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An Iterative Rational Fraction Polynomial Technique for Modal Identification*

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Abstract. The rational fraction polynomial (RFP) modal identification procedure is a well known frequency domain fitting technique. To deal with a linear problem, the RFP procedure does not directly minimize the fitting error, i.e. the difference between the experimental and the analytical frequency response function, but a frequency weighted function of it: this causes bias in the modal parameter estimates. In this paper an iteration procedure is proposed which uses the output of the RFP technique as a starting estimate, and minimizes the true fitting error, expressed as a first order Taylor expansion of the identified parameters. Results are quite satisfactory: the fitting error is notably reduced after few iterations. Moreover, less computational modes with respect to the original RFP method are needed to obtain a good fit in a given frequency band.

Sommario. Il metodo dei polinomi ortogonali (RFP) è una ben nota tecnica di identificazione modale operante nel dominio delle frequenze. Per ottenere un problema di ottimizzazione lineare, nel metodo RFP non è minimizzato direttamente l'errore di 'fitting', differenza tra la risposta sperimentale e quella analitica, ma l'errore pesato attraverso una opportuna funzione della frequenza. Tale operazione causa in generale una stima distorta dei parametri modali. In questo lavoro si propone una tecnica iterativa in cui i parametri modali identificati attraverso il metodo RFP costituiscono la stima iniziale per innescare la procedura iterativa stessa. Il metodo minimizza l'effettivo errore di fitting e i risultati ottenuti sono molto soddisfacenti. L'errore di fitting risulta notevolmente ridotto dopo poche iterazioni. Inoltre, rispetto al tradizionale metodo dei polinomi ortogonali, è necessario un minor numero di modi computazionali per ottenere una buona identificazione in un assegnato campo di frequenze.

Key words: Identification, modal analysis, curve-fitting, vibrations.

1. Introduction

Experimental modal analysis and some of its most advanced applications, such as model updating and structural modification, lie upon the ability of correctly identifying modal parameters from experimental measurements, which is nowadays almost taken for granted. In fact many good identification algorithms exist, both in the time and frequency domains, but unfortunately experimental data are not always easy to fit. In intricate situations such as noisy measurements, high modal coupling, or both, the reliability of the estimated modal parameters is poor: different methods can in fact produce very different results, and even with a single technique, depending on different reasonable operator choices, one may obtain different sets of modal parameters. Therefore any improvement to the current identification procedures is welcome.

The rational fraction polynomial (RFP) method [1] is considered a very efficient technique for modal identification in the frequency domain. It is a direct method, in the sense that the entire set of modal parameters is determined in one step, by solving a linear problem.

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Moreover, the use of orthogonal polynomials [2] largely improves the numerical efficiency of the technique, making it very robust and reliable. However, this simplicity is not obtained without difficulty: as shown in section 2, the price to pay is that the RFP method does not minimize the true fitting error, which is the difference between the experimental and fitted frequency response functions, but gives a weighted error, denoted by the product of the fitting error times a frequency dependent function. In section 3 it is shown how this procedure introduces a distortion which prevents minimizing the actual fitting error, causing bias in the parameter estimates.

Consideration of the above questions leads to minimization of the true fitting error, which is a nonlinear function of the identification parameters. This involves a nonlinear minimization technique, which is generally iterative in nature, thus requiring a starting estimate of the solution and often presenting a difficult convergence. Such drawbacks can be reduced by defining an iteration technique which uses the output of the RFP method as a starting estimate, and expresses the error as a first order Taylor expansion of the identification parameters. This procedure makes the subproblem defined at each iteration very similar to the one stated in the original RFP technique, and allows the use of the same solution algorithm, including the use of orthogonal polynomials. Continuing on from the original RFP technique, the new procedure has been named the iterative rational fraction polynomial (IRFP). Examples and comparisons are shown in section 4.

2. The Rational Fraction Polynomial Method

It is assumed that the inertance FRF, $H(s)$, is represented within a given frequency range by a viscous linear parametric model in the form:

$$H(s) = L + \sum_{k=r+1}^{r+2n} s^2 \frac{R_k}{s - \lambda_k} + Us^2 + e_M(s) \tag{1}$$

where $s = j\omega$; n is the assumed number of modes within the considered frequency band; λ_k and R_k, occurring in complex conjugate pairs, are respectively poles and residues; $e_M(s)$ is the modelling error, accounting for every discrepancy between the system and the model (choice of an erroneous number of modes in the selected frequency band, use of upper and lower residual terms U and L to describe the contribution of the out-of-band modes, nonlinearities, etc.).

Brought to a common denominator, relation (1) reads:

$$H(s) = \frac{b_q s^q + \cdots + b_1 s + b_0}{s^p + \cdots + a_1 s + a_0} + e_M(s) \tag{2}$$

where a_i and b_i are real; $p = 2n$ and $q = 2n + 2$. The name Rational Fraction Polynomial method, for the identification procedure recalled here, derives from the above expression of $H(s)$.

The experimental inertance $\tilde{H}(s)$ can be related to $H(s)$ by the expression:

$$\tilde{H}(s) = H(s) + e_E(s) \tag{3}$$

where $e_E(s)$ is the experimental error. The statistical properties of $e_E(s)$ strongly depend on the way $\tilde{H}(s)$ is actually obtained. Usually $e_E(s)$ is assumed to be derived from a white random process with zero mean.

From equations (2) and (3) the following expression can be derived:

$$e(s) = e_E(s) + e_M(s) = \tilde{H}(s) - \frac{b_q s^q + \cdots + b_1 s + b_0}{s^p + \cdots + a_1 s + a_0}, \tag{4}$$

where the fitting error $e(s)$, the difference between experimental and analytical FRF, is clearly a nonlinear function of the a_i's.

The aim of the identification process is to find a rational fraction polynomial model that 'fits' the experimental FRF, i.e. to find a set of coefficients (a_i, b_i) such that $e(s)$ is minimized in the mean square sense. Since $e(s)$ is not linearly related to the coefficients a_i, the process involves using the nonlinear minimization technique. Due to its iterative nature, the nonlinear procedure would require a starting estimate of the unknowns and could cause trouble with respect to convergence.

In the RFP technique [1], the requirement to minimize the fitting error $\tilde{e}(s)$ is relaxed in order to deal with a linear problem. Therefore a different identification problem is defined, which is equivalent to the original one only if $e(s)$ can be made identically zero: equation (4) is multiplied by $D(s) = s^p + \cdots + a_1 s + a_0$, obtaining:

$$(s^p + \cdots + a_1 s + a_0)\tilde{H}(s) - (b_q s^q + \cdots + b_1 s + b_0) = e_W(s), \tag{5}$$

where $e_W(s)$ is a weighted error term that accounts for both the experimental error e_E and the modelling error e_M:

$$e_W(s) = (s^p + \cdots + a_1 s + a_0)\,[e_M(s) + e_E(s)] = D(s)\,e(s). \tag{6}$$

Now $e_W(s)$, which depends linearly on the unknown parameters a_i and b_i, is minimized instead of $e(s)$, with the hope that the minimum of $e_W(s)$ corresponds to a reasonably low value of $e(s)$.

Equation (5) can be given a more general form by introducing two polynomial bases in s, $\{\phi^n(s)\}$ and $\{\theta^n(s)\}$, instead of the natural basis $\{s^n\}$:

$$(\phi^p + \cdots + \hat{a}_1\phi + \hat{a}_0)\tilde{H}(s) - (\hat{b}_q\theta^q + \cdots + \hat{b}_1\theta + \hat{b}_0) = e_W(s), \tag{7}$$

where the real coefficients $\hat{a}_i$ and $\hat{b}_i$ are linearly related to a_i and b_i. Coefficients $\hat{a}_i$ and $\hat{b}_i$ can now be obtained from equation (7) by minimizing e_W in the mean square sense. This can be made straightforwardly since e_W is a linear function of $\hat{a}_i$ and $\hat{b}_i$, by defining the objective function:

$$E(\hat{a}_0, \ldots, \hat{a}_{p-1}, \hat{b}_0, \ldots, \hat{b}_q) = \sum_{l=1}^{M} e_W^*(s_l) e_W(s_l) = \sum_{l=1}^{M} (D^*(s_l)e^*(s_l))(D(s_l)e(s_l)), \tag{8}$$

where M is the number of experimental frequencies and the superscript * denotes complex conjugate. With the positions:

$$\mathbf{v} = \begin{pmatrix} \phi^p(s_1)\tilde{H}(s_1) \\ \vdots \\ \phi^p(s_M)\tilde{H}(s_M) \end{pmatrix} \qquad \mathbf{Q} = \begin{bmatrix} \theta^q(s_1) & \cdots & 1 \\ \vdots & \vdots & \vdots \\ \theta^q(s_M) & \cdots & 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \phi^{p-1}(s_1)\tilde{H}(s_1) & \cdots & \tilde{H}(s_1) \\ \vdots & \vdots & \vdots \\ \phi^{p-1}(s_M)\tilde{H}(s_M) & \cdots & \tilde{H}(s_M) \end{bmatrix} \qquad \hat{\mathbf{a}} = \begin{pmatrix} \hat{a}_{p-1} \\ \vdots \\ \hat{a}_0 \end{pmatrix} \qquad \hat{\mathbf{b}} = \begin{pmatrix} \hat{b}_q \\ \vdots \\ \hat{b}_0 \end{pmatrix} \tag{9}$$

the following expression for vector $\mathbf{e}_W = [e_W(s_1) \cdots e_W(s_M)]^T$ holds:

$$\mathbf{e}_W = \mathbf{v} + [\,\mathbf{P}\,|-\mathbf{Q}\,] \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{pmatrix}. \tag{10}$$

The objective function E can be rewritten as:

$$\begin{aligned} E = \mathbf{e}_W^T \mathbf{e}_W^* = \mathbf{v}^T\mathbf{v}^* + 2\left[\hat{\mathbf{a}}^T \;\; \hat{\mathbf{b}}^T\right] \mathrm{Re}\left\{\begin{bmatrix} \mathbf{P}^T \\ -\mathbf{Q}^T \end{bmatrix} \mathbf{v}^*\right\} + \\ + \left[\hat{\mathbf{a}}^T \;\; \hat{\mathbf{b}}^T\right] \mathrm{Re}\begin{bmatrix} \mathbf{P}^T\mathbf{P}^* & -\mathbf{P}^T\mathbf{Q}^* \\ -\mathbf{Q}^T\mathbf{P}^* & \mathbf{Q}^T\mathbf{Q}^* \end{bmatrix} \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{pmatrix}. \end{aligned} \tag{11}$$

The solution is obtained by requiring that the derivative of E with respect to $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be zero:

$$\mathrm{Re}\left\{\begin{bmatrix} \mathbf{P}^T \\ -\mathbf{Q}^T \end{bmatrix} \mathbf{v}^* + \begin{bmatrix} \mathbf{P}^T\mathbf{P}^* & -\mathbf{P}^T\mathbf{Q}^* \\ -\mathbf{Q}^T\mathbf{P}^* & \mathbf{Q}^T\mathbf{Q}^* \end{bmatrix}\right\} \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{pmatrix} = \mathbf{0}. \tag{12}$$

The problem (12) is ill-conditioned for an arbitrary choice of the polynomial bases $\{\phi^n(s)\}$ and $\{\theta^n(s)\}$, and in particular when the natural basis $\{s^n\}$ is used. Ill-conditioning can be significantly reduced if the two polynomial bases are chosen in order to satisfy the orthogonality conditions:

$$\begin{aligned} \sum_{l=1}^{M} \phi^{i^*}(s_l) \left|\tilde{H}(s_l)\right|^2 \phi^j(s_l) &= \delta_{ij} \quad i,j = 1,...,p \\ \sum_{l=1}^{M} \theta^{i^*}(s_l)\theta^j(s_l) &= \delta_{ij} \quad i,j = 1,...,q. \end{aligned} \tag{13}$$

It can be shown that such polynomial bases exist, independently of the frequency spacing, which needs not to be constant ([2]). If the conditions (13) are valid, then:

$$\mathbf{P}^T\mathbf{P}^* = \mathbf{I} \qquad \mathbf{Q}^T\mathbf{Q}^* = \mathbf{I} \qquad \mathbf{P}^T\mathbf{v}^* = \mathbf{0} \tag{14}$$

and consequently the system (12) becomes:

$$\begin{bmatrix} \mathbf{I} & \mathrm{Re}(-\mathbf{P}^T\mathbf{Q}^*) \\ \mathrm{Re}(-\mathbf{Q}^T\mathbf{P}^*) & \mathbf{I} \end{bmatrix} \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{pmatrix} = \mathrm{Re}\left\{\begin{bmatrix} \mathbf{0} \\ \mathbf{Q}^T \end{bmatrix} \mathbf{v}^*\right\} \tag{15}$$

Not only the system (15) is well conditioned, but it can be solved separately with respect to $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. This greatly reduces the dimension of the problem:

$$\begin{aligned} \hat{\mathbf{a}} &= \left[\mathbf{I} - \mathrm{Re}(\mathbf{P}^T\mathbf{Q}^*)\mathrm{Re}(\mathbf{Q}^T\mathbf{P}^*)\right]^{-1} \mathrm{Re}(\mathbf{P}^T\mathbf{Q}^*)\mathrm{Re}(\mathbf{Q}^T\mathbf{v}^*) \\ \hat{\mathbf{b}} &= \mathrm{Re}(\mathbf{Q}^T\mathbf{v}^*) + \mathrm{Re}(\mathbf{Q}^T\mathbf{P}^*)\hat{\mathbf{a}}. \end{aligned} \tag{16}$$

Once $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are known, they can be transformed back to the coefficients $\mathbf{a}$ and $\mathbf{b}$ of the natural polynomial basis.

Having computed a_i and b_i, the residues R_k and poles λ_k can be evaluated in the same way as the polynomial coefficients of equation (2) are obtained from poles and residues. The poles are the roots of the polynomial:

$$z^p + \cdots + a_1 z + a_0 = 0 \tag{17}$$

whereas the residues are computed from the coefficients b_i and poles λ_k through the relations:

$$L = \frac{b_0}{a_0} \qquad U = b_q \qquad R_k = \frac{b_q\lambda_k^{q-2} + \cdots + b_1\lambda_k^{-1} + b_0\lambda_k^{-2}}{\prod_{i\neq k}(\lambda_k - \lambda_i)}. \tag{18}$$

These are obtained by equating relation (1) with (2), and noticing that $\prod_{k=1}^{p}(s - \lambda_k) = s^p + \cdots + a_1 s + a_0$.

3. The Iterative Rational Fraction Polynomial Method

3.1. Drawbacks of the RFP method

The introduction of the weighted error $e_W(s)$, according to equation (6), yields a linear expression in the parameters a_i and b_i, but introduces a distorsion which is absent only if $e_M(s)$ and $e_E(s)$ are both zero. In any other case the minimization is performed on a weighted error, with the weighting function given by the denominator $D(s)$ of the identified transfer function. $D(s)$ is *a priori* unknown, since it depends on the unknown parameters $a_1, \ldots, a_p$, but likely its roots will be not far from the system poles.

Consequently the function $D(s)$ presents two remarkable properties:

- its absolute value is very low at frequencies corresponding to the imaginary part of the identified system poles;
- its absolute value definitely increases as frequency increases, since the function is a polynomial in $s = j\omega$.

It should be recalled that, despite the minimization of the weighted error $e_W(s)$, the actual identification goal is to obtain low values for the fitting error $e(s)$. From equation (8) it can be seen that a frequency dependent weight is assigned to the norm of $e(s)$. More precisely the weight tends to zero close to the natural frequencies and tends to increase with frequency. Therefore the error $e(s)$:

- is higher close to the natural frequencies, where it reaches peak values;
- decreases with the increasing frequency, i.e. it tends to be higher at low frequencies and lower at high frequencies.

These two effects can be defined as localization and unbalance, respectively. In Figure 1, the identified $|D(s)|$ of a typical FRF measurement together with the error $|e(s)|$ are plotted versus frequency: a sort of 'complementarity' between $|D(s)|$ and $|e(s)|$ can be observed, since when the former is high the latter is low and viceversa. Since $D(s)$ and $e(s)$ have different units, labeling the Y-axis would be misleading: in fact only the relative trend of the above quantities is relevant.

The effects of such a frequency distribution of the error are easily predicted: poor fitting in the low frequency band, badly localized natural frequencies, erroneous damping estimates, wrong mode shapes.

3.2. Description of the IRFP method

The above discussion stresses the unconveniences of considering the weighted error defined by equation (6) instead of the fitting error defined by equation (3). The latter is in fact derived without any weighting function, and therefore, after the identification, it should appear balanced and uniformly distributed inside the frequency band.

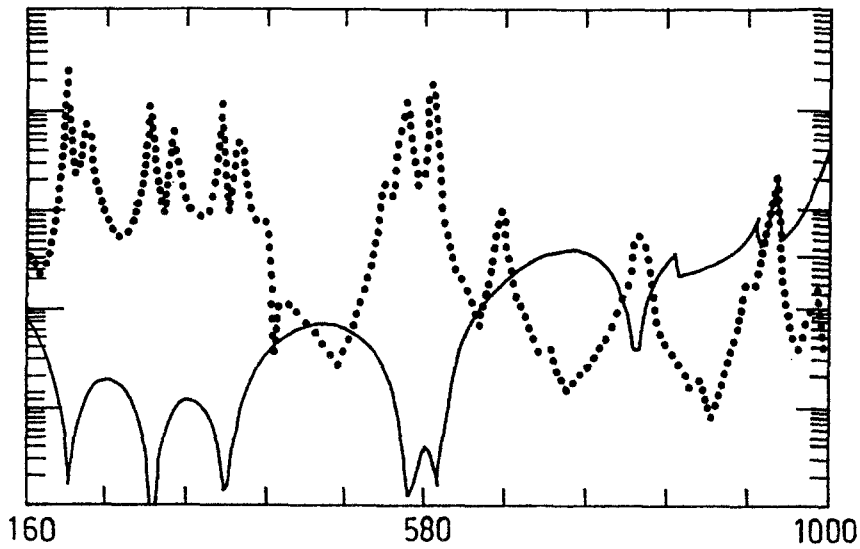

Fig. 1. Identified $|D(s)|$ (—) and error $|e(s)|$ ($\cdots$) for a typical FRF measurement.

The difficulties related to the solution of a nonlinear problem can be faced as follows. First, the RFP method is used to get a starting solution $\mathbf{a}^{(0)}$, $\mathbf{b}^{(0)}$. Equation (4) is then linearized by considering a Taylor series expansion truncated to first order terms:

$$e(s) = e(s)|_{\mathbf{a}^{(k)},\mathbf{b}^{(k)}} + \sum_{r=0}^{q} \frac{\partial e(s)}{\partial b_r}\bigg|_{\mathbf{a}^{(k)},\mathbf{b}^{(k)}} (b_r - b_r^{(k)}) + \sum_{r=0}^{p-1} \frac{\partial e(s)}{\partial a_r}\bigg|_{\mathbf{a}^{(k)},\mathbf{b}^{(k)}} (a_r - a_r^{(k)}) \quad (19)$$

where $\mathbf{a}^{(k)}$, $\mathbf{b}^{(k)}$, the starting point of the expansion, is set equal to $\mathbf{a}^{(0)}$, $\mathbf{b}^{(0)}$ at the first step. The terms at the r.h.s. of equation (19) are easily computed as:

$$e(s)|_{\mathbf{a}^{(k)},\mathbf{b}^{(k)}} = \tilde{H}(s) - \frac{b_q^{(k)} s^q + \cdots + b_1^{(k)} s + b_0^{(k)}}{s^p + \cdots + a_1^{(k)} s + a_0^{(k)}} = \tilde{H}(s) - H^{(k)}(s) = e^{(k)}(s)$$

$$\frac{\partial e(s)}{\partial b_r}\bigg|_{\mathbf{a}^{(k)},\mathbf{b}^{(k)}} = -\frac{s^r}{s^p + \cdots + a_1^{(k)} s + a_0^{(k)}} = -\frac{s^r}{D^{(k)}(s)} \quad (20)$$

$$\frac{\partial e(s)}{\partial a_r}\bigg|_{\mathbf{a}^{(k)},\mathbf{b}^{(k)}} = \frac{b_q^{(k)} s^q + \cdots + b_1^{(k)} s + b_0^{(k)}}{(s^p + \cdots + a_1^{(k)} s + a_0^{(k)})^2} s^r = \frac{H^{(k)}(s) s^r}{s^p + \cdots + a_1^{(k)} s + a_0^{(k)}}$$
$$= \frac{H^{(k)}(s)}{D^{(k)}(s)} s^r,$$

where $D^{(k)}(s) = s^p + \cdots + a_1^{(k)} s + a_0^{(k)}$.

After substituting the expressions (20) in equation (19), the following linearized expression for $e(s)$ is obtained:

$$(s) = e^{(k)}(s) - \sum_{r=0}^{q} \frac{s^r}{D^{(k)}(s)}(b_r - b_r^{(k)}) + \sum_{r=0}^{p-1} \frac{H^{(k)}(s)}{D^{(k)}(s)} s^r (a_r - a_r^{(k)})$$

$$= e^{(k)}(s) - \frac{1}{D^{(k)}(s)} \sum_{r=0}^{q} s^r \delta b_r^{(k)} + \frac{H^{(k)}(s)}{D^{(k)}(s)} \sum_{r=0}^{p-1} s^r \delta a_r^{(k)}, \tag{21}$$

where $\delta a_r^{(k)} = (a_r - a_r^{(k)})$ and $\delta b_r^{(k)} = (b_r - b_r^{(k)})$. It can be noticed that two polynomials in s, with coefficients $\delta a_r^{(k)}, \delta b_r^{(k)}$, appear in equation (21). This again can be given a more general form by introducing two polynomial bases in s, $\{\phi^{(k)}_n(s)\}$ and $\{\theta^{(k)}_n(s)\}$, instead of the natural basis $\{s^n\}$:

$$e(s) = e^{(k)}(s) - \frac{1}{D^{(k)}(s)} \sum_{r=0}^{q} \theta^{(k)}_r \, \delta \hat{b}_r^{(k)} + \frac{H^{(k)}(s)}{D^{(k)}(s)} \sum_{r=0}^{p-1} \phi^{(k)}_r \, \delta \hat{a}_r^{(k)}, \tag{22}$$

where $\delta \hat{a}_r^{(k)}$ and $\delta \hat{b}_r^{(k)}$ are the coefficients in the new polynomial bases. These coefficients have to be determined in order to minimize the norm of the error $e(s)$, i.e. the objective function:

$$E^{(k)}(\delta \hat{a}_0^{(k)}, \delta \hat{a}_1^{(k)}, \ldots, \delta \hat{a}_{p-1}^{(k)}, \delta \hat{b}_0^{(k)}, \ldots, \delta \hat{b}_q^{(k)}) = \sum_{i=1}^{M} e^*(s_i) e(s_i). \tag{23}$$

Equation (22) can be rewritten in matrix form:

$$\mathbf{e} = \mathbf{e}^{(k)} + \left[\mathbf{P}^{(k)} \middle| -\mathbf{Q}^{(k)} \right] \begin{pmatrix} \delta \hat{\mathbf{a}}^{(k)} \\ \delta \hat{\mathbf{b}}^{(k)} \end{pmatrix} \tag{24}$$

where:

$$\mathbf{e} = [e(s_1) \cdots e(s_M)]^T \qquad \mathbf{e}^{(k)} = \left[e^{(k)}(s_1) \cdots e^{(k)}(s_M)\right]^T$$

$$\mathbf{P}^{(k)} = \begin{bmatrix} \frac{H^{(k)}(s_1)}{D^{(k)}(s_1)} \phi^{(k)}_{p-1}(s_1) & \frac{H^{(k)}(s_1)}{D^{(k)}(s_1)} \phi^{(k)}_{p-2}(s_1) & \cdots & \frac{H^{(k)}(s_1)}{D^{(k)}(s_1)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{H^{(k)}(s_M)}{D^{(k)}(s_M)} \phi^{(k)}_{p-1}(s_M) & \frac{H^{(k)}(s_M)}{D^{(k)}(s_M)} \phi^{(k)}_{p-2}(s_M) & \cdots & \frac{H^{(k)}(s_M)}{D^{(k)}(s_M)} \end{bmatrix} \tag{25}$$

$$\mathbf{Q}^{(k)} = \begin{bmatrix} \frac{\theta^{(k)}_q(s_1)}{D^{(k)}(s_1)} & \cdots & \frac{1}{D^{(k)}(s_1)} \\ \vdots & \vdots & \vdots \\ \frac{\theta^{(k)}_q(s_M)}{D^{(k)}(s_M)} & \cdots & \frac{1}{D^{(k)}(s_M)} \end{bmatrix} \qquad \delta \hat{\mathbf{a}}^{(k)} = \begin{pmatrix} \delta \hat{a}_{p-1}^{(k)} \\ \vdots \\ \delta \hat{a}_0^{(k)} \end{pmatrix} \qquad \delta \hat{\mathbf{b}}^{(k)} = \begin{pmatrix} \delta \hat{b}_q^{(k)} \\ \vdots \\ \delta \hat{b}_0^{(k)} \end{pmatrix}.$$

The analogy between equations (9)–(10) and equations (24)–(25) is immediately apparent. Again, the objective function (equation (23) in this case) is minimized by requiring that its derivatives with respect to $\delta\hat{\mathbf{a}}^{(k)}$ and $\delta\hat{\mathbf{b}}^{(k)}$ be zero:

$$\mathrm{Re}\left\{\begin{bmatrix}\mathbf{P}^{(k)T}\\ -\mathbf{Q}^{(k)T}\end{bmatrix}\mathbf{e}^{(k)*}+\begin{bmatrix}\mathbf{P}^{(k)T}\mathbf{P}^{(k)*} & -\mathbf{P}^{(k)T}\mathbf{Q}^{(k)*}\\ -\mathbf{Q}^{(k)T}\mathbf{P}^{(k)*} & \mathbf{Q}^{(k)T}\mathbf{Q}^{(k)*}\end{bmatrix}\right\}\begin{pmatrix}\delta\hat{\mathbf{a}}^{(k)}\\ \delta\hat{\mathbf{b}}^{(k)}\end{pmatrix}=\mathbf{0}. \tag{26}$$

The polynomial bases $\{\phi^{(k)}_n(s)\}$ and $\{\theta^{(k)}_n(s)\}$ can now be selected so as to satisfy the following orthogonality conditions:

$$\begin{aligned}\sum_{l=1}^{M}\phi^{(k)*}_i(s_l)\left|\frac{H^{(k)}(s_l)}{D^{(k)}(s_l)}\right|^2\phi^{(k)}_i(s_l) &= \delta_{ij}\quad i,j=1,\ldots,p\\ \sum_{l=1}^{M}\theta^{(k)*}_i(s_l)\left|\frac{1}{D^{(k)}(s_l)}\right|^2\theta^{(k)}_i(s_l) &= \delta_{ij}\quad i,j=1,\ldots,q.\end{aligned} \tag{27}$$

From the above conditions it is obvious that the polynomial bases $\{\phi^{(k)}_n(s)\}$ and $\{\theta^{(k)}_n(s)\}$ must be explicitly marked with the index $^{(k)}$: they in fact depend on the starting point used in the Taylor expansion. In view of the conditions (27), equation (26) can be rewritten:

$$\begin{bmatrix}\mathbf{I} & \mathrm{Re}(-\mathbf{P}^{(k)T}\mathbf{Q}^{(k)*})\\ \mathrm{Re}(-\mathbf{Q}^{(k)T}\mathbf{P}^{(k)*}) & \mathbf{I}\end{bmatrix}\begin{pmatrix}\delta\hat{\mathbf{a}}^{(k)}\\ \delta\hat{\mathbf{b}}^{(k)}\end{pmatrix}=\mathrm{Re}\left\{\begin{bmatrix}\mathbf{P}^{(k)T}\\ -\mathbf{Q}^{(k)T}\end{bmatrix}\mathbf{e}^{(k)*}\right\} \tag{28}$$

and, after the positions:

$$\mathbf{A}^{(k)}=\mathrm{Re}(-\mathbf{Q}^{(k)T}\mathbf{P}^{(k)*})\quad \mathbf{d}_1^{(k)}=\mathrm{Re}(-\mathbf{P}^{(k)T}\mathbf{e}^{(k)*})\quad \mathbf{d}_2^{(k)}=\mathrm{Re}(-\mathbf{Q}^{(k)T}\mathbf{e}^{(k)*}) \tag{29}$$

equation (28) becomes:

$$\begin{bmatrix}\mathbf{I} & \mathbf{A}^{(k)T}\\ \mathbf{A}^{(k)} & \mathbf{I}\end{bmatrix}\begin{pmatrix}\delta\hat{\mathbf{a}}^{(k)}\\ \delta\hat{\mathbf{b}}^{(k)}\end{pmatrix}=\begin{pmatrix}\mathbf{d}_1^{(k)}\\ \mathbf{d}_2^{(k)}\end{pmatrix}. \tag{30}$$

The solution of equation (30) gives:

$$\begin{aligned}\delta\hat{\mathbf{a}}^{(k)} &= \left[\mathbf{I}-\mathbf{A}^{(k)T}\mathbf{A}^{(k)}\right]^{-1}\left[\mathbf{d}_1^{(k)}-\mathbf{A}^{(k)T}\mathbf{d}_2^{(k)}\right]\\ \delta\hat{\mathbf{b}}^{(k)} &= \mathbf{d}_2^{(k)}-\mathbf{A}^{(k)}\delta\hat{\mathbf{a}}^{(k)}.\end{aligned} \tag{31}$$

After transforming back $\delta\hat{\mathbf{a}}^{(k)}$ and $\delta\hat{\mathbf{b}}^{(k)}$ into the natural basis through two transformation matrices $\mathbf{C}_\phi^{(k)}$ and $\mathbf{C}_\theta^{(k)}$, the values of **a** and **b** can be updated as follows:

$$\begin{aligned}\mathbf{a}^{(k+1)} &= \mathbf{a}^{(k)}+\mathbf{C}_\phi^{(k)}\delta\hat{\mathbf{a}}^{(k)}\\ \mathbf{b}^{(k+1)} &= \mathbf{b}^{(k)}+\mathbf{C}_\theta^{(k)}\delta\hat{\mathbf{b}}^{(k)}.\end{aligned} \tag{32}$$

The IRFP procedure is summarized in the block diagram shown in Figure 2, which indicates that the most important tasks of the iterative part of the algorithm (generation of Forsythe polynomials and system solution) can be handled by the same routines used in the one-step RFP technique.

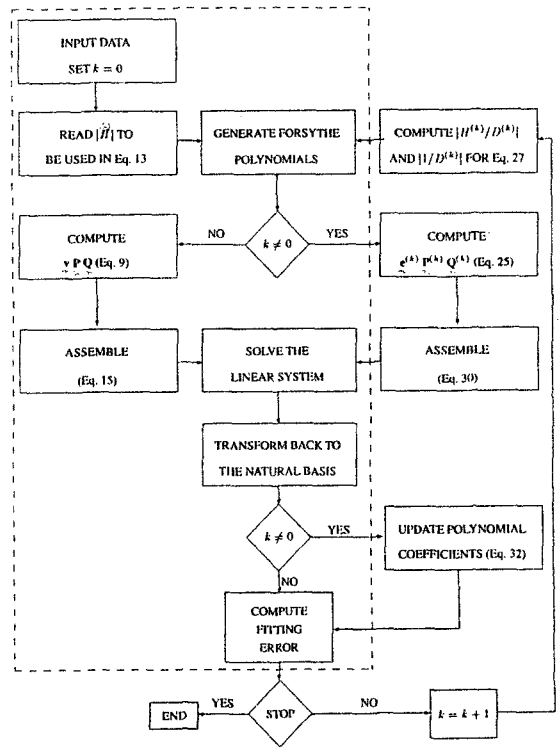

Fig. 2. Block diagram of the IRFP procedure (the RFP part is represented inside the dashed box).

4. Results

Significant numerical results are shown in the present section. In particular a comparison between the RFP and IRFP method is performed to point out some typical improvements practically obtained when using the iterative procedure. The first test concerns the identification of a theoretically computed FRF relative to an 8-dof lumped system. Figures 3 and 4 show respectively the results obtained by RFP and IRFP methods. In Figure 3 the above mentioned localization effect is particularly apparent in fitting the first and sixth modal peak: the latter is practically ignored. The result obtained by using the IRFP technique is quite satisfactory: no bias is actually present in the identified FRF.

The second test is performed on an experimental FRF from a real structure (lab frame). The identification is conducted in the frequency range 0–500 Hz, in which the FRF shows nine modal peaks. Therefore nine modes have been considered in the fitting procedure corresponding to a denominator polynomial of degree eighteen. This is practically the maximum degree available to the analyst when using commercial RFP software, and this is related, according to our experience, to many numerical problems of the algorithm beyond this limit. The result obtained by the RFP method is shown in Figure 5: again, the localization effect is very important, especially on the first and eigth modal peaks, and as in the above case the smallest peak is lost in the RFP reconstruction. Figure 6 shows the FRF identified by the IRFP procedure, which exhibits no significant distorsion.

The third test is performed on the same experimental FRF used previously, but now the identification is conducted in the frequency range 0–550 Hz, in which the FRF shows more than nine modal peaks. However, for the above stated reasons, only nine modes could be used

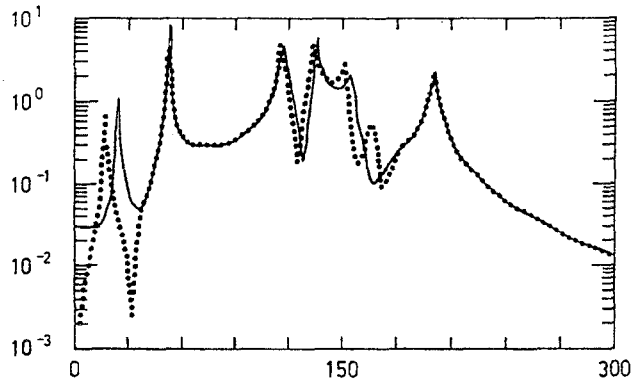

Fig. 3. RFP method: theoretical (· · ·) and identified (—) FRFs.

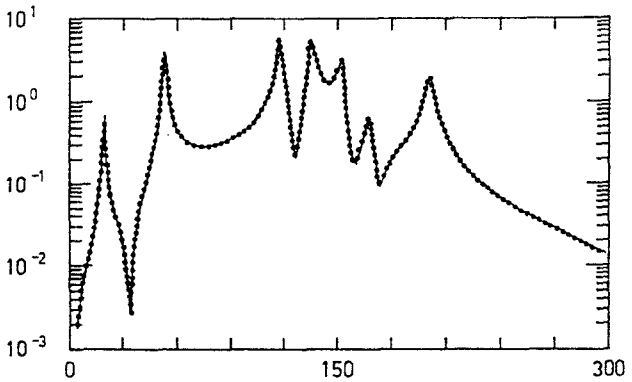

Fig. 4. IRFP method: theoretical (· · ·) and identified (—) FRFs.

in the identification. In Figure 7 the experimental FRF and the RFP fitted FRF are compared: a high distorsion appears in the whole frequency range and only three out of nine available modes (one around 120 Hz and the remaining ones close to the upper frequency limit) are correctly identified. Figure 8 shows the FRF identified by the IRFP procedure: the result is quite satisfactory since all the nine modes available to the algorithm are employed for the best in order to minimize the fitting error. According to the authors' experience, this is the

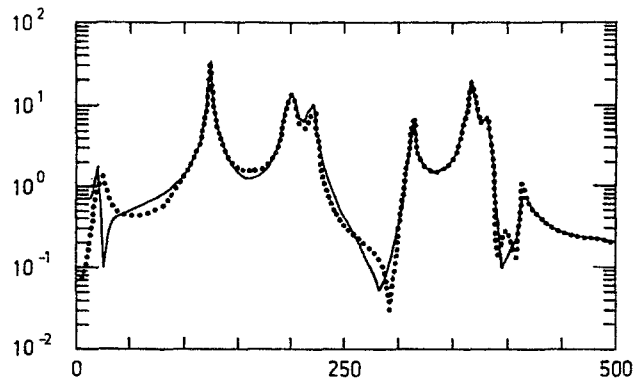

Fig. 5. RFP method: experimental (· · ·) and identified (—) FRFs (range 0–500 Hz).

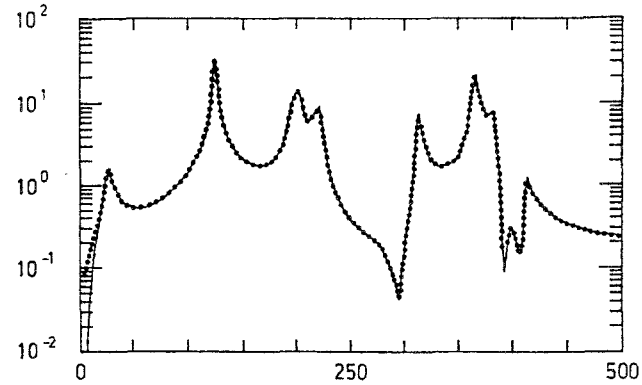

Fig. 6. IRFP method: experimental (· · ·) and identified (—) FRFs (range 0–500 Hz).

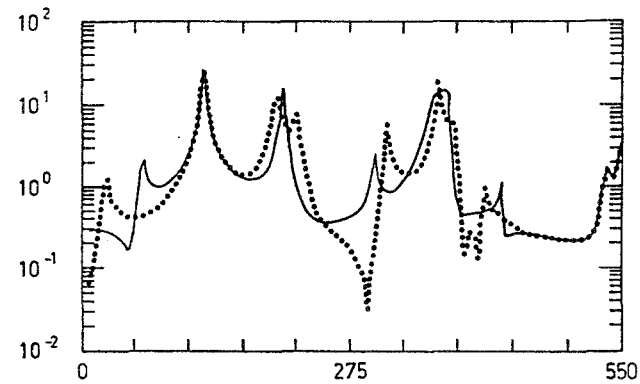

Fig. 7. RFP method: experimental (· · ·) and identified (—) FRFs (range 0–550 Hz).

general result whenever the RFP and IRFP methods are forced to use an insufficient number of modes.

The absolute value of the fitting error in the last test is shown in Figure 9: the RFP error is characterized by typical peaks in the resonance regions (localization effect), and by a large difference between low frequency and high frequency error (unbalance effect); on the contrary the IRFP error is lower and homogenously distributed within the frequency range.

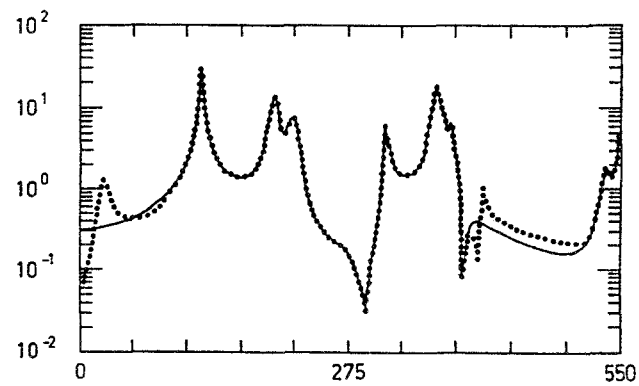

Fig. 8. IRFP method: experimental (· · ·) and identified (—) FRFs (range 0–550 Hz).

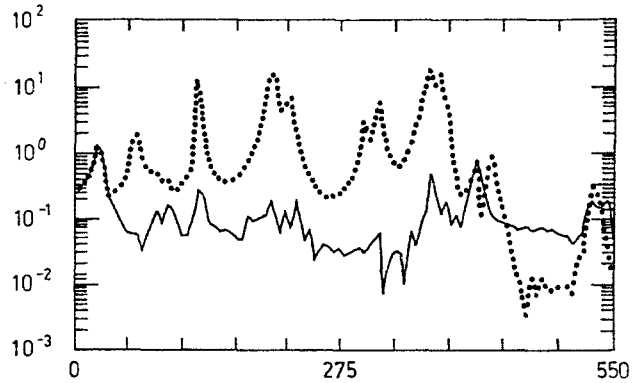

Fig. 9. Frequency trend of the fitting error in RFP (· · ·) and IRFP (—) methods.

5. Conclusions

A curve fitting procedure, the Iterative Rational Fraction Polynomial (IRFP) technique, has been developed starting from the well known Rational Fractional Polynomial method. The IRFP procedure presents the advantage of minimizing the true fitting error, defined as the difference between the experimental and the fitted FRFs, thus producing unbiased modal parameter estimates, unlike the RFP technique, which minimizes a frequency weighted function of that error.

The additional computational effort, implied by the IRFP technique to perform the iterations, turns out to be not very important. First, the method converges in the majority of cases and very rapidly, usually requiring one or two iterations at most. Whenever convergence of the IRFP method fails, one could of course be satisfied with the estimate previously given by the RFP technique, but convergence failure usually implies extremely bad results also for the RFP algorithm. Furthermore, unlike the RFP technique, the IRFP procedure can work without time-consuming tricks like the use of the so-called computational modes. In order to get acceptable results, often the RFP procedure must be run by specifying a number n of modes greater than those strictly necessary, or by specifying an order of numerator greater than $2n + 2$ (see equations (1) and (2)). The additional modes, referred to as computational modes, have no physical meaning, but yield a smaller fitting error at the price of an increasing computational burden.

The unnecessary computational modes bring another important advantage for the IRFP technique, in addition to saving computer time. Despite the use of orthogonal polynomials, which greatly improves the numerical efficiency of the technique, there is still an upper bound on the maximum degree of the polynomials that can be handled without degrading the results. This is turned into an upper bound on the number of modes that can be used inside a given frequency band. For most commercial RFP softwares the maximum number of modes in a band is nine, corresponding to denominator polynomials of degree eighteen. This limit is maintained in the IRFP procedure but, unlike in the original RFP technique, all the used modes are true vibration modes, thus providing extended fitting capabilities.

The reported results are quite satisfactory and demonstrate the very good performance of the IRFP technique, both in absolute terms and compared to the parent RFP method. Further results, which include a demonstration of the statistical behaviour with respect to external noise, are reported in reference [3].

Finally, it must be remarked that the IRFP procedure, here presented for single FRF processing, can be straightforwardly extended to become a global frequency domain modal parameter estimation technique, like the RFP method [4].

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References

1. Richardson, M.H and Formenti, D.L., 'Parameter estimation from frequency response measurements using rational fraction polynomials', *Proc. 1st International Modal Analysis Conf., Orlando (U.S.A.)*, 1982, pp. 167–186.
2. Forsythe, G.E., 'Generation and use of orthogonal polynomials for data-fitting with a digital computer', *SIAM J.*, **5** (1957) 74–88.
3. Carcaterra, A. and Sestieri, A., 'Bias in modal parameters using the direct and iterative RFP identification procedures', *Proc. 17th International Seminar on Modal Analysis, Leuven (Belgium)*, 1992, pp. 1627–1641.
4. Richardson, M.H. and Formenti, D.L., 'Global curve fitting of frequency response measurements using rational fraction polynomial method', *Proc. 3rd International Modal Analysis Conference, Orlando (U.S.A.)*, 1985.