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Rings having normality in terms of the Jacobson radical

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Abstract A ring R is defined to be J-normal if for any $a, r \in R$ and idempotent $e \in R$, $ae = 0$ implies $Rera \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R. The class of J-normal rings lies between the classes of weakly normal rings and left min-abel rings. It is proved that R is J-normal if and only if for any idempotent $e \in R$ and for any $r \in R$, $R(1-e)re \subseteq J(R)$ if and only if for any $n \geq 1$, the $n \times n$ upper triangular matrix ring $U_n(R)$ is a J-normal ring if and only if the Dorroh extension of R by $\mathbb{Z}$ is J-normal. We show that R is strongly regular if and only if R is J-normal and von Neumann regular. For a J-normal ring R, it is obtained that R is clean if and only if R is exchange. We also investigate J-normality of certain subrings of the ring of 2×2 matrices over R.

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1 Introduction

Throughout this work, every ring is associative with identity unless otherwise stated. Recently, some kinds of normality for rings have been investigated in the literature. For instance, the notion of quasi-normality of rings was defined in [13], that is, a ring R is called quasi-normal if $ae = 0$ implies $eaRe = 0$ for every nilpotent element a and idempotent e of R. On the other hand, another kind of normality was introduced in [14], namely, a ring R is said to be weakly normal if for all elements a, r and $e^2 = e$ of R, $ae = 0$ implies that $Rera$ is a nil left ideal of R. It is seen that the notion of a weakly normal ring is a generalization of that of a quasi-normal ring.

The Jacobson radical is an important tool for studying the structure of a ring. In the light of aforementioned concepts, it is a reasonable question that what kind of properties does a ring gain when it satisfies normality in terms of its Jacobson radical? This question is one of the motivations to deal with the notion of normality

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in terms of the Jacobson radical. In this direction, motivated by the works on quasi-normal rings and weakly normal rings, we introduce and study a class of rings, called J-normal rings, which is a generalization of that of weakly normal rings. We call a ring R J-normal if for all elements a, r and $e^2 = e$ of R, $ae = 0$ implies that $Rera$ is contained in the Jacobson radical of R. By [13, Theorem 2.1] and [14, Theorem 2.1], quasi-normal rings and weakly normal rings are characterized by a condition on $eR(1-e)$ for an arbitrary idempotent e. We also give a characterization of J-normality of rings in terms of $eR(1-e)$ where $e^2 = e \in R$ as follows.

$$\text{R is} \begin{cases} \text{quasi-normal} \\ \text{weakly normal} \\ \text{J-normal} \end{cases} \Leftrightarrow eR(1-e) \quad \text{is} \begin{cases} \text{a right ideal} \\ \text{contained in } N^*(R) \\ \text{contained in } J(R) \end{cases} \text{for any } e^2 = e \in R.$$

Since the Jacobson radical of R is a semiprime ideal, R is J-normal if and only if for all elements a, r and idempotent e of R, $ae = 0$ implies that $areR$ is contained in the Jacobson radical of R. We prove that some results of weakly normal rings can be extended to J-normal rings. We supply some examples to show that a J-normal ring need not be quasi-normal. We determine the position of the class of J-normal rings in the ring theory by investigating the relations between the class of J-normal rings and certain classes of rings such as abelian rings, min-abel rings, central reversible rings, central semicommutative rings, directly finite rings. We obtain characterizations of J-normal rings from different aspects. We discuss properties of J-normal rings and also give structure theorems. Moreover, we work on extensions of this class of rings such as trivial extensions, polynomial extensions and Dorroh extensions. Using inspected extensions, we present more characterizations of J-normal rings.

Furthermore, some applications of J-normal rings are performed. In this direction, this concept is considered for the ring of matrices. Morita context rings, full matrix rings, triangular matrix rings and generalized matrix rings are investigated with regard to J-normality. On the other hand, it is known that a clean ring is an exchange ring, but the converse need not hold in general. It is proved that the converse of this statement is true for weakly normal rings. This paper is an attempt to give some weaker conditions such as a J-normal ring is clean if and only if it is exchange. These connections make the concept of J-normal rings more attractive to study.

Let $M_n(R)$ denote the full matrix ring over R and $U_n(R)$ the upper triangular matrix ring over R and the subring $\{(a_{ij}) \in U_n(R) \mid \text{all diagonal entries of } (a_{ij}) \text{ are equal}\}$ is denoted by $D_n(R)$ where n is a positive integer. Also, $J(R)$, $P(R)$ and $C(R)$ stand for the Jacobson radical, the prime radical and the center of R, respectively.

2 J-normal rings

In [13], Wei and Li defined and investigated quasi-normal rings. A ring R is defined to be quasi-normal if $ae = 0$ implies $eaRe = 0$ for any nilpotent a and idempotent e of R. In [14], they defined and investigated weakly normal rings. A ring R is weakly normal if for all $a, r \in R$ and any idempotent e, $ae = 0$ implies $Rera$ is a nil left ideal of R. Clearly every quasi-normal ring is weakly normal. There exists a weakly normal ring which is not quasi-normal. See for example [14]. In this note, we study another kind of normality using the Jacobson radical of a ring.

Definition 2.1 A ring R is called J-normal if for any idempotent e and any $a, r \in R$, $ae = 0$ implies $Rera \subseteq J(R)$.

We give a characterization of J-normal rings.

Theorem 2.2 *Let R be a ring. Then, the following are equivalent.*

(1) *R is J-normal.*
(2) *$eR(1-e) \subseteq J(R)$ for any idempotent e.*
(3) *$(1-e)Re \subseteq J(R)$ for any idempotent e.*
(4) *Every idempotent is central modulo $J(R)$.*

Proof (1) $\Rightarrow$ (2) Assume that R is J-normal and $e^2 = e \in R$. Then, $(1-e)e = 0$ implies $Rer(1-e) \subseteq J(R)$ for $r \in R$. Since R has an identity, $eR(1-e) \subseteq J(R)$.

(2) $\Leftrightarrow$ (3) is clear since if e is idempotent then so is $1-e$.

(3) $\Rightarrow$ (4) Let $e^2 = e \in R$. Then, for any $x \in R$, $ex(1-e) \in J(R)$ and $(1-e)xe \in J(R)$. Hence, $ex - xe \in J(R)$.

(4) $\Rightarrow$ (1) Let $a \in R$ and $e^2 = e \in R$ with $ae = 0$. Write $\bar{R} = R/J(R)$. Then, $\bar{e} = e + J(R)$ is central in $\bar{R}$. So $\bar{R}\bar{e}\bar{r}\bar{a} = \bar{R}\bar{r}\bar{a}\bar{e} = \bar{0}$ for any $r \in R$. Hence, $Rera \subseteq J(R)$.

□

Corollary 2.3 *If $R/J(R)$ is abelian, then R is J-normal. The converse holds if idempotents lift modulo $J(R)$.*

We combine Theorem 2.2 and Corollary 2.3 as follows.

Theorem 2.4 *Let R be a ring such that the idempotents lift modulo $J(R)$. Then R is J-normal if and only if $R/J(R)$ is abelian.*

Corollary 2.5 *Let R be a ring with $J(R) = 0$. Then, R is J-normal if and only if it is abelian.*

There are J-normal rings R such that $R/J(R)$ need not be abelian.

Example 2.6 Let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$ and Q the set of quaternions over the ring R as in [11, Example 2.3]. Then, Q is a domain and, therefore, J-normal. On the other hand, $Q/J(Q)$ is isomorphic to $M_2(\mathbb{Z}_3)$. Therefore, $Q/J(Q)$ is not abelian.

Note that since $J(R)$ is a semiprime ideal, then R is J-normal if and only if for all $a, r \in R$ and any idempotent $e \in R$, $ae = 0$ implies $areR \subseteq J(R)$. Abelian rings, semicommutative rings and commutative rings are J-normal rings. There are more sources of examples for J-normal rings as the following lemma shows.

Lemma 2.7 *The following hold.*

(1) *Every weakly normal ring is J-normal.*
(2) *Every quasi-normal ring is J-normal.*
(3) *Every ring R with $R/J(R)$ J-normal is J-normal.*
(4) *Every feckly reduced ring is J-normal.*
(5) *Every weakly reversible ring is J-normal.*
(6) *Every central reversible ring is J-normal.*
(7) *Every central semicommutative ring is J-normal.*

Proof (1) Let R be a weakly normal ring, $a \in R$ and $e^2 = e \in R$ with $ae = 0$. Then, $Rera$ is a nil left ideal of R for all $r \in R$. So $Rera \subseteq J(R)$.

(2) Let R be a quasi-normal ring, by [14, Corollary 2.3], R is weakly normal. So by (1), R is J-normal.

(3) Let a, $e^2 = e \in R$ with $ae = 0$. Then, $\overline{ae} = \bar{0}$ in $\overline{R}$. So for any $\bar{r} \in \overline{R}$, $\overline{Rera} = \bar{0}$ since $R/J(R)$ is J-normal. Hence, $Rera \subseteq J(R)$.

(4) In [11], a ring R is called feckly reduced if $R/J(R)$ is a reduced ring. Let a, $e^2 = e \in R$ with $ae = 0$. In the ring $R/J(R)$, $\overline{ae} = \bar{0}$. It implies that for any $r \in R$, $\overline{era}$ is nilpotent. By hypothesis, $\overline{era} = \bar{0}$. Thus, $Rera \subseteq J(R)$ for all $r \in R$.

(5) In [5], a ring R is said to be weakly reversible if for all $a, b, r \in R$ such that $ab = 0$, $Rbra$ is a nil left ideal of R. Assume that R is a weakly reversible ring. Let $e^2 = e \in R$ and $a \in R$ be an arbitrary element with $ae = 0$. By assumption, $Rera$ is a nil left ideal. As it is contained in $J(R)$, R is a J-normal ring.

(6) In [4], a ring R is called central reversible if for any $a, b \in R$, $ab = 0$ implies ba is central in R. Let $e^2 = e \in R$ and $a \in R$ be an arbitrary element with $ae = 0$. The ring R being central reversible implies that era is central for each $r \in R$. Then, $eraRera = R(era)^2 = 0$. Hence, $(Rera)^2 = 0$. Thus, $Rera \subseteq J(R)$.

(7) In [9], a ring R is called central semicommutative if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$. Assume that R is a central semicommutative ring. Let $ae = 0$ for any idempotent $e \in R$ and any element $a \in R$. The ring R being central semicommutative implies that $eraRe$ is central in R for each $r \in R$. Then, $(Rera)^2 = (Rera)(Rera) = RraeraRe = 0$ and so $Rera \subseteq J(R)$ for each $r \in R$. Hence, R is J-normal. □

There are rings which do not satisfy the converse statements in Lemma 2.7.

Examples 2.8

(1) Let F be a field. The upper triangular matrix ring $R = U_3(F)$ is not quasi-normal by [14, Theorem 2.1]. But R is J-normal since R is weakly normal by [14, Example 2.13].

(2) Consider the ring Q in Example 2.6. Then, Q is J-normal but $Q/J(Q)$ is neither feckly reduced nor abelian nor J-normal.

Let R be a ring and I an ideal of R. The ideal I is called J-normal if it has the J-normality as a ring without identity.

Proposition 2.9 *The following hold.*

(1) *Let R be a J-normal ring. Then, every ideal I of R is J-normal.*
(2) *Any direct product of rings $\{R_i\}_{i\in I}$ is J-normal if and only if each ring R_i $(i \in I)$ is J-normal.*

Proof (1) Let I be an ideal of a J-normal ring R and $a \in I$, $e^2 = e \in I$ with $ae = 0$. Then, $Rera \subseteq J(R)$ for all $r \in R$. Since $J(I) = J(R) \cap I$, $Iexa \subseteq J(I)$ for all $x \in I$.
(2) Let $\{R_i\}_{i\in I}$ be a class of rings. Assume that for each $i \in I$, R_i is J-normal. Let $R = \Pi R_i$ and $a = (a_i)$, $e^2 = e = (e_i) \in R$ with $ae = 0$. Then, $e_i^2 = e_i$ and $a_i e_i = 0$ for each $i \in I$. By hypothesis $R_i e_i b_i a_i \subseteq J(R_i)$ for each $b_i \in R_i$. For any $b_i \in R_i$, set $b = (b_i)$. Then, $Reba \subseteq J(R)$ since $J(R) = \Pi J(R_i)$. For the converse, suppose that $R = \Pi R_i$ is a J-normal ring. Then, each R_i is an ideal of R. By (1) each R_i is a J-normal ring.

□

Theorem 2.10 *Let R be a ring. Consider the following conditions.*

(1) *R is a J-normal ring.*
(2) *For any idempotent $e \in R$ and for any $r \in R$, $R(1-e)re \subseteq J(R)$.*
(3) *For a central idempotent $e \in R$, eR and$(1-e)R$ are J-normal rings.*
(4) *For any idempotent $e \in R$, $eR(1-e) \subseteq P(R)$.*

Then, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *and* (4) $\Rightarrow$ (1).

Proof (1) $\Rightarrow$ (2) Let $e^2 = e \in R$. By Theorem 2.2, for any $r \in R$, $(1-e)re \in J(R)$. Hence, $R(1-e)re \subseteq J(R)$.
(2) $\Rightarrow$ (1) Clear.
(1) $\Rightarrow$ (3) Let $ea \in eR$ and $(fe)^2 = fe \in eR$ with $eaefe = 0$. Then, for any $er \in eR$, $eRfe(erea) \subseteq J(eR)$ since $aef = 0$ and e is central.
(3) $\Rightarrow$ (1) Note that eR and $(1-e)R$ are J-normal rings and R is a direct sum of ideals eR and $(1-e)R$. So R is J-normal by Proposition 2.9 (2).
(4) $\Rightarrow$ (1) Let $e^2 = e \in R$ and $a \in R$. Assume that $ae = 0$. Let $f = 1-e$. By (4), $(1-f)Rf \subseteq P(R)$. Then, $eR(1-e) \subseteq P(R)$. Hence, $era(1-e) \in P(R)$ for each $r \in R$. Since $P(R) \subseteq J(R)$, $Rera(1-e) \subseteq J(R)$ for each $r \in R$. So R is J-normal.

□

Recall that an idempotent e in a ring R is called left semicentral if for every element $r \in R$, $re = ere$, and e is called left minimal if the left ideal Re is a minimal left ideal of R. Also, in [12], a ring is said to be left min-abel if every left minimal idempotent is left semicentral.

Theorem 2.11 *J-normal rings are left min-abel.*

Proof Let R be a J-normal ring and e a left minimal idempotent of R. For any $r \in R$, set $x = re - ere$. Assume that $x \neq 0$. Then, $ex = 0$ and $xe = x$ and $x(1-e) = 0$. The fact that $e(1-e) = 0$ and $(1-e)e = 0$ implies $Rer(1-e) \subseteq J(R)$ and $R(1-e)re \subseteq J(R)$ for all $r \in R$. So $xe = x$ implies $Rx \subseteq Re$. Since e is left minimal idempotent, $Re = Rx$. But $Rx = R(1-e)xe \subseteq J(R)$. Hence, $e \in J(R)$. This contradicts $x \neq 0$. Thus, $re = ere$ for all $r \in R$ and R is left min-abel. □

Theorem 2.12 *Let R be a J-normal ring. Then for any maximal left or right ideal M and for any $e^2 = e \in R$, either $e \in M$ or $1-e \in M$.*

Proof Let M be a maximal left ideal of R with $e \notin M$, then $Re + M = R$. There exist $r \in R$ and $m \in M$ such that $1 = re + m$. Since R is J-normal, we have $(1-e)se \in J(R)$ for all $s \in R$. Thus, $1-e = (1-e)1 = (1-e)re + (1-e)m \in M$. □

Theorem 2.13 *J-normal rings are directly finite.*

Proof Let $a, b \in R$ with $ab = 1$. Let $e = 1 - ba$. Since R is J-normal and $ae = 0$, $Rera \subseteq J(R)$ for all $r \in R$. Since R has the identity, $ea \in J(R)$. Multiplying the latter from the right by b and using $ab = 1$, we have $e \in J(R)$. So $e = 0$ and then $1 = ba$. □

J-normal property for rings does not extend to matrix rings.

Example 2.14 We consider the rings $\mathbb{Z}_2$ and $M_2(\mathbb{Z}_2)$. Then, $\mathbb{Z}_2$ is J-normal, but $M_2(\mathbb{Z}_2)$ is not J-normal. Indeed, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$. But

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

is not contained in $J(M_2(\mathbb{Z}_2))$.

A ring R is called clean if every element in R is the sum of an idempotent and a unit, and it is called exchange provided that for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$ (see for detail [1,8]). Clean rings are always exchange. The converse holds if all idempotents of R are central. Also note that a ring R is an exchange ring if and only if idempotents can be lifted modulo every left (resp., right) ideal (see [8]). If u is an invertible element of a ring R and $r \in J(R)$, then $u + r$ is invertible. For if $a = u + r$, then $u^{-1}a - 1 = u^{-1}r \in J(R)$. As $1 + u^{-1}r$ is invertible, $u^{-1}a$ is invertible and so is a. Also for an element $a \in R$, a is invertible in R if and only if $\overline{a}$ is invertible in $R/J(R)$.

Theorem 2.15 *Let R be a J-normal ring. Then, R is clean if and only if R is exchange.*

Proof One direction is trivial from [8, Proposition 1.8]. Assume that R is an exchange ring. Then, so is $R/J(R)$. According to Corollary 2.5, $R/J(R)$ is abelian. In the light of [8], $R/J(R)$ is clean. Let $x \in R$. There exist an idempotent $\bar{e} \in R/J(R)$ and a unit $\bar{u} \in R/J(R)$ such that $\bar{x} = \bar{e} + \bar{u}$ where $\bar{e}$ refers to the element $e + J(R)$ of $R/J(R)$. By assumption, we may suppose e is an idempotent in R. So u is invertible in R and we can find $r \in J(R)$ such that $x = e + u(1 + u^{-1}r)$, where $u(1 + u^{-1}r)$ is a unit element of R. So R is clean. □

Note that in [14] an element e of a ring R is called op-idempotent if $e^2 = -e$. An op-idempotent need not be an idempotent. For example, $\overline{4}^2 = \overline{1} = -\overline{4}$ in $\mathbb{Z}_5$. However, $\overline{4}$ is not an idempotent.

Theorem 2.16 *Let R be a J-normal ring. Then, for all $a, r \in R$ and any op-idempotent $e \in R$, $ae = 0$ implies $Rera \subseteq J(R)$.*

Proof Let $a, r \in R$ and $e \in R$ an op-idempotent with $ae = 0$. Then, $e^4 = e^2$. Since R is J-normal, we have $Re^2ra \subseteq J(R)$ for all $r \in R$. Since $e^2 = -e$, $Rera \subseteq J(R)$. □

3 Extensions

In [10], generalized matrix ring $K_s(R)$ over a ring R is defined and investigated in detail. Addition in $K_s(R)$ is componentwise and multiplication is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} ax + sbz & ay + bt \\ cx + dz & dt + scy \end{bmatrix}.$$

Then, $K_s(R)$ is an associative ring if and only if s is central. And ideal structures of $K_s(R)$ are given as follows:

Lemma 3.1 [10, Lemma 4.2] *Let R be a ring and s a central element of R.*

(1) *K is an ideal of $K_s(R)$ if and only if $K = \begin{bmatrix} I_1 & I_2 \\ I_3 & I_4 \end{bmatrix}$ where each I_i is an ideal of R with $I_1 + I_4 \subseteq I_2 \cap I_3$ and $s(I_2 + I_3) \subseteq I_1 \cap I_4$.*

(2) $J(K_s(R)) = \begin{bmatrix} J(R) & (s : J(R)) \\ (s : J(R)) & J(R) \end{bmatrix}$, *where $(s : J(R)) = \{r \in R \mid sr \in J(R)\}$.*

Theorem 3.2 *If $K_s(R)$ is J-normal, then R is J-normal.*

Proof Assume that $K_s(R)$ is a J-normal ring. Let $a, e^2 = e \in R$ with $ae = 0$. Consider $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$ for any $b \in R$. We have $E^2 = E$ and $AE = 0$. By assumption, $K_s(R)EBA \subseteq J(K_s(R))$. By comparing (1, 1) entries, we have $Reba \subseteq J(R)$ for each $b \in R$. Hence, R is J-normal. □

Note that Example 2.14 shows that the converse statement of Theorem 3.2 need not be true in general for any $s \in \mathbb{Z}_2$.

Proposition 3.3 *The following hold for a ring R.*

(1) *If $R[x]$ is J-normal, then R is J-normal.*
(2) *If R is abelian, then $R[x]$ is J-normal.*

Proof (1) Assume that $R[x]$ is J-normal. Let $e^2 = e \in R$ and $a \in R$ be an arbitrary element of R with $ae = 0$. Then, $R[x]ef(x)a \subseteq J(R[x])$. Hence, $rea_0a \in J(R[x]) \cap R \subseteq J(R)$ for all $r \in R$ where a_0 is the constant term of $f(x)$, for all $a_0 \in R$. So R is J-normal.

(2) Suppose that R is abelian. By [2, Theorem 5], every idempotent of $R[x]$ is contained in R. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $e^2 = e \in R[x]$ with $f(x)e = 0$. Then, $a_i e = 0$ for $i = 0, 1, 2, \ldots, n$. By supposition, R is J-normal. This implies that $Rera_i \subseteq J(R)$ for any $r \in R$ and $i \in \{0, 1, 2, 3, \ldots, n\}$. Since x commutes with every element of R, $Rxexrxa_ix \subseteq J(R)[x]$. $J(R)[x]$ being an ideal of $R[x]$ implies $R[x]eg(x)f(x) \subseteq J(R)[x]$ for all $g(x) \in R[x]$. By Amitsur Theorem, $J(R[x]) = (J(R[x]) \cap R)[x]$ implies $J(R)[x] \subseteq J(R[x])$. This completes the proof. □

In the next result, we show that J-normality of rings is inherited by the corner rings.

Proposition 3.4 *If R is a J-normal ring, then eRe is J-normal for any $e^2 = e \in R$.*

Proof Let $eae, (efe)^2 = efe \in eRe$ with $(eae)(efe) = 0$. Since R is J-normal, $R(efe)r(eae)$ is contained in $J(R)$ for all $r \in R$. Hence, $e(R(efe)r(eae))e \subseteq eJ(R)e = J(eRe)$. So $(eRe)(efe)(ere)(eae) \subseteq J(eRe)$. Thus, eRe is J-normal. □

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. It is well known that $J(R) = \begin{bmatrix} J(S) & M \\ 0 & J(T) \end{bmatrix}$.

Proposition 3.5 *Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Then, R is J-normal if and only if S and T are J-normal.*

Proof The necessity is obvious by Proposition 3.4. Assume that S and T are J-normal. Let $A = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$, $E = \begin{bmatrix} e & n \\ 0 & f \end{bmatrix} \in R$ such that $E^2 = E$ and $AE = 0$. Then, e and f are idempotent elements of S and T, respectively. Hence, we have $ae = 0$ and $bf = 0$ in S and T, respectively. By hypothesis, $Sexa \subseteq J(S)$ and $Tfyb \subseteq J(T)$ for any $x \in S$ and $y \in T$. We get $REXA \subseteq J(R)$ for all $X \in R$. Thus, R is J-normal. □

Corollary 3.6 *R is a J-normal ring if and only if for any $n \geq 1$, the $n \times n$ upper triangular matrix ring $U_n(R)$ is a J-normal ring.*

Given a ring R and a bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$, where $r \in R$ and $m \in M$ when the usual matrix operations are used.

Corollary 3.7 *R is a J-normal ring if and only if its trivial extension is a J-normal ring.*

Corollary 3.8 *R is a J-normal ring if and only if for any $n \geq 1$, $R[x]/(x^n)$ is a J-normal ring, where (x^n) is the ideal of $R[x]$ generated by (x^n).*

Proposition 3.9 *Let R be a J-normal ring. Then, the following hold.*

(1) *If $e^2 = e \in R$ satisfies $ReR = R$, then $e = 1$.*
(2) *If $e^2 = -e \in R$ satisfies $ReR = R$, then $e = -1$.*

Proof (1) For any $e^2 = e \in R$, we have $(1-e)e = 0$. Since R is J-normal, $Rer(1-e) \subseteq J(R)$ for all $r \in R$. By assumption, $1 - e \in J(R)$ and so $1 - e = 0$. We have $e = 1$.
(2) For any $e^2 = -e \in R$, e^2 is an idempotent. So we have $(1-e^2)e^2 = 0$. Since R is J-normal, $Re^2r(1-e^2) \subseteq J(R)$ for all $r \in R$. By assumption, $R(1+e) \subseteq J(R)$. Then, $1 + e \in J(R)$ and so $e = -1$.

□

Recall that a ring R is called von Neumann regular if for any $a \in R$, there exists $b \in R$ such that $a = aba$. A ring R is said to be unit-regular if for any $a \in R$, $a = aua$ for some unit element $u \in R$. A ring R is called strongly regular if for any $a \in R$, $a = a^2b$ for some $b \in R$. Clearly,

$$\{\text{strongly regular rings}\} \subsetneqq \{\text{unit-regular rings}\} \subsetneqq \{\text{von Neumann regular rings}\}.$$

In the next result, we characterize strongly regular rings in terms of J-normality.

Theorem 3.10 *Let R be a ring. Then, the following conditions are equivalent.*

(1) *R is strongly regular.*
(2) *R is J-normal and von Neumann regular.*

Proof (1) $\Rightarrow$ (2) Assume that R is strongly regular, hence R is von Neumann regular. Also, R is reduced. Thus, R is J-normal.
(2) $\Rightarrow$ (1) Since R is von Neumann regular, for any $a \in R$ we have $a = aba$ for some $b \in R$. Write $e = ba$. Hence $a = ae$ and so $a(1-e) = 0$. Since R is J-normal, for all $r \in R$, $R(1-e)ra \subseteq J(R)$. We have $1(1-e)1a \in J(R)$. As $J(R) = 0$, $a = ea = (ba)a = ba^2$ and so R is strongly regular.

□

For any ring R, $V_n(R)$ is the subring of $M_n(R)$ where n is a positive integer:

$$V_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{bmatrix} \mid a_i \in R, 1 \le i \le n \right\}.$$

The Jacobson radicals of $V_n(R)$ and $D_n(R)$ are given by

$$J(V_n(R)) = \{(a_i) \in V_n(R) \mid a_1 \in J(R)\},$$
$$J(D_n(R)) = \left\{(a_{ij}) \in D_n(R) \mid a_{ii} \in J(R), 1 \le i \le n\right\},$$

respectively.

Let R be a ring and $D(\mathbb{Z}, R)$ denote the Dorroh extension of R by the ring of integers $\mathbb{Z}$. Then, $D(\mathbb{Z}, R)$ is the ring defined by the direct sum $\mathbb{Z} \oplus R$ with componentwise addition and multiplication $(n, r)(m, s) = (nm, ns + mr + rs)$ where $(n, r),\ (m, s) \in D(\mathbb{Z}, R)$. By [7], $J(D(\mathbb{Z}, R)) = (0, J(R))$.

Theorem 3.11 *The following are equivalent.*

(1) *R is J-normal.*
(2) *$D(\mathbb{Z}, R)$ is J-normal.*
(3) *For any positive integer n, $U_n(R)$ is J-normal.*
(4) *For any positive integer n, $D_n(R)$ is J-normal.*
(5) *For any positive integer n, $V_n(R)$ is J-normal.*

Proof (1) $\Rightarrow$ (2) Let (n,r), $(m,s)^2 = (m,s) \in D(\mathbb{Z}, R)$ and assume $(n,r)(m,s) = (0,0)$. Hence, $(nm, ns + mr + rs) = (0,0)$. Then, $nm = 0$ and $ns + mr + rs = 0$. We divide the proof into two cases.
Case I: Assume that $m = 0$. We get $s^2 = s \in R$. Thus, we have $(n1_R + r)s = 0$ in R. Since R is J-normal, $Rst(n1_R + r)$ is contained in the Jacobson radical of R for all $t \in R$. For any (l,y), $(k,x) \in D(\mathbb{Z}, R)$, $(l,y)(0,s)(k,x)(n,r) = (0, (l1_R + y)s(k1_R + x)(n1_R + r)) \in J(D(\mathbb{Z}, R))$. Hence, $D(\mathbb{Z}, R)(m,s)(k,x)(n,r) \subseteq J(D(\mathbb{Z}, R))$ for any $(k,x) \in D(\mathbb{Z}, R)$.
Case II: Assume that $m = 1$. In this case, $n = 0$, $s^2 = -s \in R$ and $r(1_R + s) = 0$. Then, $1_R + s$ is an idempotent element of R. By hypothesis, $R(1_R + s)tr \subseteq J(R)$ for any $t \in R$. For any (l,y), $(k,x) \in D(\mathbb{Z}, R)$, $(l,y)(1,s)(k,x)(0,r) = (0, (l+y)(1_R + s)(k+x)r) \in J(D(\mathbb{Z}, R))$. Hence, $D(\mathbb{Z}, R)(m,s)(k,x)(n,r) \subseteq J(D(\mathbb{Z}, R))$ for any $(k,x) \in D(\mathbb{Z}, R)$.

(2) $\Rightarrow$ (1) Suppose that $D(\mathbb{Z}, R)$ is J-normal. Note that J-normal property is preserved under a ring isomorphism. As R is isomorphic to the ideal $\{(0,r) \mid r \in R\} \subseteq D(\mathbb{Z}, R)$, we have R is J-normal.

(1) $\Rightarrow$ (3) Suppose that R is J-normal. To prove $U_n(R)$ is J-normal, let $A = (a_{ij})$, $E^2 = E = (e_{ij}) \in U_n(R)$ with $AE = 0$. Then, $a_{ii}e_{ii} = 0$ for all i with $0 \leq i \leq n$. By (1), $Re_{ii}ra_{ii} \subseteq J(R)$ for all $r \in R$ and i with $0 \leq i \leq n$. The Jacobson radical of $U_n(R)$ is

$$J(U_n(R)) = \begin{bmatrix} J(R) & R & R & \dots & R & R \\ 0 & J(R) & R & \dots & R & R \\ 0 & 0 & J(R) & \dots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J(R) & R \\ 0 & 0 & 0 & \dots & 0 & J(R) \end{bmatrix}.$$

Hence, $U_n(R)EBA \subseteq J(U_n(R))$ for all $B \in U_n(R)$.

(3) $\Rightarrow$ (4) Let $A = (a_{ij})$, $E^2 = E = (e_{ij}) \in D_n(R)$ with $AE = 0$. Set $a_{11} = a$, $e_{11} = e$. Then, $ae = 0$. By (3), $U_n(R)EXA \subseteq J(U_n(R))$ for all $X \in U_n(R)$. By comparing $(1,1)$ entries, we have $Rera \subseteq J(R)$ for all $r \in R$. Hence, $D_n(R)EXA \subseteq J(D_n(R))$ for all $X \in J(D_n(R))$.

(4) $\Rightarrow$ (5) Let $A = (a_i)$, $E^2 = E = (e_i) \in V_n(R)$ with $AE = 0$. By (4), $D_n(R)EXA \subseteq J(D_n(R))$ for all $X \in D_n(R)$. Then, the diagonal entries of matrices of $D_n(R)EXA$ are contained in $J(D_n(R))$; the same holds for the matrices of $V_n(R)EXA$ for all $X \in V_n(R)$. Hence, $V_n(R)EXA \subseteq J(V_n(R))$.

(5) $\Rightarrow$ (1) Let a, $e^2 = e \in R$ with $ae = 0$. Let A denote the matrix in $V_n(R)$ having diagonal entries a elsewhere 0 and E the idempotent matrix in $V_n(R)$ having diagonal entries e elsewhere 0. Then, $AE = 0$. By (5), $V_n(R)EXA \subseteq J(V_n(R))$ for all $X \in V_n(R)$. The diagonal entries of matrices in $V_n(R)EXA$ belong to $J(V_n(R))$. Hence, $Rexa \subseteq J(R)$ for all $x \in R$. This completes the proof. □

Proposition 3.12 *Let R be a ring. If all nilpotent elements of R are in $J(R)$, then R is J-normal.*

Proof Let a, $e^2 = e \in R$ with $ae = 0$. For any $r \in R$, $aer = 0$. We have era is a nilpotent element of R. By hypothesis, $era \in J(R)$. So $Rera$ is contained in $J(R)$ for all $r \in R$. Hence, R is J-normal. □

Let R be a ring and S a subring of R and

$$T[R,S] = \{(r_1, r_2, \dots, r_n, s, s, \dots) : r_i \in R, s \in S, n \geq 1, 1 \leq i \leq n\}.$$

Then, $T[R,S]$ is a ring under the componentwise addition and multiplication. We use the following characterization of the Jacobson radical. Let R be a ring and $r \in R$. Then, r is called right quasi-regular (r.q.r for short) in R if there exists $s \in R$ such that $r \circ s = 0$, where $r \circ s = r + s - rs$. In terms of right quasi-regular elements of R, $J(R) = \{a \in R \mid aR \text{ is right quasi-regular in } R\}$, see also [6, Definition 6.6 and Chapter 6] for details. In [3], it is shown that $J(T[R,S]) = T[J(R), J(R) \cap J(S)]$. In the following, we give necessary and sufficient conditions for $T[R,S]$ to be J-normal.

Proposition 3.13 *Let R be a ring and S a subring of R. Then, the following are equivalent.*

(1) *$T[R,S]$ is J-normal.*
(2) *R and S are J-normal.*

Proof (1) $\Rightarrow$ (2) Let $a \in R$, $e^2 = e \in R$ with $ae = 0$. Set $X = (a, 0, 0, \dots)$ and $Y = (e, 0, 0, \dots)$. Then, $XY = 0$ and $Y^2 = Y$. By (1), $T[R,S]YZX \subseteq J(T[R,S])$ for each $Z \in T[R,S]$. Let $r \in R$ and $Z = (r, 0, 0, \dots) \in T[R,S]$. For this Z, $T[R,S]YZX \subseteq J(T[R,S])$ implies $Rera \subseteq J(R)$ since $J(T[R,S]) = T[J(R), J(R) \cap J(S)]$. Let s, $f^2 = f \in S$ with $sf = 0$. Set $X_1 = (0, 0, s, s, s, s, \dots) \in$

$T[R, S]$ and $Y_1 = (0, 0, f, f, f, \ldots) \in T[R, S]$. Then Y_1 is an idempotent in $T[R, S]$ and $X_1 Y_1 = 0$. Let $s' \in S$ and $Z_1 = (0, s', s', s', \ldots)$. By (1), $T[R, S]Y_1 Z_1 X_1 \subseteq T[J(R), J(R) \cap J(S)]$. Then, $Sfs's \subseteq J(R) \cap J(S)$, in particular, $Sfs's \subseteq J(R) \cap J(S)$ for each $s' \in S$. Hence, R and S are J-normal.

(2) $\Rightarrow$ (1) Let $a = (a_1, a_2, \ldots, a_n, b, b, \cdots)$, $c^2 = c = (c_1, c_2, \ldots, c_m, d, d, \ldots) \in T[R, S]$ with $ac = 0$. Then, all components $c_1, c_2, \ldots, c_m$ of c are idempotents in R and d is an idempotent in S. We prove $T[R, S]cza \subseteq T[J(R), J(R) \cap J(S)]$ for any $z \in T[R, S]$. Let $g = (g_1, g_2, \ldots, g_t, s, s, \ldots) \in T[R, S]$. We divide the proof into some cases:
Case I: $n < m$. Then, $a_i c_i = 0$ for $1 \le i \le n$, $bc_j = 0$ for $n+1 \le j \le m$ and $bd = 0$. By (2), $Rc_i ra_i \subseteq J(R)$ for $1 \le i \le n$, $Rc_j rb \subseteq J(R)$ for $n + 1 \le j \le m$ and $Rdrb \subseteq J(R)$ for any $r \in R$. In particular, $Sdsb \subseteq J(R) \cap J(S)$ for any $s \in S$. Hence, $T[R, S]cga \subseteq J(T[R, S])$ for any $g \in T[R, S]$.
Case II: $n = m$. Then, $a_i c_i = 0$ for $1 \le i \le n$ and $bd = 0$. By (2), $Rc_i ra_i \subseteq J(R)$ for $1 \le i \le n$ and $Rdrb \subseteq J(R)$ for any $r \in R$. In particular, $Sdsb \subseteq J(R) \cap J(S)$ for any $s \in S$. Hence, $T[R, S]cga \subseteq J(T[R, S])$ for any $g \in T[R, S]$.
Case III: $n > m$. Then, $a_i c_i = 0$ for $1 \le i \le m$, $a_j d = 0$ for $m + 1 \le j \le n$ and $bd = 0$. By (2), $Rc_i ra_i \subseteq J(R)$ for $1 \le i \le m$, $Rdra_i \subseteq J(R)$ for $m + 1 \le j \le n$ and $Rdrb \subseteq J(R)$ for any $r \in R$. In particular, $Sdsb \subseteq J(R) \cap J(S)$ for any $s \in S$. Hence, $T[R, S]cga \subseteq J(T[R, S])$ for any $g \in T[R, S]$. This completes the proof. □

4 J-normality of some subrings of matrix rings

Let R be a ring, $C(R)$ be the center of R and $Inv(R)$ be the set of all invertible elements of R. Let $s \in C(R)$ and set

$$L_{(s)}(R) = \left\{ \begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in M_2(R) \mid a, b, c, d \in R \right\}$$

where the operations are defined as those in $M_2(R)$. Then, $L_{(s)}(R)$ is a subring of $M_2(R)$.

Proposition 4.1 *Let R be a ring and $s \in R$ be a central invertible element. Then, $L_{(s)}(R) \cong M_2(R)$.*

Proof The homomorphism $\alpha\colon L_{(s)}(R) \to M_2(R)$ defined by $\alpha \begin{bmatrix} a & b \\ sc & d \end{bmatrix} = \begin{bmatrix} a & sb \\ c & d \end{bmatrix}$ where $\begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in L_{(s)}(R)$ is an isomorphism. □

Lemma 4.2 *Let R be a ring and $s \in C(R) \cap J(R)$. Then, $Inv(L_{(s)}(R))$, the set of all invertible elements of $L_{(s)}(R)$, is $\mathcal{S} = \left\{ \begin{bmatrix} a & x \\ sy & d \end{bmatrix} \in M_2(R) \mid a, d \in Inv(R), x, y \in R \right\}$.*

Proof We show $Inv(L_{(s)}(R)) = \mathcal{S}$. Let $A = \begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in Inv(L_{(s)}(R))$. There exists $B = \begin{bmatrix} x & y \\ su & v \end{bmatrix} \in Inv(L_{(s)}(R))$ such that $AB = BA = I$ identity matrix. Then, $AB = I$ implies

$$\begin{aligned} ax + bsu &= 1 && (1) \\ ay + bv &= 0 && (2) \\ scx + dsu &= 0 && (3) \\ scy + dv &= 1 && (4) \end{aligned}$$

Since $s \in J(R)$, (1) implies that ax is invertible and (4) implies that dv is invertible. Similarly, $BA = I$ implies that xa and vd are invertible. Hence, a and d are invertible. So $A \in \mathcal{S}$.

Conversely, let $A = \begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in \mathcal{S}$. We prove $A \in Inv(L_{(s)}(R))$. To complete the proof, we look for a $B = \begin{bmatrix} x & y \\ su & v \end{bmatrix} \in L_{(s)}(R)$ such that $AB = BA = I$ identity matrix. Assume that such B exists and we determine entries of B in terms of entries of A and a^{-1} and d^{-1}. Note that $AB = I$ implies the Eqs. (1)–(4). Let $r = 1 - a^{-1}bd^{-1}sc$ and $t = 1 - d^{-1}sca^{-1}b$. Then, r and t are invertible in R since $s \in J(R)$. From Eqs. (1)–(4), we have $x = r^{-1}a^{-1}$, $y = -a^{-1}bt^{-1}d^{-1}$, $su = -d^{-1}scr^{-1}a^{-1}$, $v = t^{-1}d^{-1}$. Similarly, any matrix $C \in L_{(s)}(R)$ satisfying $CA = I$ has entries which are expressible in terms of entries of A and a^{-1} and d^{-1}. It follows that $A \in Inv(L_{(s)}(R))$. □

Lemma 4.3 *Let R be a ring and let $s \in C(R) \cap J(R)$. Then*

$$J(L_{(s)}(R)) = \left\{ \begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in M_2(R) \mid a, d \in J(R), b, c \in R \right\}.$$

Proof Let $A = \begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in J(L_{(s)}(R))$ and r and z be arbitrary elements in R. Set $B = \begin{bmatrix} r & 0 \\ 0 & z \end{bmatrix}$. Let I denote the 2×2 identity matrix. Then, $I - AB$ is invertible. By Lemma 4.2, $1 - ar$ and $1 - dz$ are invertible in R for each $r, s \in R$. So $a, d \in J(R)$.

For the converse inclusion, let $A = \begin{bmatrix} a & b \\ sc & d \end{bmatrix} \in L_{(s)}(R)$. Assume that $a, d \in J(R)$. Then, $1 - ar$ and $1 - dz$ are invertible in R for each $r, z \in R$. Since $s \in J(R)$, for any $B = \begin{bmatrix} r & u \\ st & z \end{bmatrix}$, $I - AB$ has $1 - ar - sbt$ and $1 - dz - scu$ as main diagonal entries which are invertible for each $r, z \in R$. By Lemma 4.2, $I - AB$ is invertible in $L_{(s)}(R)$. Hence, $A \in J(L_{(s)}(R))$. This completes the proof. □

Theorem 4.4 *Let R be a ring and let $s \in C(R) \cap J(R)$. If $L_{(s)}(R)$ is J-normal, then R is J-normal. If $s = 0$ and R is J-normal, then $L_{(0)}(R)$ is J-normal.*

Proof Suppose that $L_{(s)}(R)$ is J-normal. Let $e^2 = e$, $a \in R$ with $ae = 0$. Set $A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$. Then, $AE = 0$ and $E^2 = E$. By supposition, $L_{(s)}(R)EBA \subseteq J(L_{(s)}(R))$ for all $B \in L_{(s)}(R)$. Comparing entries and invoking Lemma 4.3, we have $Rera \subseteq J(R)$ for all $r \in R$. Assume that $s = 0$ and R is J-normal. Then $L_{(0)}(R)$ is isomorphic to $U_2(R)$. By Theorem 3.11, $L_{(0)}(R)$ is J-normal. □

The rings $L_{(s,t)}(R)$: Let R be a ring, and let $s, t \in C(R)$. Let $L_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R \right\}$, where the operations are defined as those in $M_3(R)$. Then, $L_{(s,t)}(R)$ is a subring of $M_3(R)$.

Lemma 4.5 *Let R be a ring, and let $s, t \in C(R)$. Then, the set of all invertible elements of $L_{(s,t)}(R)$ is*

$$Inv(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, d, f \in Inv(R), c, e \in R \right\}.$$

Proof Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in Inv(L_{(s,t)}(R))$ and $B = \begin{bmatrix} x & 0 & 0 \\ su & v & tz \\ 0 & 0 & r \end{bmatrix} \in L_{(s,t)}(R)$ with $AB = BA = I$ the 3×3 identity matrix over R. An easy calculation shows that $xa = ax = 1$, $vd = dv = 1$, $fr = rf = 1$. These equations show that a, d and f are invertible in R.

For the converse inclusion, let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$ with a, d and f are invertible in R. Assume that there exists a matrix $B = \begin{bmatrix} x & 0 & 0 \\ sr & u & tv \\ 0 & 0 & z \end{bmatrix} \in L_{(s,t)}(R)$ with $AB = I$ where I is the 3×3 identity matrix. The fact that $AB = I$ implies

$$ax = 1 \quad (1)$$
$$scx + dsr = 0 \quad (2)$$
$$du = 1 \quad (3)$$
$$dtv + tez = 0 \quad (4)$$
$$fz = 1 \quad (5)$$

Let a^{-1}, d^{-1} and f^{-1} denote the inverses of a, b and f, respectively. We look for solutions for these equations in terms of entries of A and $a^{-1} = x$, $d^{-1} = u$ and $f^{-1} = z$. Equations (2) and (4) give rise to $sr = -d^{-1}sca^{-1}$,

$tv = -d^{-1}tef^{-1}$. Similarly, existence of a matrix $C \in L_{(s,t)}(R)$ having diagonal entries invertible and satisfying $CA = I$ is proved. It follows that A is invertible. □

Lemma 4.6 *Let R be a ring and let $s, t \in C(R) \cap J(R)$. Then*
$$J(L_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, d, f \in J(R), c, e \in R \right\}.$$

Proof Let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in J(L_{(s,t)}(R))$ and r, u and z be arbitrary elements in R. Set $B = \begin{bmatrix} r & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & z \end{bmatrix}$. Let I denote the 3×3 identity matrix. Then, $I - AB$ is right invertible. By Lemma 4.5, $1 - ar$, $1 - du$ and $1 - fz$ are right invertible in R for each $r, u, z \in R$. So $a, d, f \in J(R)$.

For the converse inclusion, let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$. Assume that $a, d, f \in J(R)$. Then, $1 - ar$, $1 - du$ and $1 - fz$ are right invertible in R for each $r, u, z \in R$.

Let $B = \begin{bmatrix} r & 0 & 0 \\ sl & m & tp \\ 0 & 0 & q \end{bmatrix} \in L_{(s,t)}(R)$. $I - AB = \begin{bmatrix} 1 - ar & 0 & 0 \\ -scr - dsl & 1 - dm & -dtp - teq \\ 0 & 0 & 1 - fq \end{bmatrix}$ is right invertible in $L_{(s,t)}(R)$ for each $B \in L_{(s,t)}(R)$. Hence, $A \in J(L_{(s,t)}(R))$. □

Theorem 4.7 *Let R be a ring and let $s, t \in C(R) \cap J(R)$. Then, R is J-normal if and only if $L_{(s,t)}(R)$ is J-normal.*

Proof Necessity: Assume that R is J-normal and let $A = \begin{bmatrix} a & 0 & 0 \\ sc & d & te \\ 0 & 0 & f \end{bmatrix} \in L_{(s,t)}(R)$ and $E^2 = E = \begin{bmatrix} x & 0 & 0 \\ su & y & tv \\ 0 & 0 & z \end{bmatrix} \in L_{(s,t)}(R)$ with $AE = 0$. Then, $E^2 = E$ implies $x^2 = x$, $y^2 = y$ and $z^2 = z$. We have $ax = 0$, $dy = 0$ and $fz = 0$. By assumption, for any x', y' and $z' \in R$, $Rxx'a \subseteq J(R)$, $Ryy'd \subseteq J(R)$ and $Rzz'f \subseteq J(R)$. For any $B \in L_{(s,t)}(R)$, the diagonal entries of $L_{(s,t)}(R)EBA$ are contained in $J(L_{(s,t)}(R))$. By Lemma 4.6, $L_{(s,t)}(R)EBA \subseteq J(L_{(s,t)}(R))$.

Sufficiency: Assume that $L_{(s,t)}(R)$ is J-normal. Let $a \in R$ and $e^2 = e \in R$ with $ae = 0$, and $A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in L_{(s,t)}(R)$ and $E^2 = E = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in L_{(s,t)}(R)$.
Then, $AE = 0$. By assumption, $L_{(s,t)}(R)EBA \subseteq J(L_{(s,t)}(R))$ for all $B \in L_{(s,t)}(R)$.
For all $r \in R$, $\begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in J(L_{(s,t)}(R))$ for each $b \in R$. By Lemma 4.6, $Reba \subseteq J(R)$ for each $b \in R$. So R is J-normal. □

The rings $H_{(s,t)}(R)$: Let R be a ring and let $s, t \in C(R)$. Let

$$H_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}.$$

Then, $H_{(s,t)}(R)$ is a subring of $M_3(R)$.

Lemma 4.8 *Let R be a ring, and let $s \in C(R) \cap J(R)$. Then, the set of all invertible elements of $H_{(s,t)}(R)$ is*

$$Inv(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid a, d, f \in Inv(R), c, e \in R \right\}.$$

Proof Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in Inv(H_{(s,t)}(R))$ and $B = \begin{bmatrix} x & 0 & 0 \\ u & v & z \\ 0 & 0 & r \end{bmatrix} \in H_{(s,t)}(R)$ with $AB = BA = I$ the 3×3 identity matrix over R. An easy calculation shows that $xa = ax = 1$, $vd = dv = 1$, $fr = rf = 1$. Therefore, a, d and f are invertible in R.

For the converse inclusion, let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$ with a, d and f are invertible in R. Assume that there exists a matrix $B = \begin{bmatrix} x & 0 & 0 \\ r & u & v \\ 0 & 0 & z \end{bmatrix} \in H_{(s,t)}(R)$ such that $AB = I$ where I is the 3×3 identity matrix. $AB = I$ implies

$$\begin{aligned} ax &= 1 && (1) \\ cx + dr &= 0 && (2) \\ du &= 1 && (3) \\ dv + ez &= 0 && (4) \\ fz &= 1 && (5) \end{aligned}$$

Let a^{-1}, d^{-1} and f^{-1} denote the inverses of a, d and f, respectively. We look for solutions for these equations in terms of entries of A and $a^{-1} = x$, $d^{-1} = u$ and $f^{-1} = z$. Equations (2) and (4) give rise to $sr = d^{-1}sca^{-1}$, $tv = -d^{-1}tef^{-1}$. Similarly, existence of a matrix $C \in H_{(s,t)}(R)$ having diagonal entries invertible and satisfying $CA = I$ is proved. It follows that A is invertible. □

Lemma 4.9 *Let R be a ring and let $s, t \in C(R) \cap J(R)$. Then,*

$$J(H_{(s,t)}(R)) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R) \mid a, d, f \in J(R), c, e \in R \right\}.$$

Proof Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in J(H_{(s,t)}(R))$ and r, u and z be arbitrary elements in R. Set $B_1 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$, $B_2 = \begin{bmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{bmatrix}$, $B_3 = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} \in H_{(s,t)}(R)$. Let I denote the 3×3 identity matrix. Then $I - AB_i$ is invertible where $i = 1, 2, 3$. By Lemma 4.5, $1 - ar$, $1 - du$ and $1 - fz$ are invertible in R for each r, u, $z \in R$. So $a, d, f \in J(R)$.

For the converse inclusion, let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$. Assume that $a, d, f \in J(R)$. Then, $1 - ar$, $1 - du$ and $1 - fz$ are invertible in R for each $r, u, z \in R$.

Let $B = \begin{bmatrix} r & 0 & 0 \\ l & m & p \\ 0 & 0 & q \end{bmatrix} \in H_{(s,t)}(R)$. $I - AB = \begin{bmatrix} 1 - ar & 0 & 0 \\ -cr - dl & 1 - dm & -dp - eq \\ 0 & 0 & 1 - fq \end{bmatrix}$ is invertible in $H_{(s,t)}(R)$ for each $B \in H_{(s,t)}(R)$. Hence, $A \in J(H_{(s,t)}(R))$. □

Theorem 4.10 *Let R be a ring and let $s, t \in C(R) \cap J(R)$. Then, R is J-normal if and only if $H_{(s,t)}(R)$ is J-normal.*

Proof Necessity: To prove $H_{(s,t)}(R)$ is J-normal; let $A = \begin{bmatrix} a & 0 & 0 \\ b & d & e \\ 0 & 0 & f \end{bmatrix} \in H_{(s,t)}(R)$ and $E^2 = E = \begin{bmatrix} x & 0 & 0 \\ u & y & v \\ 0 & 0 & z \end{bmatrix} \in H_{(s,t)}(R)$ with $AE = 0$. Then, $E^2 = E$ implies $x^2 = x$, $y^2 = y$ and $z^2 = z$. We have $ax = 0$, $dy = 0$ and $fz = 0$. By assumption, for any x', y' and z', $Rxx'a \subseteq J(R)$, $Ryy'd \subseteq J(R)$ and $Rzz'f \subseteq J(R)$. Then, all the diagonal entries of $H_{(s,t)}(R)EBA$ belong to $J(R)$. By Lemma 4.9, $H_{(s,t)}(R)EBA \subseteq J(H_{(s,t)}(R))$. This completes the proof.

Sufficiency: Assume that $H_{(s,t)}(R)$ is J-normal. Let $a \in R$ and $e^2 = e \in R$ with $ae = 0$, and $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in H_{(s,t)}(R)$ and $E^2 = E = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} \in H_{(s,t)}(R)$.

Then, $AE = 0$. By assumption, $H_{(s,t)}(R)EBA \subseteq J(H_{(s,t)}(R))$ for all $B \in H_{(s,t)}(R)$.

For all $r \in R$, $\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix} \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in J(H_{(s,t)}(R))$ for each $b \in R$. By Lemma 4.9, $Reba \subseteq J(R)$ for each $b \in R$. So R is J-normal. □

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