mathrm{d}H_1/\mathrm{d}E_1} - \frac{1}{\mathrm{d}H_2/\mathrm{d}E_2}\right). \tag{4}$$

This can be thought as a Taylor series of the power flow up to the first order in terms of the thermodynamic temperature difference.

Equations (3) and (4) suggest a new insight into the power flow analysis of mechanical resonators. However, it must be noticed that, when trying to develop the entropy analysis of mechanical resonators, the obtained results do not coincide completely with those of classical thermodynamics, as clarified in Section 3.

The missing point is still the introduction of the resonator entropy to which the following section is addressed.

3. ENTROPY OF MECHANICAL RESONATORS

An attempt to introduce entropy in vibro-acoustics has been recently made in [8], where both the Boltzmann [1–4] and the Shannon entropy [9] have been introduced for a mechanical resonator. A more general entropy concept for mechanical oscillators, derived by revisiting the Khinchin mathematical theory of Thermodynamics [10], is proposed in this paper.

The entropy concept illustrated in the following needs the introduction of some new elements: system and sub-system definitions, macro- and micro-states of a system and the related structure and generating functions.

By these bases a function H for mechanical resonators is built satisfying the requirements stated in the previous section. H is here called entropy of the resonator.

3.1. SUB-SYSTEMS, MACRO- AND MICRO-STATES, Ω AND Φ FUNCTIONS

Let us consider a dynamical system $\mathbf{R}$ whose state variables are represented by $\mathbf{x}(t), \dot{\mathbf{x}}(t)$. These vectors give a complete information on the actual state of the system and when their value is known at $t = t_0$ they suffice to predict the system time history for any $t > t_0$. Moreover, in the following, only conservative systems are considered characterised by an invariant total energy E, i.e. $\dot{E}(\mathbf{x}, \dot{\mathbf{x}}) = 0$. This property is valid provided that the system is isolated, i.e. it is left to itself, and the internal dissipation effects are absent or negligible.

While the energy E defines global properties of the system and partial information on its actual state, the vectors $\mathbf{x}, \dot{\mathbf{x}}$ on the contrary give detailed and complete information. Let us distinguish between the micro-states of the system, associated with all the possible values of the couple $\mathbf{x}, \dot{\mathbf{x}}$ and the system macro-states associated with the value of a global function depending on $\mathbf{x}, \dot{\mathbf{x}}$, e.g. E. It is clear that infinite different states $\mathbf{x}, \dot{\mathbf{x}}$ can lead to the same value of E. In other words infinite micro-states can be associated to a single macro-state. For example, it is obvious that the elemental spring–mass oscillator can have the same energy level for different combinations of $x, \dot{x}$, i.e. of its position (related to the potential elastic energy) and its velocity (related to the kinetic energy).

The space Γ defined by the variables $x_i, \dot{x}_i$ (components of $\mathbf{x}, \dot{\mathbf{x}}$ for $i = 1, N$) is called phase space.

The infinite micro-states associated with a given energy level E_0 (macro-state) are determined simply by imposing the constraint:

$$E(\mathbf{x}, \dot{\mathbf{x}}) = E_0.$$

This equation defines a hyper-surface Σ belonging to Γ. Looking at Fig. 1, the ellipsoidal surface Σ associated to the macro-state energy is represented (for the simple case $N = 1$) in Fig. 1. The points on it are system's micro-states, while the whole set of points on it is the macro-state associated to the given energy level. When a certain energy level E characterizes the macro-state of $\mathbf{R}$ at $t = 0$, then for any $t > 0$ the micro-states are constrained over the surface Σ of Γ and the point P of co-ordinates $x_1, x_2, \ldots, x_N, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N$ describes a trajectory lying on Σ.

Let us now introduce the energy decomposition of $\mathbf{R}$ into two sub-systems $\mathbf{R_1}$ and $\mathbf{R_2}$ (being the following considerations still valid for an arbitrary number of sub-systems).

$\mathbf{R_1}$ and $\mathbf{R_2}$ are defined sub-systems of $\mathbf{R}$ if the sum of their energies E_1 and E_2 equals the energy of $\mathbf{R}$ and if E_1 and E_2 depend on two sets of variables that have an empty intersection and whose union provides the set of variables of $\mathbf{R}$.

Introducing the state vectors $\mathbf{v}$, $\mathbf{v_1}$ and $\mathbf{v_2}$ of $\mathbf{R}$, $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively, the previous statement takes the concise form:

$$E(\mathbf{v}) = E_1(\mathbf{v}_1) + E_2(\mathbf{v}_2), \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$
$$\mathbf{v} = \{\mathbf{x}, \dot{\mathbf{x}}\}^T = \{\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2\}^T, \quad \mathbf{v}_1 = \{\mathbf{x}_1, 0, \dot{\mathbf{x}}_1, 0\}^T, \quad \mathbf{v}_2 = \{0, \mathbf{x}_2, 0, \dot{\mathbf{x}}_2\}^T.$$

For a general choice of the state variables, this decomposability property is not trivial at all, even for simple linear systems. In fact it is well known that

$$E(\mathbf{v}) = E_1(\mathbf{v}_1) + E_2(\mathbf{v}_2) + E_{12}(\mathbf{v}_1, \mathbf{v}_2)$$

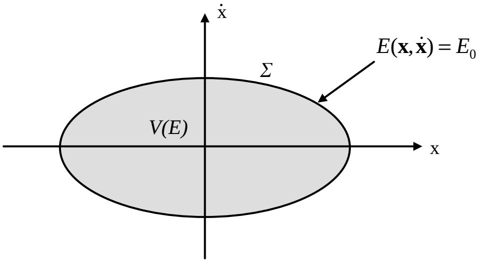

Figure 1. Volume V and surface Σ in the phase space single-dof system.

the third term being on the right-hand side the mixed energy contribution. This happens typically when $\mathbf{x}, \dot{\mathbf{x}}$ represent physical co-ordinates of the resonator, e.g. displacement and velocity of oscillating physical masses belonging to **R**. In this case the system decomposability in energy sub-systems is not strictly possible. However, in some cases, the mixed energy term is negligible in comparison with the other energy contributions, i.e.

$$E_1 >> E_{12}, \quad E_2 >> E_{12}.$$

This condition is briefly indicated in the following as weak coupling condition.

When $\mathbf{x}, \dot{\mathbf{x}}$ as well as $\mathbf{x}_1, \dot{\mathbf{x}}_1$ and $\mathbf{x}_2, \dot{\mathbf{x}}_2$ are physical co-ordinates, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be also identified as physical parts of **R**. In such a case the energy decomposability expresses an approximate property only, holding under the hypothesis of weak coupling.

However, a chance of choosing the state variable allowing for an exact energy decomposability exists. This can be done at least when considering the special case of linear systems, when **x** is the vector of the normal co-ordinates of **R**.

In this case the energy expression does not contain mixed terms, i.e. it is simply the sum of the energies related to each normal co-ordinate (modal energies). In this way any arbitrary decomposition, obtained by groups of modal co-ordinates, leads to a set of energy sub-systems in the sense specified by the previous definition. Note that, when using modal co-ordinates, the sub-systems are not necessarily physical parts of **R** but they are modes or groups of modes of **R**.

Let us now introduce the structure function and the generating function.

The equal energy surface Σ wraps the volume V_Σ. The micro-states with energy E belong to Σ so that the set of micro-states $\mathbf{x}, \dot{\mathbf{x}}$ having energy $E' < E$ fall inside the volume wrapped by Σ. The measure of this volume can be simply introduced as follows:

$$V(E) = \int_{V_\Sigma} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N \mathrm{d}\dot{x}_1 \mathrm{d}\dot{x}_2 \ldots \mathrm{d}\dot{x}_N$$

that is a function of the considered energy level E, since Σ is the energy surface associated to E. Let us consider the function $\Omega(E)$ obtained by deriving $V(E)$ with respect to the energy E, i.e.

$$\Omega(E) = \frac{\mathrm{d}V}{\mathrm{d}E}.$$

This is named structure function of **R**. Although this definition is self-contained, a more physical interpretation of it is given in Appendix A, where it is shown that the structure function provides a measure of the number of micro-states associated to a given energy level.

We are now in the position of introducing the fundamental composition property of the structure function. Let us consider the decomposition of **R** in terms of two energy sub-systems $\mathbf{R_1}$ and $\mathbf{R_2}$ whose structure functions are $\Omega_1(E_1)$ and $\Omega_2(E_2)$, respectively. The goal is the determination of the structure function $\Omega(E)$ of **R** in terms of the structure functions of the two energy sub-systems $\mathbf{R_1}$ and $\mathbf{R_2}$. It is not too difficult to demonstrate (see [10] or, following a different approach, Appendix B) that the composition rule is provided by

$$\Omega(E) = \int_{-\infty}^{+\infty} \Omega_1(E_1)\Omega_2(E - E_1)\,\mathrm{d}E_1 = \Omega_1 * \Omega_2.$$

This result represents the structure function of a composite resonator in terms of convolution between the structure functions of its energy sub-systems.

Taking the Laplace transform of the previous equation, the composition rule simplifies as

$$\Phi(s) = \Phi_1(s)\Phi_2(s)$$

where $\Phi(s)$ is referred as the generating function of $\mathbf{R}$. These conclusions hold when $\mathbf{R_1}$ and $\mathbf{R_2}$ are sub-systems of $\mathbf{R}$, i.e. if they are weakly coupled or are modal groups of $\mathbf{R}$.

The composition rule is the key to provide the two properties of entropy shown in Section 3.4.

3.2. ENTROPY OF A MECHANICAL OSCILLATOR AND ITS TEMPERATURE

In this section the definition of entropy and its implications concerning the derivation of the resonator temperature is illustrated.

Let us consider the function $f(s,E)$ defined as follows:

$$f(s,E) = \log\left\lfloor e^{sE}\Phi(s)\right\rfloor$$

associated to a given energy level E of $\mathbf{R}$. Consider $f(s,E)$ as a function of the Laplace variable s only, while E is regarded momentarily as a simple parameter. Let us search for a relative minimum of $f(s,E)$. The value $s = \sigma$ for which such a minimum holds, is simply determined as follows:

$$\frac{\partial f}{\partial s} = \frac{1}{e^{sE}\Phi(s)}[Ee^{sE}\Phi(s) + e^{sE}\Phi'(s)] = 0 \quad \text{i.e.} \quad E = -\left.\frac{\Phi'(s)}{\Phi(s)}\right|_{s=\sigma}. \tag{5}$$

This equation defines the value of σ that keeps the wished minimum and it is also clear that it depends on the energy value E, i.e. $\sigma = \sigma(E)$. When considering $f(\sigma(E),E)$, i.e. the minimum value of f, a function of the energy E only is obtained.

This is the Khinchin entropy $H(E)$ of $\mathbf{R}$. Note that by the definition of entropy, any constant value can be added to H without modifing the value of σ and equation (5). Thus entropy is

$$H(E) = \log[e^{\sigma E}\Phi(\sigma)] + const \quad \sigma \text{ being the solution of } \frac{\Phi'(\sigma)}{\Phi(\sigma)} = -E.$$

$H(E)$ so defined satisfies the two fundamental requirements outlined in Section 2, i.e. the Boltzmann's inequality and the addition property, that will be illustrated later in Section 3.4.

On the contrary, here it is useful to stress some important physical implications of the stated definition.

The basic question to which the entropy expression allows to answer is what kind of dependency exists between the thermodynamic temperature T and the energy E. After simple mathematics and by definition (3), one has

$$\frac{dH}{dE} = \frac{d\sigma}{dE}E + \sigma + \frac{\Phi'(\sigma)}{\Phi(\sigma)}\frac{d\sigma}{dE} = \sigma \quad \text{and} \quad \sigma = \frac{1}{T}.$$

This equation provides the chance of deriving a basic result. In fact, because of the relationship between σ and E given by the second of equation (5), one can write

$$\frac{\Phi'(1/T)}{\Phi(1/T)} = -E. \tag{6}$$

The focal point of the present work in centred around this equation. The quantity that actually controls the power flow, accordingly with the analysis developed in Section 2, is T and it is a function of the system energy E. Equation (6) provides implicitly the relationship between these two variables.

Thus a formal procedure to determine the expression of the thermodynamic temperature, and to derive a well founded power flow analysis is established as follows:

(a) calculate the volume V defined in Section 3.1;
(b) determine the structure function $\Omega = \mathrm{d}V/\mathrm{d}E$;
(c) calculate the generating function Φ by the Laplace transform of Ω;
(d) solve equation (6) with respect to T.

This approach is applied in Section 4 to derive a basic result concerning the power flow between N-dof linear resonators.

3.3. ENTROPY PROPERTIES OF MECHANICAL RESONATORS

Let us consider two initially separated resonators $\mathbf{R_1}$ and $\mathbf{R_2}$ characterised by an initial condition of absence of any energy interaction (*adiabatic* point), i.e. $\dot{E}_1(0) = 0, \dot{E}_2(0) = 0$, E_1 and E_2 being their respective energies. The entropy related to each resonator is expressed by:

$$H_1(E_1) = \log[\mathrm{e}^{\sigma_1 E_1} \Phi_1(\sigma_1)]\ \sigma_1 \text{ being the solution of } \frac{\Phi_1'(\sigma_1)}{\Phi_1(\sigma_1)} = -E_1$$

$$H_2(E_2) = \log[\mathrm{e}^{\sigma_2 E_2} \Phi_2(\sigma_2)]\ \sigma_2 \text{ being the solution of } \frac{\Phi_2'(\sigma_2)}{\Phi_2(\sigma_2)} = -E_2 \tag{7}$$

Let us now couple the two resonators so that $\mathbf{R_1}$ and $\mathbf{R_2}$ can be considered two energy sub-systems of the global resonator $\mathbf{R}$ obtained by their connection. The coupling produces an energy exchange between $\mathbf{R_1}$ and $\mathbf{R_2}$. The entropy $H(E)$ of $\mathbf{R}$ is

$$H(E) = \log[\mathrm{e}^{\sigma E} \Phi(\sigma)]\ \sigma \text{ being the solution of } \frac{\Phi'(\sigma)}{\Phi(\sigma)} = -E$$

Since $\mathbf{R_1}$ and $\mathbf{R_2}$ are sub-systems of $\mathbf{R}$ it follows $E = E_1 + E_2$ and $\Phi = \Phi_1 \Phi_2$. Therefore,

$$H(E) = \log\left\lfloor \mathrm{e}^{\sigma(E_1+E_2)} \Phi_1(\sigma)\Phi_2(\sigma) \right\rfloor = \log\left\lfloor \mathrm{e}^{\sigma E_1} \Phi_1(\sigma) \right\rfloor + \log\left\lfloor \mathrm{e}^{\sigma E_2} \Phi_2(\sigma) \right\rfloor$$

where the two determined contributions on the right-hand side are smaller than H_1 and H_2, respectively. The entropy definition indeed states that H_1 and H_2 are the minimum of the functions $\log\left\lfloor \mathrm{e}^{sE_1} \Phi_1(s) \right\rfloor, \log\left\lfloor \mathrm{e}^{sE_2} \Phi_2(s) \right\rfloor$ kept for $s = \sigma_1$ and $s = \sigma_2$, respectively. Therefore, it follows that when evaluating the previous functions for $s = \sigma$, one has

$$H = \log\left\lfloor \mathrm{e}^{\sigma E_1} \Phi_1(\sigma) \right\rfloor + \log\left\lfloor \mathrm{e}^{\sigma E_2} \Phi_2(\sigma) \right\rfloor \geq H_1 + H_2. \tag{8}$$

This result states: when coupling two resonators $\mathbf{R_1}$ and $\mathbf{R_2}$, such that they are sub-systems of the obtained composite resonator $\mathbf{R}$, the entropy of $\mathbf{R}$ is always greater than or equal to the sum of the individual entropy of $\mathbf{R_1}$ and $\mathbf{R_2}$ before the coupling. This is the entropy inequality property. Now the addition property is illustrated.

Preliminarily we introduce the concept of thermodynamic equilibrium between resonators: two coupled resonators $\mathbf{R_1}$ and $\mathbf{R_2}$ are in thermodynamic equilibrium if $\sigma_1 = \sigma_2$ (*equilibrium* point), i.e., accordingly with the definition given in the previous section, when they have the same thermodynamic temperature T.

Let us consider two coupled energy sub-systems in thermodynamic equilibrium that form the composite resonator $\mathbf{R}$ characterised by the quantities E, H, σ, Φ.

Due to the equilibrium it is: $\sigma_1 = \sigma_2 = \sigma^*$. If the respective energies of $\mathbf{R_1}$ and $\mathbf{R_2}$ after the coupling are E_1^*, E_2^*, one has:

$$E_1^* = -\frac{\Phi_1'(\sigma^*)}{\Phi_1(\sigma^*)}, \quad E_2^* = -\frac{\Phi_2'(\sigma^*)}{\Phi_2(\sigma^*)}$$

$$E_1^* + E_2^* = -\frac{\Phi_1'(\sigma^*)}{\Phi_1(\sigma^*)} - \frac{\Phi_2'(\sigma^*)}{\Phi_2(\sigma^*)} = -\frac{(\mathrm{d}/\mathrm{d}\sigma^*)[\Phi_1(\sigma^*)\Phi_2(\sigma^*)]}{\Phi_1(\sigma^*)\Phi_2(\sigma^*)}.$$

Since the two resonators are energy sub-systems of $\mathbf{R}$, then $E = E_1^* + E_2^*$ and $\Phi(\sigma) = \Phi_1(\sigma)\Phi_2(\sigma)$ and, from the previous written equation, follows that $\sigma = \sigma^*$. Then the entropy of $\mathbf{R}$ is

$$H(E) = \log[\mathrm{e}^{\sigma E}\Phi(\sigma)] = \log\left\lfloor \mathrm{e}^{\sigma^* E_1^*}\Phi_1(\sigma^*)\right\rfloor + \log\left\lfloor \mathrm{e}^{\sigma^* E_2^*}\Phi_2(\sigma^*)\right\rfloor = H_1^* + H_2^*. \tag{9}$$

This equation reads: the entropy of a composite resonator equals the sum of the entropy of its energy sub-systems when they are in thermodynamic equilibrium.

By combining equations (8) and (9) the following inequality is obtained:

$$H_1^* + H_2^* \geq H_1 + H_2 \Rightarrow \Delta H_1 + \Delta H_2 \geq 0 \Rightarrow \Delta H \geq 0 \tag{9a}$$

i.e. an entropy increase is expected when passing from an *adiabatic* (A) to an *equilibrium* (E) point of the resonators (call it $A \Rightarrow E$ transition).

When considering an elemental energy exchange $\mathrm{d}E$ and the time $\mathrm{d}t$ (of course always positive) this energy transfer takes, it follows $\mathrm{d}H/\mathrm{d}t \geq 0$, that corresponds to the relationship named in Section 2 Boltzmann's inequality. This implies that the equivalence between equations (9a) and (1) is allowed only if the states A and E are close enough. When this requirements is not satisfied, the entropy inequality holds only in the incremental form stated by equation (9a) but not in the differential fashion given by equation (1). This means that an increasing average trend of the mechanical entropy is certainly expected along any finite time interval Δt, as demonstrated by equation (9a), but the instantaneous entropy rate $\mathrm{d}H/\mathrm{d}t$ is not necessarily always positive.

However, a correct interpretation of equation (9a) needs further comments. Equation (9a) has been obtained assuming that the system is proceeding towards the equilibrium point, i.e. the considered process is the $A \Rightarrow E$ transition. Nevertheless, in the frame of our analysis, no reason induces to exclude transitions of the type $E \Rightarrow A$. It is a matter of fact that, e.g. for mechanical resonators exhibiting free oscillations periodicity, a periodic entropy is expected; thus a periodic sequence of transitions $\{A \Rightarrow E \Rightarrow A \ldots\}$ is also expected. In this case the entropy property reads:

$$\Delta H_{A\Rightarrow E} \geq 0, \quad \Delta H_{E\Rightarrow A} \leq 0 \tag{9b}$$

clearly derived by equation (9a). It means that, for mechanical systems, $E \Rightarrow A$ transitions can be also observed implying an entropy decrease. The previous fact has a physical implication: when mechanical oscillators are coupled a positive energy flow from the resonator having lower energy to the one having higher energy can be observed (transition $E \Rightarrow A$ with $\Delta H_{E\Rightarrow A} \leq 0$).

Considering the arguments presented in Section 2 and equations (9b), equation (2) takes the two-fold form:

$$\left[\frac{\mathrm{d}H_1}{\mathrm{d}E_1} - \frac{\mathrm{d}H_2}{\mathrm{d}E_2}\right]\dot{W} \geq 0 \text{ along } A \Rightarrow E, \quad \left[\frac{\mathrm{d}H_1}{\mathrm{d}E_1} - \frac{\mathrm{d}H_2}{\mathrm{d}E_2}\right]\dot{W} \leq 0 \quad \text{along } E \Rightarrow A.$$

However, again it can be concluded that the entropy rate vs energy is the quantity controlling the energy flow in both the considered cases. Rather, the difference between the transition $A \Rightarrow E$ and the transition $E \Rightarrow A$, is related to the way the entropy rate controls the energy flow. A simple solution to this problem consists in changing the sign of the right-hand side of equation (3) when considering a $E \Rightarrow A$ transition, i.e. $1/T = -\mathrm{d}H/\mathrm{d}E$.

However, it must be noticed that although the periodicity in the free response of a linear mechanical resonator is possible and depending on the values of its natural frequencies, it is intuitive that, in general, as the number N of modes increases, the chance of a periodic free response tends to be lost, or in other words, the probability to observe a periodicity decreases. In particular as N tends to infinity, i.e. for complex resonators, the response tends to be non-periodic (i.e. the probability of a periodic response tends to zero). Thus, for resonators complex enough, the periodic sequence transition $A \Rightarrow E \Rightarrow A \dots$ is not actually observed, it being replaced by a single infinitely long $A \Rightarrow E$ transition. Thus, the energy process considered by inequality (9a) can be thought as an $A \Rightarrow E$ transition in which the *equilibrium* point is reached after an infinitely long time. It means that in the present approach, the hypothesis of complex resonator equipped with a large number of modes is implicitly accepted and a $E \Rightarrow A$ transition becomes largely improbable. Under these conditions, taking the form (9a) only is legitimate. This is what happens in Statistical Thermodynamics that deals with systems characterised by a great number of degrees of freedom (e.g. the atomic lattice).

A final remark concerns the absence of external forces in the previous analysis. It can be stated that if we accept that the constitutive relationship [see equations (4) and (11)] providing the energy flow expression depends only on the state variables of the resonators but not explicitly on the driving force, this expression can be determined irrespective of the nature of the force acting on the system. This intrinsic nature of the constitutive relationship is highly reasonable and common to many fields of physics such as elasticity, fluid-dynamics, electro-dynamics, etc. Thus, the analysis of energy sharing between freely interacting oscillators developed in the previous sections provides a constitutive relationship [see equation (11)] that should be still valid even in the forced case.

4. ENTROPY ANALYSIS OF ENERGY FLOW BETWEEN *N*-DOF RESONATORS

The aim of the present section is to apply the outlined entropy methodology to the analysis of the energy exchange between two very general systems leading to a simple equation expressing the energy flow between them.

Consider an N-dof linear resonators whose kinetic and potential energies are T and U, respectively. The procedure given at the end of Section 3.2, consisting of the steps (a)–(d), is applied to determine the thermodynamic temperature of the resonator in a fashion similar to that used in [10] to derive the statistical theory of the ideal monatomic gas.

Let us start with step (a). The equal energy surface Σ, associated with the energy level E, is represented in the phase space Γ by the equation

$$\tfrac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}} + \tfrac{1}{2}\mathbf{q}^T\Lambda\mathbf{q} - E = 0, \quad \text{being } T = \tfrac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}}, \quad U = \tfrac{1}{2}\mathbf{q}^T\Lambda\mathbf{q} \qquad (10)$$

where $\mathbf{q}$ is the vector of the normal co-ordinates of the resonator and $\mathbf{\Lambda}$ the diagonal eigenvalues matrix. It must be underlined that the energy of the resonator has been assumed to be dependent only on the state variables of the resonators itself. No mixed energy terms are considered, thus implicitly assuming a weak interaction of the considered system with others.

The first of equations (10) represents an ellipsoid in Γ, whose semi-axes lengths are easily determined. In fact, the intersections of this ellipsoid with the q_i and $\dot{q}_i$ axes are given by solving the previous equation by setting to zero the whole set of co-ordinates except one, i.e.

$$\tfrac{1}{2}q_i^2\lambda_i = E, \quad q_i = \sqrt{2E/\lambda_i}$$

$$\tfrac{1}{2}\dot{q}_i^2 = E, \quad \dot{q}_i = \sqrt{2E} \quad i = 1, \ldots, N$$

where N is the number of the degrees of freedom and also the number of modes.

Since the volume V of an ellipsoid, of given semi-axes $q_i = \sqrt{2E/\lambda_i}$ and $\dot{q}_i = \sqrt{2E}$, is simply proportional to the products of all the semi-axis lengths, one has

$$V = A(2E)^{N/2}\frac{(2E)^{N/2}}{\prod_{i=1}^{N}\lambda_i} = 2^N A\frac{E^N}{\prod_{i=1}^{N}\lambda_i}$$

A being a proportionality constant that does not play any role in the following.

Following steps (b) and (c), the structure function Ω and the associated generating function Φ(s) are:

$$\Omega = \frac{\mathrm{d}V}{\mathrm{d}E} = 2^N A\frac{NE^{N-1}}{\prod_{i=1}^{N}\lambda_i}, \quad \Phi(s) = 2^N A\frac{N}{\prod_{i=1}^{N}\lambda_i}\frac{(N-1)!}{s^N} = 2^N A\frac{N!}{\prod_{i=1}^{N}\lambda_i}\frac{1}{s^N}.$$

Finally, step (d), leading to the evaluation of the thermodynamic temperature of the resonator, is achieved by equation (6), i.e.

$$\left.\frac{\Phi'(s)}{\Phi(s)}\right|_{s=\sigma=1/T} = -E, \Rightarrow -\left.\frac{N}{s}\right|_{s=\sigma=1/T} = -E \Rightarrow T = \frac{E}{N}.$$

This is a key result of this paper: the thermodynamic temperature of a linear system is proportional to the average modal energy E/N. Thus, accordingly with equation (4), the power flow between two resonators $\mathbf{R_1}$ and $\mathbf{R_2}$, can be expressed by

$$\dot{W} \propto \left(\frac{E_1}{N_1} - \frac{E_2}{N_2}\right). \tag{11}$$

As mentioned before, a hypothesis of weak coupling has been implicitely assumed, since mixed energy terms in equations (10) have been neglected.

In this way, a demonstration of one of the most important results of SEA is provided on the basis of the entropy concept.

5. A NUMERICAL EXAMPLE

To check the result expressed by inequality (9a), that is the founding element of the entropy approach, a numerical analysis of the entropy time history of a two-dof linear system is here presented.

The system consists of two identical oscillators with mass $m = 1\,\mathrm{kg}$ and stiffness $k = 1\,\mathrm{N/m}$ (Fig. 2). They initially vibrate separately and no energy interaction takes place, i.e. $\dot{E}_1(0) = 0$ and $\dot{E}_2(0) = 0$ (*adiabatic* point A). Moreover, they initially have different energies, e.g. $E_1(0) = E_0$ and $E_2(0) = 0$, and at $t = 0$ they are coupled by a spring of constant $k = 0.1\,\mathrm{N/m}$ (weak coupling). Thus inequality (9) should predict that the maximum entropy is reached when the resonators find the *equilibrium* point $E_1 = E_2 = E_0/2$, producing a net energy transfer. When considering the results of Sections 3 and 4 with $N = 1$, the expressions for each of the two resonators

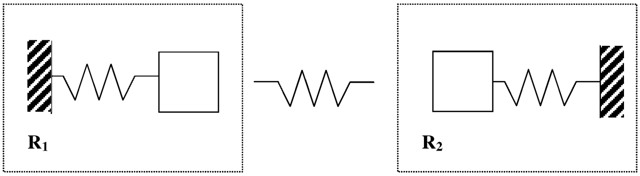

Figure 2. Illustration of the coupling of the two resonators.

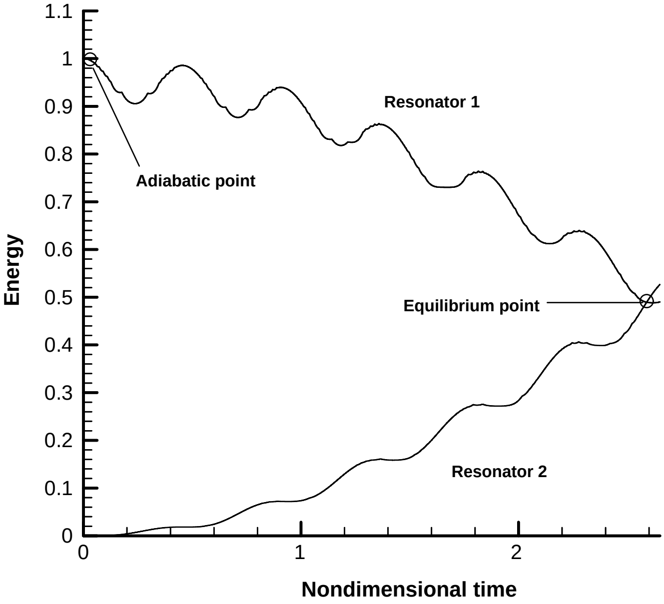

Figure 3. Energy of the two resonators proceeding towards the equilibrium point.

follow:

$$\Phi(s) = \frac{2}{\sqrt{mk}}\frac{1}{s}, \qquad \sigma = \frac{1}{T}, \qquad T = E$$

$$\Phi(E) = \frac{2}{\sqrt{mk}}E, \qquad H(E) = 1 + \log\left(\frac{2}{\sqrt{mk}}E\right) + const.$$

By letting the arbitrary constant equal to $-1 - \log(E_0/\sqrt{mk})$, the right-hand side of inequality (9a) reads:

$$H_1 + H_2 = \log\left(\frac{4E_1E_2}{E_0^2}\right).$$

One can just verify that $H_1 + H_2 \to -\infty$ at the *adiabatic* point, while at the *equilibrium* point $H_1^* + H_2^* = 0$, simply confirming inequality (9a).

In Fig. 3 the time history of both the dimensionless energies E_1/E_0 and E_2/E_0 is given vs the dimensionless time $2\pi t\sqrt{k/m}$ during the interval leading from the initial adiabatic

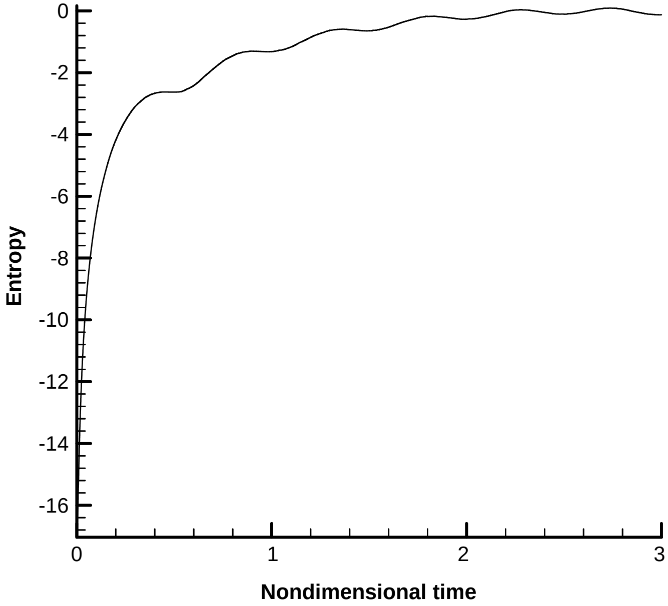

Figure 4. Total entropy of the coupled resonators proceeding towards the equilibrium point.

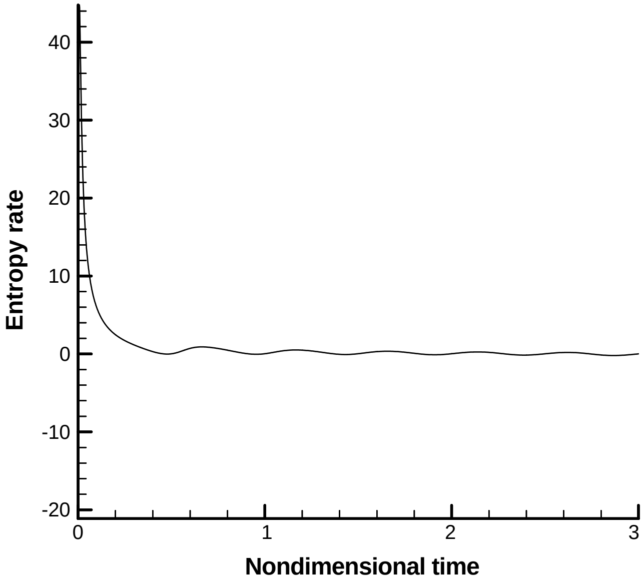

Figure 5. Entropy rate time history.

point to the first equilibrium condition (obvious numerical reasons suggest to let $E_1(0) = (1 - a)E_0$ and $E_2(0) = aE_0$, $a = 10^{-8}$ instead of zero $E_1(0) = E_0$, $E_2(0) = 0$). In Fig. 4 the time history of $H_1 + H_2$ clearly shows the increasing trend predicted by (9a). Finally in Fig. 5 the entropy rate $\dot{H}$ is given, it being positive until the condition of energy equilibrium is reached.

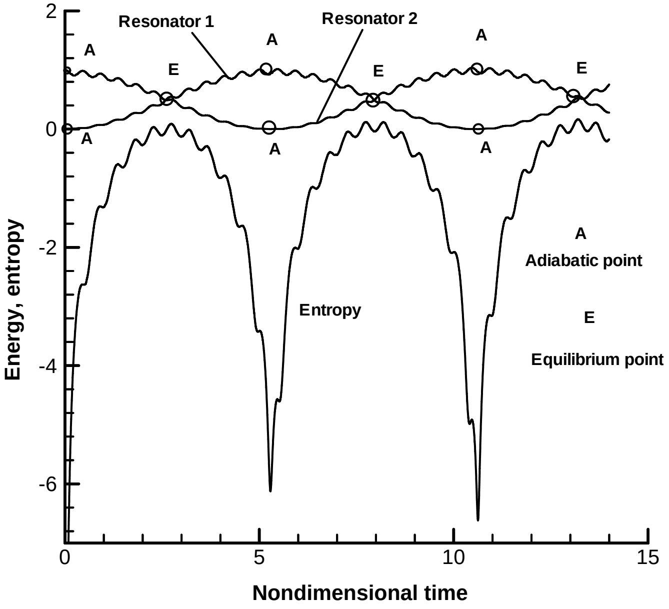

Figure 6. A–E–A. . .tranformation sequence.

However, in Section 3 it has been specified that beside the $A \Rightarrow E$ transition, the $E \Rightarrow A$ transition can also be observed for mechanical resonators. Thus investigating a larger time scale, sequential transitions $\{A \Rightarrow E \Rightarrow A \ldots\}$ are discovered, as shown in Fig. 6, where the upper curves represent the energies of the two resonators, while the lower curve is the total entropy of the system obeying inequalities (9a) and (9b).

6. CONCLUSIONS

In this paper a suitable entropy-based formulation of power flow analysis between mechanical resonators is presented, accounting both for the first and the second principle of thermodynamics. This analysis allows a rational approach to the problem of energy sharing in mechanical systems. The systematic application of the proposed methodology to the case of linear weakly coupled multi-dof resonators leads to a general demonstration of one of the basic statement of SEA. However, the entropy approach seems to have the chance of interesting generalisations to a wider class of resonators such as non-linear systems. In fact the linearity of elastic forces is not claimed by this approach and even in this case the thermodynamic temperature should be obtained following the procedure given in Section 3.2.

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APPENDIX A

$\Omega(E)$ is actually related to a suitable measure of the set of micro-states associated with a given energy level. The following considerations show how the function $\Omega(E)$ keeps this information.

Let us introduce a small energy band B_E centred around the value E i.e. $B_E \equiv [E - \Delta E/2, E + \Delta E/2]$. The two energy levels $E - \Delta E/2, E + \Delta E/2$ define two energy surfaces Σ_- and Σ_+, respectively. The gap volume $V_{\Sigma(-,+)}$ enclosed between Σ_- and Σ_+, contains the set of micro-states whose energy E' is:

$$E - \Delta E/2 < E' < E + \Delta E/2.$$

Thus, when varying the energy level E (being the bandwidth definitely assigned), the set of micro-states trapped between Σ_- and Σ_+ varies too. In this way a correspondence between the macro-states defined by E and the micro-states contained inside $V_{\Sigma(-,+)}$ is established. These are the micro-states corresponding to the small energy band B_E located around E, or more simply associated to E itself, once the bandwidth is assigned.

Again a natural way to introduce the measure of this set of micro-states can be obtained by the measure $V_{(-,+)}$ of $V_{\Sigma(-,+)}$. This can be evaluated by the difference between the volumes contained inside the surfaces Σ_+ and Σ_- respectively, i.e.

$$V_{(-,+)} = V(E + \Delta E/2) - V(E - \Delta E/2).$$

Thus,

$$V_{(-,+)} = \frac{V(E + \Delta E/2) - V(E - \Delta E/2)}{\Delta E} \approx \frac{\mathrm{d}V}{\mathrm{d}E}\Delta E = \Omega(E)\Delta E$$

is valid for a bandwidth small enough.

Since the bandwidth is constant, the previous relationship legitimate the conclusion that the structure function $\Omega(E)$ provides a measure of the set of micro-states associated to a small energy band around E.

This measure can be further specified by introducing the concept of number of states. If the whole phase space Γ is partitioned into small cells of constant volume $\Delta\Gamma = \Delta x_1, \Delta x_2, \ldots, \Delta x_N, \Delta\dot{x}_1, \Delta\dot{x}_2, \ldots, \Delta\dot{x}_N$ and all the micro-states belonging to the same cell are considered equivalent (i.e. they are actually the same micro-state), then a finite number N_S of micro-states (or cells) fall inside the volume $V_{\Sigma(-,+)}$. More precisely

$$N_S = \frac{V_{(-,+)}}{\Delta\Gamma} = \Omega\frac{\Delta E}{\Delta\Gamma}.$$

Of course, $\Delta\Gamma$ as well ΔE are arbitrary yet small. However, the ratio $\Delta E/\Delta\Gamma$ is a proportionality constant (i.e. it does not depend on E) between N_S and the structure function Ω. This fact justifies the interpretation of the structure function as the number of micro-states having energy in the small band B_E, or, more simply, associated to the energy level E, once the bandwidth is assigned.

This result is useful to prove the convolution property of Ω given in Appendix B.

APPENDIX B

Let us assign the energy E of $\mathbf{R}$; since $\mathbf{R_1}$ and $\mathbf{R_2}$ are energy decompositions of $\mathbf{R}$, then:

$$E = E_1(\mathbf{v}_1) + E_2(\mathbf{v}_2) \Rightarrow E_2(\mathbf{v}_2) = E - E_1(\mathbf{v}_1).$$

It means that once E is assigned, the energies E_1 and E_2 cannot vary arbitrarily, because $E_1 + E_2 = E$ and the obvious constraints $E_1 < E, E_2 < E$ hold. These requirements are satisfied by varying in an arbitrary fashion E_1 in the range $[0, E]$ and computing consequently E_2 as $E_2 = E - E_1$. Thus for any given energy $E_1 \in [0, E]$ two macro-states E_1 and $E_2 = E - E_1$ for $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively, are defined. The values of the structure functions $\Omega_1(E_1)$ and $\Omega_2(E - E_1)$ are associated to them and, in force of the previously stated arguments, the number of micro-states:

$$N_1(E_1) = \Omega_1(E_1)\frac{\Delta E_1}{\Delta\Gamma_1} \quad \text{and} \quad N_2(E - E_1) = \Omega_2(E - E_1)\frac{\Delta E_2}{\Delta\Gamma_2}.$$

All the combinations of the N_1 micro-states of $\mathbf{R}_1$ and of the N_2 micro-states of $\mathbf{R_2}$ are given by:

$$N(E_1, E) = N_1(E_1)N_2(E - E_1).$$

where $N(E_1, E)$ is the number of possible micro-states of $\mathbf{R}$ when choosing for $\mathbf{R_1}$ the energy level E_1 and the total energy equal to E. The chance of using this formula is provided by the fact that the two sub-systems have no common state variables. Since E_1 can range in the interval $[0, E]$, the total number $N(E)$ of micro-states of $\mathbf{R}$ having energy E is found by summing $N(E_1^{(i)}, E)$ for $i = 1, N_E$, where $E_1^{(i)} = i\Delta E_1$ and $N_E = E/\Delta E_1$. Thus:

$$N(E) = \sum_{i=1}^{N_E} N_1(E_1^{(i)})N_2(E - E_1^{(i)})\Delta E_1.$$

This equation can be also expressed by using the structure functions, i.e.

$$N_1(E_1)N_2(E - E_1) = \Omega_1(E_1)\Omega_2(E - E_1)\frac{\Delta E_1}{\Delta\Gamma_1}\frac{\Delta E_2}{\Delta\Gamma_2}$$

$$N(E) = \Omega(E)\frac{\Delta E}{\Delta\Gamma}.$$

Thus,

$$\Omega(E)\frac{\Delta E}{\Delta\Gamma} = \sum_{i=1}^{N_E} \Omega_1(E_1^{(i)})\Omega_2(E - E_1^{(i)})\frac{\Delta E_1}{\Delta\Gamma_1}\frac{\Delta E_2}{\Delta\Gamma_2}.$$

Since $\mathbf{R_1}$ and $\mathbf{R_2}$ are an energy decomposition of $\mathbf{R},$ the union of their set of variables, i.e. $\mathbf{x}_1, \dot{\mathbf{x}}_1$ and $\mathbf{x}_2, \dot{\mathbf{x}}_2$, provides just the set of variables of $\mathbf{R}$ i.e. $\mathbf{x}, \dot{\mathbf{x}}$; moreover, the two sets have an empty intersection. This hypothesis allows to write:

$$\Delta\Gamma_1 = \Delta x_1, \Delta x_2, \ldots, \Delta x_M, \Delta\dot{x}_1, \Delta\dot{x}_2, \ldots, \Delta\dot{x}_M$$

$$\Delta\Gamma_2 = \Delta x_{M+1}, \Delta x_{M+2}, \ldots, \Delta x_N, \Delta\dot{x}_{M+1}, \Delta\dot{x}_{M+2}, \ldots, \Delta\dot{x}_N$$

$$\Delta\Gamma = \Delta x_1, \Delta x_2, \ldots, \Delta x_M, \Delta x_{M+1}, \Delta x_{M+2}, \ldots, \Delta x_N, \Delta\dot{x}_1, \Delta\dot{x}_2, \ldots, \Delta\dot{x}_M, \Delta\dot{x}_{M+1}, \Delta\dot{x}_{M+2}, \ldots, \Delta\dot{x}_N$$

so that:

$$\Delta\Gamma = \Delta\Gamma_1 \Delta\Gamma_2.$$

Moreover, the bandwidths $\Delta E, \Delta E_1, \Delta E_2$, being arbitrary, can be set equal (note however, that any other choice does not alter relevantly the mathematical derivation of results.

Therefore,

$$\Omega(E) = \sum_{i=1}^{N_E} \Omega_1(E_1^{(i)})\Omega_2(E - E_1^{(i)})\Delta E_1 \approx \int_0^E \Omega_1(E_1)\Omega_2(E - E_1)\,\mathrm{d}E_1.$$

Since in the previous integral the two structure functions $\Omega_1(E_1)$ and $\Omega_2(E - E_1)$ have arguments ranging inside $[0, E]$ only, without any loss of generality, we can set:

$$\Omega_i(E_i) = 0 \text{ for } E_i < 0,\ E_i > E$$

and rewrite the integral by using the equivalent form:

$$\Omega(E) = \int_{-\infty}^{+\infty} \Omega_1(E_1)\Omega_2(E - E_1)\,\mathrm{d}E_1 = \Omega_1 * \Omega_2.$$