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RESEARCH ARTICLE

Flat Rotational Surfaces with Pointwise 1-Type Gauss Map Via Generalized Quaternions

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Abstract In this paper, we determine a rotational surface by means of generalized quaternions and study this flat rotational surface with pointwise 1-type Gauss map in four-dimensional generalized space $\mathbb{E}^4_{\alpha\beta}$. Also, for some special cases of α and β, we obtain the characterizations of flat rotational surfaces with pointwise 1-type Gauss map in four-dimensional Euclidean space $\mathbb{E}^4$ and four-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$.

Keywords Quaternions · Gauss map · Pointwise 1-type Gauss map · Rotational surface

Mathematics Subject Classification 53B25 · 53C40

1 Introduction

Quaternions first introduced by Hamilton are a number system that is a generalization of the complex numbers in four-dimensional space. A real quaternion q is defined as $q = q_0 + q_1 i + q_2 j + q_3 k$ where q_0, q_1, q_2, q_3 are real numbers and $1, i, j, k$ are the basis elements which satisfy $i^2 = j^2 = k^2 = ijk = -1$. The set of quaternions H with these basis elements $\{1, i, j, k\}$ is isomorphic to four-dimensional vector space $\mathbb{R}^4$. There are three fundamental operations on H: addition, scalar multiplication and quaternion multiplication. The addition and scalar multiplication are defined same as the addition and scalar multiplication on $\mathbb{R}^4$ but the quaternion multiplication is determined by distributive law and the multiplication rule between the basis elements of H. The set of quaternions H is a real vector space with these addition and scalar multiplication. Also, it is an associative and non-commutative four-dimensional Clifford algebra with together quaternion multiplication.

The set of all unit quaternions forms 3-sphere S^3. It is a Lie group that is isomorphic to the group $SU(2)$ and double covering the group $SO(3)$, the group of three-dimensional rotations. On the other hand, any quaternions can be represented as the terms of 4×4 real matrices. The matrix representation of a unit quaternion is a real orthogonal 4×4 matrix of determinant 1. So, a unit quaternion could be used to represent the rotations in $\mathbb{R}^4$. Since the rotations in three-dimensional space and four-dimensional space can be expressed by quaternions, they are commonly used in computer graphics, computer vision, robotics, computer simulations, orbital mechanics, etc.

Quaternions were generalized, and a brief introduction of generalized quaternions was given by Pottman and Wallner [1]. Recently, their some algebraic properties were studied by Jafari [2]. Jafari and Yaylı [3, 4] described the rotations in three-dimensional generalized linear space $\mathbb{E}^3_{\alpha\beta}$ and four-dimensional generalized linear space $\mathbb{E}^4_{\alpha\beta}$ by means of generalized quaternions. Also, Arslan et al. [5] studied rotational surfaces in n-dimensional Euclidean space.

Let $G(n, m)$ be a Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$.

The Gauss map G of an n-dimensional submanifold M of m-dimensional Euclidean space $\mathbb{E}^m$ is a smooth map

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which carries a point p in M into the n-plane through the origin in $\mathbb{E}^m$ obtained by translating parallelly the tangent space at p of M, that is, it is a smooth map which carries a point p in M into $G(n, m)$. The Grassmannian manifold $G(n, m)$ is canonically embedded in $\wedge^n \mathbb{E}^m \cong \mathbb{E}^N, N = \binom{m}{n}$. Hence, the Gauss map is defined by $G : M \to G(n,m) \subset E^N$, $G(p) = (e_1 \wedge \ldots \wedge e_n)(p)$. Chen and Piccinni [6] studied submanifolds with finite type Gauss map.

A submanifold M of a Euclidean or pseudo-Euclidean space is said to have pointwise 1-type Gauss map if satisfies

$$\Delta G = f(G + C) \tag{1}$$

for some nonzero smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in Eq. (1) is zero vector. Otherwise, pointwise 1-type Gauss map is said to be of second kind. Rotational surfaces in Euclidean space and pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied [7–10]. Also tensor product surfaces with pointwise 1-type Gauss map were recently studied [11].

In this paper, we determine a rotational surface via generalized quaternions and study this flat rotational surface with pointwise 1-type Gauss map in four-dimensional generalized linear space $\mathbb{E}^4_{\alpha\beta}$. Also, for some special cases of α and β, we obtain the characterizations of flat rotational surfaces with pointwise 1-type Gauss map in four-dimensional Euclidean space $\mathbb{E}^4$ and four-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ which are given by Aksoyak and Yaylı [12, 13].

2 Preliminaries

The set of generalized quaternions, denoted by $H_{\alpha\beta}$, is defined by

$$H_{\alpha\beta} = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ q_t \in \mathbb{R},\ t = 0,1,2,3\},$$

where i, j, k are quaternionic units which satisfy the equalities

$$i^2 = -\alpha, j^2 = -\beta, k^2 = -\alpha\beta,$$
$$ij = k = -ji, jk = \beta i = -kj, ki = \alpha j = -ik \text{ and } \alpha, \beta \in \mathbb{R}.$$

By choosing α and β there are the following special cases:

1. If $\alpha = \beta = 1$ is considered, then $H_{\alpha\beta}$ is the algebra of real quaternions.
2. If $\alpha = 1$, $\beta = -1$ is considered, then $H_{\alpha\beta}$ is the algebra of split quaternions.
3. If $\alpha = 1$, $\beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of semi-quaternions.
4. If $\alpha = -1$, $\beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of split semi-quaternions.
5. If $\alpha = \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of $\frac{1}{4}$−quaternions.

For any $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$ in $H_{\alpha\beta}$, the addition rule for generalized quaternions is defined as:

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k$$

and the multiplication of a generalized quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ by a real scalar c is defined as:

$$cq = cq_0 + cq_1 i + cq_2 j + cq_3 k.$$

$H_{\alpha\beta}$ is a real vector space according to this addition and scalar multiplication.

Generalized quaternion product is defined as:

$$\begin{aligned} pq = {} & (p_0q_0 - \alpha p_1q_1 - \beta p_2q_2 - \alpha\beta p_3q_3) \\ & + (p_1q_0 + p_0q_1 - \beta p_3q_2 + \beta p_2q_3)i \\ & + (p_2q_0 + \alpha p_3q_1 + p_0q_2 - \alpha p_1q_3)j \\ & + (p_3q_0 - p_2q_1 + p_1q_2 + p_0q_3)k \end{aligned}$$

or it could be expressed as:

$$pq = \begin{bmatrix} p_0 & -\alpha p_1 & -\beta p_2 & -\alpha\beta p_3 \\ p_1 & p_0 & -\beta p_3 & \beta p_2 \\ p_2 & \alpha p_3 & p_0 & -\alpha p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \tag{2}$$

The generalized quaternion product has an associative and distributive property on the addition, but it has not the commutative property in general.

The conjugate of a generalized quaternion q is denoted by $\bar{q}$ and defined by $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$. The norm of a generalized quaternion q is defined as: $N_q = q\bar{q} = q_0^2 + \alpha q_1^2 + \beta q_2^2 + \alpha\beta q_3^2$.

Let $u = (u_0, u_1, u_2, u_3)$, $v = (v_0, v_1, v_2, v_3) \in \mathbb{R}^4$ and α, $\beta \in \mathbb{R}$. The generalized inner product u and v is defined by

$$g(u,v) = \langle u,v \rangle_{\alpha\beta} = u_0v_0 + \alpha u_1v_1 + \beta u_2v_2 + \alpha\beta u_3v_3$$

or it could be written

$$g(u,v) = u^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix} v = u^t G v.$$

So the vector space on $\mathbb{R}^4$ equipped with generalized scalar product is called four-dimensional generalized space and denoted by $\mathbb{E}^4_{\alpha\beta} = \left(\mathbb{R}^4, \langle , \rangle_{\alpha\beta}\right)$ [3].

If $\alpha = \beta = 1$, then $\mathbb{E}^4_{\alpha\beta}$ is four-dimensional Euclidean space $\mathbb{E}^4$.

If $\alpha = 1$, $\beta = -1$, then $\mathbb{E}^4_{\alpha\beta}$ is four-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$.

A matrix $A_{4\times4}$ is called semi-orthogonal matrix in four-dimensional generalized space $\mathbb{E}^4_{\alpha\beta}$ if $A^T GA = G$ and $\det A = 1$. The set of all semi-orthogonal matrices is called rotational group in $\mathbb{E}^4_{\alpha\beta}$ [3].

Let $\mathbb{E}^4_{\alpha\beta}$ be four-dimensional generalized space. Then, the metric tensor g in $\mathbb{E}^4_{\alpha\beta}$ has the form

$$g = \mathrm{d}x_0^2 + \alpha dx_1^2 + \beta \mathrm{d}x_2^2 + \alpha\beta \mathrm{d}x_3^2,$$

where (x_0, x_1, x_2, x_3) is a standard rectangular coordinate system in $\mathbb{E}^4_{\alpha\beta}$.

Let M be a two-dimensional submanifold of four-dimensional generalized space $\mathbb{E}^4_{\alpha\beta}$. We denote Levi-Civita connections of $\mathbb{E}^4_{\alpha\beta}$ and M by $\tilde{\nabla}$ and ∇, respectively. Let e_1, e_2, e_3, e_4 be an adapted local orthonormal frame in $\mathbb{E}^4_{\alpha\beta}$ such that e_1, e_2 are tangent to M and e_3, e_4 normal to M. We use the following convention on the ranges of indices: $1 \le i, j, k, \ldots \le 2$, $3 \le r, s, t, \ldots \le 4$, $1 \le A, B, C, \ldots \le 4$.

Let ω_A be the dual-1 form of e_A defined by $\omega_A(X) = \langle e_A, X\rangle$ and $\varepsilon_A = \langle e_A, e_A\rangle = \pm 1$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \varepsilon_B \omega_{AB} e_B,$$

where $\omega_{AB} + \omega_{BA} = 0$. Then, we have

$$\tilde{\nabla}^{e_i}_{e_k} = \sum_{j=1}^{2} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=3}^{4} \varepsilon_r h^r_{ik} e_r \tag{3}$$

and

$$\tilde{\nabla}^{e_s}_{e_k} = -\sum_{j=1}^{2} \varepsilon_j h^s_{kj} e_j + D^{e_s}_{e_k},\ D^{e_s}_{e_k} = \sum_{r=3}^{4} \varepsilon_r \omega_{sr}(e_k) e_r, \tag{4}$$

where D is the normal connection and h^r_{ik} are the coefficients of the second fundamental form h.

If we define a covariant differentiation $\bar{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^{\perp}M$ of M by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M, then we have the Codazzi equation

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) \tag{5}$$

and the Gauss equation is given by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{6}$$

where the vectors X, Y, Z and W are tangent to M and R is the curvature tensor associated with ∇. The curvature tensor R associated with ∇ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For any real function f on M, the Laplacian Δf of f is given by

$$\Delta f = -\varepsilon_i \sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla^{e_i}_{e_i}} f \right). \tag{7}$$

The Gaussian curvature K of M in $\mathbb{E}^4_{\alpha\beta}$ is given by

$$K = \sum_{s=3}^{4} \varepsilon_s \left(h^s_{11} h^s_{22} - h^s_{12} h^s_{21} \right). \tag{8}$$

Also if Gaussian curvature of M vanishes identically, i.e., $K = 0$, the surface M is called flat.

3 Flat Rotation Surfaces with Pointwise 1-Type Gauss Map Via Generalized Quaternions

In this section, by using generalized quaternions we determine a rotational surface in four-dimensional generalized space $\mathbb{E}^4_{\alpha\beta}$. If we choose generalized quaternions p and q in Eq. (2) as $p = \cos t + i \frac{1}{\sqrt{\alpha}} \sin t$ and $q = x(s) + jy(s)$, we obtain following rotational surface in $\mathbb{E}^4_{\alpha\beta}$.

$$X(t, s) = \begin{pmatrix} \cos t & -\frac{\alpha}{\sqrt{\alpha}} \sin t & 0 & 0 \\ \frac{1}{\sqrt{\alpha}} \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\frac{\alpha}{\sqrt{\alpha}} \sin t \\ 0 & 0 & \frac{1}{\sqrt{\alpha}} \sin t & \cos t \end{pmatrix} \begin{pmatrix} x(s) \\ 0 \\ y(s) \\ 0 \end{pmatrix},$$

$$M : X(t, s) = \left(x(s) \cos t, \frac{1}{\sqrt{\alpha}} x(s) \sin t, y(s) \cos t, \frac{1}{\sqrt{\alpha}} y(s) \sin t \right), \tag{9}$$

where α is positive real constant and $\varphi(s) = (x(s), 0, y(s), 0)$ is the profile curve of M. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as follows:

$$e_1 = \frac{1}{\sqrt{\varepsilon_1(x^2(s)+\beta y^2(s))}}\Big(-x(s)\sin t, \frac{1}{\sqrt{\alpha}}x(s)\cos t, -y(s)\sin t, \frac{1}{\sqrt{\alpha}}y(s)\cos t\Big),$$
$$e_2 = \frac{1}{\sqrt{\varepsilon_2\Big((x'(s))^2+\beta(y'(s))^2\Big)}}\Big(x'(s)\cos t, \frac{1}{\sqrt{\alpha}}x'(s)\sin t, y'(s)\cos t, \frac{1}{\sqrt{\alpha}}y'(s)\sin t\Big),$$
$$e_3 = \frac{1}{\sqrt{\varepsilon_3\beta\Big((x'(s))^2+\beta(y'(s))^2\Big)}}\Big(-\beta y'(s)\cos t, -\frac{\beta}{\sqrt{\alpha}}y'(s)\sin t, x'(s)\cos t, \frac{1}{\sqrt{\alpha}}x'(s)\sin t\Big),$$
$$e_4 = \frac{1}{\sqrt{\varepsilon_4\beta(x^2(s)+\beta y^2(s))}}\Big(-\beta y(s)\sin t, \frac{\beta}{\sqrt{\alpha}}y(s)\cos t, x(s)\sin t, -\frac{1}{\sqrt{\alpha}}x(s)\cos t\Big),$$

where β is nonzero real constant. It is easily seen that

$$\langle e_1, e_1\rangle = \varepsilon_1, \langle e_2, e_2\rangle = \varepsilon_2, \langle e_3, e_3\rangle = \varepsilon_3 = \varepsilon\varepsilon_2, \langle e_4, e_4\rangle = \varepsilon_4 = \varepsilon\varepsilon_1,$$

where ε_1, ε_2 and ε are signatures of $x^2(s)+\beta y^2(s)$, $(x'(s))^2+\beta(y'(s))^2$ and β, respectively. Then, we have the dual 1-forms as:

$$\omega_1 = \varepsilon_1\sqrt{\varepsilon_1(x^2(s)+\beta y^2(s))}dt \text{ and}$$
$$\omega_2 = \varepsilon_2\sqrt{\varepsilon_2\Big((x'(s))^2+\beta(y'(s))^2\Big)}ds.$$

By a direct computation, we can obtain coefficients of the second fundamental form and the connection forms as:

$$\begin{aligned} &h_{11}^3 = \beta\varkappa\lambda b(s),\ h_{12}^3 = 0,\ h_{22}^3 = -\beta\varkappa\lambda c(s),\\ &h_{11}^4 = 0,\ h_{12}^4 = -\beta\varkappa\lambda b(s),\ h_{22}^4 = 0. \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\omega_{12} = -\varepsilon_1\varkappa a(s)\omega_1, \omega_{13} = \varepsilon_1\beta\varkappa\lambda b(s)\omega_1,\\ &\omega_{14} = -\varepsilon_2\beta\varkappa\lambda b(s)\omega_2,\ \omega_{23} = -\varepsilon_2\beta\varkappa\lambda c(s)\omega_2,\\ &\omega_{24} = -\varepsilon_1\beta\varkappa\lambda b(s)\omega_1,\ \omega_{34} = -\varepsilon_1\beta\varkappa\lambda^2 a(s)\omega_1. \end{aligned} \tag{11}$$

Moreover, combining Eqs. (3), (4), (10) and (11) we have covariant differentiation with respect to e_1 and e_2 as follows:

$$\begin{aligned} &\tilde{\nabla}_{e_1}e_1 = -\varepsilon_2\varkappa a(s)e_2 + \varepsilon\varepsilon_2\beta\varkappa\lambda b(s)e_3,\\ &\tilde{\nabla}_{e_2}e_1 = -\varepsilon\varepsilon_1\beta\varkappa\lambda b(s)e_4,\\ &\tilde{\nabla}_{e_1}e_2 = \varepsilon_1\varkappa a(s)e_1 - \varepsilon\varepsilon_1\beta\varkappa\lambda b(s)e_4,\\ &\tilde{\nabla}_{e_2}e_2 = -\varepsilon\varepsilon_2\beta\varkappa\lambda c(s)e_3,\\ &\tilde{\nabla}_{e_1}e_3 = -\varepsilon_1\beta\varkappa\lambda b(s)e_1 - \varepsilon\varepsilon_1\beta\varkappa\lambda^2 a(s)e_4,\\ &\tilde{\nabla}_{e_2}e_3 = \varepsilon_2\beta\varkappa\lambda c(s)e_2,\\ &\tilde{\nabla}_{e_1}e_4 = \varepsilon_2\beta\varkappa\lambda b(s)e_2 + \varepsilon\varepsilon_2\beta\varkappa\lambda^2 a(s)e_3,\\ &\tilde{\nabla}_{e_2}e_4 = \varepsilon_1\beta\varkappa\lambda b(s)e_1, \end{aligned} \tag{12}$$

where

$$a(s) = \frac{x(s)x'(s)+\beta y(s)y'(s)}{\varepsilon_1(x^2(s)+\beta y^2(s))}, \tag{13}$$

$$b(s) = \frac{x(s)y'(s)-x'(s)y(s)}{\varepsilon_1(x^2(s)+\beta y^2(s))}, \tag{14}$$

$$c(s) = \frac{x''(s)y'(s)-x'(s)y''(s)}{\varepsilon_2\Big((x'(s))^2+\beta(y'(s))^2\Big)}, \tag{15}$$

$$\begin{aligned} &\varkappa(s) = \frac{1}{\sqrt{\varepsilon_2\Big((x'(s))^2+\beta(y'(s))^2\Big)}},\\ &\lambda(s) = \frac{1}{\sqrt{\varepsilon\beta}}. \end{aligned} \tag{16}$$

Without loss of generality, we assume that the profile curve φ is parameterized by its arc-length, that is,

$$(x'(s))^2+\beta(y'(s))^2 = 1. \tag{17}$$

In that case, we have that $\varepsilon_2 = 1$. From Eqs. (2) and (10), we obtain Gaussian curvature K of M as:

$$K = -\beta b(s)(c(s)+\varepsilon_1 b(s)). \tag{18}$$

Furthermore, by using Eqs. (5), (6) and after some computations we have Gauss and Codazzi equations for

$$a'(s) + \varepsilon_1 a^2(s) = \beta b(s)(c(s)+\varepsilon_1 b(s)) \tag{19}$$

and

$$b'(s) = -(2\varepsilon_1 a(s)b(s) + a(s)c(s)), \tag{20}$$

respectively.

By using Eqs. (7), (12) and with straightforward computation, the Laplacian ΔG of the Gauss map $G = e_1 \wedge e_2$ is computed as:

$$\begin{aligned} \Delta G = {} & \beta\big(3b^2(s) + c^2(s)\big)e_1 \wedge e_2\\ & + \varepsilon\varepsilon_1\beta\lambda(-b'(s) + \varepsilon_1 c'(s))e_1 \wedge e_3\\ & - \varepsilon\varepsilon_1\beta\lambda a(s)(\varepsilon_1 b(s) - c(s))e_2 \wedge e_4\\ & + 2\varepsilon\varepsilon_1\beta b(s)(\varepsilon_1 b(s) + c(s))e_3 \wedge e_4 \end{aligned} \tag{21}$$

Now, we determine the flat rotational surfaces in $\mathbb{E}^4_{\alpha\beta}$ with the pointwise 1-type Gauss map.

Suppose that the rotational surface M given by the parameterization (9) is flat. From Eq. (18), we obtain that $b(s) = 0$ or $\varepsilon_1 b(s) + c(s) = 0$. We assume that $\varepsilon_1 b(s) + c(s) \neq 0$. Then, $b(s)$ is equal to zero and Eq. (20) implies that $a(s)c(s) = 0$. Since $\varepsilon_1 b(s) + c(s) \neq 0$, it implies that $c(s)$ is not equal to zero. Then, we obtain as $a(s) = 0$. In that case, by using Eqs. (13) and (14) we obtain that $\varphi(s) = (x(s), 0, y(s), 0)$ is a constant vector. This is a contradiction. Therefore, $c(s) = -\varepsilon_1 b(s)$ for all s. From Eq. (19), we get

$$a'(s) + \varepsilon_1 a^2(s) = 0 \tag{22}$$

whose trivial solution and non-trivial solution are

$$a(s) = 0$$

and

$$a(s) = \frac{1}{\varepsilon_1 s + s_0},$$

respectively.

3.1 Case $a(s) = 0$

We assume that $a(s) = 0$. By Eq. (20), we obtain that $b = b_0$ is a constant. So, we have $c = -\varepsilon_1 b_0$. In that case, by using Eqs. (13), (14) and (15), x and y satisfy the following differential equations:

$$x^2(s) + \beta y^2(s) = \mu \quad \mu \text{ is a constant}, \tag{23}$$

$$x(s)y'(s) - x'(s)y(s) = \varepsilon_1 b_0 \mu, \tag{24}$$

$$x''(s)y'(s) - x'(s)y''(s) = -\varepsilon_1 b_0. \tag{25}$$

3.1.1 *Case $\beta > 0$*

We assume that $\beta > 0$. Then, we obtain a Riemannian metric. Equation (23) is always positive. In that case, we have $\varepsilon_1 = 1$. We can choose μ in Eq. (23) as $\mu = \mu_0^2$, where μ_0 is a nonzero real constant. So, by using Eq. (23) we can put

$$x(s) = \mu_0 \cos\theta(s), \quad y(s) = \frac{\mu_0}{\sqrt{\beta}} \sin\theta(s). \tag{26}$$

By differentiating Eq. (26), we get

$$x'(s) = -\sqrt{\beta}\theta'(s)y(s) \text{ and } y'(s) = \frac{1}{\sqrt{\beta}}\theta'(s)x(s), \tag{27}$$

where $\theta(s)$ is some angle function. By substituting Eqs. (26) and (27) into Eq. (24), we have

$$\theta(s) = \sqrt{\beta} b_0 s + \delta, \quad \delta = \text{const}.$$

On the other hand, since the curve φ is a unit speed curve, from Eq. (17) we have

$$\beta b_0^2 \mu_0^2 = 1.$$

Then, we can write components of the curve φ as:

$$\begin{aligned} x(s) &= \mu_0 \cos\left(\sqrt{\beta} b_0 s + \delta\right), \\ y(s) &= \frac{\mu_0}{\sqrt{\beta}} \sin\left(\sqrt{\beta} b_0 s + \delta\right), \quad \beta b_0^2 \mu_0^2 = 1. \end{aligned} \tag{28}$$

Hence, we obtain that the profile curve φ is a family of ellipse. On the other hand, by using Eq. (21) we can rewrite the Laplacian of the Gauss map G with $a(s) = 0$ and $b = -\varepsilon_1 c = b_0$

$$\Delta G = 4\beta b_0^2 G,$$

that is, the flat surface M is pointwise 1-type Gauss map with the function $f = 4\beta b_0^2$ and $C = 0$, even if it is a pointwise 1-type Gauss map of the first kind.

Remark 1 If we consider as $\alpha = \beta = 1$, then we get four-dimensional Euclidean space $\mathbb{E}^4$. Also for $\alpha = \beta = 1$, the profile curve φ of flat rotational surface with pointwise 1-type Gauss map which is parameterized by Eq. (28) becomes a circle and we obtain the results which are given by Aksoyak and Yaylı[13]. Hence, the Case 3.1.1 can be considered as a generalization of that study.

3.1.2 *Case $\beta < 0$*

We suppose that $\beta < 0$. In that case, we obtain a semi-Riemannian metric. If we consider Eq. (23) with $\beta < 0$, we can put

$$\begin{aligned} x(s) &= \frac{1}{2}\xi\left(\mu_2 e^{\theta(s)} + \mu_1 e^{-\theta(s)}\right), \\ y(s) &= \frac{1}{2}\xi\left(\frac{\mu_2}{\sqrt{-\beta}} e^{\theta(s)} - \frac{\mu_1}{\sqrt{-\beta}} e^{-\theta(s)}\right), \end{aligned} \tag{29}$$

where $\theta(s)$ is some smooth function, $\xi = \pm 1$ and $\mu = \mu_1 \mu_2$. Differentiating Eq. (29) with respect to s, we have

$$x'(s) = \theta'(s)\sqrt{-\beta} y(s), \; y'(s) = \frac{\theta'(s)}{\sqrt{-\beta}} x(s). \tag{30}$$

By substituting Eqs. (29) and (30) into Eq. (24), we get

$$\theta(s) = \varepsilon_1 \sqrt{-\beta} b_0 s + \delta, \; \delta = \text{const}.$$

And since the curve φ is a unit speed curve, we have

$$\beta b_0^2 \mu = 1.$$

Since $\beta < 0$. then $\mu < 0$. So $x^2(s) + \beta y^2(s) < 0$. In that case, we obtain that $\varepsilon_1 = -1$. Then, we can write components of the curve φ as:

$$x(s) = \frac{1}{2}\xi\left(\mu_2 e^{-\sqrt{-\beta}b_0 s+\delta} + \mu_1 e^{-(-\sqrt{-\beta}b_0 s+\delta)}\right),$$
$$y(s) = \frac{1}{2}\xi\left(\frac{\mu_2}{\sqrt{-\beta}}e^{-\sqrt{-\beta}b_0 s+\delta} - \frac{\mu_1}{\sqrt{-\beta}}e^{-\left(-\sqrt{-\beta}b_0 s+\delta\right)}\right), \beta b_0^2\mu_1\mu_2 = 1. \tag{31}$$

Hence, we have that the profile curve φ is a family of hyperbolas. On the other hand, by using Eq. (21) we can rewrite the Laplacian of the Gauss map G with $a(s) = 0$ and $b = -\varepsilon_1 c = b_0$ as follows:

$$\Delta G = 4\beta b_0^2 e_1 \wedge e_2,$$

that is, the flat surface M is pointwise 1-type Gauss map with the function $f = 4\beta b_0^2$ and $C = 0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Remark 2 If we consider as $\alpha = 1$ and $\beta = -1$, then we get four-dimensional semi-Euclidean space $\mathbb{E}_2^4$. Also for $\alpha = 1$ and $\beta = -1$, the profile curve φ of flat rotational surface with pointwise 1-type Gauss map which is parameterized by Eq. (31) coincides the profile curve of flat rotational surface with pointwise 1-type Gauss map in $\mathbb{E}_2^4$ which is obtained by Aksoyak and Yaylı [12]. So, the Case 3.1.2 can be considered as a generalization of that study.

3.2 Case $a(s) = \frac{1}{\varepsilon_1 s + s_0}$

In this part, we give a common proof for the cases which β is positive or negative. Now, we assume that $a(s) = \frac{1}{\varepsilon_1 s + s_0}$. By using $c(s) = -\varepsilon_1 b(s)$ and Eq. (20), we get

$$b'(s) = -\varepsilon_1 a(s) b(s) \tag{32}$$

or we can write

$$\frac{b'(s)}{b(s)} = \frac{-\varepsilon_1}{\varepsilon_1 s + s_0},$$

whose solution is given by

$$b(s) = \frac{\gamma}{|\varepsilon_1 s + s_0|}, \quad \gamma \text{ is a constant.} \tag{33}$$

By using Eq. (21), we can rewrite the Laplacian of the Gauss map G with the equalities $c(s) = -\varepsilon_1 b(s), b'(s) = -\varepsilon_1 a(s) b(s)$ and $a'(s) = -\varepsilon_1 a^2(s)$ as follows:

$$\Delta G = 4\beta b^2(s) e_1 \wedge e_2 + 2\varepsilon\beta\lambda a(s) b(s) e_1 \wedge e_3 - 2\varepsilon\beta\lambda a(s) b(s) e_2 \wedge e_4. \tag{34}$$

We suppose that the flat rotational surface M has pointwise 1-type Gauss map. From Eqs. (1) and (34), we get

$$4\varepsilon_1\beta b^2(s) = f\varepsilon_1 + f\langle C, e_1 \wedge e_2\rangle, \tag{35}$$

$$2\varepsilon_1\beta\lambda a(s) b(s) = f\langle C, e_1 \wedge e_3\rangle, \tag{36}$$

$$-2\varepsilon_1\beta\lambda a(s) b(s) = f\langle C, e_2 \wedge e_4\rangle. \tag{37}$$

Then, we have

$$\langle C, e_1 \wedge e_4\rangle = 0, \quad \langle C, e_2 \wedge e_3\rangle = 0, \quad \langle C, e_3 \wedge e_4\rangle = 0. \tag{38}$$

By using Eqs. (36) and (37), we obtain

$$\langle C, e_1 \wedge e_3\rangle = -\langle C, e_2 \wedge e_4\rangle. \tag{39}$$

By differentiating the first equation in Eq. (38) with respect to e_1 and by using Eq. (12), the third equation in Eqs. (38) and (39), we get

$$2a(s)\langle C, e_1 \wedge e_3\rangle + \beta\lambda b(s)\langle C, e_1 \wedge e_2\rangle = 0. \tag{40}$$

Combining Eqs. (35), (36) and (40), we have

$$\gamma\left(4a^2(s) + 4\beta b^2(s) - f\right) = 0. \tag{41}$$

Firstly, we consider $4a^2(s) + 4\beta b^2(s) - f \neq 0$. Then, we get $\gamma = 0$ and it implies that $b = c = 0$. In that case, the surface M becomes totally geodesic and has harmonic Gauss map, that is, $\Delta G = 0$. Now, we assume that the fuction f satisfies the following equation:

$$f = 4a^2(s) + 4\beta b^2(s), \tag{42}$$

where f depends only on s. By differentiating f with respect to s and by using Eqs. (22), (32) and (42), we get

$$f' = -2\varepsilon_1 a(s) f. \tag{43}$$

By differentiating Eq. (36) with respect to e_2 and by using Eqs. (12), (22), (32), (35), (36), (42), (43) and the third equation in Eq. (38), we have

$$a^2 b = 0$$

or from Eq. (33) we can write

$$\gamma a^3 = 0.$$

Since $a(s) \neq 0$, it follows that $\gamma = 0$. Then, we obtain that $b = c = 0$ again. We obtain that the Gauss map of M is harmonic.

Theorem 1 *Let M be the flat rotational surface given by the parameterization (9). Then, M has pointwise 1-type Gauss map if and only if M is either totally geodesic or parameterized by one of the following*

(1)

$$X(t,s) = \begin{pmatrix} \mu_0 \cos\theta(s)\cos t, & \frac{1}{\sqrt{\alpha}}\mu_0 \cos\theta(s)\sin t, \\ \frac{\mu_0}{\sqrt{\beta}}\sin\theta(s)\cos t, & \frac{1}{\sqrt{\alpha}}\frac{\mu_0}{\sqrt{\beta}}\sin\theta(s)\sin t \end{pmatrix},$$

where $\theta(s) = \sqrt{\beta}b_0 s + \delta$ and $\beta b_0^2\mu_0^2 = 1$.

(2)

$$X(t,s)=\begin{pmatrix} \frac{1}{2}\xi\left(\mu_2 e^{\theta(s)}+\mu_1 e^{-\theta(s)}\right)\cos t, \\ \frac{1}{\sqrt{\alpha}}\frac{1}{2}\xi\left(\mu_2 e^{\theta(s)}+\mu_1 e^{-\theta(s)}\right)\sin t, \\ \frac{1}{2}\xi\left(\frac{\mu_2}{\sqrt{-\beta}}e^{\theta(s)}-\frac{\mu_1}{\sqrt{-\beta}}e^{-\theta(s)}\right)\cos t, \\ \frac{1}{\sqrt{\alpha}}\frac{1}{2}\xi\left(\frac{\mu_2}{\sqrt{-\beta}}e^{\theta(s)}-\frac{\mu_1}{\sqrt{-\beta}}e^{-\theta(s)}\right)\sin t \end{pmatrix},$$

where $\theta(s)=-\sqrt{-\beta}b_0 s+\delta$ and $\beta b_0^2\mu_1\mu_2=1$.

Corollary 1 *Let M be non-totally geodesic flat rotational surface given by the parameterization (9). If M has pointwise 1-type Gauss map, then the Gauss map G on M is pointwise 1-type Gauss map of the first kind.*

Corollary 2 *Let M be non-totally geodesic flat rotational surface given by the parametrization (9). If M has pointwise 1-type Gauss map, then the profile curves of M are circles in four-dimensional generalized space* $\mathbb{E}^4_{\alpha\beta}$. *These curves are Euclidean ellipses or hyperbolas.*

Corollary 3 *Let M be non-totally geodesic flat rotational surface given by the parametrization (9). If M has pointwise 1-type Gauss map, then it is a part of sphere in four-dimensional generalized space* $\mathbb{E}^4_{\alpha\beta}$. *This sphere is a Euclidean ellipsoid or hyperboloid.*

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