Study on general origamis and Veech groups of flat surfaces

著者	Kumagai Shun
学位授与機関	Tohoku University
学位授与番号	11301甲第20468号
URL	http://hdl.handle.net/10097/00135837

Study on general origamis and Veech groups of flat surfaces

A thesis submitted for the degree of
Doctor of Philosophy

by

Shun Kumagai

Research Center for Pure and Applied Mathematics,
Graduate School of Information Sciences,
Tohoku University

Supervisor:	Prof. Toshiyuki Sugawa
Examiners:	Prof. Hitoshi Murakami
	Prof. Hiroki Shimakura
	Prof. Kei Funano
Date of submission:	20, January 2022

Abstract

In this century, an origami (a square-tiled translation surface) is intensively studied as an object with special properties of its translation structure and its $PSL(2,\mathbb{R})$-orbit embedded in the moduli space, particularly in the context of the study of the absolute Galois group and the Teichmüller geodesic flow.

We formulate the concept of origamis generalized in the language of flat surfaces arising naturally in the Teichmüller theory. The family of flat surfaces with two cylindrical directions that induce a fixed origami as a combinatorial structure is parametrized in the Euclidian space. The $PSL(2,\mathbb{R})$-orbits of such flat surfaces are observed in terms of origamis.

Furthermore, we present some calculation results on origamis and discuss the Galois conjugacy of the $PSL(2,\mathbb{R})$-orbits of origamis.

Acknowledgements

I would like to show my first and greatest appreciation to the supervisor in my Ph.D. course, Professor Toshiyuki Sugawa. His insightful comments and sophisticated instruction have been very supportive. I am deeply grateful to the supervisor in my master course, Professor Hiroshige Shiga. He tolerantly gave me a lot of guidance and led me to my current field of research.

The discussion with Professor Fumiharu Kato was very illuminating. I was able to broaden my knowledge and was very much motivated. I note that he suggested "oriorigami" as the name for the "general origami" in this thesis. I thank Professor Yoshihiko Shinomiya and Yuta Hatasa for meaningful discussions. Professor Hitoshi Murakami, Professor Hiroki Shimakura, and Professor Kei Funano are examiners of my Ph.D. thesis. I thank them for their helpful comments.

In Weihnachtsworkshop 2019 at Karlsruhe, I had a lot of significant discussions on my research. I would like to thank Professor Frank Herrlich, Professor Gabriela Weitze-Schmithüsen, and Professor Martin Möller for their expert advices and comments. I thank Sven Caspart for helpful discussions.

Lastly, I wish to thank my family for their support.

The computation in this thesis was carried out using the computer resource offered under the category of General Projects by Cyber Science Center, Tohoku University. I thank the Cyber Science Center staff for their support of my program. This work was supported by JASSO Scholarship, Global Hagi Scholarship, and JSPS KAKENHI Grant Number JP21J12260.

Table of Contents

List of Figures

List of Tables

Nomenclature

Notation

Symbol	Description
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$\bar{\mathbb{Q}}$	the algebraic closure of $\mathbb{Q}$, the field of algebraic numbers over $\mathbb{Q}$
$\mathbb{R}$	the field of real numbers
$\mathbb{K}_{>0}$	$\{x \in \mathbb{K} \mid x > 0\}$; the subset defined by the subscript condition
$\mathfrak{i}$	$\sqrt{-1}$; the imaginary unit
$\mathbb{C}$	the complex plane, the field of complex numbers
$\hat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$; the Riemann sphere
$\mathrm{Re}(z)$	the real part of $z \in \mathbb{C}$
$\mathrm{Im}(z)$	the imaginary part of $z \in \mathbb{C}$
$\bar{z}$	the complex conjugate of $z \in \mathbb{C}$
$\mathbb{H}$	$\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$; the upper half plane
$\mathbb{L}$	$\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$; the lower half plane
$[\cdot]$	assignment of the class or the quotient under given equivalence
$SL(2, G)$	$\{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in G,\ ad - bc = 1\}$; the special linear group $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$
$PSL(2, G)$	$\{[A] = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mid a, b, c, d \in G,\ ad - bc = 1\}$; the projective special linear group
$\bar{A}$, $\bar{G}$	the mirror conjugate of a matrix and a group (e.g. $\bar{A} = JAJ$)
F_2	$\langle x, y\rangle$; the free group of rank 2

I_d	$\{1, 2, \dots, d\}$; the set of d indices
$\bar{I}_d$	$\{\pm 1, \pm 2, \dots, \pm d\}$; the double of the set of d indices
$\mathfrak{S}_d$	$\mathrm{Sym}(I_d)$; the symmetric group
$\bar{\mathfrak{S}}_d$	$\{\bar{\sigma} \in \mathrm{Sym}(\bar{I}_d) \mid \bar{\sigma}(-i) = -\bar{\sigma}(i),\ i \in \bar{I}_d\}$; the rotation-symmetric permutation group

Chapter 1

Introduction

Background

A Riemann surface is a connected, complex 1-dimensional manifold. A Riemann surface is called analytically finite type (g, n) if it has genus g and precisely n boundary components that are points. Poincaré-Klein-Koebe's uniformization theorem [36–39, 50] in 1907 classifies the complex structures of universal Riemann surfaces into the three cases: the Riemann sphere $\hat{\mathbb{C}}$ ('elliptic'), the complex plane $\mathbb{C}$ ('parabolic'), and the upper plane $\mathbb{H}$ or equivalently the unit disk $\mathbb{D}$ ('hyperbolic'). These three cases are distinguished by the type of each Riemann surface. In most cases where $2g - 2 + n > 0$, Riemann surfaces are hyperbolic. This thesis thinks of hyperbolic Riemann surfaces of analytically finite type. Such a Riemann surface is represented by a Fuchsian model $\mathbb{H}/\Gamma$ where Γ is a group acting on $\mathbb{H}$ properly discontinuously by the Möbius transformations.

Biholomorphism classes of Riemann surfaces are called moduli. Analysis on the space $M_{g,n}$ of moduli of Riemann surfaces of type (g, n) is worked out in terms of its universal covering space $T_{g,n}$, the Teichmüller space introduced by Teichmüller in the 1930s. The Teichmüller space $T_{g,n}$ parameterizes deformations of a fixed Riemann surface of type (g, n) under quasiconformal mappings. A quasiconformal mapping is a homeomorphism defined by the Beltrami equation $f_{\bar{z}} = \mu f_z$, where μ is a bounded measurable $(1, -1)$-form with norm less than one called the Beltrami differential. Ahlfors and Bers [2] showed the existence and uniqueness theorem for the Beltrami equation on the Riemann sphere. Simultaneous uniformization leads to the Bers embedding of the Teichmüller space $T_{g,n}$ into a complex Banach space of dimension $3g - 3 + n$. The embedded image of Teichmüller space $T_{g,n}$ in the complex Euclidian space $\mathbb{C}^{3g-3+n}$ is a bounded domain homeomorphic to the ambient space. The covering transformation group of the universal covering $T_{g,n} \to M_{g,n}$ is given by the action of quasiconformal self-mappings of a type (g, n)-surface R up to homotopically-trivial mappings. Such a group $\mathrm{Mod}_{g,n}$ is called the Teichmüller-modular group or the mapping class group of type (g, n).

Holomorphic quadratic differentials on Riemann surfaces plays a significant role in Teichmüller theory. Consider the Banach space $\mathcal{Q}^{\infty}(R)$ of uniformly bounded holomrphic quadratic differentials or (in analytically finite case) equivalently the space $\mathcal{Q}(R)$ of integrable holomorphic quadratic differentials (admitting simple poles at the boundary) on a Riemann surface R. Then the space $\mathcal{Q}^{\infty}(R)$ embeds the Teichmüller space $T_{g,n}$ and the space $\mathcal{Q}(R)$ appears to be the cotangent space $T_{g,n}$ as a dual of $\mathcal{Q}^{\infty}(R)$. The space $\mathcal{Q}(R)$ is naturally stratified in terms of specified data of singular orders and primitivity. Each stratum is a complex analytic orbifold parametrized on the cohomology group relative to singularities by the period coordinates [4, 30, 57].

A Riemann surface R together with an integrable holomorphic quadratic differential ϕ is called a flat surface. The coordinates defined by line integral in the locally defined differential $\sqrt{\phi}$ form an atlas any of whose transition map is half-translation. Such a structure induces the notion of locally-affine geometry with a flat metric with finitely many conical singularities. Teichmüller's theorem states that every quasiconformal mapping is uniquely represented by some flat structures as an extremal affine deformation that attains the bound of norm of its Beltrami differential. For each fixed flat surface (R, ϕ), we obtain a holomorphic and isometric embedding $\hat{\iota}_{\phi} : \mathbb{D} \hookrightarrow T_{g,n}$ of the upper half plane into the Teichmüller space. The action of Teichmüller-modular group on the embedded disk is described by the Möbius transformation of derivative of locally affine self-mapping of (R, ϕ). The group of such action is called the Veech group $\Gamma(R, \phi)$, and the embedded disk projects into the moduli space as an orbifold $\mathbb{H}/\Gamma(R, \phi)$. If $\Gamma(R, \phi)$ has a finite covolume, the orbifold $\mathbb{H}/\Gamma(R, \phi)$ is an algebraic curve, and we obtain a curve family embedded in the moduli space $M_{g,n}$ as an algebraic curve called the Teichmüller curve induced from (R, ϕ).

Veech groups were introduced by Veech in the context of the study of the geodesic flow on a flat surface. In his paper [56] in 1989, Veech showed a dichotomy of billiard (reflecting at boundary) geodesic flows on polygonal regions on the plane. He also presented the first nontrivial example of Veech group. Earle and Gardiner [10] reformulated the theory of flat surfaces in terms of the Teichmüller spaces. Veech groups of flat surfaces are studied in terms of combinatorial objects invariant under affine self-mappings, such as [7, 11, 51, 54].

An (abelian, or formerly known as "oriented") origami [25] is a typical example of a flat surface with a combinatorial structure. It is a finite covering of the unit square torus branched over precisely one point, which comes down to a combinatorial structure. An origami has a Veech group as a subgroup of $SL(2,\mathbb{Z})$ of finite index, and induces a Teichmüller curve defined over $\bar{\mathbb{Q}}$ called an origami curve. Schmithüsen [51] showed that the universal Veech group $SL(2,\mathbb{Z})$ of an abelian origami $\mathcal{O}$ acts automorphically on the free group F_2 and its Veech group is the stabilizer of the fundamental group of $\mathcal{O}$ under this action. She also presented an algorithm to calculate the Veech group of given origami using the Reidemeister-Schreier method [43]. Ellenberg and McReynolds [12] showed a sufficient condition for a group to be the Veech group of an abelian origami. A significant aspect of origami is the compatibility of the Galois actions on embedding origami curves.

The absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the group of field automorphisms of an algebraic closure $\bar{\mathbb{Q}}$ of the field of rational numbers has been intensively studied in many areas for a long time. A remarkable progress of this study started from the theorem of Belyĭ [5] in 1979. He showed a purely combinatorial or analytical condition 'three-times branched covering of the projective line' (Belyĭ covering) for an algebraic curve to be defined over the field $\bar{\mathbb{Q}}$. By the fundamental theory of coverings, every Belyĭ covering is identified with a combinatorial object called a dessin d'enfants. The absolute Galois group $G_{\mathbb{Q}}$ acts faithfully the set of dessins, and some $G_{\mathbb{Q}}$-invariants are visualized in terms of dessins. Using the idea of dessins to approach the absolute Galois group $G_{\mathbb{Q}}$ is a project contained in the Grothendieck's programme [18] in 1984. Drinfel'd [8] and Ihara [31] showed that the Grothendieck-Teichmüller group $\widehat{GT}$ realizes the absolute Galois group $G_{\mathbb{Q}}$ as a subgroup, the "$\widehat{GT}$-relation". The Grothendieck-Teichmüller group $\widehat{GT}$ is defined pure combinatorially and is related to the study of profinite mapping class groups [53].

Origamis are well-behaved in this context. In 2005, Möller [45] showed that the $G_{\mathbb{Q}}$-action on origamis respects the embedding origami curves into the moduli spaces. In his paper, Möller presented an application to the $\widehat{GT}$-relation by considering the $G_{\mathbb{Q}}$-action on an origami curve induced from a degree 4 origami. The $G_{\mathbb{Q}}$-action on the embedding origami curve is compared with the action on a Teichmüller tower [24] that is defined by a collection of mapping class groups linked by certain natural homomorphisms coming from inclusions of underlying surfaces. The faithfulness of the $G_{\mathbb{Q}}$-action on origamis are shown by the "M-origami" construction [45, 48] using dessins.

Results of this thesis

In this thesis, we consider flat surfaces whose Veech groups can be dealt with similarly to origamis. To do this, we introduce a generalization of origamis in the category of flat surfaces. Such a general origami is regarded as a covering with a prescribed branching behavior like an abelian origami, but the situation is a bit different from abelian case in the sense of combinatorial characterizations and its Veech group. Theorem 5.1.7 shows combinatorial characterizations and equivalence of general origamis. A characterization is given by regluing an abelian origami after inverting prescribed squares is a better tool for the calculation of Veech group.

Earle and Gardiner [10] showed that every flat surface with two finite-cylindrical directions (Jenkins-Strebel directions) admits a decomposition into finitely many aligned parallelograms. We may regard this decomposition as an origami-like decomposition given by aligned parallelograms of specified moduli replacing unit square cells of a general origami. In terms of Theorem 5.1.7, we observe that a general origami gives a finite system of linear equations whose solution space presents the set of moduli for which the replacement of cells successes. Theorem 5.2.3 shows that there is a one-to-one correspondence between flat surfaces with fixed two finite-cylindrical directions and origamis with compatible moduli lists up to equivalences. It leads to Corollary 5.2.4 that the family of flat surfaces inducing a general origami in prescribed directions is parametrized in the quotient of the solution space of the system of linear equations by a finite group.

As a consequence of Theorem 5.2.3, we may compare two origami-like decompositions of a flat surface to determine whether each matrix belongs to the Veech group, as in the way shown in Corollary 5.3.1. Every origami-like decomposition combinatorially projects via a covering with a prescribed branching behavior concerning the singularities, and thus the inclusion of Veech groups holds for origami-like flat surfaces in such a covering relation. Theorem 5.3.5 observes such a situation as the Veech group of the lower surface acting on the set of monodromies, where the Veech group of the upper surface is the stabilizer of the class of original monodromy. This result is a generalization of [51, 52, 54].

We present a set of algorithms summarized in Theorem 6.1.7 for calculating the Veech groups of general origamis. It has a different structure than [51, 54], which computes the Veech group from a single object. Our algorithm makes an exhaustive and simulatenous calculation of the Veech groups of all origamis of a fixed degree. We create a list of equivalence classes of general origamis according to Theorem 5.1.7 and compute the action of the universal Veech group $PSL(2,\mathbb{Z})$ on these classes defined by the re-decomposition of the origami-like decomposition. Each $PSL(2,\mathbb{Z})$-orbit corresponds to the square-tiled points of the Teichmüller curves induced from origamis, and elements of Veech groups are represented by closed cycles.

We have calculated up to $d = 7$, classified the result by Galois invariants, and summarized the possibility of Galois conjugation as in Theorem 6.2.1.

Chapter 2

Concepts in Complex Analysis

2.1 Covering

In this section, we present basic theory of coverings of topological surfaces. In Section 2.2 a branched covering appears as a morphism in the category of Riemann surfaces. The theory of covering is an important tool in discussions about dessins (Section 3.2) and origami (Section 4.4.2 and 5.1). This section is based on [14, 17, 49].

We say that a *topological surface* is a connected, oriented, two-dimensional real manifold.

Definition 2.1.1. A mapping $f : R \longrightarrow S$ between two topological surfaces R and S is called a *branched covering* if there exists nowhere dense subset $B \subset S$ and a mapping $\nu : R \longrightarrow \mathbb{Z}_{\geq 0}$ with $R \setminus \mathrm{supp}(\nu - 1) = f^{-1}(B)$ such that:

(1) every point $p \in S \setminus B$ has an *evenly covered* neighborhood, an open neighborhood $U \subset S \setminus B$ such that $f^{-1}(U)$ is the disjoint union of open subsets in R, each of which restricts f to a homeomorphism onto U.

(2) arround every point $p \in R$, f is locally represented by the form $z \mapsto z^{\nu(p)}$.

We say that $\mathrm{Br}(f) = B \subset S$ is the set of *branched points* and $\mathrm{Crit}(f) = f^{-1}(B) \subset R$ is the set of *critical points*. The integer $\nu(p) =: \mathrm{mult}_p(f)$ is called the *multiplicity* of f at $p \in R$. The mapping f is called an *unbranched covering* (or simply a *covering*) if $\mathrm{Br}(f) = \emptyset$. The restriction of a branched covering to the unbranched region $R^* = R \setminus f^{-1}(B)$ is an unbranched covering.

Definition 2.1.2. Two coverings $f_i : R_i \longrightarrow S_i$ $(i = 1, 2)$ are called *equivalent* if there exists two homeomorphisms $\varphi : R_1 \longrightarrow R_2$ and $\psi : S_1 \longrightarrow S_2$ such that $\psi \circ f_1 = f_2 \circ \varphi$.

Definition 2.1.3. Let R, S be topological surfaces.

(1) A *path* in R is a continous mapping $\gamma : [0, 1] \to R$. The points $\gamma(0), \gamma(1) \in R$ are the *endpoints* of γ. A path is called a *loop* if the two endpoints coincide.

(2) A *homotopy* joining two continous mappings $f_0, f_1 : R \to S$ is a continous mapping $F : R \times [0, 1] \to S$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for any $x \in R$. We say that f_0, f_1 are *homotopic* if there exists a homotopy joining them. By *fixed-endpoint homotopy* we mean a homotopy F of paths such that every $F(t, s)$, $s \in [0, 1]$ has the same endpoints.

A covering $\pi_R : \tilde{R} \to R$ with $\tilde{R}$ simply connected is called a *universal covering* of R. The name 'universal' comes from the *universal property* that for any covering $f : R' \to R$ there exists a covering $\pi' : \tilde{R} \to R'$ such that $\pi' = \pi \circ f$.

Lemma 2.1.4. *For every topological surface R, one can construct a universal covering $\pi_R : \tilde{R} \to R$ of R in the following way.*

(i) fix a base point p on R.

(ii) let $\tilde{R}$ be the space of all paths $\gamma : [0, 1] \to R$ with $\gamma(0) = p$ up to fixed-endpoint homotopy.

(iii) define $\pi_R : \tilde{R} \to R$ by $[\gamma] \mapsto \gamma(1)$.

Proposition 2.1.5 (lifting property)**.** *Let R and S be topological surfaces. Then we have the following:*

(1) *For any path $\gamma : [0, 1] \to R$ and $\tilde{p} \in \tilde{R}$ with $\gamma(0) = \pi_R(\tilde{p})$, there exists a unique path $\tilde{\gamma} : [0, 1] \to \tilde{R}$ such that $\pi_R \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{p}$.*

(2) *For any continous mapping $f : R \to S$ and $\tilde{p} \in \tilde{R}$, $\tilde{q} \in \tilde{S}$ with $f(\pi_R(\tilde{p})) = \pi_S(\tilde{q})$, there exists a unique continous mapping $\tilde{f} : \tilde{R} \to \tilde{S}$ such that $f \circ \pi_R = \pi_S \circ \tilde{f}$ and $\tilde{f}(\tilde{p}) = \tilde{q}$.*

We say that $\tilde{\gamma}$ ($\tilde{f}$, respectively) is a lift of γ (f, respectively).

As a consequence of Proposition 2.1.5, the cardinality of $f^{-1}(p)$ defines the *degree* $\deg(f)$ of a covering $f : R \to S$ independent of choices of $p \in R$. If $\deg(f) = d \in \mathbb{N}$, we say that f is a *finite covering* or a *d-fold covering*.

Definition 2.1.6. Let R, S be topological surfaces and $f : R \to S$ be a covering. Fix base points $p \in R$ and $q \in S$.

(1) The *fundamental group* $\pi_1(R, p)$ of R with base point p is the group of homotopy classes of all loops on R based on p, with the group structure defined by post-composition of loops.

(2) The *monodromy map* of f is a homeomorphism $m_f : \pi_1(S, q) \to \mathrm{Sym}(f^{-1}(q))$ that assigns each loop $\gamma \in \pi_1(S, q)$ to the permutation defined by the lift of γ starting from each point in $f^{-1}(q)$. The image of the monodromy map is called the *monodromy group* of f. Note that m_f can be defined as the homomorphism into the symmetric group $\mathfrak{S}_{\deg(f)}$ up to conjugacy, independent of the choices of base point $q \in S$.

(3) A *covering transformation* (or *deck transformation*) of f is a homeomorphism $\varphi : R \to R$ satisfying that $f \circ \varphi = f$. The covering transformations of f form a group under post-composition, denoted by $\mathrm{Deck}(f)$ or $\mathrm{Deck}(R/S)$ (if f is understood).

Remark 2.1.7. Let $f : R \to S$ be a covering and fix $p \in R$, $q = f(p) \in S$. Then the stabilizer $\mathrm{Stab}_{m_f}(p)$ is the image of $\pi_1(R, p)$ embedded in $\pi_1(S, q)$, via the natural homomorphism $f_\#$ induced from f. Once a subgroup $H < \pi_1(S, q)$ given, one can reconstruct a covering $f : R_H \to S$ so that the embedded image of $\pi_1(R_H, p)$ in $\pi_1(S, q)$ is H, in a similar way to Lemma 2.1.4. In this construction, the monodromy m_f is identified with the action of $H < \pi_1(S, q)$ on the coset representatives in $\pi_1(S, q)/H$, where $\mathrm{Stab}_{m_f}(1) = H$.

Note about the equivalence as follows. Suppose that two equivalent coverings $f_i : R_i \to S_i$ ($i = 1, 2$) are joined via homeomorphisms $\varphi : R_1 \to R_2$ and $\psi : S_1 \to S_2$ as in Definition 2.1.2. Fix a base point $p_1 \in R_1$ and let $p_2 = \varphi(p_1) \in R_2$, $q_1 = f_1(p_1) \in S_1$, and $q_2 = f_2(p_2)$. Then, we have isomorphisms $\varphi^\# : \pi_1(R_1, p_1) \to \pi_1(R_2, p_2)$ and $\psi^\# : \pi_1(S_1, q_1) \to \pi_1(S_2, q_2)$ commutative with the embeddings $f_i^\# : \pi_1(R_i, p_i) \hookrightarrow \pi_1(S_i, q_i)$ ($i = 1, 2$). We say that f_i ($i = 1, 2$) are in *covering equivalence over* a topological surface S if $S_1 = S_2 = S$ and $\psi = id$. In this case, the isomorphism φ is a covering transformation and the embedded image of $\pi_1(R_i, p_i), i = 1, 2$ are conjugated in $\pi_1(S_1, q_1)$. The monodromy homomorphisms $m_{f_i}, i = 1, 2$ are identified by the pullback of the isomorphism $\varphi^\#$.

Let $\pi : \tilde{R} \to R$ be the universal covering constructed as in Lemma 2.1.4. Then every covering transformation of π is represented by the action of $\pi_1(R, p)$ on $\tilde{R}$ by the pre-compositions of paths. (See [32, p.31].)

Proposition 2.1.8. *The following are equivalent for a covering $f : R \to S$; in this case we say that f is a Galois covering.*

(1) $\mathrm{Deck}(R/S)$ *acts transitively on $f^{-1}(p) \subset R$ for a fixed base point $p \in S$.*

(2) *The monodromy map m_f is an isomorphism onto the monodromy group.*

(3) *The fundamental group of S embeds the fundamental group of R as a normal subgroup via the natural homomorphism induced from f.*

Lemma 2.1.9. *Let $f : R \to S$ be a Galois covering. Then* $\mathrm{Deck}(R/S)$ *acts properly discontinuously on R, that is, for any compact subset $K \subset R$, there are at most finitely many elements $\varphi \in$* $\mathrm{Deck}(R/S)$ *such that $\varphi(K) \cap K \neq \emptyset$. Furthermore, each element in* $\mathrm{Deck}(R/S)$ *except for the identity has no fixed points.*

Let R be a topological surface and G be a group of self-homeomorphisms of R acting properly discontinuously on R. Then the natural projection $\pi_G : R \to R/G$ induces a topological structure on the quotient space R/G. With this structure, π_G becomes a Galois covering with $\mathrm{Deck}(\pi_G) = G$.

Proposition 2.1.10. *A Galois covering $f : R \to S$ is equivalent to $\pi_{\mathrm{Deck}(f)}$.*

2.2 Riemann surface

This section is based on [32].

A one-dimensional connected complex manifold is called a *Riemann surface*. In other words, a Riemann surface is a connected Hausdorff space R with a one-dimensional complex structure, a maximal atlas $\mathcal{A}_R = \{(U_i, \varphi_i)\}_{i \in I}$ satisfying the following properties.

(1) Every U_i is an open subset of R and $R = \bigcup_{i \in I} U_i$.

(2) Every φ_i is a homeomorphism of U_i onto an open subset of $\mathbb{C}$.

(3) For every U_i, U_j with $U_i \cap U_j \neq \emptyset$, the transition mapping

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \tag{2.1}$$

is a biholomorphism on the plane.

Example 1. The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $\mathcal{A}_{\hat{\mathbb{C}}} = \left\{(\mathbb{C}, z), \left(\hat{\mathbb{C}} \setminus \{0\}, \frac{1}{z}\right)\right\}$.

Example 2. A torus $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ $(\tau \in \mathbb{H})$ with

$$\mathcal{A}_{E_\tau} = \left\{ \left(\left\{ [z] \;\middle|\; |z - z_0| < \min\left\{0, \frac{|\tau|}{2}\right\} \right\}, z \right) \;\middle|\; z_0 \in \mathbb{C} \right\}. \tag{2.2}$$

Definition 2.2.1. Let R, S be Riemann surfaces.

(1) A continous mapping $f : R \to S$ is called *holomorphic* if for every $(U, \varphi) \in \mathcal{A}_R$ and $(V, \psi) \in \mathcal{A}_S$, the local representation

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V) \tag{2.3}$$

is a holomorphic mapping on the plane.

(2) A bijective holomorphic mapping $f : R \to S$ is called a *biholomorphism*. We say that R, S are *biholomorphically equivalent* (R is *biholomorphic* to S) if there exists a biholomorphism $f : R \to S$.

(3) A Riemann surface R is called of *analytically finite type* (g, n) if R is obtained from a compact Riemann surface of genus g by removing n points. The removed points are called the *marked points*.

Local analysis on Riemann surfaces is reduced to analysis on domains in the plane via their complex structures. Local properties of holomorphic functions on the plane [16, III.3], such as the open mapping theorem, the identity theorem, and the maximum principle also hold for holomorphic mappings on Riemann surfaces.

Lemma 2.2.2. *Let R, S be Riemann surfaces. Then a non-constant holomorphic mapping $f : R \to S$ is a branched covering.*

Proof. Let $p \in R$ be an arbitrary point and $(V, \varphi_V) \in \mathcal{A}_S$ be a chart arround $f(p) \in S$. Since f is holomorphic, we may take a chart $(U, \varphi') \in \mathcal{A}_S$ arround $p \in R$ so that f is locally represented by a convergent Taylor series

$$\varphi_V \circ f \circ \varphi'^{-1}(z) = \sum_{i=n}^{\infty} a_i z^i = z^n g(z), \quad z \in U, \tag{2.4}$$

where $a_i \in \mathbb{C}$, $i = n, n+i, \dots$ with $a_n \neq 0$. By taking U sufficiently small, the restriction $g \restriction_{\varphi'(U)}$ admits a holomorphic, non-zero branch of n-th root. If we set

$$\varphi_U = \varphi' \cdot \sqrt[n]{g \circ \varphi'} \;\text{ on } U, \tag{2.5}$$

the chart (U, φ_U) has a biholomorphic transition with (U, φ') by the inverse function theorem. Thus everywhere the mapping f is locally represented as $\varphi_V \circ f \circ \varphi_U^{-1}(z) = z^n$, and the claim follows. $\square$

The complex structure of a Riemann surface naturally lifts via a covering. The arguments on coverings in Section 2.1 is translated in the language of holomorphic mappings on Riemann surfaces.

Theorem 2.2.3 (Uniformization theorem, Poincaré-Klein-Koebe [36–39, 50])**.** *Every simply connected Riemann surface is biholomorphically equivalent to one of the three Riemann surfaces $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}$.*

Thus every Riemann surface R has a universal covering $\pi : \tilde{R} \to R$ where $\tilde{R}$ is biholomorphically equivalent to either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}$. By Theorem 2.1.10, R is represented by $\tilde{R}/\Gamma$ for a subgroup Γ acting biholomorphically on $\tilde{R}$. The following classification of universal Riemann surfaces holds.

Lemma 2.2.4. *Let $\pi : \tilde{R} \to R$ be the universal covering of a Riemann surface R. We denote by* $\mathrm{Aut}(\tilde{R})$ *the group of all automorphisms (self-biholomorphisms) on $\tilde{R}$.*

(a) *$\tilde{R} = \hat{\mathbb{C}}$ if and only if R is biholomorphically equivalent to $\hat{\mathbb{C}}$. Moreover,* $\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d, \in \mathbb{C},\ ad-bc=1 \right\} \cong PSL(2,\mathbb{C})$ *holds.*

(b) *$\tilde{R} = \mathbb{C}$ if and only if R is biholomorphically equivalent to one of $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, or tori. Moreover,* $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az+b \mid a,b \in \mathbb{C}, a \neq 0\}$ *holds.*

In particular, for a Riemann surface of analytically finite type (g,n), its universal covering space is $\mathbb{H}$ if and only if $2g-2+n > 0$. In this case, It follows that $\mathrm{Aut}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d, \in \mathbb{R},\ ad-bc=1 \right\} \cong PSL(2,\mathbb{R})$.

Let $z_i \in \hat{\mathbb{C}}$ $(i=1,2,3)$ be pairwise distinct three points. Then, the mapping $\gamma_{z_1,z_2,z_3} \in \mathrm{Aut}(\hat{\mathbb{C}})$ defined by

$$\gamma_{z_1,z_2,z_3}(z) = \frac{(z_1-z_2)(z-z_1)}{(z_3-z_2)(z-z_3)}\ ,\ z \in \hat{\mathbb{C}} \tag{2.6}$$

maps (z_1,z_2,z_3) to $(0,1,\infty)$. If $z_i \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ $(i=1,2,3)$, the mapping γ_{z_1,z_2,z_3} belongs to $\mathrm{Aut}(\mathbb{H})$. In this way, $\mathrm{Aut}(\hat{\mathbb{C}})$ ($\mathrm{Aut}(\mathbb{H})$, respectively) acts sharply 3-transitively on $\hat{\mathbb{C}}$ ($\partial\mathbb{H} = \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, respectively).

From now on in this chapter, we consider a Riemann surface of analytically finite type (g,n) with $2g-2+n > 0$. Such a Riemann surface is represented by the quotient of $\tilde{R} = \mathbb{H}$ by a subgroup of $PSL(2,\mathbb{R})$ acting properly discontinuously on $\mathbb{H}$, by the formula:

$$A \cdot z = \frac{az+b}{cz+d},\ \ A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL(2,\mathbb{R}),\ z \in \mathbb{H}. \tag{2.7}$$

Definition 2.2.5. The automorphism on $\mathbb{H}$ defined by the formula (2.7) is called a *Möbius transformation*. A subgroup of $PSL(2,\mathbb{R})$ acting properly discontinuously on $\mathbb{H}$ is called a *Fuchsian group*. A Fuchsian group representing a Riemann surface R is called a *Fuchsian model* of R.

A biholomorphism $f : R \to S$ lifts to an automorphism on $\mathbb{H}$ via the universal coverings of R and S. For each biholomorphism class of a Riemann surface R, its Fuchsian model Γ_R is uniquely determined up to conjugacy in $PSL(2,\mathbb{R})$.

Lemma 2.2.6 (classification of Möbius transformations, [32, Section 2.3.3]). *Let* $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL(2,\mathbb{R})$ *and* $\mathrm{tr}(A) = a + d$. *Then, one of the following holds for the Möbius transformation* γ_A *of* A *up to conjugacy in* $PSL(2,\mathbb{R})$.

(a) *(parabolic)* γ_A *is conjugated to* $z \mapsto z+c$ *for some* $c \in \mathbb{C}\setminus\{0\}$ *if and only if* $\mathrm{tr}(A)^2 = 4$.

(b) *(elliptic)* γ_A *is conjugated to* $z \mapsto e^{\mathrm{i}\theta}z$ *for some* $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ *if and only if* $0 \leq \mathrm{tr}(A)^2 < 4$.

(c) *(hyperbolic)* γ_A *is conjugated to* $z \mapsto kz$ *for some* $k \in \mathbb{R}_{>0}$ *if and only if* $\mathrm{tr}(A)^2 > 4$.

(d) *(loxodromic)* γ_A *is conjugated to* $z \mapsto \lambda z$ *for some* $\lambda \in \mathbb{C},\ |\lambda| \neq 1,\ \lambda \notin [0,\infty)$ *if and only if* $\mathrm{tr}(A)^2 \in \mathbb{C} \setminus [0,4]$.

The upper half plane $\mathbb{H}$ and the unit disk $\mathbb{D}$ are biholomorphic under the mapping

$$h : \mathbb{H} \to \mathbb{D} : z \mapsto \frac{z - \mathrm{i}}{z + \mathrm{i}}. \tag{2.8}$$

Each of them admits a hyperbolic metric

$$ds_{\mathbb{D}} = \frac{|dz|}{1 - |z|^2}, \quad z \in \mathbb{D}, \tag{2.9}$$

$$ds_{\mathbb{H}} = h^* ds_{\mathbb{D}} = \frac{|dz|}{2\mathrm{Im}z}, \quad z \in \mathbb{H}, \tag{2.10}$$

for which every automorphism is an isometry. The distance function on $\mathbb{D}$ is given by

$$d_{\mathbb{D}}(z_1, z_2) = \log\left\{\left(1 + \left|\frac{z_2 - z_1}{1 - \overline{z_1}z_2}\right|\right)\left(1 - \left|\frac{z_2 - z_1}{1 - \overline{z_1}z_2}\right|\right)^{-1}\right\}, \quad z_1, z_2 \in \mathbb{D}. \tag{2.11}$$

2.3 Teichmüller space

This section is based on [1, 32, 47].

Let $L^\infty(D)$ denote the complex Banach space of all uniformly bounded measurable functions on a domain $D \subset \hat{\mathbb{C}}$ with the norm

$$\|\mu\|_\infty = \operatorname*{ess.sup}_{z \in D}|\mu(z)| < \infty, \quad \mu \in L^\infty(D). \tag{2.12}$$

Definition 2.3.1. Let $D \subset \hat{\mathbb{C}}$ be a domain. An orientation-preserving homeomorphism f on D into $\hat{\mathbb{C}}$ is called a *quasiconformal mapping* if the following holds.

(1) The mapping f is absolutely continous on lines (ACL): f is continous on every horizontal or vertical local segments on D. Or equivalently, f admits distributional partial derivatives $f_z, f_{\bar{z}}$ almost everywhere on D.

(2) The Beltrami equation $f_{\bar{z}} = \mu f_z$ holds almost everywhere on D, for some $\mu \in L^\infty(D)$ with $0 \leq \|\mu\|_\infty < 1$.

The function $\mu_f = \mu = \dfrac{f_{\bar{z}}}{f_z} \in L^\infty(D)$ is called the *Beltrami coefficient* of f. The constant $K(f) = \dfrac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty} \geq 1$ is called the *maximal dilatation* of f.

Example 3. For $0 \leq k < 1$, the mapping $f(z) = \dfrac{z + k\bar{z}}{1-k}$ defined on the plane is a quasiconformal mapping such that $f(i) = i$, $\mu_f = k$, and $K(f) = \dfrac{1+k}{1-k}$.

Lemma 2.3.2. *Let $f : D_1 \to D_2$, $g : D_2 \to D_3$ be quasiconformal mappings. Then the following holds:*

$$\mu_g \circ f \cdot \frac{\overline{f_z}}{f_z} = \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu_f}\mu_{g\circ f}}. \tag{2.13}$$

In particular, for biholomorphisms $\varphi : D_1 \to D_2$ and $\psi : D_2 \to D_3$, it follows that

$$\mu_{g\circ\varphi} = \mu_g \circ \varphi \cdot \frac{\overline{\varphi_z}}{\varphi_z}, \quad \mu_{\psi\circ f} = \mu_f. \tag{2.14}$$

As a consequence of Lemma 2.3.2, for two quasiconformal mappings $f_i : D_1 \to D_2$ $(i = 1, 2)$, we have

$$\begin{aligned} \|\mu_{f_2\circ f_1^{-1}}\|_\infty &= \operatorname*{ess.sup}_{f_1^{-1}(\mathbb{H})=\mathbb{H}} \left| \frac{\mu_{f_2} - \mu_{f_1}}{1 - \overline{\mu_{f_1}}\mu_{f_2}} \right|, \text{ and} \\ \log K(f_2 \circ f_1^{-1}) &= \operatorname*{ess.sup}_{\mathbb{H}} \log \left\{ \left(1 + \left| \frac{\mu_{f_2} - \mu_{f_1}}{1 - \overline{\mu_{f_1}}\mu_{f_2}} \right| \right) \left(1 - \left| \frac{\mu_{f_2} - \mu_{f_1}}{1 + \overline{\mu_{f_1}}\mu_{f_2}} \right| \right)^{-1} \right\} \\ &= \operatorname*{ess.sup}_{\mathbb{H}} d_{\mathbb{D}}(\mu_{f_1}, \mu_{f_2}). \end{aligned} \tag{2.15}$$

Lemma 2.3.3. *Let $f : D_1 \to D_2$, $g : D_2 \to D_3$ be two quasiconformal mappings. Then,*

(1) *$g \circ f : D_1 \to D_3$ is a quasiconformal mapping with $K(g \circ f) \leq K(g)K(f)$,*

(2) *$K(f) = 1$ if and only if f is a biholomorphism.*

Note that the concept of quasiconformal mappings naturally extends to Riemann surfaces. The maximal dilatation is an invariant under biholomorphisms.

Let R be a Riemann surface of analytically finite type (g, n) with $2g - 2 + n > 0$. We consider an arbitrary tuple (R, S, f) (or (S, f) if R is understood) of a Riemann surface S and a quasiconformal mapping $f : R \to S$, called a *marked Riemann surface* based on R.

Every f lifts via the universal coverings to a quasiconformal mapping $\tilde{f} : \mathbb{H} \to \mathbb{H}$ such that $\tilde{f} \circ \gamma \circ \tilde{f}^{-1} \in \Gamma_S$ for any $\gamma \in \Gamma_R$, since the following holds on $\mathbb{H}$:

$$\begin{aligned} \pi_S \circ \tilde{f} \circ \gamma \circ \tilde{f}^{-1} &= f \circ \pi_R \circ \gamma \circ \tilde{f}^{-1} \\ &= f \circ \pi_R \circ \tilde{f}^{-1} \\ &= f \circ f^{-1} \circ \pi_S \\ &= \pi_S. \end{aligned} \tag{2.16}$$

The conjugation $\theta_{\tilde{f}}(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ defines a group isomorphism $\theta_{\tilde{f}} : \Gamma_R \to \Gamma_S$. It depends on the way choosing a lift $\tilde{f}$, that is unique up to pre-compoistions in Γ_R and post-compoistions in Γ_S. We say that two isomorphisms $\theta_i : \Gamma_1 \to \Gamma_2$ $(i = 1, 2)$ are *equivalent* if they arise from essentially the same quasiconformal mappings. That is, for some $\delta_1 \in \Gamma_1$, $\delta_2 \in \Gamma_2$,

$$\theta_2(\gamma) = \delta_2^* \circ \theta_1 \circ \delta_1^*(\gamma), \quad \gamma \in \Gamma_1. \tag{2.17}$$

Lemma 2.3.4. *Two marked Riemann surfaces (R, S, f_i) $(i = 1, 2)$ determine equivalent isomorphisms if and only if f_1, f_2 are homotopic.*

Proof. Let $\tilde{f}_i : \mathbb{H} \to \mathbb{H}$ be a lift of f_i $(i = 1, 2)$. If f_1 and f_2 are homotopic, $\tilde{f}_1$ and $\tilde{f}_2$ are also joined by a homotopy $\tilde{F}_t$, $t \in [0, 1]$. For each $z \in \mathbb{H}$ and $t \in [0, 1]$, the orbit of z under $\theta_{\tilde{F}_t}(\Gamma_S) = \Gamma_S$ is a discrete set invariant for t. It cannot be moved continously and we have $\theta_{\tilde{F}_t} = id_{\Gamma_S}$.
Conversely, if f_1 and f_2 determine equivalent isomorphisms, we may choose $\tilde{f}_i$ $(i = 1, 2)$ so that $\theta_{\tilde{f}_1} = \theta_{\tilde{f}_2} =: \theta$ holds. For each $z \in \mathbb{H}$ and $t \in [0, 1]$, define $\tilde{F}_t(z)$ as the point on the hyperbolic geodesic between $\tilde{f}_1(z)$ and $\tilde{f}_2(z)$ dividing by the ratio $t : (1 - t)$. Then we have

$$\tilde{F}_t \circ \gamma(z) = \theta(\gamma) \circ \tilde{F}_t(z), \quad z \in \mathbb{H}, \gamma \in \Gamma_R, \tag{2.18}$$

and thus the mapping $\tilde{F}_t$ projects to a homotopy joining f_1 and f_2. $\square$

Definition 2.3.5. Fix a Riemann surface R of analytically finite type (g, n) with $2g-2+n > 0$.

(1) We say that two marked Riemann surfaces $(S_1, f_1), (S_2, f_2)$ are *Teichmüller equivalent* if there exists a biholomorphism $\varphi : S_1 \to S_2$ homotopic to $f_2 \circ f_1^{-1} : S_1 \to S_2$. We call the set of Teichmüller equivalence classes of all marked Riemann surfaces based on R the *Teichmüller space* of R and denote by $T(R)$.

(2) We say that two marked Riemann surfaces $(S_1, f_1), (S_2, f_2)$ are *biholomorphically equivalent* if $f_2 \circ f_1^{-1} : S_1 \to S_2$ is a biholomorphism. We call the set of biholomorphically equivalence classes of all marked Riemann surfaces the *moduli space* of R and denote by $M(R)$.

For any two points $x_i = [R, S_i, f_i] \in T(R)$ $(i = 1, 2)$, we set

$$d_T(x_1, x_2) := \inf\{\log K(g) \mid g : S_1 \to S_2 \text{ is homotopic to } f_2 \circ f_1^{-1}\}. \tag{2.19}$$

It follows from (2.15) and Lemma 2.3.3 that the mapping d_T defines a distance on $T(R)$, called the *Teichmüller distance*.

For every quasiconformal mapping $g : R_1 \to R_2$, the pre-composition

$$\rho_g([R_1, S, f]) := [R_2, S, f \circ g^{-1}], \quad [R_1, S, f] \in T(R_1) \tag{2.20}$$

defines a isometry $T(R_1) \to T(R_2)$ called a *geometric isomorphism*. In this way, quasiconformal self-mappings of R acts on the Teichmüller space $T(R)$. The group of all geometric automorphisms of $T(R)$ is called the *Teichmüller modular group* of R and denoted by $\mathrm{Mod}(R)$. Every biholomorphically equivalent marked Riemann surfaces are related by a suitable geometric automorphism, and $M(R) = T(R)/\mathrm{Mod}(R)$ holds.

Remark 2.3.6. The action of quasiconformal self-mappings on the Teichmüller space is faithful up to factor of homotopically trivial mappings except for few exceptional types $(2, 0)$, $(1, 2)$, $(1, 1)$, $(0, 4)$, and $(0, 3)$ [47, Proposition 2.3.10]. Quasiconformal mappings approximate orientation preserving homeomorphisms. Teichmüller spaces are equivalently defined by the space of classes of orientation preserving homeomorphisms, and then the Teichmüller modular group $\mathrm{Mod}(R)$ is identified with the *mapping class group* of R.

Via the geometric isomorphisms, $T(R)$ are mutually homeomorphic for all Riemann surfaces R of the same analytically finite type (g, n). The Teichmüller space of such R is independent of the base point, and we denote such a space by $T_{g,n}$. Similarly we may denote by $\mathrm{Mod}_{g,n}$ and $M_{g,n}$ up to identifications of spaces under the change of base points.

Theorem 2.3.7 (measurable Riemann mapping theorem, Ahlfors-Bers, [2])**.** *For any $\mu \in L^\infty(\hat{\mathbb{C}})$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal mapping $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with Beltrami coefficient μ that leaves* $0, 1, \infty$ *fixed.*

For $D = \hat{\mathbb{C}}$, $\mathbb{H}$, or $\mathbb{L}$, let $B_1(D)$ denote the unit ball in $L^\infty(D)$. By Lemma 2.3.2, for a marked Riemann surface (R, S, f), it follows that

$$\mu_f = \mu_{f\circ\gamma} = \mu_f \circ \gamma \cdot \frac{\overline{\gamma'}}{\gamma'} \text{ a.e. on } \mathbb{H}, \text{ for every } \gamma \in \Gamma_R. \tag{2.21}$$

We say that $B_1(D, \Gamma_R) := \{\mu \in B_1(D) \mid \mu = \mu \circ \gamma \cdot \frac{\overline{\gamma'}}{\gamma'} \text{ a.e. on } \mathbb{H}, \text{ for every } \gamma \in \Gamma_R\}$ is the set of *Beltrami differentials* on D with respect to Γ_R. Every $\mu \in \mathrm{B}_1(\mathbb{H}, \Gamma_R)$ extends to $B_1(\hat{\mathbb{C}}, \Gamma_R)$ in the following two ways:

(1) (symmetric extension) $\mu = 0$ on $\hat{\mathbb{R}}$ and $\mu(t) = \overline{\mu(\bar{t})}$ for each $t \in \mathbb{L}$,

(2) (holomorphic extension) $\mu = 0$ on $\mathbb{L}$.

Let f^μ (f_μ, respectively) be the unique quasiconformal mapping given by Theorem 2.3.7 corresponding to the extension (1) ((2), respectively) of μ. As the compositions with each element in Γ_R gives an another normalized solution of the Beltrami equation, f^μ and f_μ are compatible with the action of Γ_R.

Lemma 2.3.8. *$f^\mu = f^\nu$ holds on $\hat{\mathbb{R}}$ if and only if $f_\mu = f_\nu$ holds on $\mathbb{L}$.*

Proof. If $f^\mu = f^\nu$ holds on $\hat{\mathbb{R}}$, the mapping $f = (f^\mu)^{-1} \circ f^\nu : \mathbb{H} \to \mathbb{H}$ extends by the identity on $\hat{\mathbb{R}} \cup \mathbb{L}$ as to be ACL. Then, the mapping $g = f^\mu \circ f \circ (f^\nu)^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal mapping whose Beltrami coefficient vanishes, and hence a Möbius transformation by Lemma 2.3.3. By Theorem 2.3.7 it must be identity and thus $f_\mu = f_\nu$ holds on $\mathbb{L}$.

Conversely, if $f_\mu = f_\nu$ holds on $\mathbb{L}$, then $f_\mu = f_\nu$ still holds on $\hat{\mathbb{R}} \cup \mathbb{L}$. As before, the mapping $h = f^\mu \circ (f_\mu)^{-1} \circ f_\nu \circ (f^\nu)^{-1} : \mathbb{H} \to \mathbb{H}$ extends by the identity on $\hat{\mathbb{R}} \cup \mathbb{L}$ to be a quasiconformal mapping whose Beltrami coefficient vanishes. Again by Theorem 2.3.7, it must be identity and thus $f^\mu = f^\nu$ holds on $\hat{\mathbb{R}}$. □

By definition, the mapping f_μ is biholomorphic on the lower half-plane $\mathbb{L}$. For each biholomorphism f on $\mathbb{L}$, we define the *Schwarzian derivative* $\mathcal{S}(f)$ of f by

$$\mathcal{S}(f)(z) := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{2.22}$$

$$= (\log f'(z))'' - \frac{1}{2}\{(\log f'(z))'\}^2, \quad z \in \mathbb{L}. \tag{2.23}$$

Lemma 2.3.9. *The following holds for any holomorphic mappings f, g on $\mathbb{L}$:*

(1) *$\mathcal{S}(f) = 0$ if and only if f is a Möbius transformation,*

(2) *$\mathcal{S}(g \circ f)(z) = \mathcal{S}(g)(f(z)) \cdot f'(z)^2 + \mathcal{S}(f)(z)$, $z \in \mathbb{L}$.*

Proof. By soloving $\mathcal{S}(f) = 0$ with equation (2.23), we will observe that such a f is a Möbius transformation. The rest follows from a direct calculation. □

Thus the Schwarzian derivative of f_μ, $\mu \in B_1(\mathbb{H}, \Gamma_R)$ is regarded as a bounded *holomorphic quadratic differential* (see Definition 4.1.1) on $\mathbb{L}/\Gamma_R$ that is the mirror image of R. The space $\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$ of hyperbolically bounded, holomorphic quadratic differentials on $\mathbb{L}/\Gamma_R$ is a complex Banach space equipped with the hyperbolic L^∞ norm (of weight -2), whose dimension is $3g-3+n$ by Riemann-Roch theorem. The mapping $\mathcal{B} : T(R) \to \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$ defined by $\mathcal{B}([S, g]) := S(f_{\mu_g})$ is injective by Lemma 2.3.8 and Lemma 2.3.9. It induces a complex structure from $\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R) \cong \mathbb{C}^{3g-3+n}$, and the mapping $\mathcal{B}$ is called the *Bers' embedding*.

The composition of the surjective mapping $\mathcal{P} : B_1(\mathbb{H}, \Gamma_R) \to T(R) : \mu \mapsto [f^\mu(R), f^\mu]$ and the Bers' embedding is called the *Bers' projection* $\Phi = \mathcal{B} \circ \mathcal{P} : B_1(\mathbb{H}, \Gamma_R) \to \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$. The above is summarized in the following diagram, where $QC(\hat{\mathbb{C}}, \Gamma_R)$ denotes the group of quasiconformal mappings of $\hat{\mathbb{C}}$ compatible with Γ_R.

$$\begin{tikzcd}
B_1(\mathbb{H},\Gamma_R) \arrow[r, "\mathcal{P}"] \arrow[d, phantom, sloped, "\in"] & T(R) \arrow[r] \arrow[rrr, bend left, "\mathcal{B}"] \arrow[d, phantom, sloped, "\in"] & B_1(\mathbb{H},\Gamma_R) \arrow[r] \arrow[d, phantom, sloped, "\in"] & QC(\hat{\mathbb{C}},\Gamma_R) \arrow[r] \arrow[d, phantom, sloped, "\in"] & \mathcal{Q}^\infty(\mathbb{L},\Gamma_R) \arrow[d, phantom, sloped, "\in"] \\
\mu \arrow[r, mapsto] & {[f^\mu(R), f^\mu = g]} \arrow[r, mapsto] & \mu_g \arrow[r, mapsto] & f_{\mu_g} \arrow[r, mapsto] & \mathcal{S}(f_{\mu_g})
\end{tikzcd}$$

It is proved by Ahlfors and Weill [3] [32, Theorem 6.9] that the *harmonic Beltrami differential* operator defined by

$$\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R) \to B_1(\mathbb{H}, \Gamma_R) : \alpha \mapsto \big(\mu_\alpha(z) = -2(\operatorname{Im} z)^2 \alpha(\bar{z}), \quad z \in \mathbb{H}\big) \tag{2.24}$$

gives a local inverse mapping of the Bers' projection.

The following analysis of the projection Φ is due to Bers [6] [47, Chapter 3]. The derivative of the Bers' projection Φ in the direction $\nu \in L^\infty(\mathbb{H}, \Gamma_R)$ at $\mu \in B_1(\mathbb{H}, \Gamma_R)$ is defined by

$$\dot{\Phi}_\mu[\nu] = \lim_{t \to 0} \frac{\Phi(\mu + t\nu) - \Phi(\mu)}{t}. \tag{2.25}$$

The limit (2.25) does exist and is represented as a bounded linear mapping on $B_1(\mathbb{H}, \Gamma_R)$ by the formula [32, Section 6.2.2]:

$$\dot{\Phi}_\mu[\nu] = \left(z \mapsto -\frac{6}{\pi} \iint_{\mathbb{H}} \frac{\nu(\zeta) \left(\frac{df_\mu}{d\zeta}(\zeta)\right)^2}{\big(f_\mu(\zeta) - f_\mu(z)\big)^4} d\xi d\eta \cdot f'_\mu(z)^2, \quad z \in \mathbb{L} \right). \tag{2.26}$$

The mapping (2.26) surjects onto $\mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$ and its kernel is a direct summand in $B_1(\mathbb{H}, \Gamma_R)$. Finally, the Bers' projection Φ is a holomorphic *submersion*; everywhere the mapping $\Phi : B_1(\mathbb{H}, \Gamma_R) \to \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$ locally is a projection

$$U_1 \times U_2 \to U_1 \hookrightarrow V, \quad U_1, U_2 \subset B_1(\mathbb{H}, \Gamma_R), \ V \subset \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R). \tag{2.27}$$

Then, the Banach space $L^\infty(\mathbb{H}, \Gamma_R)$ splits into the kernel $\ker \dot{\Phi}_\mu$ and the tangent space $T_{\mathcal{P}(\mu)} T(R) = \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)$. The dual space $T^*_{\mathcal{P}(\mu)} T(R) = \mathcal{Q}^\infty(\mathbb{L}, \Gamma_R)^*$ is represented [47, Theorem 1.4.3] as to be the space $\mathcal{Q}(\mathbb{L}, \Gamma_R) = \mathcal{Q}(\bar{R})$ of integrable holomorphic quadratic differentials (Definition 4.1.2), by the following *Weil-Peterson inner product* pairing:

$$(\alpha, \phi)_{\mathbb{L}/\Gamma_R} = \iint_{\mathbb{L}/\Gamma_R} \alpha(z)\overline{\phi(z)}\frac{|dz|}{(2\mathrm{Im}z)^2}, \quad \alpha \in \mathcal{Q}^{\infty}(\mathbb{L}, \Gamma_R), \quad \phi \in \mathcal{Q}(\mathbb{L}, \Gamma_R), \tag{2.28}$$

where the integral refers to any fundamental domain for Γ_R in $\mathbb{L}$. Remark that the spaces $\mathcal{Q}(R)$ and $\mathcal{Q}^{\infty}(R)$ are equal for a Riemann surface R of analytically finite type. We have the following.

Proposition 2.3.10. *Let R be a Riemann surface and $[S, f] \in T(R)$. Then, the cotangent space $T^*_{[S,f]}T(R)$ is isomorphic to the space $\mathcal{Q}(\bar{R})$ of integrable holomorphic quadratic differentials on $\bar{R}$.*

Chapter 3

Concepts in Algebraic Geometry

3.1 Variety

In this section, we we make some notes to grasp an outline of the algebraic-geometric aspects of the subject described in chapter 2. This section is based on [21].

Let $\mathbb{K}$ be a field and $\mathbb{K}[x_1, \ldots x_n]$ be the polynominal ring in n variables over $\mathbb{K}$. Assume that $\mathbb{K}$ is an *algebraically closed* field i.e. every non-constant polynomial in $\mathbb{K}[x]$ has a root in $\mathbb{K}$. For a subset $T \subset \mathbb{K}[x_1, \ldots x_n]$, Let $Z(T)$ denote the set of common zeros of all elements in T:

$$Z(T) = \{(a_1, \ldots a_n) \in \mathbb{K}^n \mid f(a_1, \ldots a_n) = 0 \text{ for all } f \in T\}. \tag{3.1}$$

A subset $Y \subset \mathbb{K}^n$ is called an *algebraic set* if $Y = Z(T)$ for some $T \subset \mathbb{K}[x_1, \ldots x_n]$. Every algebraic set $Z(T), T \subset \mathbb{K}[x_1, \ldots x_n]$ is uniquely represented by the ideal generated by T.

The collection of algebraic sets in $\mathbb{K}^n$ satisfies the axiom of closed sets. We consider the *Zariski topology* on $\mathbb{K}^n$ defined by taking the closed subsets to be the algebraic sets. An irreducible closed subset in $\mathbb{K}^n$ is called an *affine variety*. An open subset in an affine variety is called a *quasi-affine variety*.

Definition 3.1.1. For a subset $Y \subset \mathbb{K}^n$, define the *ideal* $I(Y)$ of Y by

$$I(Y) = \{f \in \mathbb{K}[x_1, \ldots x_n] \mid f(a_1, \ldots a_n) = 0 \text{ for all } (a_1, \ldots a_n) \in Y\}. \tag{3.2}$$

Theorem 3.1.2 (Hilbert's Nullstellensats [21, Theorem 1.3A])**.** *Let $\mathbb{K}$ be an algebraically closed field, let $\mathfrak{a} \subset \mathbb{K}[x_1, \ldots x_n]$ be an ideal, and $f \in \mathbb{K}[x_1, \ldots x_n]$ be a polynominal which vanishes at all points of $Z(\mathfrak{a})$. Then,the polynomial f belongs to the radical ideal*

$$\sqrt{\mathfrak{a}} = \{g \in \mathbb{K}[x] \mid g^r \in \mathfrak{a} \textit{ for some integer } r > 0\}. \tag{3.3}$$

Corollary 3.1.3 ([21, Corollary 1.4])**.** *There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{K}^n$ and radical ideals in $\mathbb{K}[x_1, \dots x_n]$, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.*

Proof. The correspondences I and Z give inclusion-reversing by their definitions. Let $\mathfrak{a} \subset \mathbb{K}[x_1, \dots x_n]$ be an arbitrary ideal. Theorem 3.1.2 implies $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. For any algebraic set $Y = Z(\mathfrak{a})$, applying Z to $\mathfrak{a} \subset \sqrt{\mathfrak{a}} = I(Z(\mathfrak{a}))$, we obtain $Y \supset Z(I(Y))$. The converse inclusion clearly holds.
Let $Y \subset \mathbb{K}^n$ be an irreducible algebraic set and $fg \in I(Y)$. Then Y is written by

$$Y = Y \cap (Z(fg)) = Y \cap (Z(f) \cup Z(g)) = (Y \cap Z(f)) \cup (Y \cap Z(g)), \tag{3.4}$$

which is the union of two proper closed subsets. By irreducibility, we have either $Y \subset Z(f)$ or $Y \subset Z(f)$, and hence either $f \in I(Y)$ or $g \in I(Y)$. Conversely, suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$ for a prime ideal $\mathfrak{p} \subset \mathbb{K}[x_1, \dots x_n]$. Then $\mathfrak{p} = I(Y_1 \cup Y_2) = I(Y_1) \cup I(Y_2)$, so either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$ holds. By applying Z we conclude the irreducibility. □

The affine space $\mathbb{K}^n$ is known to be a *noetherian* topology space, that is, any (strictly) descending chain $Y_1 \supset Y_2 \supset \cdots$ of closed subsets stops in finite index: $Y_n = Y_{n+1} = \cdots$ for some integer n. In particular, a maximal ideal $\mathfrak{m} \subset \mathbb{K}[x_1, \dots x_n]$ corresponds to a minimal irreducible component of $\mathbb{K}^n$ that must be a point.

Definition 3.1.4. Let $Y \in \mathbb{K}^n$ be an algebraic set. The *(Krull) dimension* $\dim(Y)$ of Y is the supremum of the height h of a descending chain $\mathfrak{p}_1 \supset \mathfrak{p}_2 \supset \dots \supset \mathfrak{p}_h$ in $\mathrm{Spec}(I(Y))$. The variety Y is called a *curve* if it has dimension 1. The *affine coordinate ring* $A(Y)$ of Y is the quotient ring $\mathbb{K}[x_1, \dots x_n]/I(Y)$.

Definition 3.1.5. Let Y be an affine variety.

(1) A function $f : Y \to \mathbb{K}$ on Y is called *regular* at $p \in Y$ if there exists a open neighborhood $U \subset Y$ of p and polynominals $g, h \in \mathbb{K}[x_1, \dots x_n]$ such that h is nowhere zero on U, and $f = g/h$ on U. Let $\mathcal{O}(Y)$ be the ring of all regular functions on Y.

(2) A *germ* of a regular function at $p \in Y$ is the class of a pair (U, f) of a open neighborhood $U \subset Y$ of p and a regular function $f : U \to \mathbb{K}$ under the equivalence relation defined by

$$(U, f) \sim_p (V, g) \Leftrightarrow f = g \text{ on } U \cap V. \tag{3.5}$$

For each $p \in Y$, the ring $\mathcal{O}_p$ of all germs of regular functions at p is called the *local ring* of p on Y.

(3) A *rational function* on Y is the class of a pair (U, f) of a nonempty open set $U \subset Y$ and a regular function $f : U \to \mathbb{K}$ under the equivalence relation defined by

$$(U, f) \sim (V, g) \Leftrightarrow f = g \ \text{ on } \ U \cap V. \tag{3.6}$$

The field $K(Y)$ of all rational functions on Y is called the *function field* on Y.

A continuous mapping $\varphi : Y_1 \to Y_2$ between two affine varieties is called a *morphism* if $\varphi \circ f : U \to \mathbb{K}$ is regular for any regular function $f : U \to \mathbb{K}$ on an open set $U \subset Y_1$. A *rational map* is the class of a pair (U, φ) of a nonempty open set $U \subset Y_1$ and a morphism $\varphi : Y_1 \to Y_2$ under the equivalence relation defined by

$$(U, \varphi) \sim (V, \psi) \Leftrightarrow \varphi = \psi \ \text{ on } \ U \cap V. \tag{3.7}$$

A rational map $[U, \varphi]$ is called *dominant* if for some (and hence every) representative (U, φ), the image $\varphi(U) \subset Y$ is dense.

Theorem 3.1.6 ([21, Theorem 3.2])**.** *Let $Y \subset \mathbb{K}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then the following holds:*

(1) *The ring $\mathcal{O}(Y)$ is isomorphic to the affine coordinate ring $A(Y)$.*

(2) *For each point $p \in Y$, let $\mathfrak{m}_p \subset A(Y)$ be the ideal of polynominals vanishing at p. Then, $p \mapsto \mathfrak{m}_p$ gives a one-to-one correspondence between the points of Y and maximal ideals of $A(Y)$.*

(3) *For each point $p \in Y$, $\mathcal{O}_p \cong A(Y)/\mathfrak{m}_p$ and* $\dim(\mathcal{O}_p) = \dim(Y)$ *hold.*

(4) *The function field $K(Y)$ is isomorphic to the quotient field of $A(Y)$, and hence $K(Y)$ is a finitely generated extension field of $\mathbb{K}$, of transcendence degree* $\dim(Y)$.

An affine variety $Y \subset \mathbb{K}^n$ is called *nonsingular* at $p \in Y$ if

$$\operatorname{rank}\left(\frac{\partial f_j}{\partial x_i}(p)\right)_{\substack{i=1,\ldots m\\ j=1,\ldots n}} = n - \dim(Y), \tag{3.8}$$

where $\{f_1, \ldots f_m\}$ is a generating system of the ideal $I(Y)$ of Y. The variety Y is called nonsingular if it is nonsingular at any point $p \in Y$. If $\mathbb{K} = \mathbb{C}$, a nonsingular variety Y is a complex manifold of dimension $\dim(Y)$.

For an affine variety $Y \subset \mathbb{K}^n$, we may take the *projective* model defined in the projective n-space $\mathbb{K}^{n+1}/\mathbb{K}^\times$ and in terms of homogenous polynominals. There is a similar discussion above for the projective varieties.

Theorem 3.1.7 ([21, Corollary 6.12])**.** *The following three categories are equivalent:*

(1) *the category of nonsingular projective curves with dominant rational maps,*

(2) *the category of compact Riemann surfaces with holomorphic mappings, and*

(3) *the category of function fields of dimension* 1 *over* $\mathbb{C}$ *with* $\mathbb{C}$*-algebra homomorphisms.*

For a commutative ring R, the set of all prime ideals of R is called the *spectrum* of R and denoted by $\mathrm{Spec}(R)$. By Corollary 3.1.3, the spectrum $\mathrm{Spec}(R)$ of R is identified with the set of all irreducible algebraic sets on which every $f \in R$ vanishes. The Zariski topology on $\mathrm{Spec}(R)$ is defined by an open basis $\left\{U_f = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\} \,\middle|\, f \in R\right\}$.

Definition 3.1.8 (localization)**.** For a commutative ring R and a multiplicative closed set S, we define an R-module $S^{-1}R$ as follows.

(1) Let $S^{-1}R$ be the quotient of $S \times R$ by the equivalence relation:

$$(s_1, r_1) \sim (s_2, r_2) \Leftrightarrow {}^{\exists}m \in S \text{ s.t. } m(s_1 r_2 - s_2 r_1) = 0. \tag{3.9}$$

We denote the equivalence class of $(s, r) \in S^{-1}R$ by r/s.

(2) For $r_1/s_1,\ r_2/s_2 \in S^{-1}R$ and $c \in R$, define an addition and a scalar multiplication by

$$r_1/s_1 + r_2/s_2 := (r_1 s_2 + r_2 s_1)/(s_1 s_2), \tag{3.10}$$

$$c \cdot r_1/s_1 := (c r_1)/s_1. \tag{3.11}$$

For each $\mathfrak{p} \in \mathrm{Spec}(R)$, the R-module $R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R$ has a unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ and is called the local ring of $\mathrm{Spec}(R)$ at $\mathfrak{p}$. The quotient field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is called the *residue field* of $\mathrm{Spec}(R)$ at $\mathfrak{p}$.

There exists a unique unique way to glue the local rings of $\mathrm{Spec}(R)$ to form the ring $R_f := \{1, f, f^2, \ldots\}^{-1} R = R[x]/(fx - 1)$ on U_f, $f \in R$ [21, Proposition 2.2]. A spectrum with the structure (*sheaf* of local rings) constructed in this way is called an *affine scheme*.

Example 4. The spectrum of a field $\mathbb{K}$ is the one point scheme $\mathrm{Spec}(\mathbb{K}) = \{0\}$. Its local ring is $\mathcal{O}_{\{0\}} = \mathbb{K}$.

Let $\mathbb{K}_1 \subset \mathbb{K}$ be a subfield. Fix an embedding $\varphi : \mathbb{K}_1 \hookrightarrow \mathbb{K}[x_1, \ldots x_n]$. Then, for $f \in \mathbb{K}[x_1, \ldots x_n]$ and $(p_1, \ldots p_n) \in \mathbb{K}_1^n$, the mapping

$$A(U_f) = \mathbb{K}[x_1, \ldots x_n]/(f) \to \mathbb{K} : x_j \mapsto p_j, \quad j = 1, \ldots n \tag{3.12}$$

is well-defined as to be compatible with the embedding $\mathbb{K}_1 \hookrightarrow \mathbb{K}$ if and only if $f(p_1, \dots p_n)$ vanishes. We may reproduce the algebraic set $Z(f)$ from the the affine coordinate ring $A(U_f)$ in this way. More generally, if we introduce the notion of *schemes* that are defined by gluing locally defined 'spaces with local rings' each of which isomorphic to an affine scheme, every scheme S with a fixed morphism φ to a fixed affine scheme $\mathrm{Spec}(\mathbb{K})$ is locally represented by a variety over $\mathbb{K}$. Now, the embedding φ controls the coefficients arising in the local representation by Theorem 3.1.6. Each of the categories stated in Theorem 3.1.7 is identified with the category of schemes. We say that a scheme S admitting a morphism to $\mathrm{Spec}(\mathbb{K})$ is *defined over* $\mathbb{K}$, and a fixed morphism $\varphi : S \to \mathrm{Spec}(\mathbb{K})$ is called a *structure morphism*.

3.2 Dessin d'enfant

This section is based on [28, 33].

The *absolute Galois group* $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the group of field automorphisms on the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. The absolute Galois group $G_{\mathbb{Q}}$ acts on the category of nonsingular projective curves defined over $\bar{\mathbb{Q}}$ by post-composing with fixed structure morphism.

Theorem 3.2.1 (Belyĭ [5], 1979)**.** *A nonsingular projective curve C is defined over $\bar{\mathbb{Q}}$ if and only if it admits a covering $\beta : C \to \hat{\mathbb{C}}$ branched at most over three points.*

Definition 3.2.2. A compact Riemann surface R admitting a meromorphic function $\beta : R \to \hat{\mathbb{C}}$ branched over at most three points $0, 1, \infty \in \hat{\mathbb{C}}$ is called a *Belyĭ surface*. Such β is called a *Belyĭ covering* and a pair (R, β) is called a *Belyĭ pair*.

The if-part of Theorem 3.2.1 is sketched in Belyĭ's paper [5], and known as an "obvious part" to those who are familiar with the results of Weil's paper [58]. Köck [35] reformulated the proof of this part. The only-if-part of Theorem 3.2.1 is worked out by the following *Belyĭ's algorithm* [33, Section 1.4.4]:

(1) Take a projection $\pi : C \to \hat{\mathbb{C}}$. Since C is nonsingular, the mapping π is holomorphic and has finite set of blanched points $B := \mathrm{Br}(\pi) \subset \bar{\mathbb{Q}} \cup \{\infty\}$.

(2) Take a minimal polynominal $P \in \mathbb{Q}[x]$ of B. As P and P' are invariant under the $G_{\mathbb{Q}}$-action, so are $\mathrm{Crit}(P)$ and $\mathrm{Br}(P)$. Let $B' := \mathrm{Br}(P \circ \pi) \subset P(\mathrm{Br}(\pi)) \cup \mathrm{Br}(P)$. The number of branched points outside $\mathbb{Q}$ can be strictly reduced by repeatedly composing a minimal polynominal of the branched set.

(3) Let $i = 0$, $P_0 = P$, and $\mathrm{Br}(P_i \circ \pi) = \{p_1, \dots p_n\} \subset \mathbb{Q}$.

(4) Take a branched point $p \in \mathrm{Br}(P_i \circ \pi) \subset \mathbb{Q}$. We may assume $p \in [0,1] \cap \mathbb{Q}$ up to the automorphisms $z \mapsto 1-z$, $z \mapsto z^{-1}$ that preserve $\{0,1,\infty\}$. For $p = \frac{n}{m} \in [0,1] \cap \mathbb{Q}$, let $P_j := Q_{m,n} \circ P_{j-1}$ where

$$Q_{m,n}(z) := \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n, \quad z \in \mathbb{C}. \tag{3.13}$$

The polynominal $Q_{m,n}$ sends $0, 1 \mapsto 0$, $p \mapsto 1$, $\infty \mapsto \infty$, and $\mathrm{Crit}(Q_{m,n})$ leaves in $\{0, 1, \infty, p\}$.

(5) As the cardinality of $\mathrm{Br}(P_j \circ \pi) \setminus \{0,1,\infty\} \subset Q(\mathrm{Br}(\pi)) \cup \mathrm{Br}(P) \subset \mathrm{Br}(P_j \circ \pi)$ strictly decreases for j, we can repeat the step (4) in finite time n so that $\mathrm{Br}(P_n \circ \pi) \subset \{0,1,\infty\}$. Then, $P_n \circ \pi$ is a Belyĭ covering.

Definition 3.2.3. A *dessin d'enfants* (or simply a *dessin*) is a bipartite, connected, filling graph-embedding. i.e. an embedding a pair of finite sets $\mathcal{G} = (\mathcal{V} = \mathcal{V}_\circ \sqcup \mathcal{V}_\bullet, \mathcal{E})$ into a topological surface R as follows:

(1) the image of every *vertex* $v \in \mathcal{V}$ is a point on R,

(2) the image of every *edge* $e \in \mathcal{E}$ is a path intersecting no edges,

(3) every edge has one endpoint in $\mathcal{V}_\circ$ and the other in $\mathcal{V}_\bullet$,

(4) every two vertices are connected via finitely many edges, and

(5) every components of $R \setminus \mathcal{G}$ (*faces*) are homeomorphic to an open disk.

The number of edges is called the *degree* of a dessin d'enfants. We say that two dessins d'enfants $f_i : \mathcal{G}_i \hookrightarrow R_i$ $(i = 1, 2)$ are *equivalent* if there exists a homeomorphism $g : R_1 \to R_2$ such that $f_2 = g \circ f_1$.

Example 5. By assigning $\mathcal{V}_\circ = \{0\}$, $\mathcal{V}_\bullet = \{1\}$, $\mathcal{E} = \{[0,1]\}$, we obtain the trivial dessin on the Riemann sphere $\hat{\mathbb{C}}$. Every Belyĭ covering $\beta : R \to \hat{\mathbb{C}}$ induces a dessin d'enfants on R as the pullback of the trivial dessin (see Fig. 3.A).

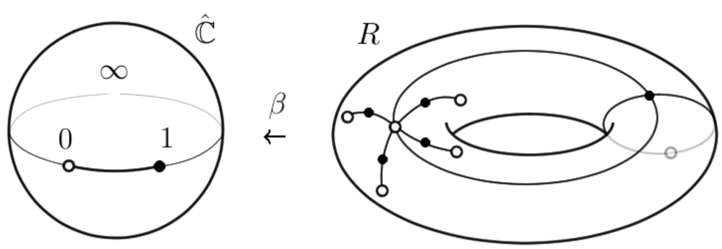

Fig. 3.A Example of a dessin d'enfants induced from a Belyĭ covering $\beta : R \to \hat{\mathbb{C}}$.

Proposition 3.2.4. *A dessin d'enfant of degree d is up to equivalence (in brackets) uniquely determined by each of the following.*

(a) *A Belyĭ pair (R, β) of degree d [up to covering equivalence over $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$].*

(b) *A pair of two permutations $x, y \in \mathfrak{S}_d$ generating a transitive permutation group [up to conjugation in $\mathfrak{S}_d$].*

(c) *A subgroup H of the free group F_2 of index d [up to conjugation in F_2].*

Proof. As we have seen in Remark 2.1.7, the objects in (a-c) are in a one-to-one correspondence up to equivalence. For a Belyĭ pair (R, β), the pullback of the trivial dessin under β gives a dessin on R. The covering equivalence of Belyĭ pairs corresponds to the equivalence of dessins given by this construction. Conversely, once a dessin $(\mathcal{V}_\circ \sqcup \mathcal{V}_\bullet, \mathcal{E}) \hookrightarrow R$ of degree d given up to equivalence, we may define two permutations $x \in \mathrm{Sym}(\mathcal{E})$ ($y \in \mathrm{Sym}(\mathcal{E})$, respectively) by the anti-clockwise permutation of edges arround all the vertices in $\mathcal{V}_\circ$ ($\mathcal{V}_\bullet$, respectively). We obtain a permutation group as in (b) by numbering edges, the way of which is unique up to conjugation in $\mathfrak{S}_d$. □

Definition 3.2.5. We say that a branched covering $f : R \to S$ has the *valency list*

$$(k_1^{p_1}, \ldots, k_{l_1}^{p_1} \mid \ldots \mid k_1^{p_n}, \ldots k_{l_n}^{p_n}), \quad k_1^{p_\cdot} \leq k_2^{p_\cdot} \leq \ldots \leq k_{l_\cdot}^{p_\cdot} \tag{3.14}$$

if $\mathrm{Br}(f) = \{p_1, \ldots p_n\}$ and the pullback $f^{-1}(p)$ consists precisely of n_p points of orders $k_1^p, \ldots, k_{n_p}^p$ for every $p \in \mathrm{Br}(f)$.

For a Belyĭ pair (R, β), the multiplicity function $\mathrm{mult}_\bullet(\beta) : R \to \mathbb{Z}_{\geq 0}$ locally is represented by the multiplicity of a raional function, and it is invariant under the $G_{\mathbb{Q}}$-action. Thus the valency list of a Belyĭ covering β is a $G_{\mathbb{Q}}$-invariant.

The genus g of R is obtained by Euler charasteristic calculation as

$$g = 1 + \frac{l_0 + l_1 + l_\infty - d}{2}, \tag{3.15}$$

where d is the degree of β and $(k_1^0, \ldots k_{l_0}^0 \mid k_1^1, \ldots k_{l_1}^1 \mid k_1^\infty, \ldots k_{l_\infty}^\infty)$ is the valency list of β.

For a dessin $D = (x, y)$, $x, y \in \mathfrak{S}_d$, its *automorphism group* $\mathrm{Aut}(D)$ is defined by the centralizer $\mathrm{Cent}_{\mathfrak{S}_d}\langle x, y\rangle$. Every $\sigma \in \mathrm{Aut}(D)$ corresponds to an orientation-preserving self-homeomorphism respecting the graph embedding and an automorphism of the Belyĭ surface compatible with Belyĭ covering. In particular, the isomorphism class of the automorphism group of a dessin is a $G_{\mathbb{Q}}$-invariant. The realization of a group in terms of automorphism group of dessin is studied by Jones [34] and Hidalgo [29].

Example 6. Fig. 3.B shows the two dessins $\mathcal{D}_i$ $(i = 1, 2)$ defined by the Belyĭ pairs

$$y^2 = x(x-1)\left(x - (-1)^i\sqrt{2}\right), \quad \beta(x,y) = \frac{4}{x^2}\left(1 - \frac{1}{x^2}\right), \ i = 1, 2. \tag{3.16}$$

They are conjugated by the element $(\sqrt{2} \mapsto -\sqrt{2}) \in G_{\mathbb{Q}}$, and have the same valency list $(1^2, 2, 4 \mid 2^2, 4 \mid 8)$.

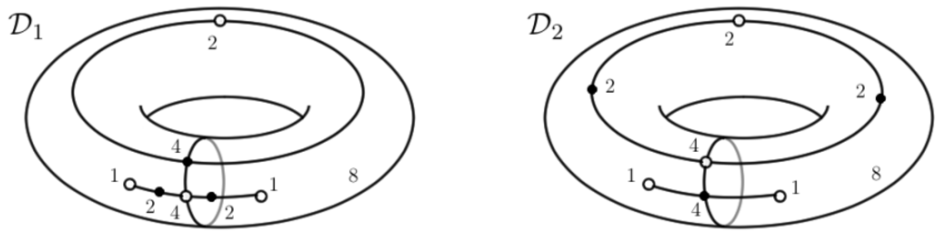

Fig. 3.B Example of Galois conjugate dessins.

Example 7. Let $d \in \mathbb{Z}_{>0}$. The *cycle graph* with d edges defines a dessin $\mathcal{C}_d = (x, y)$ where

$$x = (1\ 2)(3\ 4)\cdots(2d-1\ 2d), \quad y = (2\ 3)(4\ 5)\cdots(2d\ 1) \in \mathfrak{S}_d. \tag{3.17}$$

Its automorphism group is generated by a cyclic permutation $(1\ 3\ \cdots\ 2d-1)(2\ 4\ \cdots\ 2d)$ and an involutive permutation $(1\ 2d)(2\ 2d-1)(3\ 2d-2)\cdots(d\ d+1)$. So the automorphism group is $\mathrm{Aut}(\mathcal{C}_d) \cong \{\pm 1\} \rtimes C_d$ where C_d is the cyclic group of order d.

We mention to the following formulation of the $G_{\mathbb{Q}}$-action on dessins [31, Appendix]. Let $\mathcal{N}$ be the set of finite index normal subgroups of the free group F_2. The *profinite free group* $\hat{F}_2$ is defined by

$$\hat{F}_2 = \{(g_N N)_{N \in \mathcal{N}} \mid g_N \in F_2,\ g_N N = g_{N'} N \text{ for any } N, N' \in \mathcal{N} \text{ with } N > N'\}. \tag{3.18}$$

Fix a tangential base point $\vec{u} = \vec{01}$ on $\hat{\mathbb{C}}$. For each Belyĭ pair (R, β), fix a point $v \in \beta^{-1}(0)$ of order m and choose a local coordinate z arround v such that $\beta(z) = z^m$. Consider the map

$$\varphi_v(f) = \sum_{n=-k}^{\infty} a_n z^{\frac{n}{m}}, \quad f \in \bar{\mathbb{Q}}(R), \tag{3.19}$$

where $\sum_{n=-k}^{\infty} a_n z^n$ is the Laurent series expantion of f arround 0. Arround the point v, $\varphi_v(f)$ has precisely m branches

$$\sum_{n=k}^{\infty} a_n \zeta_m^{kn} z^{\frac{n}{m}}, \quad k = 1, \ldots m, \tag{3.20}$$

where ζ_m is a m-th root of unity. The mapping $\varphi = \varphi_v$ defines an embedding of $\bar{\mathbb{Q}}(R)$ into the space $P_{\vec{u}}$ of *convergent Puiseux series* of the form (3.20) with coefficients in $\bar{\mathbb{Q}}$ based on $\vec{u}$. The monodromy x arround $0 \in \hat{\mathbb{C}}$ acts on the image by the rotational permutation of these branches. The absolute Galois group $G_{\mathbb{Q}}$ acts on $P_{\vec{u}}$ as in the following diagram: for $\sigma \in G_{\mathbb{Q}}$,

$$\begin{tikzcd}
\bar{\mathbb{Q}}(R) \arrow[r, hook, "\varphi"] & P_{\vec{u}} & & P_{\vec{u}} & \bar{\mathbb{Q}}(R) \arrow[l, hook', "x(\varphi)"'] \\[-15pt]
\Cup & \Cup & & \Cup & \Cup \\[-15pt]
f \arrow[r, mapsto] & \sum_{n=k}^{\infty} a_n z^{\frac{n}{m}} \arrow[rr, mapsto, "x"] \arrow[d, mapsto, "\sigma"] & & \sum_{n=k}^{\infty} a_n \zeta_m^n z^{\frac{n}{m}} \arrow[d, mapsto, "\sigma"] & f \arrow[l, mapsto] \\
f^{\sigma} \arrow[r, mapsto] & \sum_{n=k}^{\infty} \sigma(a_n) z^{\frac{n}{m}} \arrow[rr, mapsto, "\sigma\cdot x"] & & \sum_{n=k}^{\infty} \sigma(a_n)\sigma(\zeta_m)^n z^{\frac{n}{m}} & f^{\sigma} \arrow[l, mapsto] \\[-15pt]
\Cap & \Cap & & \Cap & \Cap \\[-15pt]
\bar{\mathbb{Q}}(R^{\sigma}) \arrow[r, hook, "\varphi^{\sigma}"] & P_{\vec{u}}^{\sigma} & & P_{\vec{u}}^{\sigma} & \bar{\mathbb{Q}}(R^{\sigma}) \arrow[l, hook', "\sigma\cdot x(\varphi^{\sigma})"']
\end{tikzcd}$$

In particular, the σ-image of the monodromy x is described by the formula

$$\sigma \cdot x = x^{\lambda_\sigma}, \tag{3.21}$$

where $\zeta_m^{\lambda_\sigma} = \sigma(\zeta_m)$. For the monodromy y arround $1 \in \hat{\mathbb{C}}$, we can make the same argument except that there is a conjugate path t connecting 0 to 1. The path t can be interpreted as acting on embeddings $\bar{\mathbb{Q}}(R) \hookrightarrow \mathcal{P}_{\vec{u}}$. By working with the "fundamental groupoid", we can calculate the action of σ on t, and $f_\sigma \in t^{-1}(\sigma \cdot t)$ belongs to F_2. The σ-image of the monodromy y is described by the formula

$$\begin{aligned}
\sigma \cdot y &= \sigma \cdot (t^{-1}xt) \\
&= \left((\sigma \cdot t^{-1})t\right)\left(t^{-1}(\sigma \cdot x)t\right)\left({}^{-1}(\sigma \cdot t)\right) \\
&= f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma.
\end{aligned} \tag{3.22}$$

The *Grothendieck-Teichmüller group* $\widehat{GT}$ introduced by Drinfel'd [8] is a group acting on $\hat{F}_2$ in the same way to $G_{\mathbb{Q}}$ (formulae (3.21) and (3.22)). The group $\widehat{GT}$ is defined in a purely combinatorial way, and is known as the automorphism group of the "tower" of profinite mapping class groups [24, 53].

Theorem 3.2.6 ($\widehat{GT}$-relation, Drinfel'd [8] and Ihara [31])**.** *The absolute Galois group $G_{\mathbb{Q}}$ is embedded in* $\mathrm{Aut}(\hat{F}_2)$ *as a subgroup of the Grothendieck-Teichmüller group $\widehat{GT}$.*

Chapter 4

Flat surfaces and origamis

4.1 Flat surface

Let R be a Riemann surface of finite analytic type (g, n) with $2g - 2 + n > 0$.

Definition 4.1.1. A *holomorphic quadratic differential* on R is a family $\phi = \{\phi_\alpha\}$ such that:

(1) for each $\alpha = (U, \varphi) \in \mathcal{A}$, $\phi_\alpha : \varphi(U) \to \mathbb{C}$ is a nonconstant holomorphic mapping,

(2) for each $\alpha_i = (U_i, \varphi_i) \in \mathcal{A}_R$ $(i = 1, 2)$ with $U_1 \cap U_2 \neq \emptyset$,

$$\phi_{\alpha_1}(z) = \phi_{\alpha_2} \circ \varphi_1^{-1}(z) \cdot \left(\frac{d\varphi_2 \circ \varphi_1^{-1}(z)}{dz} \right)^2, \quad \text{for any } z \in \varphi_1(U_1 \cap U_2). \tag{4.1}$$

A *holomorphic abelian differential* on R is defined similarly but (2) replaced by

(2') for each $\alpha_i = (U_i, \varphi_i) \in \mathcal{A}_R$ $(i = 1, 2)$ with $U_1 \cap U_2 \neq \emptyset$,

$$\phi_{\alpha_1}(z) = \phi_{\alpha_2} \circ \varphi_1^{-1}(z) \cdot \left(\frac{d\varphi_2 \circ \varphi_1^{-1}(z)}{dz} \right), \quad \text{for any } z \in \varphi_1(U_1 \cap U_2). \tag{4.2}$$

A pair (R, ϕ) of Riemann surface R and a holomorphic quadratic differential ϕ on R is called a *flat surface*. We say that the set $\mathrm{Sing}(R, \phi)$ of the marked points of R and the zeros of ϕ is the set of the *singularities* of (R, ϕ). Define the *order* of ϕ by

$$\mathrm{ord}_p(\phi) := \begin{cases} \mathrm{mult}_p(\phi) & \text{if } p \in \mathrm{Sing}(R, \phi) \\ 0 & \text{otherwise} \end{cases}, \quad \text{for each } p \in R. \tag{4.3}$$

Let $p_0 \in R^* = R \setminus \mathrm{Sing}(R, \phi)$ and $\alpha = (U, z) \in \mathcal{A}_R$ be a chart around p_0. Then ϕ defines a natural coordinate of ϕ (*ϕ-coordinate*) defined by

$$\zeta_\phi(p) = \int_{p_0}^{p} \sqrt{\phi_\alpha(z)}dz, \;\; p \in U, \;\; (U, z) \in \mathcal{A}_R, \tag{4.4}$$

for which $\phi = (d\zeta_\phi)^2$ holds. The ϕ-coordinates give an atlas $\mathcal{A}_\phi$ on R^* any of whose transition map is a *half-translation* $\zeta \mapsto \pm\zeta + c$ ($c \in \mathbb{C}$). Such a structure, an atlas any of whose coordinate transformation is a half-translation is called a *flat structure*. The atlas $\mathcal{A}_\phi$ extends to each singularity $p_0 \in \mathrm{Sing}(R, \phi)$ of multiplicity m, with local representation

$$\zeta_\phi(p) = \int_{p_0}^{p} \sqrt{z^m}dz = z(p)^{\frac{m}{2}+1}, \;\; p \in U \setminus \{p_0\}, \tag{4.5}$$

for a suitable chart $(U, z) \in \mathcal{A}_R$ arround p_0.

A flat structure on R defines a holomorphic quadratic differential $\phi = (dz)^2$ conversely. Note that $\mathcal{A}_\phi$ is boholomorphically equivalent to $\mathcal{A}_R$ as a complex structure on R. In this way, we identify ϕ with the flat surface (R, ϕ).

Definition 4.1.2.

(1) We say that two flat surfaces (R, ϕ) and (S, ψ) are *isomorphic* if there exists a homeomorphism $f : (R, \phi) \to (S, \psi)$ that is locally a half-translation.

(2) We say that a flat surface (R, ϕ) is *abelian* if ϕ becomes the square of an abelian differential on R and otherwise *non-abelian* (or *primitive*).

(3) Let $g \geq 0, m_1, \ldots, m_n$ be integers and R be a Riemann surface. Define as follows.

$$\begin{aligned} \mathcal{Q}(R) &:= \{\phi : \text{holomorphic quadratic differential on } R\}, \\ \mathcal{Q}_g &:= \{\phi \in \mathcal{Q}(S) \mid S : \text{Riemann surface of genus } g, \int_R |\phi| < \infty\}, \\ \mathcal{Q}_g(m_1, \ldots, m_n) &:= \{\phi \in \mathcal{Q}_g \mid \phi \text{ has precisely } n \text{ singularities of orders } m_1, \ldots, m_n\}. \end{aligned}$$

Let $\mathcal{Q}^a$ ($\mathcal{Q}^p$, respectively) be the symbols that assign a set of abelian (non-abelian, respectively) differentials in place of $\mathcal{Q}$. (e.g. $\mathcal{Q}^a(R) = \{\phi \in \mathcal{Q}(R) : \text{abelian}\}$.)

Definition 4.1.3. Let (R, ϕ), (S, ψ) be flat surfaces in $\mathcal{Q}_g$.

(1) For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, we denote by $[A] = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the quotient class of A in $PSL(2, \mathbb{R})$. We define an affine map on the plane as follows:

$$T_A(x + \mathrm{i}y) = (ax + cy) + \mathrm{i}(bx + dy), \;\; x, y \in \mathbb{R}. \tag{4.6}$$

(2) A homeomorphism $f : (R, \phi) \to (S, \psi)$ is called *locally affine* or an *affine deformation* if everywhere f is locally represented by $z \mapsto T_A(z) + c$, for some $A \in SL(2, \mathbb{R})$ and $c \in \mathbb{R}^2$ with respect to the natural coordinates of ϕ and ψ, arround everywhere on R^*.

(3) For a locally affine homeomorphism $f : (R, \phi) \to (S, \psi)$, the local derivative A is constant up to a factor $\mathbb{R}_{\neq 0}$, independent of the natural coordinates of ϕ and ψ. We call $D(f) := [A] \in PSL(2, \mathbb{R})$ the *derivative* of f.

(4) The group $\mathrm{Aff}^+(R, \phi) := \{f : (R, \phi) \to (R, \phi) : \text{locally affine}\}$ is called the *affine group* of (R, ϕ). The group $\Gamma(R, \phi) = \{D(f) \in PSL(2, \mathbb{R}) \mid f \in \mathrm{Aff}^+(R, \phi)\}$ is called the *Veech group* of (R, ϕ).

For every affine deformation $f : (R, \phi) \to (S, \psi)$,

Theorem 4.1.4 (Teichmüller's existence & uniqueness theorem, [47, Theorem 2.6.4])**.** *Let $2g - 2 + n > 0$ and $[R_1, R_2, f] \in T_{g,n}$. Then, there exist $0 \le k < 1$, $\phi, \psi \in \mathcal{Q}_g$, and a quasiconformal mapping $f_T : R_1 \to R_2$ such that the following holds.*

(1) *$\phi \in \mathcal{Q}(R_1)$ and $\psi = f_T^* \phi \in \mathcal{Q}(R_2)$.*

(2) *f_T is homotopic to f.*

(3) *f_T is an affine deformation locally represented as $\zeta_\psi \circ f_T \circ \zeta_\phi^{-1}(z) = \dfrac{z + k\bar{z}}{1 - k}$.*

Moreover, f_T is a unique mapping that attains the minimal dilatation $K(f_T) = \dfrac{1 + k}{1 - k}$ in the homotopy class of f.

By Lemma 2.3.2, the Beltrami differential of the mapping f_T in Theorem 4.1.4 is given by

$$\mu_{f_T} = \mu_{\zeta_\psi^{-1} \circ \left(z \mapsto \frac{z + k\bar{z}}{1-k}\right) \circ \zeta_\phi} = \mu_{\left(z \mapsto \frac{z+k\bar{z}}{1-k}\right)} \circ \zeta_\phi \cdot \frac{(\overline{\zeta_\phi})_z}{(\zeta_\phi)_z} = k \frac{\overline{\phi}}{|\phi|}. \tag{4.7}$$

Let $(R, \phi) \in \mathcal{Q}_g$ be a flat surface and $t \in \mathbb{H}$. Consider the flat surface (R, ϕ_t) whose natural coordinates are deformed by the formula

$$\zeta_{\phi_t} = \mathrm{Re}(\zeta_\phi) + t \cdot \mathrm{Im}(\zeta_\phi) = \frac{1 - \mathrm{i}t}{2} \zeta_\phi + \frac{1 + \mathrm{i}t}{2} \overline{\zeta_\phi}. \tag{4.8}$$

Define $\hat{\iota}_\phi : \mathbb{H} \to T(R)$ by $\hat{\iota}_\phi(t) := [R_t = (R_{\mathrm{top}}, \mathcal{A}_{\phi_t}), f_t = id_{R_{\mathrm{top}}}]$, where R_{top} is the underlying topological surface of R. The mapping $f_t : R \to R_t$ is an affine deformation locally represented by the formula (4.8) with Beltrami coefficient $h(t) \dfrac{\overline{\phi_1}}{|\phi_1|}$ where $h(t) = \dfrac{t - \mathrm{i}}{t + \mathrm{i}} \in \mathbb{D}$. It follows from equation (2.15) that $\hat{\iota}_\phi$ is an isometric embedding with respect to the hyperbolic metric and the Teichmüller metric. Since the mapping $\hat{\iota}_\phi$ is the composition of the holomorphic mapping $\mathbb{H} \to B_1(\mathbb{R}, \Gamma_R)$ and the projection $\mathcal{P} : B_1(\mathbb{R}, \Gamma_R) \to T(R)$, it is holomorphic with respect to the complex structure of $T(R)$.

The embedded image $D_\phi := \hat{\iota}_\phi(\mathbb{H}) \subset T(R)$ is called the *Teichmüller disk* induced from ϕ.

Lemma 4.1.5 ([9, Lemma10.1]). *Let $g : R_1 \to R_2$ be a quasiconformal mapping between Riemann surfaces of analytically finite type and $\phi \in \mathcal{Q}(R_1)$. If the image of D_ϕ under $\rho_g : T(R_1) \to T(R_2)$ contains the base point of $T(R_2)$, then there exists $s \in \mathbb{H}$, $\psi \in \mathcal{Q}(R_2)$ and an affine deformation $f : R_1 \to R_2$ such that the following holds.*

(1) $\rho_g = \rho_f$,

(2) $\psi = g^*\phi$,

(3) $\mu_g = -h(s)\dfrac{\overline{\phi}}{|\phi|}$,

(4) $\rho_g(\hat{\iota}_\phi(t)) = \hat{\iota}_\psi\left(h^{-1}\left(\dfrac{h(t) - h(s)}{1 - h(t)\overline{h(s)}}\right)\right)$ *for any* $t \in \mathbb{H}$.

In particular, $g(D_\phi) = D_\psi$.

As a consequence of Lemma 4.1.5, the maximal subgroup of the Teichmüller-modular group acting on D_ϕ is the affine group $\mathrm{Aff}^+(R, \phi)$. The action of $f \in \mathrm{Aff}^+(R, \phi)$ with derivative $D(f) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is described by

$$\rho_f(\hat{\iota}_\phi(t)) = \hat{\iota}_\phi\left(\frac{-at + b}{ct - d}\right), \quad t \in \mathbb{H}, \tag{4.9}$$

which is the Möbius transformation of $\overline{D(f)} = J^{-1}D(f)J$, $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Also by Lemma 4.1.5, the embedding $\hat{\iota}_\phi : \mathbb{D} \hookrightarrow T(R)$ is unique up to the natural $\mathbb{C} \setminus \{0\}$-action for each ϕ. The mapping $\iota_\phi : \mathbb{H} \xrightarrow{\hat{\iota}_\phi} T(R) \xrightarrow{\text{proj.}} M(R)$ results in an orbifold $C_\phi \cong \mathbb{H}/\overline{\Gamma(R, \phi)}$ embedded in the moduli space $M(R)$. In the case that $\Gamma(R, \phi)$ has a finite covolume, $C_\phi \subset M(R)$ is an algebraic curve called the *Teichmüller curve* induced from (R, ϕ). The above is summarized in the following diagram.

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\hat{\iota}_\phi", "\cong"'] \arrow[d, "\text{proj.}"] & D_\phi \arrow[r, hook, "\text{incl.}"] \arrow[d, "\text{proj.}"] & T(R) \arrow[d, "\text{proj.}"] \\
\mathbb{H}/\overline{\Gamma(R, \phi)} \arrow[r, "\iota_\phi", "\cong"'] & C_\phi \arrow[r, hook, "\text{incl.}"] & M(R)
\end{tikzcd}$$

4.2 Stratum of holomorphic quadratic differentials

Let R be a Riemann surface of analytically finite type (g, n) with $2g - 2 + n > 0$. By Definition 4.1.2, we can say the following about the orders of singularities of a flat surface:

$$\begin{cases} m_j > 0 \text{ and even} & \text{if } (R, \phi) \in \mathcal{Q}_g^a(m_1, \ldots m_k), \\ m_j \geq -1, \neq 0 & \text{if } (R, \phi) \in \mathcal{Q}_g^p(m_1, \ldots m_k). \end{cases} \tag{4.10}$$

Note that any singularity of order -1 is regarded as a marked point. For a *geometric triangulation*, a triangulation of R such that the vertices are singularities and the edges are saddle connections, by Euler characteristic calculation it follows that

$$\sum_{j=1}^{n} m_j = 4g - 4. \tag{4.11}$$

The set $\mathcal{Q}_g(m_1, \ldots m_k)$ splits into the disjoint union

$$\begin{aligned} \mathcal{Q}_g &= \mathcal{Q}_g^a \sqcup \mathcal{Q}_g^p \\ &= \left(\bigsqcup \mathcal{Q}_g^a(m_1, \ldots m_k) \right) \sqcup \left(\bigsqcup \mathcal{Q}_g^p(m_1, \ldots m_k) \right), \end{aligned} \tag{4.12}$$

where $(m_1, \ldots m_k)$ runs possible tuples of integers satisfying (4.10) and (4.11). The equality (4.12) is called a *stratification* into strata $\mathcal{Q}_g^\bullet(m_1, \ldots m_k)$. Two different strata contain a common flat surface only if they differ by nonsingular marked points.

It is observed [30, 57] that every stratum $\mathcal{Q}_g^a(m_1, \ldots m_n)$ is complex analytic and of dimension $2g - 1 + n$. Its parametrization is given in terms of relative homology [23, Section 2.1]. The homology group of the genus g surface S_g $(= R_{\text{top}})$ *relative to* n points $p_1, \ldots p_n \in S_g$ is defined by

$$H_m(S_g, \{p_1, \ldots p_n\}, \mathbb{Z}) = \operatorname{im}(\partial_{m+1}) / \ker(\partial_m), \quad m = 0, 1, \ldots, \tag{4.13}$$

where $C_m(S_g, n) = C_m(S_g)/C_m(\{p_1, \ldots p_n\})$ is the quotient group of m-simplices and $\partial_m : C_m(S_g, n) \to C_{m-1}(S_g, n)$ is the boundary homomorphism. The trivial element in $H_m(S_g, \{p_1, \ldots p_n\}, \mathbb{Z})$ is represented by the form $\partial\gamma + \gamma'$, for some $\gamma \in C_{m+1}(S_g)$ and $\gamma' \in C_m(\{p_1, \ldots p_n\})$.

The first relative homology group $H_1(S_g, \{p_1, \ldots p_n\}, \mathbb{Z})$ is a free abelian group of dimension $2g - 1 + n$, whose basis can be taken as a standard basis $\{A_1, B_1, \ldots A_g, B_g\}$ together with a choice of paths $\gamma_1, \ldots \gamma_{n-1}$ joining n marked points as shown in Fig. 4.A. Local coor-

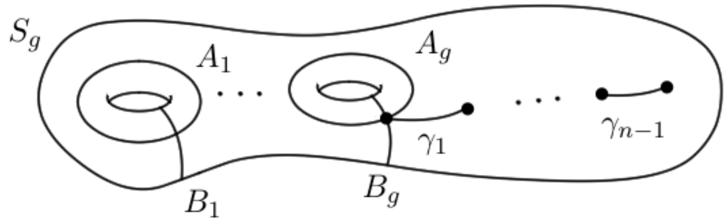

Fig. 4.A A basis of the first relative homology group $H_1(S_g, \{p_1, \dots p_n\}, \mathbb{Z})$. The black dots indicate the points $p_1, \dots, p_n$, and the path γ_i joins p_i and p_{i+1}.

dinates of a stratum of abelian differentials are given by the *period map* Ψ of $\mathcal{Q}_g^a(m_1, \dots m_k)$ to the first relative cohomology group $H^1(S_g, \{p_1, \dots p_n\}, \mathbb{C}) \cong \mathbb{C}^{2g-1+n}$ defined by

$$\Psi(R, \omega^2) = \left([\gamma_j] \mapsto \int_{\gamma_j} \omega \right), \quad j = 1, \dots, 2g-1+n, \quad (R, \omega^2) \in \mathcal{Q}_g^a(m_1, \dots m_k), \tag{4.14}$$

where $\gamma_1, \dots, \gamma_{2g-1+n}$ is a fixed basis of $H_1(S_g, \{p_1, \dots p_n\}, \mathbb{Z})$.

Proposition 4.2.1 ([42, Construction1]). *For a flat surface $(R, \phi) \in \mathcal{Q}_g^p$, the analytic continuation of the two branches of $\sqrt{\phi}$ gives a branched covering $\pi_\phi : (\hat{R}, \psi) \to (R, \phi)$ such that:*

(1) *$\psi = \pi_\phi^* \phi$ is abelian,*

(2) *the branched points of π_ϕ are percisely the singularities of ϕ of odd orders, and*

(3) *π_ϕ is the minimal coveing in the sense of (1).*

We say that $(\hat{R}, \psi)$ is the *canonical double* and π_ϕ is the *canonical double covering* of $(\hat{R}, \psi)$. The canonical double admits an involution $\tau : \hat{R} \to \hat{R}$ interchanging every two preimages of π_ϕ.

For a singularity p of (R, ϕ), it follows that

$$\pi_\phi^{-1}(p) \text{ consists of } \begin{cases} \text{two points of order} & \mathrm{ord}_p(\phi) & \text{if } \mathrm{ord}_p(\phi) \text{ is even,} \\ \text{one point of order} & 2\mathrm{ord}_p(\phi) + 2 & \text{if } \mathrm{ord}_p(\phi) \text{ is odd.} \end{cases} \tag{4.15}$$

If there are l singularities of odd order on (R, ϕ), it follows from equation (4.11) that the genus $\hat{g}$ of $\hat{R}$ satisfies that

$$\begin{aligned} 4\hat{g} - 4 &= 2 \sum_{\text{even order}} \mathrm{ord}_p(\phi) + \sum_{\text{odd order}} (2\mathrm{ord}_p(\phi) + 2) \\ &= 2(4g - 4) + 2l. \end{aligned} \tag{4.16}$$

Thus $(\hat{R}, \psi)$ has genus $\hat{g} = 2g - 2 + \frac{l}{2}$ and $\hat{n} = 2n - l$ singularities.

The mapping τ induces an involutive linear map τ^* on $H^1(\hat{R}, \{\hat{p}_1, \dots \hat{p}_{\hat{n}}\})$. The image of a neighborhood U of $(\hat{R}, \psi)$ in $\mathcal{Q}^a_{\hat{g}}$ decomposes into two eigenspaces $E_{\pm 1}$ with eigenvectors ± 1 for τ^*. As any abelian differential has eigenvalue -1, U is bijectively mapped into $E_{-1} \cong \mathbb{C}^{2g-2+n}$, and so locally is the non-abelian stratum $\mathcal{Q}^p_g$.

4.3 ϕ-metric

Let $(R, \phi) \in \mathcal{Q}_g$ be a flat surface. The Euclidian metric lifts via ϕ-coordinates to a flat metric on R, called the *ϕ-metric*. A geodesic of ϕ-metric is called a *ϕ-geodesic*. Via the ϕ-coordinates, a ϕ-geodesic is locally a line segment on the plane whose *direction* is uniquely determined in $\mathbb{R}/\pi\mathbb{Z}$.

Definition 4.3.1. Let $(R, \phi) \in \mathcal{Q}_g$ be a flat surface.

(1) A ϕ-geodesic joining two singularities is called a *saddle connection* on (R, ϕ).

(2) The *ϕ-cylinder* generated by a ϕ-geodesic γ is the union of all ϕ-geodesics parallel (with same direction) and free homotopic to γ. We define the direction of a ϕ-geodesic by the one of its generator.

(3) $\theta \in \mathbb{R}/\pi\mathbb{Z}$ is called *Jenkins-Strebel direction* of (R, ϕ) if almost every point in R lies on some closed ϕ-geodesic in the direction θ. Let $J(R, \phi)$ denote the set of Jenkins-Strebel directions of (R, ϕ).

A ϕ-cylinder C on (R, ϕ) admits an isomorphism $C \to \{0 < |\mathrm{Im}(z)| < h\}/\langle z \mapsto z + w\rangle$ for some *height* $h > 0$ and *width* $w > 0$. (See Section 5.2.) Note that any Jenkins-Strebel direction of flat surface in $\mathcal{Q}_g$ is *finite*, namely there are at most finitely many ϕ-cylinders of that direction in R.

We say that a system $\gamma = (\gamma_1, \dots, \gamma_p)$ of Jordan curves on R is *admissible* if none of the curves is homotopically trivial and any two distinct γ_i, γ_j are neither crossing nor (freely) homotopic. For the existence of a holomorphic quadratic differential with one Jenkins-Strebel direction, the following result is known.

Theorem 4.3.2 (Strebel, [55, Theorem 21.1])**.** *Let $\gamma = (\gamma_1, \dots, \gamma_p)$ be an admissible curve system on R, which satisfies bounded moduli condition for γ. Then for any $b = (b_1, \dots, b_p) \in \mathbb{R}^p_{>0}$, there exists a holomrphic quadratic differential ϕ on R such that $0 \in J(R, \phi)$ and (R, ϕ) is decomposed into cylinders $(V_1, ..., V_p)$, where each V_j has homotopy type γ_j and height b_j.*

Let $f \in \mathrm{Aff}^+(R, \phi)$ be an affine mapping with derivative $D(f) = [A] = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL(2, \mathbb{R})$. Then, f maps any line segment in the direction $\theta \in \mathbb{R}/\pi\mathbb{Z}$ to a line segment in the direction $A\theta := \arg(T_A(e^{i\theta}))$. We may observe that f maps a ϕ-cylinder of modulus M to a ϕ-cylinder of modulus $M/\sqrt{a^2 + c^2}$. Since the list of moduli of ϕ-cylinders of one direction are uniquely determined up to order, the following holds.

Lemma 4.3.3. *Let $J(R, \phi) \neq \emptyset$ and $(M_1^\theta, \dots, M_{n_\theta}^\theta) \in \mathbb{R}_{>0}^{n_\theta}$ be the list of moduli of the ϕ-cylinders in the direction $\theta \in J(R, \phi)$ sorted in ascending order. If $[A] = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL(2, \mathbb{R})$ belongs to $\Gamma(R, \phi)$, for any $\theta \in J(R, \phi)$, it follows that*

(1) *$A\theta \in J(R, \phi)$,*

(2) *$n_\theta = n_{A\theta} = n \in \mathbb{Z}_{>0}$, and*

(3) *$M_j^{A\theta} = M_j^\theta/\sqrt{a^2 + c^2}$ for $j = 1, \dots, n$.*

4.4 Origami

For an abelian flat surface $(R, \phi^2) \in \mathcal{Q}_g^a$, the holomorphic abelian differential ϕ on R defines natural coordinates without taking square-roots. These coordinates form an atlas any of whose transition map is a translation $\zeta \mapsto \zeta + c$ $(c \in \mathbb{C})$, called a *translation structure*. In this case, the derivative of an affine deformation is defined by distinguishing a factor of $\{\pm 1\}$, and the Veech group is defined as a subgroup of $SL(2, \mathbb{Z})$. An isomorphism of translation surfaces is defined similarly.

In this section, we will present an introduction to origamis and some results on them related to the absolute Galois group $G_{\mathbb{Q}}$. An (abelian) origami is a special example of abelian flat surface that induces a Teichmüller curve defined over $\bar{\mathbb{Q}}$. In 2005, Möller [45] proved that the $G_{\bar{\mathbb{Q}}}$-action on origamis respects the embedding of the Teichmüller curve in the moduli space, leading to an application to the $\widehat{GT}$-relation.

Definition 4.4.1. An *(abelian) origami* of degree d is an abelian flat surface obtained from d copies of the Euclidian unit squares by gluing them in such a way that

- each left edge is glued to a right edge,
- each upper edge is glued to a lower edge, and
- the resulting closed surface is connected.

The above gluing rule is called an *origami-rule*. An *origami* refers to an abelian origami in this section.

Every origami has a natural cellular decomposition. If an origami $\mathcal{O}$ has degree d and n singularities, it follows from Euler charasteristic calculation that the genus g of $\mathcal{O}$ is

$$g = 1 + \frac{d-n}{2}. \tag{4.17}$$

Lemma 4.4.2. *An origami of degree d is up to equivalence (in brackets) uniquely determined by each of the following.*

(a) *A connected, oriented graph $(\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = d$ such that each vertex has precisely one incoming edge and one outgoing edge labelled one with x and one with y, respectively [up to equivalence of the natural filling graph embedding].*

(b) *A d-fold branched covering $p : R \to E$ of a torus E branched at most over one point $\infty \in E$ [up to covering equivalence over $E \setminus \{\infty\}$].*

(c) *A pair of two permutations $x, y \in S_d$ generating a transitive permutation group [up to conjugation in S_d].*

(d) *A subgroup H of the free group F_2 of index d [up to conjugation in F_2].*

Proof. An origami naturally corresponds to a graph (a), by assigning the unit square cells to vertices and the adjacency of cells to edges reffering to the directions on the plane. By defining permutations $x, y \in S_d$ by the permutations of vertices along edges labelled with x, y. Transitivity follows from the connectedness of the resulting surface. As we have seen in Remark 2.1.7, the objects in (b-d) are in a one-to-one correspondence up to suitable equivalence where $\pi_1(E^*, \cdot) \cong F_2$. A covering (b) induces an abelian differential $\phi = p^* dz$ on R, which makes (R, ϕ) an origami. □

Let $\mathcal{O} = (p : (R, \phi) \to (E, dz))$ be an origami and $\psi = \pi_{R^*}^* \phi = \pi_{E^*}^* dz$ be the abelian differential on $\mathbb{H}$ induced from the universal covering. We fix a continuation of ψ-coordinates on $\mathbb{H}$ leading to a mapping $\zeta : (\mathbb{H}, \psi) \to \mathbb{C} \setminus \mathbb{Z} + \mathrm{i}\mathbb{Z}$. Every lift of $f \in \mathrm{Aff}^+(R, \phi)$ on $\mathbb{H}$ projects through ζ to an affine mapping $z \mapsto T_A(z) + c$ for some $A \in SL(2, \mathbb{R})$, $c \in \mathbb{C}$. As it is continuated to an affine mapping f^{dev} on $\mathbb{C} \setminus \mathbb{Z} + \mathrm{i}\mathbb{Z}$, it follows that $A \in SL(2, \mathbb{Z})$ and $c \in \mathbb{Z} + \mathrm{i}\mathbb{Z}$. We say that $f \in \mathrm{Aff}^+(R, \phi)$ is *developed* to $f^{\mathrm{dev}} \in \mathrm{Aff}^+(\mathbb{C} \setminus \mathbb{Z} + \mathrm{i}\mathbb{Z})$. Note that every affine mappping on (R, ϕ) is developed in this way and $\Gamma(\mathbb{H}, \psi) = \Gamma(E, dz) = SL(2, \mathbb{Z})$ holds.

Lemma 4.4.3 (Schmithüsen [51, Lemma 2.8])**.** *Let $\mathcal{O} = (R, \phi) = (p : R \to E)$ be an origami, and*

$$\mathrm{Aut}^+(F_2) = \{\sigma : \textit{orientation-preserving automorphism of } F_2 \cong \pi_1(E^*, \cdot)\}, \tag{4.18}$$

$$\mathrm{Inn}(F_2) = \{z^* = (w \mapsto z^{-1} w z) \in \mathrm{Aut}^+(F_2) \mid z \in F_2\} \cong F_2, \tag{4.19}$$

$$\mathrm{Out}^+(F_2) = \mathrm{Aut}^+(F_2)/\mathrm{Inn}(F_2). \tag{4.20}$$

Then, we have the following exact (in horizontal direction) and commutative diagram.

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Deck}(\mathbb{H}/E^*) \arrow[r, hook] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Aff}^+(\mathbb{H},\psi) \arrow[r, "D"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{SL}_2(\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & \mathrm{Inn}(F_2) \arrow[r, hook] & \mathrm{Aut}^+(F_2) \arrow[r] & \mathrm{Out}^+(F_2) \arrow[r] & 1 \end{tikzcd}$$

Furthermore, the subgroup of $\mathrm{Aut}^+(F_2)$ *that corresponds to* $\mathrm{Aff}^+(\mathcal{O}) < \mathrm{Aff}^+(\mathbb{H},\psi)$ *in this diagram is the stabilizer* $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}[H]$ *of the class of the fundamental group* $H < F_2$ *of the origami* $\mathcal{O}$*.*

Proof. Take an arbitrary $f \in \mathrm{Aff}^+(\mathbb{H},\psi)$ with $D(f) = I$. The developed mapping f^{dev} should be a translation in $\mathbb{Z} + \mathrm{i}\mathbb{Z}$, and thus a covering transformation of $\pi_{E^*} : \mathbb{H} \to E^*$. Converly, as every covering transformation preserves the induced translation $\psi = \pi_{E^*}^* dz$ on $\mathbb{H}$, it is a translation on $(\mathbb{H},\psi)$. So the group $\mathrm{Deck}(\mathbb{H}/E^*) \cong \pi_1(E^*,\cdot) \cong F_2$ is embedded in $f \in \mathrm{Aff}^+(\mathbb{H},\psi)$ as the kernel of the derivative D. Next, for each $f \in \mathrm{Aff}^+(\mathbb{H},\psi)$ we define

$$f^\star(g) = f^{-1} \circ g \circ f, \quad g \in \mathrm{Deck}(\mathbb{H}/E^*). \tag{4.21}$$

The mapping $f^\star(g)$ is an affine mapping with derivative I, and so $\star : f \mapsto f^\star$ defines a homomorphism of $\mathrm{Aff}^+(\mathbb{H},\psi)$ into $\mathrm{Aut}^+(F_2)$. We observe that $f^\star$ is isomorphism by showing the commutativity. The left half diagram commutes by definition. For the right half diagram, the group $\mathrm{Aut}^+(F_2)$ surjects onto $SL(2,\mathbb{Z})$ by the homomorphism defined by

$$\beta : \mathrm{Aut}^+(F_2) \to SL(2,\mathbb{Z}) : \gamma \mapsto \begin{pmatrix} \#_x\gamma(x) & \#_x\gamma(y) \\ \#_y\gamma(x) & \#_y\gamma(y) \end{pmatrix}, \tag{4.22}$$

where $\#_\bullet w$, $w \in F_2$ counts the (signed) number of x or y appearing in w. Indeed, the two generators $T \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $SL(2,\mathbb{Z})$ have pullback $\gamma_T, \gamma_S \in \mathrm{Aut}^+(F_2)$ defined by

$$\gamma_T(x,y) = (x, xy), \quad \gamma_S(x,y) = (y, x^{-1}). \tag{4.23}$$

One can observe that for an arbitrary $g \in \mathrm{Deck}(\mathbb{H}/E^*)$, the images of the two vector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ under the developed affine mapping $(f^\star(g))^{\mathrm{dev}}$ is precisely given by the matrix $\beta(f^\star)$. Thus β commutes with $\star$, the kernel of β is $\mathrm{Inn}(F_2)$, and hence the claim follows.

Finally, we show about the discription of $\mathrm{Aff}^+(\mathcal{O}) < \mathrm{Aff}^+(\mathbb{H},\psi)$ in $\mathrm{Aut}^+(F_2)$. An affine mapping $f \in \mathrm{Aff}^+(\mathbb{H},\psi)$ projects via the universal covering $\pi_{R^*} : \mathbb{H} \to R^*$ if and only if there exists an automorphism σ of $\mathrm{Deck}(\mathbb{H}/R^*) = \pi_1(R^*,\cdot)$ such that

$$f \circ \gamma = \sigma(\gamma) \circ f, \tag{4.24}$$

for any $\gamma \in \mathrm{Deck}(\mathbb{H}/R^*)$. On the other hand, the automorphism $\sigma = f^\star \in \mathrm{Aut}^+(F_2)$ satisfies (4.24) for any $\gamma \in \mathrm{Deck}(\mathbb{H}/E^*)$. It follows from Lemma 2.1.9 that the automorphism σ have to be of the form $f^\star$. Thus the claim follows. $\square$

By the fact that the Veech group of an origami is a stabilizer of a finite-index subgroup of F_2, the following finiteness follows. (See Corollary 5.3.1 for the proof.)

Lemma 4.4.4. *The Veech group of an origami is a subgroup of $SL(2,\mathbb{Z})$ of finite index.*

In her paper [51], Schmithüsen presented the following algorithm for finding the Veech group of an origami based on Lemma 4.4.3.

Algorithm 4.4.5 (Schmithüsen [51, Algorithm 1-4])**.** Let $\mathcal{O} = (R, \phi) = (x, y)$ be an origami of degree d and fix a generating system $C \subset F_2$ of the fundamental group $\pi_1(R^*) = H < F_2$. We obtain a generating system **Gen** and a representative set **Rep** of the Veech group $\Gamma(\mathcal{O}) < SL(2,\mathbb{Z})$ in the following steps:

(1) Let $m = n = N = 0$ and $R_0 = I$.

(2) For a word $W = W(T, S)$ in T and S, let $\gamma_W = W(\gamma_T, \gamma_S) \in \mathrm{Aut}^+(F_2)$ be the composition of γ_T, γ_S according to W. For each $n' \le N$, check whether $\gamma_{R_n T R_{n'}^{-1}}(c) \in F_2$ defines a closed path in $\mathcal{O}$ starting at some $i \in I_d$ (i.e. it defines a permutation z with $z(i) = i$) for all $c \in C$. If there exists such n', let $G_{m+1} = R_n T R_{n'}^{-1}$ and increment m by one. Otherwise let $R_{N+1} = R_n T$ and increment N by one.

(3) Do the same as (2) for S instead of T.

(4) If $n < N$ then go back to (2) for the next n. Otherwise finish the loop and let **Gen** $= \{G_1, \ldots G_m\}$ and **Rep** $= \{R_1, \ldots R_n\}$.

Remark 4.4.6. The surface $\mathbb{H}/PSL(2,\mathbb{Z})$ is an orbifold of genus 0 and with three singularities of order 2 at $[\mathrm{i}]$, 3 at $[\rho] = [e^{\pi \mathrm{i}/3}]$, and ∞ at the infinity. Lemma 4.4.4 implies that an arbitrary origami $\mathcal{O} = (p : R \to E)$ induces a Teichmüller curve $C(\mathcal{O})$ as a Belyĭ surface, and thus defined over $\bar{\mathbb{Q}}$. If $[\Gamma(\mathcal{O})]$ denotes the projected image of $\Gamma(\mathcal{O})$ in $PSL(2,\mathbb{Z})$, the Belyĭ covering is given by the projection $\mathbb{H}/[\Gamma(\mathcal{O})] \to \mathbb{H}/PSL(2,\mathbb{Z})$. Its monodromy is given by the natural action of $[\Gamma(\mathcal{O})]$ on the coset representatives in $SL(2,\mathbb{Z})$ (see Remark 2.1.7 and Fig. 4.B). We have the same formula as (4.17) for the genus g of Teichmüller curve $C(\mathcal{O})$, where d is the index $[PSL(2,\mathbb{Z}) : [\Gamma(\mathcal{O})]]$ and n is the number of the singularities. On the other hands, an origami $\mathcal{O}$ itself is a Belyĭ surface as a covering of the Belyĭ pair $(C_0 = \{y = 4x^3 - x\}, \beta_0(x, y) = 4x^2)$.

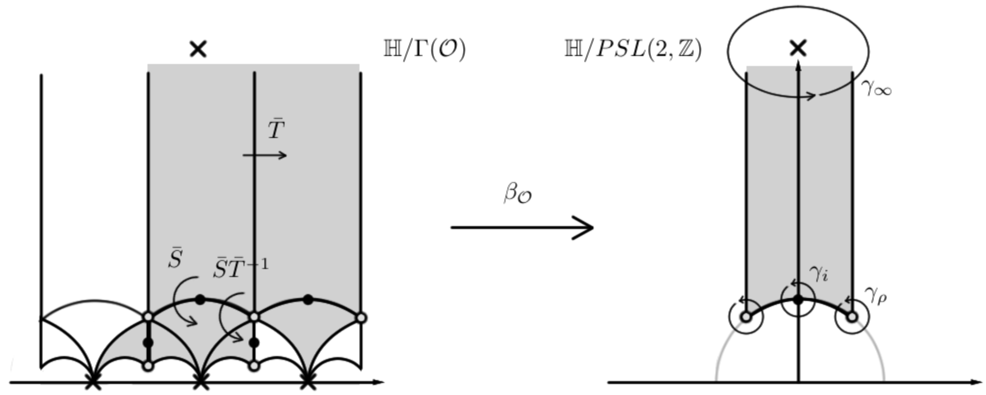

Fig. 4.B The Belyĭ covering $\beta_{\mathcal{O}}$ of the Teichmüller curve induced from an origami $\mathcal{O}$. The monodromy arround $[\mathrm{i}]$ ($[\rho], [\infty]$, respectively) is given by the action of matrix $[S]$ ($[ST^{-1}], [T]$, respectively) on the coset representatives of $PSL(2,\mathbb{Z})/[\Gamma(\mathcal{O})]$.

Proposition 4.4.7 (Möller, [45]). *Let $\sigma \in G_{\mathbb{Q}}$ and $\mathcal{O}$ be an origami of genus g. Then, $\mathcal{O}^\sigma$ is again an origami and the Belyĭ surfaces $C(\mathcal{O})$ and $C(\mathcal{O}^\sigma)$ are conjugated by σ as embedded curves in M_g.*

We mention to the $G_{\mathbb{Q}}$-conjugacy of embedded curves in $M_{g,n}$ as follows. The moduli space $M_{g,n}$ has a structure of *stack* for which $M_{g,n}$ satisfies the universal property among families of schemes of type (g,n). It parametrizes all schemes of type (g,n) with structure morphism to each assigned scheme as a contravariant functor $(\text{Schemes}/\mathbb{Z}) \to (\text{Sets})$. Restricted to the subcategory of schemes defined over $\bar{\mathbb{Q}}$, say $M_{g,n}^{\mathbb{Q}} \otimes \bar{\mathbb{Q}}$ where $M_{g,n}^{\mathbb{Q}} = M_{g,n}(\mathrm{Spec}(\mathbb{Q}))$, its *algebraic* fundamental group $\pi_1^{\mathrm{alg}}(M_{g,n}^{\mathbb{Q}} \otimes \bar{\mathbb{Q}}, \cdot)$ is known to be the profinite mapping class group $\widehat{\mathrm{Mod}}_{g,n}$. It gives an exact sequence

$$1 \to \pi_1^{\mathrm{alg}}(M_{g,n}^{\mathbb{Q}} \otimes \bar{\mathbb{Q}}, \cdot) \to \pi_1^{\mathrm{alg}}(M_{g,n}^{\mathbb{Q}}, \cdot) \to G_{\mathbb{Q}} \to 1 \tag{4.25}$$

and enables us to relate the absolute Galois group $G_{\mathbb{Q}}$ to the moduli spaces in terms of profinite mapping class groups. See [44, Section 4] and [15] for instance.

In his paper [45], Möller observed the $G_{\mathbb{Q}}$-action on the Teichmüller curve induced from a degree 4 origami called the two-steps origami. The (orbifold) fundamental group of the Teichmüller curve is embedded in the profinite mapping class group $\widehat{\mathrm{Mod}}_{2,0}$. The actions of $\widehat{GT}$ and $G_{\mathbb{Q}}$ on $\widehat{\mathrm{Mod}}_{2,0}$ were compared, and the compatibility with the Teichmüller curve induces a relation of the elements in $G_{\mathbb{Q}}$ embedded in $\widehat{GT}$.

Chapter 5

Main results

This chapter is based on [40, 41].

5.1 General origamis

Definition 5.1.1. A *general origami* of degree d is a flat surface obtained from d copies of the Euclidian unit squares by gluing along edges. An *origami* refers to a general origami in this chapter.

In the non-abelian case, similar arguments to Section 4.4.2 are valid for the genus (formula (4.17)), the development of affine maps (Corollary 4.4.4), and the Teichmüller curves (Remark 4.4.6). For Proposition 4.4.7, non-abelian origamis are mentioned but reduced from the argument by the existence of the canonical double cover which are abelian origamis. Note that non-abelian origamis are well expected to satisfy the same statement as Proposition 4.4.7.

Example 8. The *pillowcase sphere* $P = \mathbb{C}/\langle z+2, z+2\mathfrak{i}, -z\rangle$ is a degree 2, non-abelian origami in the stratum $\mathcal{Q}_0^p(-1^4)$. It is isomorphic to the elliptic involution quotient of the unit square torus and represented by the algebraic curve $C_0 : y = 4x^3 - x$. By Lemma 4.4.2, every abelian origami of degree d is a $2d$-fold covering $R \to E \to P$ of the pillowcase sphere.

Fig. 5.A shows an example of a non-abelian origami. We will observe in Theorem 5.1.7 that every origami of degree d is a $2d$-fold covering of the sphere over four points with the valency list $(2^d \mid 2^d \mid 2^d \mid *)$ over $\{[0], [1], [\mathfrak{i}], [1+\mathfrak{i}]\} \subset P$. Every critical point over the three branched points, say $[1], [\mathfrak{i}], [1+\mathfrak{i}] \in P$, has multiplicity two and so is nonsingular. The rest one branched point $[0] \in P$ pulls back the singularities of origami.

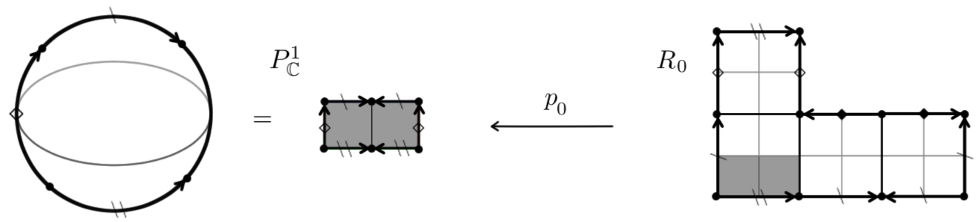

Fig. 5.A An origami of degree 4: edges with the same character are glued so that the arrows match. It admits a 8-fold covering of the pillowcase sphere P with valency list $(2^4 \mid 2^4 \mid 2^4 \mid 1^2, 3^2)$.

Notation 5.1.2. Let $I_d = \{1, \dots d\}$ be the set of d indices and $\bar{I}_d = \{\pm 1, \dots \pm d\}$ be its double. Let $\mathcal{E}_d := \{\varepsilon \in \{\pm 1\}^{\bar{I}_d} \mid \varepsilon(-i) = \varepsilon(i),\ i \in \bar{I}_d\}$ be the set of symmetric signs on $\bar{I}_d$. Let $\bar{\mathfrak{S}}_d := \{\bar{\sigma} \in \mathrm{Sym}(\bar{I}_d) \mid \bar{\sigma}(-i) = -\bar{\sigma}(i),\ i \in \bar{I}_d\}$ be the group of permutations with rotational symmetry, which naturally embeds the symmetric group $\mathfrak{S}_d$. For each $x \in \mathfrak{S}_d$ and $\varepsilon \in \mathcal{E}_d$, define a mapping $x^\varepsilon : \bar{I}_d \to \bar{I}_d$ by

$$x^\varepsilon(i) = \begin{cases} x(i) & \text{if } \varepsilon(i) = +1 \\ x^{-1}(i) & \text{if } \varepsilon(i) = -1 \end{cases} \quad \text{for each } i \in \bar{I}_d. \tag{5.1}$$

Definition 5.1.3. Let $\Omega_d := \mathfrak{S}_d \times \mathfrak{S}_d$ be the set of (possibly disconnected) abelian origamis of degree d, $\Omega^0_{2d} := \{\mathcal{O} \in \Omega_{2d} \mid \text{there exists an origami whose canonical double is } \mathcal{O}\}$, and $\tilde{\Omega}_d := \Omega_d \times \mathcal{E}_d$. For each $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$, define $(\mathbf{x}_{\mathcal{O}}, \mathbf{y}_{\mathcal{O}}) \in \Omega_{2d}$ by

$$\begin{cases} \mathbf{x}_{\mathcal{O}}(i) &= x^{\mathrm{sign}}(i) \\ \mathbf{y}_{\mathcal{O}}(i) &= \varepsilon(i) \cdot y^\varepsilon(i) \cdot \varepsilon(y^\varepsilon(i)) \end{cases} \quad \text{for each } i \in \bar{I}_d. \tag{5.2}$$

Every monodromy $\mathbf{z} \in \langle \mathbf{x}_{\mathcal{O}}, \mathbf{y}_{\mathcal{O}} \rangle < \bar{\mathfrak{S}}_d$ satisfies the following rotational symmetry with respect to the canonical double:

$$\mathbf{z}(-i) = -\mathbf{z}^{-1}(i) \text{ for each } i \in \bar{I}_d. \tag{5.3}$$

Lemma 5.1.4. *Any $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ corresponds to an origami of degree d whose canonical double covering is the abelian origami $(\mathbf{x}_{\mathcal{O}}, \mathbf{y}_{\mathcal{O}}) \in \Omega_{2d}$. In particular, $\mathcal{O} \mapsto (\mathbf{x}_{\mathcal{O}}, \mathbf{y}_{\mathcal{O}})$ gives a 1-1 correspondence $\tilde{\Omega}_d \to \Omega^0_{2d}$ up to equivalence.*

Proof. Consider the following construction for given $\mathcal{O} \in \tilde{\Omega}_d$:

(A1) Cut the resulting surface of $\mathcal{O}$ at all edges (with the edge-pairings remembered).

(A2) Apply the vertical reflection to all cells with $\varepsilon = -1$.

(A3) Glue all paired edges in such a way that with the natural coordinates, the quadratic differential $(dz)^2$ is globally defined on the resulting surface.

This produces a new origami which can be non-abelian as shown in Fig. 5.B. By taking double and gluing to the half-rotated copy in place of each cell with $\varepsilon = -1$ at (3), we obtain the canonical double represented by $(\mathbf{x}_{\mathcal{O}}, \mathbf{y}_{\mathcal{O}})$.

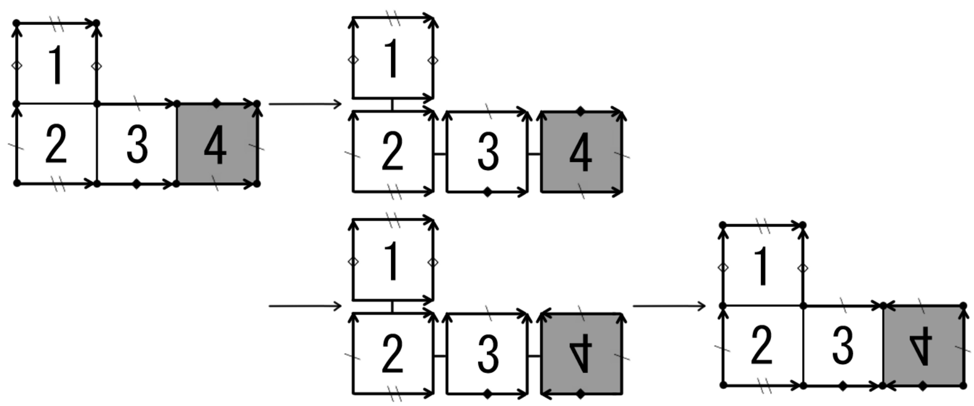

Fig. 5.B The construction of the origami in Fig. 5.A. It is given by (x, y, ε) where $x = (1)(2\ 3\ 4)$, $y = (1\ 2)(3\ 4)$, $\varepsilon = (+, +, +, -)$. We obtain the origami by regluing the abelian origami (x, y) after inverting squares of negative sign.

Conversely, consider the following construction for given $(\mathbf{x}, \mathbf{y}) \in \Omega^0_{2d}$:

(B1) Fix orientations of all horizontal and vertical cylinders in the resulting surface. For each $i \in \bar{I}_d$, let h_i (v_i, respectively) be an oriented, horizontal (vertical, respectively) closed geodesic crossing the cell with label i.

(B2) Define

$$\varepsilon(i) = \begin{cases} 1 & \text{if } v_i \text{ intersects to } h_i \text{ in positive crossing} \\ -1 & \text{if } v_i \text{ intersects to } h_i \text{ in negative crossing} \end{cases}, \ i \in \bar{I}_d.$$

(B3) Do the same operation as (A2-A3). (i.e. cut all edges, apply reflections to all cells such that $\varepsilon = -1$, and reglue them.)

It will be shown in Lemma 5.1.5 that the above procedure recovers $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ from $(\mathbf{x}_{\mathcal{O}}, \mathbf{y}_{\mathcal{O}})$. Note that ε at (B2) depends on the choice of directions at (B1), but the resulting surface is uniquely determined up to half-translation. □

Lemma 5.1.5. *Let $\mathbf{y} \in \bar{\mathfrak{S}}_d$ and $\mathbf{y} = c_1 c'_1 \cdots c_n c'_n$ be a cycle decomposition of $\mathbf{y}$, where $(a_1\ a_2\ \cdots\ a_m)' := (-a_1\ \ -a_2\ \ \cdots\ \ -a_m)^{-1}$. For $j = 1, \dots, n$, define $\varepsilon_j \in \mathcal{E}_d$ so that $\varepsilon_j(i) \cdot i$ belongs to the cycle c_j for each $i \in \bar{I}_d$. Then, the pair (y, ε) that correspond to $\mathbf{y}_{\mathcal{O}}$ under the formula (5.2) is given by*

$$\begin{cases} y & = \bar{\mathbf{y}} := |c_1| \cdots |c_n| \in \bar{\mathfrak{S}}_d \\ \varepsilon & = \varepsilon_{\mathbf{y}} := \varepsilon_1 \cdots \varepsilon_n \in \mathcal{E}_d, \end{cases} \tag{5.4}$$

where $|(a_1\, a_2 \cdots a_m)| := (|a_1|\, |a_2| \cdots |a_m|)$*. In particular, the inverse image of* $(\mathbf{x}, \mathbf{y}) \in \Omega^0_{2d}$ *under (A1-A3) in the proof of Lemma 5.1.4 is* $(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \varepsilon_{\mathbf{y}}) \in \tilde{\Omega}_d$.

Proof. Suppose $n = 1$. We denote $\mathbf{y} = cc' = (a_1\ a_2\ \cdots\ a_d)(-a_1\ \ -a_2\ \ldots\ \ -a_d)^{-1}$, $y = (|a_1|\ |a_2| \cdots\ |a_d|)(-|a_1|\ -|a_2| \cdots\ -|a_d|)$, and $a_{d+1} = a_1$. By definition, we have $\varepsilon(a_i) = 1$ and $\mathbf{y}(a_i) = \varepsilon(|a_{i+1}|)|a_{i+1}|$, for all $i \in I_d$. We will show that $\mathbf{y}(y, \varepsilon) := \varepsilon \cdot y^{\varepsilon} \cdot \varepsilon(y^{\varepsilon})$ equals to $\mathbf{y}$. For each $i \in I_d$,

$$\begin{aligned}
\mathbf{y}(y, \varepsilon)(a_i) &= \varepsilon(a_i) \cdot y^{\varepsilon(a_i)}(a_i) \cdot \varepsilon(y^{\varepsilon(a_i)}(a_i)) \\
&= y(\mathrm{sign}(a_i)|a_i|) \cdot \varepsilon(y(\mathrm{sign}(a_i)|a_i|)) \\
&= \mathrm{sign}(a_i)|a_{i+1}| \cdot \mathrm{sign}(a_i)\varepsilon(|a_{i+1}|) \\
&= \varepsilon(|a_{i+1}|)|a_{i+1}| \\
&= \mathbf{y}(a_i).
\end{aligned} \tag{5.5}$$

Applying this result to each cycle in $\mathbf{y} = c_1 c_1' \cdots c_n c_n'$, we obtain the claim for general n. □

Proposition 5.1.6. *Let* $\mathcal{O}_j = (x_j, y_j, \varepsilon_j) \in \tilde{\Omega}_d$ $(j = 1, 2)$ *be two origamis. Then* $\mathcal{O}_1, \mathcal{O}_2$ *are isomorphic as flat surfaces if and only if there exists* $\bar{\sigma} = \delta\sigma \in \bar{\mathfrak{S}}_d$ $(\delta = \mathrm{sign}(\bar{\sigma}) \in \{\pm 1\}^{\bar{I}_d}$, $\sigma \in \mathfrak{S}_d)$ *such that the following holds on* I_d*:*

(1) $\delta = \delta \circ x_1$,

(2) $x_2 = \sigma^{\#}(x_1^{\delta})$,

(3) $\xi(y_2, \delta \circ \sigma^{-1} \cdot \varepsilon_1 \circ \sigma^{-1} \cdot \varepsilon_2) = 1$ *where* $\xi(\tau, \lambda) := \lambda \cdot \lambda(\tau) \in \mathcal{E}_d$,

(4) $y_2 = \sigma^{\#}(y_1^{\delta \cdot \varepsilon_1 \cdot \varepsilon_2 \circ \sigma})$.

Proof. Assume the existence of an isomorphism between $\mathcal{O}_1$ and $\mathcal{O}_2$. Then its lift via their canonical double coverings induces a cell-to-cell correspondence $\bar{\sigma} \in \bar{S}_d$, such that $\mathbf{x}_2(i) = \bar{\sigma}^{\#}\mathbf{x}_1(i)$ and $\mathbf{y}_2(i) = \bar{\sigma}^{\#}\mathbf{y}_1(i)$ for $i \in I_d$. By the symmetry of $\mathbf{x}_1, \mathbf{x}_2$, it follows that $\mathbf{x}_2(\bar{\sigma}(-i)) = \mathbf{x}_2(-\bar{\sigma}(i))$ for each $i \in I_d$ and thus $\bar{\sigma} \in \bar{S}_d$. For $i \in I_d$ and $\varepsilon \in \{\pm 1\}$, we have the following:

$$\begin{aligned}
\bar{\sigma}(\mathbf{x}_1)(\varepsilon i) &= (\delta\sigma)(x_1^{\mathrm{sign}(\varepsilon i)}(\varepsilon i)) \\
&= \varepsilon\delta(x_1^{\varepsilon}(i))\sigma(x^{\varepsilon}(i)), && \cdots(a) \\
\mathbf{x}_2(\bar{\sigma}(\varepsilon i)) &= x_2^{\mathrm{sign}(\sigma(\varepsilon i))}(\varepsilon\delta(i)\sigma(i)) \\
&= \varepsilon\delta(i)x_2^{\varepsilon\delta(i)}(\sigma(i)), && \cdots(b) \\
\bar{\sigma}(\mathbf{y}_1)(\varepsilon i) &= (\delta\sigma)(\xi(y_1,\varepsilon_1)(\varepsilon i)\cdot y_1^{\varepsilon_1}(\varepsilon i)) \\
&= \varepsilon\xi(y_1^{\varepsilon\varepsilon_1},\varepsilon_1)(i)\cdot\delta(y_1^{\varepsilon\varepsilon_1(i)}(i))\cdot\sigma(y_1^{\varepsilon\varepsilon_1(i)}(i)), && \cdots(c) \\
\mathbf{y}_2(\bar{\sigma}(\varepsilon i)) &= \xi(y_2^{\varepsilon_2},\varepsilon_2)(\varepsilon\delta(i)\sigma(i))\cdot y_2^{\varepsilon_2}(\varepsilon\delta(i)\sigma(i)) \\
&= \varepsilon\delta(i)\xi(y_2^{\varepsilon\delta\circ\sigma^{-1}\varepsilon_2},\varepsilon_2)(\sigma(i))\cdot y_2^{\varepsilon\delta(i)\varepsilon_2(\sigma(i))}(\sigma(i)). && \cdots(d)
\end{aligned}$$

By comparing both sides of $\mathbf{x}_2(\bar{\sigma}(\varepsilon\sigma^{-1}(i))) = \bar{\sigma}(\mathbf{x}_1(\varepsilon\sigma^{-1}(i)))$, we obtain (1) and (2). Similarly for $\mathbf{y}_1, \mathbf{y}_2$, setting $\varepsilon = \varepsilon_2(i)\cdot\delta\circ\sigma^{-1}(i)$, we obtain (4) and the following:

$$\begin{aligned}
\delta\circ\sigma^{-1}(i)\cdot\xi(y_2^{\varepsilon\delta\circ\sigma^{-1}\varepsilon_2},\varepsilon_2)(i) &= \xi(y_1^{\varepsilon\varepsilon_1},\varepsilon_1)(\sigma^{-1}(i))\cdot\delta(y_1^{\varepsilon\varepsilon_1}(\sigma^{-1}(i))) \\
&= \xi(\sigma^{\#}y_1^{\varepsilon\varepsilon_1},\varepsilon_1\circ\sigma^{-1})(i)\cdot\delta\circ\sigma^{-1}(\sigma^{\#}y_1^{\varepsilon\varepsilon_1(i)}(i)).
\end{aligned}$$

With (4) $\sigma^{\#}y_1^{\varepsilon\varepsilon_1(i)}(i) = y_2(i)$, we conclude (3).

Suppose (1)-(4) conversely. Then for each $i \in I_d$, we have $(a) = (b)$ and $(c) = (d)$ for one of $\varepsilon \in \{\pm 1\}$. We may fill the equations for the other $\varepsilon \in \{\pm 1\}$ as follows. First, the signs of (a), (b) coincide by (1). The equality of the other parts of (a), (b) follows from (2) taking inverse mappings of both sides. We can say the same for the other parts of (c), (d). Finally, the equality of the signs of (c), (d) follows from (3) for each $y_2^{-1}(i) = \sigma^{\#}(y_1^{-\varepsilon_1\cdot\varepsilon_2\circ\sigma\cdot\delta})(i)$, $i \in I_d$. The above observation completes the proof. ◻

Theorem 5.1.7. *An origami of degree d is up to equivalence (mentioned in Remark 5.1.8) uniquely determined by each of the following.*

(a) *A $2d$-fold covering $p : R \to P^1_{\mathbb{C}}$ with the valency list $(2^d \mid 2^d \mid 2^d \mid *)$.*

(b) *A pair of abelian origami of degree d and a d-tuples of signs.*

(c) *A connected tripartite graph $(\mathcal{V} = \mathcal{V}_c \sqcup \mathcal{V}_h \sqcup \mathcal{V}_v, \mathcal{E})$ with $|\mathcal{V}_c| = |\mathcal{V}_h| = |\mathcal{V}_v| = d$ such that each edge connects vertices in $\mathcal{V}_c$ and either $\mathcal{V}_h$ or $\mathcal{V}_v$, and each vertex in $\mathcal{V}_c, \mathcal{V}_h, \mathcal{V}_v$ has valency $4, 2, 2$ respectively.*

(d) *A pair of permutations $\mu, \nu \in \mathrm{Sym}(\bar{I}_d)$ which are fixed-point-free, of order 2, and together with sign inversion generate a transitive permutation group.*

Proof. (origami $\Leftrightarrow$ (a) $\Leftrightarrow$ (b)) A covering (a) uniquely lifts the flat structure of the pillowcase sphere. The equivalence between origamis and (b) follows from Proposition 5.1.6. The construction (A1-A3) in the proof of Lemma 5.1.4 shows that by inverting some vertical monodromies of an abelian origami $R' \to E \to P$, one obtains a covering (a).

(origami $\Leftrightarrow$ (c) $\Leftrightarrow$ (d)) A graph (c) defines an origami by assigning a unit square cell to each vertex in $\mathcal{V}_c$, a horizontal edge to each vertex in $\mathcal{V}_h$, a vertical edge to each vertex in $\mathcal{V}_h$, and the adjacency between a cell and an edge to each edge in $\mathcal{E}$. Conversely, by composing a covering (a) to the Belyĭ pair $(C_0 = \{y = 4x^3 - x\}, \beta_0(x, y) = 4x^2)$, one obtains a dessin d'enfants on R as a graph (c). The rest of proof follows from Proposition 3.2.4. $\square$

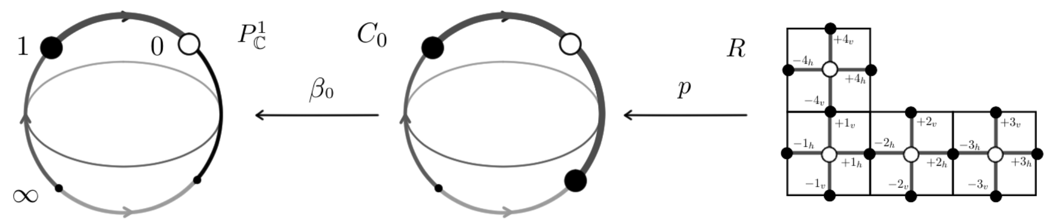

Fig. 5.C (The origami in Fig. 5.A, 5.B.) Suppose the bipartite graph $\beta^{-1}([0, 1])$ embedded in R. The monodromy group of β is generated by the two permutation ι, σ of edges around the white, brack vertices respectively. Each edge is labelled by the index of the square it belongs to and its direction. For example, the horizontal edge adjacent to the right (left, respectively) side of i-th square is labelled by $+i_h$ ($-i_h$, respectively).

Fig. 5.C shows an example of the $4d$-fold Belyĭ covering $\beta = \beta_0 \circ p : R \to P^1_{\mathbb{C}}$. The monodromy group of β is generated by two permutations $\iota, \sigma = \sigma_{\mu,\nu} \in \mathrm{Sym}(\bar{I}^h_d \sqcup \bar{I}^v_d)$ defined by

$$\begin{cases} \iota(\pm i_h) = \pm i_v & \iota(\pm i_v) = \mp i_h \\ \sigma(\pm i_h) = \mu(\pm i)_h & \sigma(\pm i_v) = \nu(\pm i)_v \end{cases} \quad \text{for each } i \in I_d, \tag{5.6}$$

where $\bar{I}^\bullet_d = \{\pm 1_\bullet, \dots, \pm d_\bullet\}$ denotes a copy of $\bar{I}_d$. The permutation $\iota\sigma$ arranges the edges clockwise around each of the centers of cells in $R \setminus \beta^{-1}([0, 1])$, which are the singularities of (R, ϕ). The permutation $\iota\sigma$ has even order, as does that of the pillowcase sphere.

Remark 5.1.8. Note about Theorem 5.1.7 as follows. The equivalence of origamis is defined by an isomorphism of flat surfaces. It corresponds to the covering equivalence of (a) over $P^1_{\mathbb{C}} \setminus \{0, \pm 1, \infty\}$. Lemma 5.1.6 presents a formula to determine the equivalence of origamis in terms of their canonical double coverings, which will be used in Section 6.1.

The equivalence of graph embeddings respecting the notion of 'horizontal, vertical' (i.e. the coloring of vertices) gives the equivalence of (c). In terms of dessins (d), it is described by the conjugacy in $\tilde{\mathfrak{S}}_d$.

We may observe that the Veech group of an origami is a finite-index subgroup of $PSL(2,\mathbb{Z})$ by the same arguments as in Section 4.4.

Theorem 5.1.9 (=Theorem 6.1.7)**.** *There exists two permutations $\sigma_T, \sigma_S \in \mathfrak{S}_d$ such that the Veech group of an origami of degree d is the stabilizer of its equivalence class under the action of $PSL(2,\mathbb{Z})$ on $\tilde{\Omega}_d$ defined by $[A](x,y,\varepsilon) := \theta^{-1}(\sigma_A^* \gamma_A(\theta(\mathcal{O})))$, $A = T, S$.*

5.2 Origami with moduli list

In this section, we consider a flat surface $(R,\phi) \in \mathcal{Q}_g$ with two distinct finite Jenkins-Strebel directions $\theta_1, \theta_2 \in J(R,\phi)$. Then R is obtained by finite collections of parallelograms in the way presented in [10, Theorem2], in which we conclude R is finite analytic type even for more general settings. We review that construction.

Let $i = 1,2$ and $\alpha_i = e^{\mathrm{i}\theta_i} \in \mathbb{R}/\pi\mathbb{Z}$. We have a decomposition of R into the ϕ-cylinders $W_1^i, \ldots, W_{n_i}^i$ in direction θ_i. For each i,j, an analytic continuation of local inverse of ϕ-coordinates gives a holomorphic covering $F_j^i : S_j^i \to W_j^i$ on a strip region $S_j^i = \{0 < \mathrm{Im} z < h_j^i\} \subset \mathbb{C}$ and $\mathrm{Deck}(F_j^i) = \langle z \mapsto z + c_j^i \rangle$ for some $h_j^i, c_j^i > 0$ (see Fig. 5.D). We denote by z_j, w_j the local ϕ-coordinates in W_j^1, W_j^2. By construction, $F_j^{1*}(\alpha_i\phi) = {dz_j}^2$ and $F_j^{2*}(\alpha_i\phi) = {dw_j}^2$ hold.

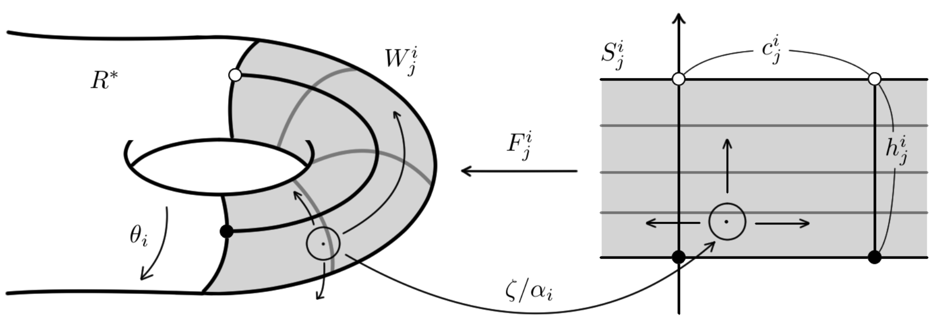

Fig. 5.D The ϕ-cylinder W_j^i and the covering F_j^i.

For any $p \in S_j^1$, there is a neighborhood U in which $F_j^1 = F_k^2 \circ f$ for some k and some holomorphic function $f : U \to S_k^2$. By the formula

$$f^* {dw_k}^2 = f^*(F_k^{2*}(\alpha_2\phi)) = F_j^{1*}(\alpha_2\phi) = (\alpha_2/\alpha_1){dz_j}^2, \tag{5.7}$$

f is continuated on S_j^1 by the form $f(z_j^1) = \alpha z_j^1 + \beta$ where $\alpha = \pm\sqrt{\alpha_2/\alpha_1}$ and $\beta \in \mathbb{C}$ (see Fig. 5.E). The intersection $V_{j,k} = S_j^1 \cap f^{-1}(S_k^2)$ is a parallelogram isometrically mapped to $W_j^1 \cap W_k^2$. The collection $(V_{j,k})_{j=1}^{n_1}$ fills the strip region S_j^1 by translations in $\mathrm{Deck}(F_j^1) = \langle z \mapsto z + c_j^1 \rangle$ $(j = 1, \ldots, n_1)$. The same can be said for $(f^{-1}(V_{j,k}))_{k=1}^{n_2}$ filling the strip region S_k^2.

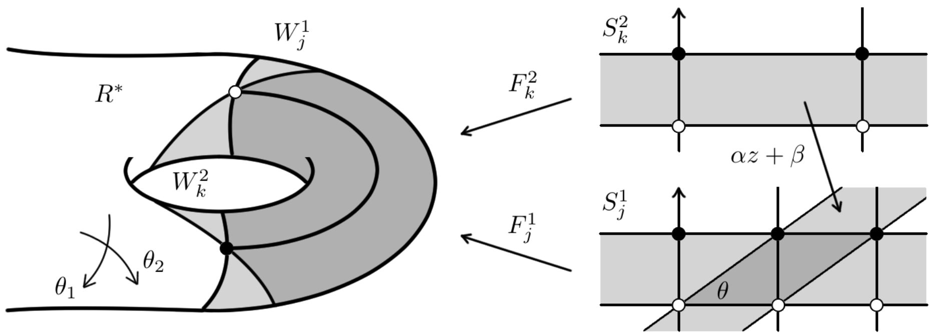

Fig. 5.E A parallelogram as an intersection of two ϕ-cylinders.

Thus the surface R is decomposed into the collection of regions $(W_j^1 \cap W_k^2)_{j,k}$, each of which is empty or isomorphic to a parallelogram on the plane. Suppose (j,k) in the latter case. Such a parallelogram $V_{j,k}$ is uniquely determined up to half-translations. We call them the (θ_1, θ_2)-*parallelograms* of (R, ϕ). Via F_j^1 and F_k^2, the isomorphism between $W_j^1 \cap W_k^2$ and $V_{j,k}$ is continued over the boundary. Thus (R, ϕ) is isomorphic to the surface obtained by gluing (θ_1, θ_2)-parallelograms along boundary edges in the way that respects the adjacencies determined by the continuations of the local isomorphisms.

A (θ_1, θ_2)-parallelogram $V_{j,k}$ has boundary edges in the directions θ_1, θ_2 and a modulus $M(V_{j,k}) = (h_j^1/h_k^2)\sin|\theta_1 - \theta_2|$. On the plane, an affine map with derivative $A \in SL(2,\mathbb{Z})$ maps a (θ_1, θ_2)-parallelogram to an an $(A\theta_1, A\theta_2)$-parallelogram whose modulus is a scalar multiple of $\rho_{A,\theta_1,\theta_2} := |T_A(e^{i\theta_2})|/|T_A(e^{i\theta_1})|$. The same holds for (R, ϕ) and we have the following in connection with Lemma 4.3.3.

Lemma 5.2.1. *Let $(R, \phi) \in \mathcal{Q}_g$, θ_1, θ_2 be two distinct directions in $J(R, \phi)$, and $\{V_i\}_{i=1}^d$ be the (θ_1, θ_2)-parallelograms of (R, ϕ). If $[A] \in PSL(2,\mathbb{R})$ belongs to $\Gamma(R, \phi)$ then $M(f(V_j)) = \rho_{A,\theta_1,\theta_2}(M(V_1)$ holds for $j = 1, \ldots, d$.*

Stretching and rotating ϕ-cylinders leads to a homeomorphism from (R, ϕ) to an origami which respects the markings determined by boundaries of parallelograms. In this way, (R, ϕ) and $\theta_1, \theta_2 \in J(R, \phi)$ determines a unique origami with additional data of moduli list $\mathbf{M} = (M_i)_{i=1}^d$ of the (θ_1, θ_2)-parallelograms and directions θ_1, θ_2. Conversely, an origami

$\mathcal{O}$ and a moduli list $\mathbf{M} = (M_i)_{i=1}^d$ compatible with $\mathcal{O}$ is supposed to give a flat surface with a decomposition as above for each pair (θ_1, θ_2) of distinct directions asigined.

Recall that an origami can be seen as a dessin given by a pair of arbitrary $\mu, \nu \in \bar{\mathfrak{S}}_d$, by Theorem 5.1.7. We will define the compatibility of $\mathbf{M} = (M_i)_{i=1}^d \in \mathbb{R}_{>0}^d$ with an origami $\mathcal{O} = (\mu, \nu)$, which purposes that we can glue d rectangles $V_1, \ldots, V_d$ with $M(V_i) = M_i$ along edges to form a flat surface (R, ϕ) in the same way as $\mathcal{O}$.

Let $|\kappa| = i$ for each $\kappa = \pm i_\cdot \in \{\pm 1_h, \ldots, \pm d_h\} \sqcup \{\pm 1_v, \ldots, \pm d_v\}$. Then $|\mu(\kappa)|$ ($|\nu(\kappa)|$, respectively) represents the rectangle adjacent to the right (upper, respectively) side of $|\kappa|$-th rectangle. Then the lengths of their horizontal (vertical, respectively) edges should be related by a factor of $K_{\kappa,\mu} = M_{|\kappa|}/M_{|\mu(\kappa)|}$ ($K_{\kappa,\nu} = M_{|\nu(\kappa)|}/M_{|\kappa|}$, respectively). When we go along a path γ on R^* joining two rectangles, indices of rectangles we pass through and directions of entry are interpreted as a path in the bipartite graph $\beta^{-1}([0,1])$. It is described in terms of monodromy of the form $\left(\iota^{k_1}\sigma\right) \cdots \left(\iota^{k_m}\sigma\right) \in \bar{\mathfrak{S}}_d$, which is a word of $\iota^k \sigma_k$ $(k = 0, 1, 2, 3)$. We may set starting edge as $+i_h$, then we have $\sigma_k = \mu$ for $k_j = 0, 2$ and $\sigma_k = \nu$ for $k_j = 1, 3$. We define as follows.

$$K_{\mathcal{O}}(\gamma, \mathbf{M}) := \prod_{j=1}^{m} K_{(\iota^{k_1}\sigma)\cdots(\iota^{k_j}\sigma)(+i_h),\ \sigma_{k_j}} \tag{5.8}$$

Definition 5.2.2. Let $\mathcal{O} = (\mu, \nu)$, $\mathcal{O}_i = (\mu_i, \nu_i)$ be origamis of degree d.

(1) We call $\mathbf{M} = (M_i)_{i=1}^d \in \mathbb{R}_{>0}^d$ a *moduli list compatible with* $\mathcal{O}$ if $K_{\mathcal{O}}(\gamma, \mathbf{M}) = 1$ for any $\gamma \in \pi_1(\mathcal{O}^*)$. ($\mathcal{O}^*$ is the flat surface $\mathcal{O}$ punctured at all the singularities.)

(2) Let $\mathbf{M}_i = (M_i^i)_{i=1}^d \in \mathbb{R}_{>0}^d$ be a moduli list compatible with $\mathcal{O}_i$ for $i = 1, 2$. We say that $(\mathcal{O}_1, \mathbf{M}_1)$ and $(\mathcal{O}_2, \mathbf{M}_2)$ are equivalent if there exists $\tau \in \bar{\mathfrak{S}}_d$ such that the following holds for $i = 1, \ldots, d$.

$$\mu_1 = \tau^* \mu_2,\ \nu_1 = \tau^* \nu_2,\ M_i^1 = M_{|\tau(i)|}^2 \tag{5.9}$$

Observe that an isomorphism between two flat surfaces with two finite Jenkins-Strebel directions naturally induces an equivalence between two origamis with compatible moduli lists. The mapping $K_{\mathcal{O}}(\,\cdot\,, \mathbf{M})$ defines a group homomorphism $\pi_1(\mathcal{O}^*) \to \mathbb{R}_{>0}$. The compatibility of lengths of the rectangles placed along a path γ on R^* fails only when γ contains a loop. So we may determine the compatibility from finite generator of $\pi_1(\mathcal{O}^*)$.

From above, we can conclude the following.

Theorem 5.2.3. *Let $\theta_1, \theta_2 \in \mathbb{R}/\pi\mathbb{Z}$ be two distinct directions. A flat surface (R, ϕ) such that $\theta_1, \theta_2 \in J(R, \phi)$ is up to equivalence uniquely determined by an origami with a compatible moduli list.*

We say that a flat surface (R,ϕ) is *origami-like* if $J(R,\phi)$ has cardinality at least 2. Let $\mathcal{Q}^{2JS}$ be the symbol that assign the set of origami-like flat surfaces in place of $\mathcal{Q}$ in Definition 4.1.2. For each $(R,\phi) \in \mathcal{Q}_g^{2JS}$ and $\theta_1, \theta_2 \in J(R,\phi)$, let $P(R,\phi,\theta_1,\theta_2)$ be the origami with compatible moduli list given by Theorem 5.2.3. For two distinct directions $\theta_1, \theta_2 \in \mathbb{R}/\pi\mathbb{Z}$ and an origami $\mathcal{O} = (\mu,\nu) \in \tilde{\Omega}_d$, consider the set

$$\mathcal{Q}^{2JS}_{\theta_1,\theta_2}(\mathcal{O}) = \left\{(R,\phi) \in \mathcal{Q}_g^{2JS} \;\middle|\; \theta_1,\theta_2 \in JS(R,\phi),\;\; P(R,\phi,\theta_1,\theta_2) = (\mathcal{O},\cdot)\right\}. \tag{5.10}$$

The maping $K_{\mathcal{O}}(\gamma,\cdot)$ can be regarded as a linear map via the conjugation by the logarithm. By taking a basis of the fundamental group $\pi_1(R^*,\cdot)$, we obtain an integer matrix $A_{\mathcal{O}}$ with d rows representing a finite system of linear equations to ensure compatibility. Thus we may define

$$o : \mathbb{R}^d \supset \ker A_{\mathcal{O}} \to \mathcal{Q}^{2JS}_{\theta_1,\theta_2}(\mathcal{O}) : (x_1,\ldots x_d) \mapsto (\mathcal{O},(e^{x_1},\ldots e^{x_d})). \tag{5.11}$$

By Theorem 5.2.3, the mapping o is bijective up to the factor $\mathrm{Stab}_{\mathcal{O}} := \mathrm{Cent}_{\bar{\mathfrak{S}}_d}\langle\mu,\nu\rangle$. The group $\mathrm{Stab}_{\mathcal{O}}$ equals the automorphism group of the natural dessin $(\iota,\sigma_{\mu,\nu})$ of origami $\mathcal{O}$. In particular, the list of isomorphism classes of $\mathrm{Stab}_{A\mathcal{O}}$, $A \in PSL(2,\mathbb{Z})/\Gamma(\mathcal{O})$ is a $G_{\mathbb{Q}}$-invariant of the Teichmüller curve $C(\mathcal{O})$. Note that the group $\mathrm{Stab}_{\mathcal{O}}$ is invariant under the deformations by two matrices J, S since their images are the origamis given by $(\nu\circ\iota,\mu)$, $(\mu\circ\iota,\nu)$ respectively.

As the group $\mathrm{Stab}_{\mathcal{O}}$ trivially conjugates the mapping $K_{\mathcal{O}}(\gamma,\cdot)$, it acts on $\mathbb{R}^d$ by permutations of coordinates compatible with $A_{\mathcal{O}}$. We summarize as follows.

Corollary 5.2.4. *Let $\theta_1,\theta_2 \in \mathbb{R}/\pi\mathbb{Z}$ be two distinct directions and $\mathcal{O} = (\mu,\nu)$ be an origami of degree d. Then, the family $\mathcal{Q}^{2JS}_{\theta_1,\theta_2}(\mathcal{O})$ is globally parametrized in $\ker A_{\mathcal{O}}/\mathrm{Stab}_{\mathcal{O}}$.*

The group $\mathrm{Cent}_{\mathrm{Sym}(\bar{I}_d)}\langle\mu,\nu\rangle$ is the automorphism group of the (possibly disconnected) dessin (μ,ν) of degree $2d$. The graph of (μ,ν) is the disjoint union of cycle graphs (Example 7) each components of which corresponds to the ϕ-cylinders in the direction $[\frac{3}{2}\pi] \in \mathbb{R}/\pi\mathbb{Z}$, as shown in Fig. 5.F. The automorphism group $\mathrm{Aut}(\mu,\nu)$ is generated by finitely many groups of the form $\{\pm 1\} \rtimes C_l$, $l \in \mathbb{N}$ and permutations of cycles of the same lengths. The group $\mathrm{Stab}_{\mathcal{O}}$ is given by the intersection $\mathrm{Aut}(\mu,\nu) \cap \bar{\mathfrak{S}}_d$.

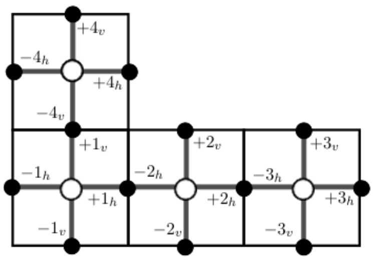

Fig. 5.F The dessin (μ, ν) is the disjoint union of cycle graphs given by the ϕ-cylinders in the direction $[\frac{3}{2}\pi] \in \mathbb{R}/\pi\mathbb{Z}$. We identify the two edges κ_h, κ_v for each $\kappa \in \bar{I}_d$.

5.3 Veech groups in terms of origamis

Corollary 5.3.1 (to Theorem 5.2.3)**.** *Let $(R, \phi) \in \mathcal{Q}^{2JS}$ be an origami-like flat surface with $\theta_1, \theta_2 \in J(R, \phi)$. $[A] \in PSL(2, \mathbb{R})$ belongs to $\Gamma(R, \phi)$ if and only if the following holds.*

(1) *$A\theta_1, A\theta_2$ belongs to $J(R, \phi)$.*

(2) *Let $(\mathcal{O}, \mathbf{M})$, $(\mathcal{O}_A, \mathbf{M}_A)$ be the origamis with compatible moduli lists given by the decomposition of (R, ϕ) in (θ_1, θ_2) and $(A\theta_1, A\theta_2)$ respectively. Then $(\mathcal{O}, \mathbf{M})$ is equivalent to $(\mathcal{O}_A, \rho_{A,\theta_1,\theta_2} \cdot \mathbf{M}_A)$.*

Proof. An affine map f on (R, ϕ) with derivative $[A]$ maps the (θ_1, θ_2)-parallelograms to the $(A\theta_1, A\theta_2)$-parallelograms with their adjacency preserved. As their moduli change by the constant multiple in the way described in Lemma 5.2.1, the equivalence follows. Conversely, (a) and (b) imply that (R, ϕ) is represented by two flat surfaces one of which is obtained from the other by the natural affine deformation with derivative $[A]$. □

For abelian origamis, the *quaternion origami* and the *Ornithorynque origami* in Fig.5.G are known as nontrivial origamis of small degree (8 and 12, respectively) with the maximal Veech group $SL(2, \mathbb{Z})$. The quaternion origami has been studied for its intrinsic properties in the moduli space [26, 27, 46]. The Ornithorynque origami was focused on [13, 44] in the context of the Teichmüller geodesic flow, the genodesic flow in the Teichmüller space defined by the contractive affine deformation of the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $t \in \mathbb{R}$.
The following is obtained by the calculation that we will state in Section 6.1.

Proposition 5.3.2. *The origami $\mathcal{D} = ((1, 2, 3, 4, 5, 6), (1, 2, 5, 6, 3, 4), (-, +, -, +, -, +))$ in Fig.5.H is the unique nontrivial origami with the maximal Veech group $PSL(2, \mathbb{Z})$ of the smallest degree* 6*, which is non-abelian. The canonical double of $\mathcal{D}$ is the Ornithorynque origami.*

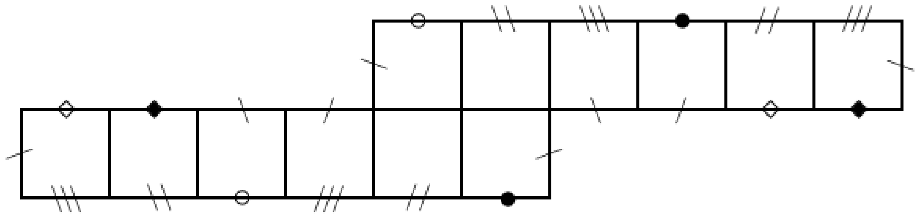

Fig. 5.G The Ornithorynque origami.

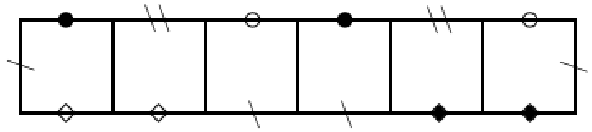

Fig. 5.H The origami $\mathcal{D}$: $(x, y, \varepsilon) = ((1, 2, 3, 4, 5, 6), (1, 2, 5, 6, 3, 4), (-, +, -, +, -, +))$.

Definition 5.3.3. Let (R, ϕ) and S, ψ be origami-like flat surfaces. We say that a finite branched covering $f : (S, \psi) \to (R, \phi)$ is an *unbranched covering of origami-like flat surfaces* if $\psi = f_* \phi$ and $\mathrm{Crit}(f) \subset f^{-1}(\mathrm{Crit}(\phi)) \subset \mathrm{Crit}(\psi)$ holds.

The condition $\mathrm{Crit}(f) \subset f^{-1}(\mathrm{Crit}(\phi))$ implies that f branches at most over the singularities of (R, ϕ). The condition $f^{-1}(\mathrm{Crit}(\phi)) \subset \mathrm{Crit}(\psi)$ implies that no singularity on (R, ϕ) is canceled when pulled back. The latter can be replaced by removing all such points, the doubled points of singularities of order -1 on (R, ϕ). For flat surfaces in covering relation, the commensurability of the Veech groups is known [19]. More strongly, the following holds in our situation.

Lemma 5.3.4. *Let $f : (S, \psi) \to (R, \phi)$ be an unbranched covering of origami-like flat surfaces. Then $\Gamma(S, \psi)$ is a finite index subgroup of $\Gamma(R, \phi)$.*

Proof. Let $\theta_1, \theta_2 \in J(R, \phi)$, $p \in R \setminus \mathrm{Crit}(\phi)$, and γ be a closed ϕ-geodesic in the direction θ_1 through p. Then any lift of γ is a ϕ-geodesic joining points in $f^{-1}(p)$ in the direction θ_1. Finite collection of such lifts form a closed ψ-geodesic and any closed ψ-geodesic is of this form. Since $\psi = f_* \phi$ where no singularity on (R, ϕ) is canceled, any pullbacks of a ϕ-cylinder are not laminated together to make a wider cylinder. Thus $J(R, \phi) = J(S, \psi)$ and each (θ_1, θ_2)-parallelogram are invariant on the plane.

Let $\mathcal{O}_R$ ($\mathcal{O}_S$, respectively) the origami determined by the decomposition $P(R, \phi, \theta_1, \theta_2)$ ($P(S, \psi, \theta_1, \theta_2)$, respectively). We can see that $\mathcal{O}_S$ is obtained from finite copies of $\mathcal{O}_R$ by regluing along their edges according to the monodromy of f. Furthermore f induces a projection from $\mathcal{O}_S$ to $\mathcal{O}_R$ which respects adjacency of squares up to the copies. So if (S, ψ)

satisfies the condition (b) in Corollary 5.3.1, then the same holds for (R,ϕ). Conversely, for $[A]\in\Gamma(R,\phi)$, the origami determined by (S,ψ) with $(A\theta_1, A\theta_2)$ is similarly constructed as $\mathcal{O}_S$ up to difference of monodromy. As it has finitely many possibilities, it coincides with $\mathcal{O}_S$ up to finite representatives. The same can be said for the decomposition of (R,ϕ) into parallelograms. □

Theorem 5.3.5. *Let $f:(S,\psi)\to(R,\phi)$ be an N-fold, unbranched covering of origami-like flat surfaces with $(\theta_1,\theta_2)\in J(R,\phi)$. Fix a base point $p\in R^*$ and a generating system $\mathcal{F}$ of $\pi_1(R^*,\cdot)$. Define the action of the Veech group $\Gamma(R,\mu)<PSL(2,\mathbb{R})$ on $\mathcal{M}=(\mathfrak{S}_N)^{\mathcal{F}}$ so that $[A]\in\Gamma(R,\mu)$ transforms the monodromy of f by taking the new decomposition in $A^{-1}(\theta_1,\theta_2)$. Then, $\Gamma(\hat{R},\psi)$ is the stabilizer of $\tau_f=m_f(\mathcal{F})\in\mathcal{M}$ under the equivalence defined by*

(1) *relabeling of sheets of f (i.e. conjugacy in $\mathfrak{S}_N$), and*

(2) *simultaneous conjugation in* $\mathrm{Stab}_{\mathcal{O}_R}$.

Proof. As we have seen in the proof of Lemma 5.3.4, it follows that any lift of $\gamma\in\pi_1(R^*,\cdot)$ connects two copies of $P(R,\phi,A^{-1}\theta_1,A^{-1}\theta_2)=P(R,\phi,\theta_1,\theta_2)$ in $P(S,\psi,A^{-1}\theta_1,A^{-1}\theta_2)$ for each $[A]\in\Gamma(R,\phi)$. One obatins the decomposition $P(S,\psi,A^{-1}\theta_1,A^{-1}\theta_2)$ by patching copies of $P(R,\phi,\theta_1,\theta_2)$ according to the new monodromy data $[A]\tau_f$. It follows from Corollary 5.3.1 that the stabilizer represents the Veech group. □

Example 9. Let $f:\mathcal{O}\to\mathcal{D}$ be an N-fold, unbranched covering of the origami $\mathcal{D}$ in Proposition 5.3.2. Then, τ_f runs over any element of $\mathcal{M}=(\mathfrak{S}_N)^7$, $\mathrm{Stab}_{\mathcal{D}}=\langle(1\,3\,5)(2\,4\,6)\rangle\cong C_3$, and the action of $\Gamma(\mathcal{D})=PSL(2,\mathbb{Z})=\langle[T],[S]\rangle$ on $\mathcal{M}$ is defined by

$$[T](\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6)=(\tau_0,\tau_1,\tau_2,\tau_3,\tau_5,\tau_6,\tau_4^{-1}\tau_0^{-1},), \tag{5.12}$$

$$[S](\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6)=(\tau_2^{-1}\tau_6\tau_3^{-1}\tau_5^{-1}\tau_1^{-1}\tau_4,\tau_2,\tau_3,\tau_1,\tau_3\tau_6^{-1}\tau_2,\tau_5\tau_3\tau_6^{-1},\tau_1\tau_5\tau_3\tau_0), \tag{5.13}$$

for each $(\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\tau_6)\in\mathcal{M}$. The Veech group $\Gamma(\mathcal{O})$ is the stabilizer of the equivalence class of τ_f under the action of $PSL(2,\mathbb{Z})$.

The formulae (5.12) and (5.13) are obtained as follows. First, fix a generating system $\mathcal{F}=\{\tau_i\}_{i=0}^6$ of $\pi_1(R^*,\cdot)$ and label the cells of the origami $\mathcal{D}$ as shown in Fig. 5.I. Then, if we fix directions $(0,\frac{\pi}{2})$ of decomposition, the covering $f:\mathcal{O}\to\mathcal{D}$ is uniquely determined by a monodromy $\tau_f\in(\mathfrak{S}_N)^7$ up to equivalence mentioned in Theorem 5.3.5. For a matrix $[A]\in\Gamma(\mathcal{D})=PSL(2,\mathbb{Z})$, $A=T^{-1},S$, the decomposition $P(\mathcal{O},A(0,\frac{\pi}{2}))$ is tiled by the decomposition $[A]\mathcal{D}=P(\mathcal{D},A(0,\frac{\pi}{2}))\cong\mathcal{D}$ as shown in Fig. 5.J and 5.K. It is also uniquely determined by some monodromy $[A^{-1}]\tau_f \in (\mathfrak{S}_N)^7$ up to equivalence. By labelling parallelogram cells in the directions $A(0, \frac{\pi}{2})$ according to sheets of the original decomposition $P(\mathcal{O}, (0, \frac{\pi}{2}))$, we obtain the transformation $\tau_f \mapsto [A^{-1}]\tau_f$ by the formulae (5.12) and (5.13).

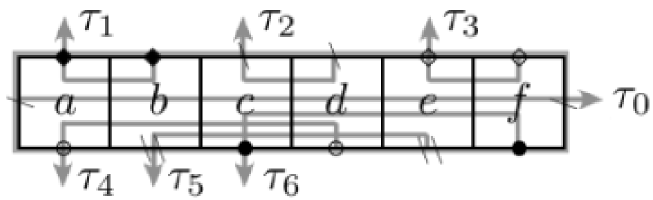

Fig. 5.I Fixed generating system $\mathcal{F} = \{\tau_i\}_{i=0}^{6}$ of $\pi_1(R^*, \cdot)$ and labeling of cells of $\mathcal{D}$.

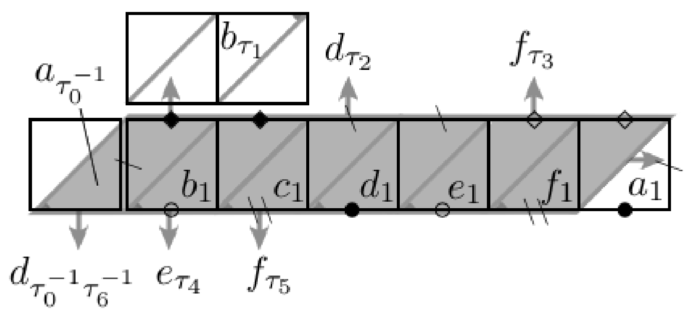

Fig. 5.J The decomposition $P(\mathcal{O}, T^{-1}(0, \frac{\pi}{2}))$ is tiled by $[T^{-1}]\mathcal{D} \cong \mathcal{D}$ (shaded). Each parallelogram cells are labelled according to the sheets of $\mathcal{D}$. We obtain the formula $[T^{-1}](\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = (\tau_0, \tau_1, \tau_2, \tau_3, \tau_0^{-1}\tau_6^{-1}, \tau_4, \tau_5)$, which leads to the formula (5.12).

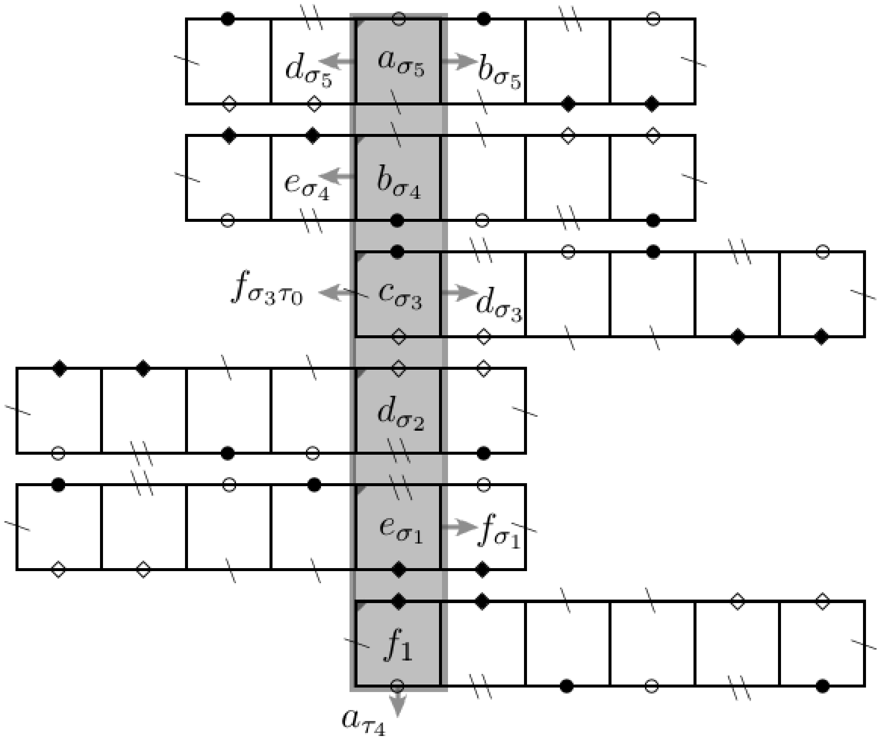

Fig. 5.K The decomposition $P(\mathcal{O}, S(0, \frac{\pi}{2}))$ is tiled by $[S]\mathcal{D} \cong \mathcal{D}$ (shaded). Each parallelogram cells s are labelled according to the sheets of $\mathcal{D}$. The symbols $\sigma_\bullet$ represent permutations as follows: $\sigma_1 = \tau_1$, $\sigma_2 = \sigma_1\tau_5$, $\sigma_3 = \sigma_2\tau_3$, $\sigma_4 = \sigma_3\tau_6^{-1}$, and $\sigma_5 = \sigma_4\tau_2$. We obtain the formula (5.13).

Chapter 6

Calculation on origamis

This chapter is based on [40]. Throughout this chapter, an *origami* refers to a general origami.

6.1 Classification of origamis into components of Teichmüller curves

This section observes Theorem 5.1.9 and states a concrete procedure for implementation.

A *partition* of d is a finite sequence of weakly decreasing positive integers that sum to d. The *partition number* $p(d)$, which counts the number of partitions of d, defines the following rapidly increasing sequence.

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, \ldots$$

(cf. http://oeis.org/A000041.) The following asymptotic formula [20] is known:

$$p(d) \quad \sim \tfrac{1}{4d\sqrt{3}} \cdot e^{\sqrt{2d/3}}.$$

The algorithm in [22] constructs all partitions of given integer. We will accept $P(d) = \{(j_1, j_2, ..., j_d) : \text{partition of } d\}$ as a known data.

To describe the isomorphism class of each origami $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$, we enumerate all the conjugators $\bar{\sigma} = \delta\sigma \in \bar{S}_d$ ($\delta = \mathrm{sign}(\bar{\sigma}) \in \{\pm 1\}^{\bar{I}_d}$, $\sigma \in S_d$) satisfying the conditions in Lemma 5.1.6. By (2), up to isomorphisms, we only have to think of x with the normalized cycle decompositions

$$x = (1 \ldots j_1)(j_1 + 1 \ldots j_1 + j_2) \ldots (j_1 + 1 \ldots \sum_{k=1}^{n} j_k = d) \tag{6.1}$$

according to the partition $(j_1, j_2, \ldots j_n) \in P(d)$ determined by its cycle lengths.

We will consider the *restricted class* of an origami, the set of origamis with the same 'x' and isomorphic to it. By Lemma 5.1.6, the restricted class is the conjugacy class in $\mathrm{Stab}(x) := \{\bar{\sigma} = \delta\sigma \in \bar{S}_d \mid \delta = \delta \circ x \text{ and } x = \sigma^{\#}(x^{\delta}) \text{ on } I_d\}$. Remark that for general $y \in \bar{S}_d$ and $\varepsilon \in \mathcal{E}_d$, the mapping y^{ε} does not belong to $\bar{S}_d$. It will be confirmed in (2) of Algorithm 6.1.1.

First, we present an algorithm for describing the $\mathrm{Stab}(x)$-conjugacy class of (y, ε) for each $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$ satisfying the conditions in Lemma 5.1.6.

Algorithm 6.1.1. For each $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$, we construct its restricted class $[\mathcal{O}] = \{(x, y', \varepsilon') \in \tilde{\Omega}_d \mid (x, y', \varepsilon') \sim (x, y, \varepsilon)\}$ in the following steps:

(1) Take $\bar{\sigma} = \delta\sigma \in \mathrm{Stab}(x)$: with $\delta = \delta \circ x$ and $x = \sigma^{\#}(x^{\delta})$ on I_d.

(2) For each $\varepsilon' \in \mathcal{E}_d$, let $y_{\bar{\sigma},\varepsilon'} := \sigma^{\#}(y^{\varepsilon\cdot\varepsilon'\circ\sigma\cdot\delta})$. Verify $\varepsilon' \in \mathcal{E}_d$ such that $y_{\bar{\sigma},\varepsilon'} \in S_d$ and $\xi(y_{\bar{\sigma},\varepsilon'}, \delta \circ \sigma^{-1} \cdot \varepsilon \circ \sigma^{-1} \cdot \varepsilon') = 1$ on I_d.

(3) Let $C_{\bar{\sigma}} := \{(x, y_{\bar{\sigma},\varepsilon'}, \varepsilon') \mid \varepsilon' \text{ passes the test in (2)}\}$.

(4) Go back to (1) for some other leftover $\sigma \in \mathrm{Stab}(x)$. When we have been through all elements in $\mathrm{Stab}(x)$, finish the algorithm and we conclude that $[\mathcal{O}] = \bigcup_{\bar{\sigma}\in\mathrm{Stab}(x)} C_{\bar{\sigma}}$.

Algorithm 6.1.2. Let $P(d) = \{(j_1, j_2, ..., j_d) : \text{partition of } d\}$. We obtain the set $C\tilde{\Omega}_d$ of the restricted classes of all origamis of degree d in the following steps.

(1) $C\tilde{\Omega}_d := \emptyset$

(2) Take $j = (j_1, j_2, ..., j_d) \in P(d)$. Define as follows:

$$\begin{aligned} d'_j &:= \max\{k \mid j_k > 0\}, \\ x_j &:= (1\ 2\ \ldots\ j_1)(j_1+1\ \ j_1+2\ \ldots\ j_1+j_2)\cdots(\textstyle\sum_{k=1}^{d'-1} j_k + 1\ \ldots\ d) \in S_d, \\ R_j &:= S_d \times \mathcal{E}_d. \end{aligned}$$

(3) Take $(y, \varepsilon) \in R_j$. Apply Algorithm 6.1.1 to $(x_j, y, \varepsilon) \in \tilde{\Omega}_d$ to get $[(x_j, y, \varepsilon)]$.

(4) Add $[(x_j, y, \varepsilon)]$ to $C\tilde{\Omega}_d$. After that, remove $(y(\mathcal{O}), \varepsilon(\mathcal{O}))$ from R_j for every $\mathcal{O} = (x_j, y(\mathcal{O}), \varepsilon(\mathcal{O})) \in [(x_j, y, \varepsilon)]$.

(5) Go back to (3) until $R_j = \emptyset$. If so, go to the next step.

(6) Go back to (2) for other leftover $j \in P(d)$. When we have been through all elements in $P(d)$, finish the algorithm.

Next, we calculate the permutations $\varphi_T, \varphi_S \in \mathrm{Sym}(C\tilde{\Omega}_d)$ which correspond to $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) \in PSL(2,\mathbb{Z})$ acting on $\tilde{\Omega}_d$ as decomposing origamis into pairs of directions $T(0,\frac{\pi}{2}) = (0,\frac{\pi}{4})$ and $S(0,\frac{\pi}{2}) = (-\frac{\pi}{2},0)$, respectively. Recall that the two automorphisms $\gamma_T, \gamma_S \in \mathrm{Aut}^+(F_2)$ in Lemma 4.4.3 are defined by:

$$\gamma_T(x,y) = (x,xy), \quad \gamma_S(x,y) = (y,x^{-1}). \tag{6.2}$$

Let $C\tilde{\Omega}_d$ be the output in Algorithm 6.1.2.

1. To obtain the permutation φ_T, we consider as follows:

$$(x,y,\varepsilon) \xmapsto{\text{Def.5.1.3}} (\mathbf{x},\mathbf{y}) \xmapsto{\gamma_T} \xmapsto{\text{conj.}} (\mathbf{x}_T,\mathbf{y}_T) \xmapsto{\text{Lem.5.1.5}} (x_T,y_T,\varepsilon_T).$$

To apply Lemma 5.1.5 we calculate $\varepsilon_{\mathbf{y}_T}$ and a cycle decomposition of $\mathbf{y}_T$. Remark that the decomposition into $T(0,\frac{\pi}{2}) = (0,\frac{\pi}{4})$ is given by γ_T and the conjugation in $(-i \mapsto ix^{-1}(i) \mid i \in I_d)$ as shown in Fig. 6.A.

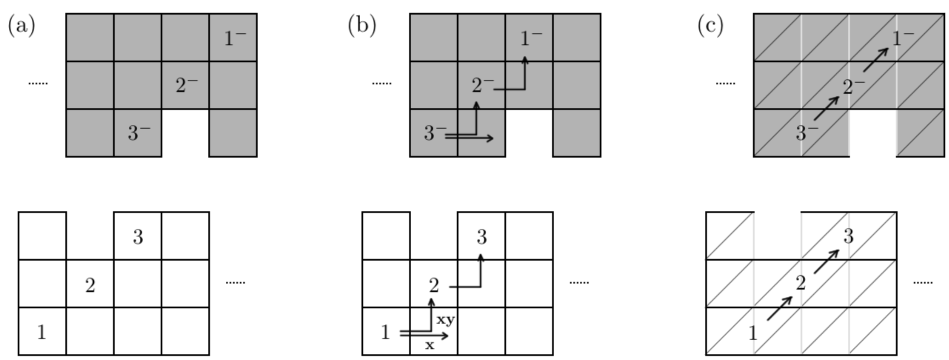

Fig. 6.A Decomposition of origami (a) into $T(0,\frac{\pi}{2}) = (0,\frac{\pi}{4})$: The disired decomposition (c) is obtained from (b) applying γ_T and the conjugation in $(-i \mapsto -x^{-1}(i) \mid i \in I_d)$.

For $\mathcal{O} = (x,y,\varepsilon) \in \tilde{\Omega}_d$, $a \in I_d$, and $\varepsilon' \in \{\pm 1\}$, we have:

$$\begin{aligned}
\gamma_T(\mathbf{y}_\mathcal{O})(\varepsilon' a) &= \mathbf{y}_\mathcal{O} \circ \mathbf{x}_\mathcal{O}(\varepsilon' a) \\
&= \mathbf{y}_\mathcal{O}(\varepsilon' x^{\varepsilon'}(a)) \\
&= \varepsilon(\varepsilon' x^{\varepsilon'}(a))) \cdot y^{\varepsilon(\varepsilon' x^{\varepsilon'}(a)))}(\varepsilon' x^{\varepsilon'}(a))) \cdot \varepsilon(y^{\varepsilon(\varepsilon' x^{\varepsilon'}(a)))}(\varepsilon' x^{\varepsilon'}(a))) \\
&= \varepsilon'\varepsilon(x^{\varepsilon'}(a))) \cdot \varepsilon' y^{\varepsilon'\varepsilon(x^{\varepsilon'}(a)))}(x^{\varepsilon'}(a)) \cdot \varepsilon'\varepsilon(y^{\varepsilon'\varepsilon(x^{\varepsilon'}(a)))}(x^{\varepsilon'}(a))) \\
&= \varepsilon'\varepsilon(x^{\varepsilon'}(a))) \cdot \varepsilon(y^{\varepsilon'\varepsilon(x^{\varepsilon'}(a)))}(x^{\varepsilon'}(a))) \cdot y^{\varepsilon'\varepsilon(x^{\varepsilon'}(a)))}(x^{\varepsilon'}(a)).
\end{aligned} \tag{6.3}$$

Algorithm 6.1.3. Let $C = [(x, y, \varepsilon)] \in C\tilde{\Omega}_d$ be a restricted class. By (6.3), we obtain $\varphi_T(C)$ in the following steps:

(1) $I'_d := I_d$, $j := 0$.

(2) $a_{0,j} := \min(I'_d)$, $\varepsilon'_{0,j} := 1$, $i := 0$.

(3) $b_{i,j} := x^{\varepsilon'_{i,j}}(a_{i,j})$, $a'_{i+1,j} := y^{\varepsilon'_{i,j}\varepsilon(b_{i,j})}(b_{i,j})$, $\varepsilon'_{i+1,j} := \varepsilon'_{i,j}\varepsilon(b_{i,j})\varepsilon(a_{i+1,j})$.

(4) Remove $a_{i,j}$ from I'_d. Define:

$$a_{i+1,j} := \begin{cases} a'_{i+1,j} & \text{if } \varepsilon'_{i+1,j} = 1, \\ x^{-1}(a'_{i+1.j}) & \text{otherwise.} \end{cases}$$

(5) If $a_{i+1,j} = a_{0,j}$, let $c_j := (a_{0,j}\ a_{1,j}\ \ldots\ a_{i,j})$. Otherwise, go back to (3) for the next i.

(6) If $I'_d \neq \emptyset$, go back to (2) for the next j. Otherwise, finish the loop and let $x_T = x$, $y_T := c_1 c_2 \cdots c_j$, and $\varepsilon_T := (a_{i,j} \mapsto \varepsilon'_{i,j})$.

(7) Seach for the isomorphism class $C_T \in C\tilde{\Omega}_d$ represented by $(x_T, y_T, \varepsilon_T)$ and we conclude that $\varphi_T(C) = C_T$.

2. To obtain the permutation φ_S, we consider as follows:

$$(x, y, \varepsilon) \xmapsto{\text{Def.5.1.3}} (\mathbf{x}, \mathbf{y}) \xmapsto{\gamma_S} \xmapsto{\text{conj.}} (\mathbf{x}_S, \mathbf{y}_S) \xmapsto{\text{Lem.5.1.5}} \xmapsto{\text{conj.}} (x_S, y_S, \varepsilon_S).$$

We use two conjugators in $\bar{S}_d$: the former collects signs of cells in each vertical cylinder to apply Lemma 5.1.5, and the latter makes x_S to be the normalized form (6.1). The former conjugator is given by $\sigma_\delta := (\pm i \mapsto \pm\delta(i)i \mid i \in I_d) \in \bar{S}_d$ where $\delta \in \mathcal{E}_d$ satisfies that for every cycle c in $\mathbf{x}$, $\{\delta(|i|)i \mid i \in c\}$ forms a cycle either c or c'.

For $\mathcal{O} = (x, y, \varepsilon) \in \tilde{\Omega}_d$, $a \in I_d$, and $\delta' \in \{\pm 1\}$, we have:

$$\begin{aligned} \gamma_S(\mathbf{x})(\delta' a) &= \mathbf{y}(\delta' a) \\ &= \varepsilon(\delta' a) \cdot y^{\varepsilon(\delta' a)}(\delta' a) \cdot \varepsilon(y^{\varepsilon(\delta' a)}(\delta' a)) \\ &= \delta'\varepsilon(a) \cdot \delta' y^{\delta'\varepsilon(a)}(a) \cdot \delta'\varepsilon(y^{\delta'\varepsilon(a)}(a)) \\ &= \delta'\varepsilon(a)\varepsilon(y^{\delta'\varepsilon(a)}(a)) \cdot y^{\delta'\varepsilon(a)}(a). \end{aligned} \tag{6.4}$$

Algorithm 6.1.4. By (6.4), we obtain δ in the following steps:

(1) $I'_d := I_d$, $j := 0$.

(2) $a_{0,j} := \min(I'_d)$, $\delta_{0,j} := 1$, $i := 0$.

(3) $a_{i+1,j} := y^{\delta_{i,j}\varepsilon(a_{i,j})}(a_{i,j})$, $\delta_{i+1,j} := \delta_{i,j}\varepsilon(a_{i,j})\varepsilon(a_{i+1,j})$

(4) Remove $a_{i,j}$ from I'_d.

(5) If $a_{i+1,j} = a_{0,j}$, let $c_j := (a_{0,j}\ a_{1,j}\ \dots\ a_{i,j})$. Otherwise go back to (3) for the next i.

(6) If $I'_d \neq \emptyset$ then go back to (2) for the next j. Otherwise finish the loop and let $x'_S := c_1 c_2 \cdots c_j$ and $\delta := (a_{i,j} \mapsto \delta_{i,j})$.

To apply Lemma 5.1.5, we will calculate $\varepsilon_{\sigma_\delta{}^\# \mathbf{y}_S}$ and a cycle decomposition of $\sigma_\delta{}^\# \mathbf{y}_S$. After that, we apply the conjugator which makes x_S to the normalized form (6.1). So in advance, we will prepare the list $\{\sigma^\# x_p \mid \sigma \in S_d\}$ equipped with information of conjugator for each $p \in P(d)$. Note that 'x's of any isomorphic two origamis share the same partition by Lemma 5.1.6. Hence the restricted classes calculated from Algorithm 6.1.1 with this list exhausts all the patterns of origamis.

For $(x, y, \varepsilon) \in \tilde{\Omega}_d$, $a \in I_d$ and $\varepsilon' \in \{\pm 1\}$, we have:

$$\begin{aligned}\delta^\#(\mathbf{y}_S)(\varepsilon' a) &= \delta(\mathbf{x}^{-1}(\delta(|\varepsilon' a|)\varepsilon' a)) \\ &= \delta(|\mathbf{x}^{-1}(\varepsilon'\delta(a)a)|) \cdot \mathbf{x}^{-1}(\varepsilon'\delta(a)a) \\ &= \varepsilon'\delta(a) \cdot \delta(x^{-\varepsilon'\delta(a)}(a)) \cdot x^{-\varepsilon'\delta(a)}(a). \end{aligned} \tag{6.5}$$

Algorithm 6.1.5. Let $C = [(x, y, \varepsilon)] \in C\tilde{\Omega}_d$ be a restricted class. By (6.5), we obtain $\varphi_S(C)$ in the following steps:

(1) $I'_d := I_d$, $j := 0$.

(2) $a_{0,j} := \min(I'_d)$, $\varepsilon'_{0,j} := 1$, $i := 0$

(3) Remove $a_{i,j}$ from I'_d. Let $a_{i+1,j} := x^{-\varepsilon'_{i,j}\delta(a_{i,j})}(a_{i,j})$, $\varepsilon'_{i+1,j} := \varepsilon'_{i,j}\delta(a_{i,j})\delta(a_{i+1,j})$.

(4) If $a_{i+1,j} = a_{0,j}$, let $c_j := (a_{0,j}\ a_{1,j}\ \dots\ a_{i,j})$. Otherwise go back to (3) for the next i.

(5) If $I'_d \neq \emptyset$, go back to (2) for the next j. Otherwise finish the loop and let $x'_S := \delta^\# x_S$, $y'_S := c_1 c_2 \cdots c_j$ and $\varepsilon'_S := (a_{i,j} \mapsto \varepsilon'_{i,j})$.

(6) Seach for the conjugator $\sigma \in S_d$ such that $\sigma^\# x'_S$ is of normalized form. Let $(x_S, y_S, \varepsilon_S) := (\sigma^\# x'_S, \sigma^\# y'_S, \varepsilon'_S \circ \sigma^{-1})$.

(7) Seach for the isomorphism class $C_S \in C\tilde{\Omega}_d$ represented by $(x_S, y_S, \varepsilon_S)$ and we conclude that $\varphi_S(C) = C_S$.

Finally, we present an algorithm for a simultaneous calculation of the Veech groups of origamis in $\tilde{\Omega}_d$.

Algorithm 6.1.6. Let $\varphi_T, \varphi_S \in \mathrm{Sym}(C\tilde{\Omega}_d)$. We obtain the $\langle \varphi_T^{-1}, \varphi_S^{-1} \rangle$-orbit decomposition of $C\tilde{\Omega}_d$ in the following steps.

(1) $I'_N := I_N$.

(2) For $t \in \mathbb{N}$, $O_t := \emptyset$.

(3) Take $i \in I'_N$ and add i to O_t.

(4) Take $j \in O_t$ and let $O(j) := \{\varphi_T^{-k}(j), \varphi_S^{-k}(j) \mid k \in \mathbb{N}\}$.

(5) Add all elements in $O(j)$ to O_t and remove them from I'_N.

(6) Go back to (4) for other leftover $j \in O_t$. When we have been through all elements in O_t, go th the next step.

(7) Go back to (2) for the next t until $I'_N = \emptyset$. If so, finish the algorithm.

Theorem 6.1.7 ([41])**.** *For each $d \in \mathbb{N}$, Algorithm 6.1.1-6.1.6 outputs the orbit decomposition of the action $PSL(2,\mathbb{Z})$ on $C\tilde{\Omega}_d$. Moreover, for each origami $\mathcal{O} \in \tilde{\Omega}_d$, the Veech group is the stabilizer* $\mathrm{Stab}_{PSL(2,\mathbb{Z})}[\mathcal{O}]$.

Proof. Let $\mathcal{O} \in \tilde{\Omega}_d$. As seen in Remark 4.4.6, the inclusion $\Gamma(\mathcal{O}) < PSL(2,\mathbb{Z})$ induces a Belyĭ covering $C(\mathcal{O}) = \mathbb{H}/\Gamma(\mathcal{O}) \to P^1_{\mathbb{C}}$. $PSL(2,\mathbb{Z})$ acts on $\tilde{\Omega}_d$ by linear deformation of the natural coordinates of origamis, which respects the $PSL(2,\mathbb{Z})$-action on the Teichmüller disk $D_{\mathcal{O}}$. Since the two decompositions $P(A\mathcal{O}, \Theta_0)$ and $P(\mathcal{O}, A^{-1}\Theta_0)$ are equal for $\Theta_0 = (0, \frac{\pi}{2})$ and every $[A] \in PSL(2,\mathbb{Z})$, the homomorphism

$$PSL(2,\mathbb{Z}) \to \mathrm{Sym}(C\tilde{\Omega}_d) : ([T],[S]) \mapsto (\varphi_T^{-1}, \varphi_S^{-1}) \tag{6.6}$$

represents the projected action on $C\tilde{\Omega}_d \hookrightarrow C_{\mathcal{O}}$. Algorithm 6.1.1 specifies the equivalence class of each origami by Lemma 5.1.6. The last part follows from Corollary 5.3.1. □

Note that we may combine the Reidemeister-Schreier method [43, 51] with Algorithm 6.1.6 to obtain the list of generators and the list of representatives of the Veech group of each origami.

6.2 Teichmüller curves and Galois conjugacy

In the following, we show some calculation results obtained by the algorithms stated in the previous section. For each degree d, we number classes of origamis according to Algorithm 6.1.2 (i.e. lexicographic order with respect to permutations and signs). We first note that all classes representing disconnected origamis are removed from the results.

We describe Teichmüller curves in the same way to [51]. Teichmüller curves of origamis are coverings of $\mathbb{H}/PSL(2,\mathbb{Z})$, and we denote copies of the standard fundamental domain of $PSL(2,\mathbb{Z})$ by isosceles triangles where the keen vertices correspond to the cusp. Every two edges with the same symbol are glued so that the cusps match. Every edge with no symbol is glued individually, making a conical point of angle $\pi/2$.

Fig.6.B and Fig.6.C show all classes of origamis of degree 4 and their positions in Teichmüller curves. There are 26 classes of abelian origamis summing up to 5 components and 34 classes of non-abelian origamis summing up to 6 components.

Table.6.D shows the number of classes of origamis, the number of components of Teichmüller curves, the range of genus of Teichmüller curves, and the number of classes of possible $G_{\mathbb{Q}}$-conjugacy for degree $1 \leq d \leq 7$. Here the possibility of $G_{\mathbb{Q}}$-conjugacy is checked by the information of degree, genus, valency list, and stratum of origamis.

	abelian				non-abelian			
d	$\#\{\mathcal{O}\}$	$\#\{C(\mathcal{O})\}$	$g(C(\mathcal{O}))$	possible $G_{\mathbb{Q}}$-conjugacy	$\#\{\mathcal{O}\}$	$\#\{C(\mathcal{O})\}$	$g(C(\mathcal{O}))$	possible $G_{\mathbb{Q}}$-conjugacy
1	1	1	0	none	0	0	0	none
2	2	1	0	〃	1	1	0	〃
3	7	2	0	〃	4	1	0	〃
4	26	5	0	〃	34	6	0	〃
5	91	8	0	〃	227	13	0	〃
6	490	28	0	1 class	2316	88	0	13 classes
7	2773	41	$0 \sim 1$	5 classes	26586	88	$0 \sim 11$	3 classes

Table 6.D Summary of the result for degree $1 \leq d \leq 7$

Theorem 6.2.1. *All Teichmüller curves induced from origamis of degree $d \leq 7$ except for the 13 cases in Table.6.E and the 9 cases in Table.6.G are distinguished by Galois invariants. Fig.6.F and Fig.6.H shows origamis that induce Teichmüller curves in each of the exceptional cases.*

Remark 6.2.2. The mirror relation implies a Galois conjugacy which induces complex conjugacy. The situation 'one pair of mirror-symmetric curves, mirroring each other' is caused by such a Galois conjugacy modifying only the embeddings of Teichmüller curves into the moduli space.

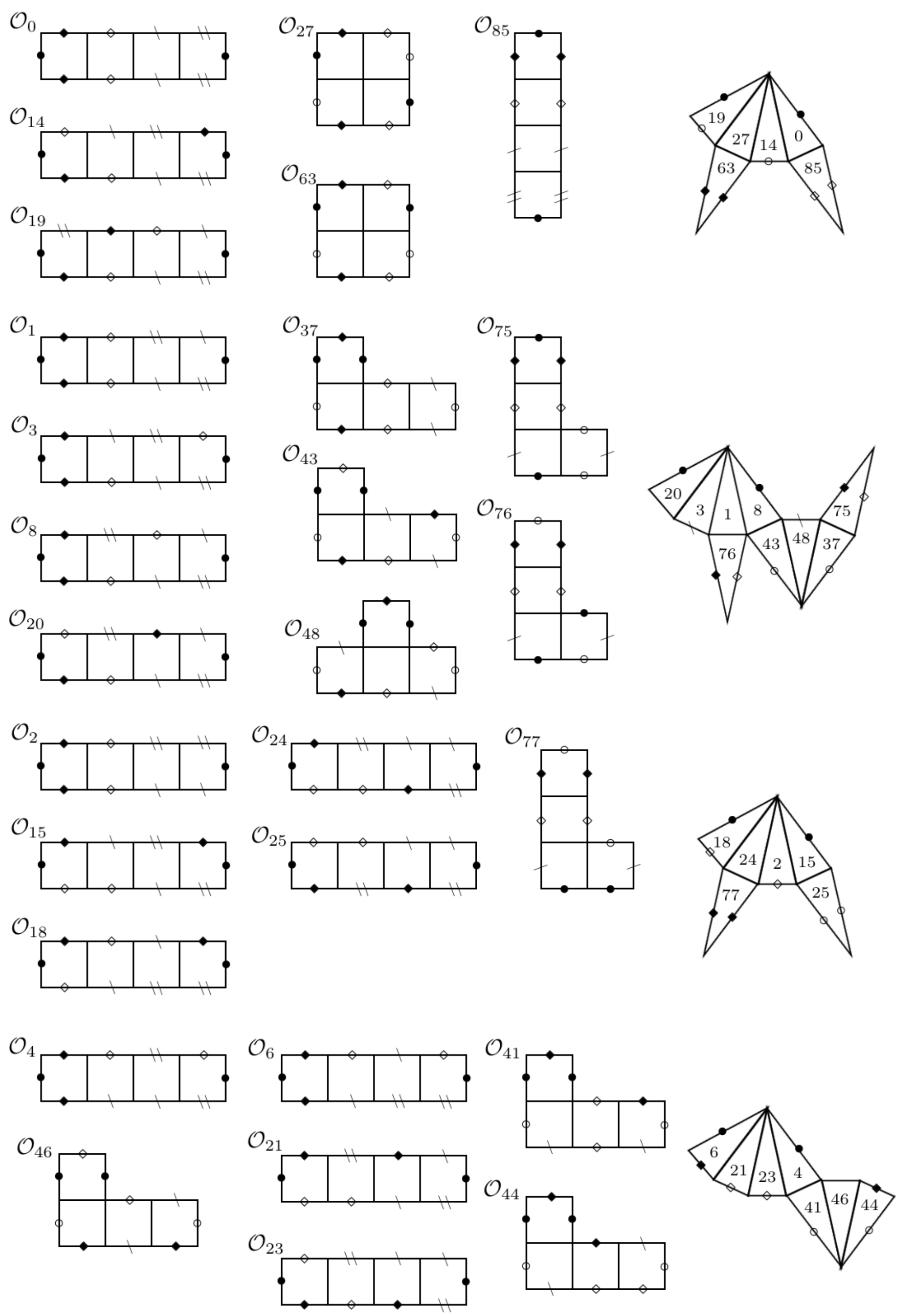

Fig. 6.B (Part 1/2) All classes of origamis of degree 4 and their positions in Teichmüller curves.

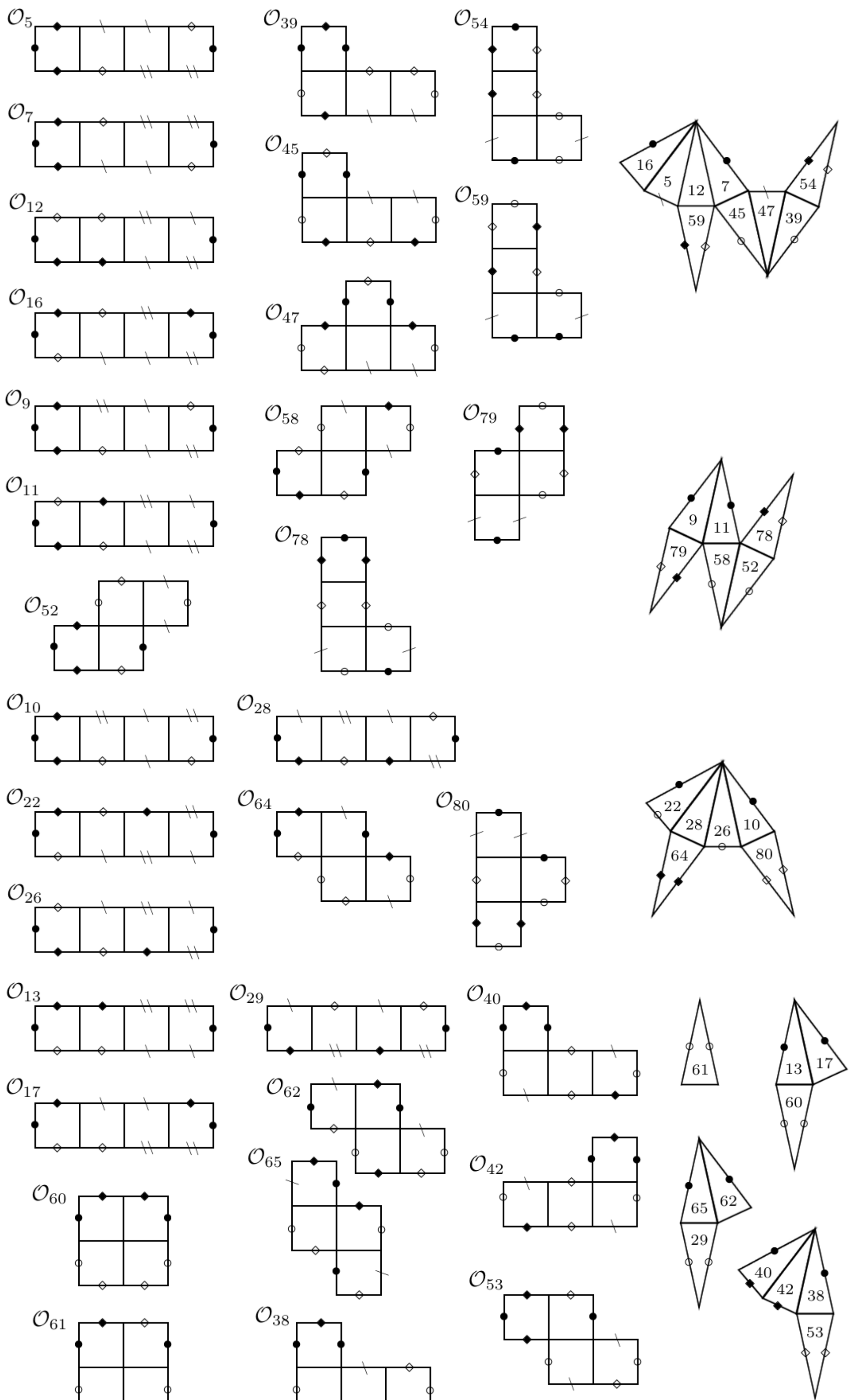

Fig. 6.C (Part 2/2) All classes of origamis of degree 4 and their positions in Teichmüller curves.

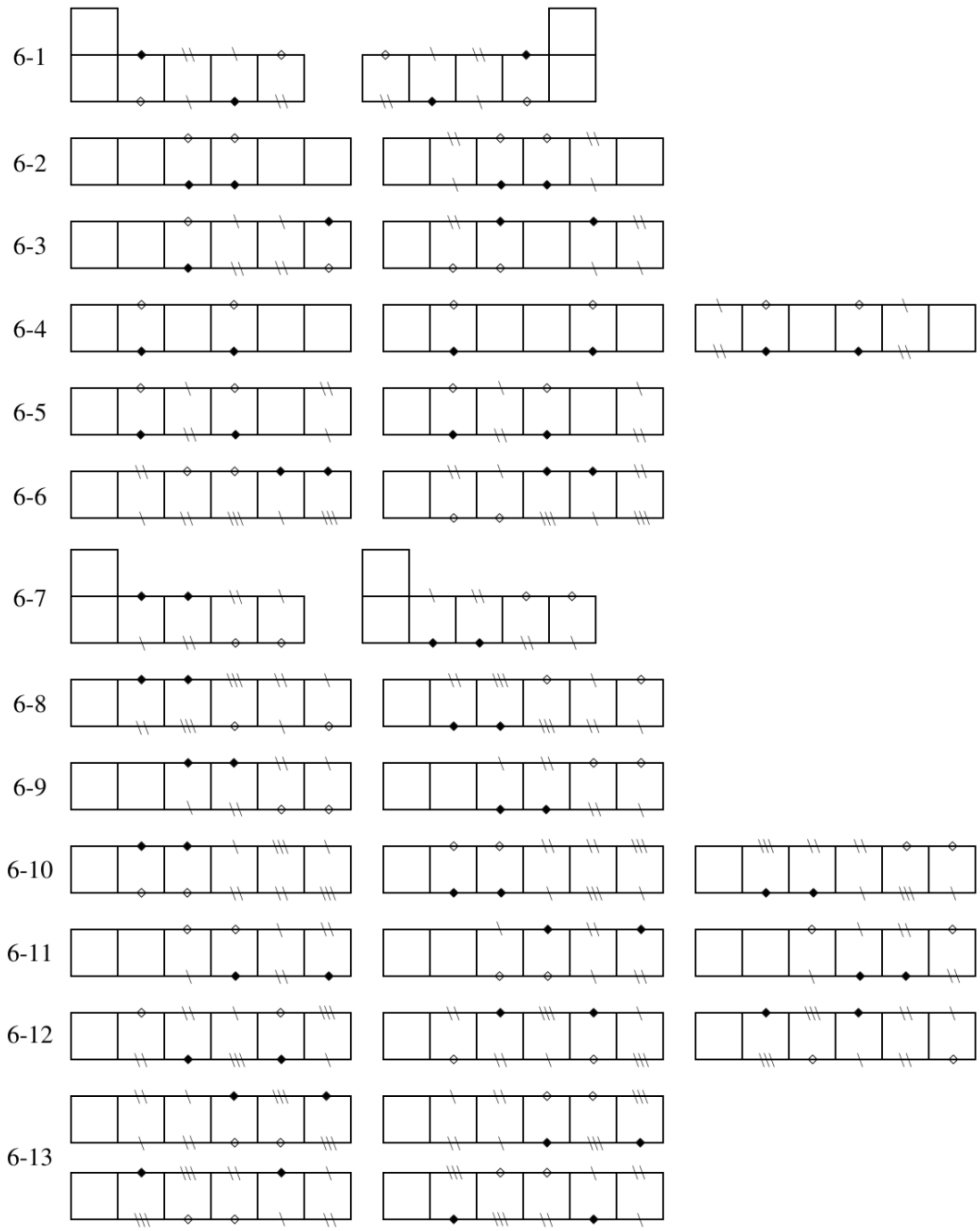

Fig. 6.F Origamis that induce Teichmüller curves in Table.6.E: unmarked edges are glued with the opposite.

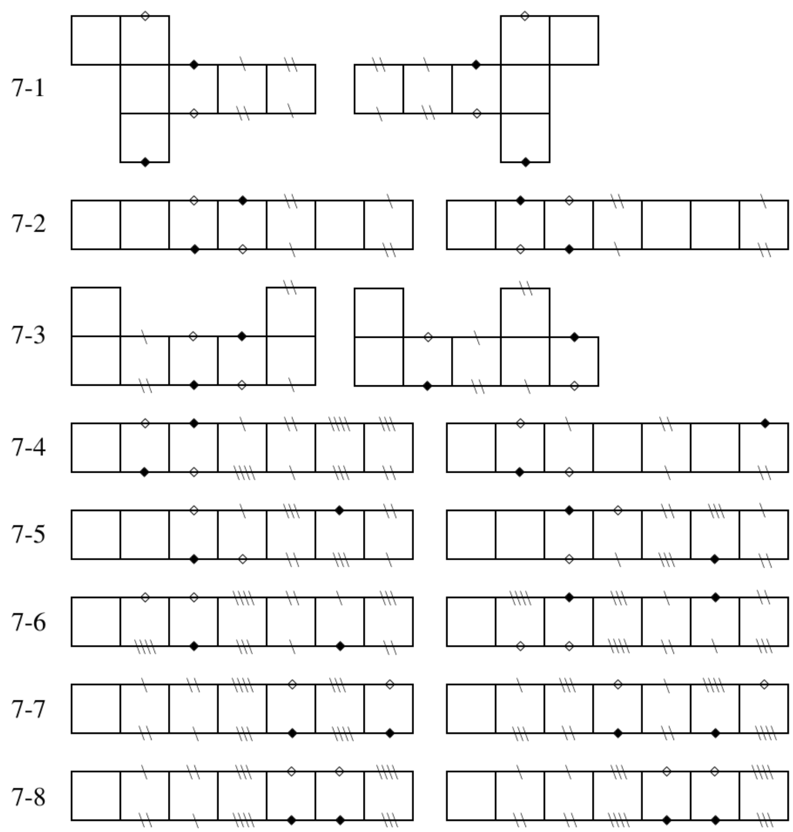

Fig. 6.H Origamis that induce Teichmüller curves in Table.6.G: unmarked edges are glued with the opposite.

No.	stratum	index	valency list of $C(\mathcal{O})$	relationship between $C(\mathcal{O})$
6-1	$\mathcal{Q}_3^a(0,8)$	15	$(3^5 \mid 2^7, 1 \mid 5, 4, 3^2)$	one pair of mirror-symmetric curves, mirroring each other
6-2	$\mathcal{Q}_1^p(-1^2, 0^3, 2)$	12	$(3^4 \mid 2^6 \mid 6, 3, 2, 1)$	two identical, mirror-closed curves
6-3	$\mathcal{Q}_2^p(-1^2, 0, 6)$	12	$(3^4 \mid 2^6 \mid 6, 3, 2, 1)$	two identical, mirror-closed curves
6-4	$\mathcal{Q}_2^p(0^2, 2^2)$	12	$(3^4 \mid 2^6 \mid 6, 3, 2, 1)$	three identical, mirror-closed curves
6-5	$\mathcal{Q}_3^p(2,6)$	12	$(3^4 \mid 2^6 \mid 6, 3, 2, 1)$	two identical, mirror-closed curves
6-6	$\mathcal{Q}_2^p(-1^2, 3^2)$	15	$(3^5 \mid 2^7, 1 \mid 6, 5, 3, 1)$	two distinct, mirror-closed curves
6-7	$\mathcal{Q}_2^p(-1^2, 3^2)$	15	$(3^5 \mid 2^7, 1 \mid 5, 4, 3^2)$	one pair of mirror-symmetric curves, mirroring each other
6-8	$\mathcal{Q}_3^p(-1, 9)$	22	$(3^7, 1 \mid 2^{11} \mid 6, 5, 4^2, 3)$	one pair of mirror-symmetric curves, mirroring each other
6-9	$\mathcal{Q}_2^p(-1^2, 0, 6)$	24	$(3^8 \mid 2^{12} \mid 6, 5, 4^2, 3, 2)$	one mirror-conjugate pair
6-10	$\mathcal{Q}_2^p(-1^3, 7)$	27	$(3^9 \mid 2^{13}, 1 \mid 6^2, 5, 4, 3^2)$	one mirror-conjugate pair & one mirror-closed curve
6-11	$\mathcal{Q}_2^p(-1, 0, 1, 4)$	36	$(3^{12} \mid 2^{18} \mid 6^2, 5^2, 4^2, 3^2)$	one mirror-conjugate pair & one mirror-closed curve
6-12	$\mathcal{Q}_3^p(1,7)$	54	$(3^{18} \mid 2^{27} \mid 6^4, 5^3, 4^3, 3)$	one mirror-conjugate pair & one mirror-closed curve
6-13	$\mathcal{Q}_3^p(-1, 9)$	66	$(3^{22} \mid 2^{33} \mid 6^6, 5^3, 4^3, 3)$	two mirror-conjugate pairs

Table 6.E Classes of possible $G_{\mathbb{Q}}$-conjugacy for degree 6

No.	stratum	index	valency list of $C(\mathcal{O})$	relationship between $C(\mathcal{O})$
7-1	$\mathcal{Q}_4^a(12)$	7	$(3^2, 1 \mid 2^3, 1 \mid 4, 3)$	one pair of mirror-symmetric curves, mirroring each other
7-2	$\mathcal{Q}_3^a(0, 2, 6)$	16	$(3^5, 1 \mid 2^8 \mid 7, 4, 3, 2)$	two distinct, mirror-closed curves
7-3	$\mathcal{Q}_4^a(12)$	21	$(3^7 \mid 2^{11} \mid 6, 5, 4, 3^2)$	two distinct, mirror-closed curves
7-4	$\mathcal{Q}_4^a(12)$	42	$(3^{14} \mid 2^{21} \mid 7^2, 5^2, 4^3, 3^2)$	two distinct, mirror-closed curves
7-5	$\mathcal{Q}_3^a(0, 2, 6)$	48	$(3^{16} \mid 2^{24} \mid 7^2, 6, 5^2, 4^3, 3^2)$	one mirror-conjugate pair
7-6	$\mathcal{Q}_2^p(-1, 1^3, 2)$	16	$(3^5, 1 \mid 2^8 \mid 7, 6, 2, 1)$	one mirror-conjugate pair
7-7	$\mathcal{Q}_4^p(12)$	28	$(3^9, 1 \mid 2^{14} \mid 7^2, 6, 3^2, 2)$	two distinct, mirror-closed curves
7-8	$\mathcal{Q}_3^p(-1^2, 10)$	36	$(3^{12} \mid 2^{18} \mid 7^3, 6, 3^2, 2, 1)$	two distinct, mirror-closed curves

Table 6.G Classes of possible $G_{\mathbb{Q}}$-conjugacy for degree 7

References

[1] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[2] Lars V. Ahlfors and Lipman Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math. (2)*, 72:385–404, 1960.

[3] Lars V. Ahlfors and G. Weill. A uniqueness theorem for Beltrami equations. *Proc. Amer. Math. Soc.*, 13:975–978, 1962.

[4] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Möller. Strata of k-differentials. *Algebr. Geom.*, 6(2):196–233, 2019.

[5] G. V. Belyĭ. On extensions of the maximal cyclotomic field having a given classical Galois group. *J. Reine Angew. Math.*, 341:147–156, 1983.

[6] Lipman Bers. A non-standard integral equation with applications to quasiconformal mappings. *Acta Math.*, 116:113–134, 1966.

[7] Joshua P. Bowman. Teichmüller geodesics, Delaunay triangulations, and Veech groups. In *Teichmüller theory and moduli problem*, volume 10 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 113–129. Ramanujan Math. Soc., Mysore, 2010.

[8] V. G. Drinfel'd. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990.

[9] Clifford J. Earle and Frederick P. Gardiner. Geometric isomorphisms between infinite-dimensional Teichmüller spaces. *Trans. Amer. Math. Soc.*, 348(3):1163–1190, 1996.

[10] Clifford J. Earle and Frederick P. Gardiner. Teichmüller disks and Veech's $\mathscr{F}$-structures. In *Extremal Riemann surfaces (San Francisco, CA, 1995)*, volume 201 of *Contemp. Math.*, pages 165–189. Amer. Math. Soc., Providence, RI, 1997.

[11] Brandon Edwards, Slade Sanderson, and Thomas A. Schmidt. Canonical translation surfaces for computing Veech groups. arXiv:2012.12444, 2014.

[12] Jordan S. Ellenberg and D. B. McReynolds. Arithmetic Veech sublattices of $\mathrm{SL}(2,\mathbf{Z})$. *Duke Math. J.*, 161(3):415–429, 2012.

[13] Giovanni Forni, Carlos Matheus, and Anton Zorich. Zero Lyapunov exponents of the Hodge bundle. *Comment. Math. Helv.*, 89(2):489–535, 2014.

[14] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981. Translated from the German by Bruce Gilligan.

[15] Paola Frediani and Frank Neumann. Étale homotopy types of moduli stacks of algebraic curves with symmetries. volume 30, pages 315–340. 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part IV.

[16] E. Freitag and R. Busam. *Complex Analysis*. Universitext (1979). Springer, 2005.

[17] William Fulton. *Algebraic topology*, volume 153 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course.

[18] Alexandre Grothendieck. Esquisse d'un programme. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 5–48. Cambridge Univ. Press, Cambridge, 1997. With an English translation on pp. 243–283.

[19] Eugene Gutkin and Chris Judge. Affine mappings of translation surfaces: geometry and arithmetic. *Duke Math. J.*, 103(2):191–213, 2000.

[20] G. H. Hardy and S. Ramanujan. Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2) **17** (1918), 75–115]. In *Collected papers of Srinivasa Ramanujan*, pages 276–309. AMS Chelsea Publ., Providence, RI, 2000.

[21] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[22] Hiroki Hashiguchi, Naoto Niki, and Shigekazu Nakagawa. Algorithms for constructing Young tableaux. Number 848, pages 38–48. 1993. Theory and applications in computer algebra (Japanese) (Kyoto, 1992).

[23] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[24] Allen Hatcher, Pierre Lochak, and Leila Schneps. On the Teichmüller tower of mapping class groups. *J. Reine Angew. Math.*, 521:1–24, 2000.

[25] Frank Herrlich. Introduction to origamis in Teichmüller space. In *Strasbourg master class on geometry*, volume 18 of *IRMA Lect. Math. Theor. Phys.*, pages 233–253. Eur. Math. Soc., Zürich, 2012.

[26] Frank Herrlich and Gabriela Schmithüsen. A comb of origami curves in the moduli space M_3 with three dimensional closure. *Geom. Dedicata*, 124:69–94, 2007.

[27] Frank Herrlich and Gabriela Schmithüsen. An extraordinary origami curve. *Math. Nachr.*, 281(2):219–237, 2008.

[28] Frank Herrlich and Gabriela Schmithüsen. Dessins d'enfants and origami curves. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 767–809. Eur. Math. Soc., Zürich, 2009.

[29] Rubén A. Hidalgo. Automorphism groups of dessins d'enfants. *Arch. Math. (Basel)*, 112(1):13–18, 2019.

[30] John Hubbard and Howard Masur. Quadratic differentials and foliations. *Acta Math.*, 142(3-4):221–274, 1979.

[31] Yasutaka Ihara. On the embedding of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) into $\widehat{\mathrm{GT}}$. In *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.*, pages 289–321. Cambridge Univ. Press, Cambridge, 1994. With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem and Pierre Lochak.

[32] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.

[33] Gareth A. Jones and Jürgen Wolfart. *Dessins d'enfants on Riemann surfaces*. Springer Monographs in Mathematics. Springer, Cham, 2016.

[34] Gareth Aneurin Jones. Automorphism groups of maps, hypermaps and dessins. *Art Discrete Appl. Math.*, 3(1):Paper No. 1.06, 14, 2020.

[35] Bernhard Köck. Belyi's theorem revisited. *Beiträge Algebra Geom.*, 45(1):253–265, 2004.

[36] Paul Koebe. Über die Uniformisierung der algebraischen Kurven. I. *Math. Ann.*, 67(2):145–224, 1909.

[37] Paul Koebe. Über die Uniformisierung der algebraischen Kurven. II. *Math. Ann.*, 69(1):1–81, 1910.

[38] Paul Koebe. Über die Uniformisierung der algebraischen Kurven. III. *Math. Ann.*, 72(4):437–516, 1912.

[39] Paul Koebe. Über die Uniformisierung der algebraischen Kurven. IV. *Math. Ann.*, 75(1):42–129, 1914.

[40] Shun Kumagai. An algorithm for classifying origamis into components of Teichmüller curves. arXiv:2006.00905, 2021.

[41] Shun Kumagai. General origamis and Veech groups of flat surfaces. arXiv:2111.09654, 2021.

[42] Erwan Lanneau. Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities. *Comment. Math. Helv.*, 79(3):471–501, 2004.

[43] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications, Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.

[44] Makoto Matsumoto. Arithmetic fundamental groups and moduli of curves. In *School on Algebraic Geometry (Trieste, 1999)*, volume 1 of *ICTP Lect. Notes*, pages 355–383. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000.

[45] Martin Möller. Teichmüller curves, Galois actions and $\widehat{GT}$-relations. *Math. Nachr.*, 278(9):1061–1077, 2005.

[46] Martin Möller. Variations of Hodge structures of a Teichmüller curve. *J. Amer. Math. Soc.*, 19(2):327–344, 2006.

[47] Subhashis Nag. *The complex analytic theory of Teichmüller spaces*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1988. A Wiley-Interscience Publication.

[48] Florian Nisbach. The Galois action on M-origamis and their Teichmüller curves. arXiv:1408.6769, 2014.

[49] Kôtaro Oikawa. *Riemann surface. (Japanese) [Kyoritsu Publ. Co., Tokyo, 1944].* 1987.

[50] H. Poincaré. Sur l'uniformisation des fonctions analytiques. *Acta Math.*, 31(1):1–63, 1908.

[51] Gabriela Schmithüsen. An algorithm for finding the Veech group of an origami. *Experiment. Math.*, 13(4):459–472, 2004.

[52] Gabriela Schmithüsen. Origamis with non congruence Veech groups. In *Proceedings of 34th Symposium on Transformation Groups*, pages 31–55. Wing Co., Wakayama, 2007.

[53] Leila Schneps. The Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$: a survey. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 183–203. Cambridge Univ. Press, Cambridge, 1997.

[54] Yoshihiko Shinomiya. Veech groups of flat structures on Riemann surfaces. In *Quasiconformal mappings, Riemann surfaces, and Teichmüller spaces*, volume 575 of *Contemp. Math.*, pages 343–362. Amer. Math. Soc., Providence, RI, 2012.

[55] Kurt Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1984.

[56] William A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.*, 97(3):553–583, 1989.

[57] William A. Veech. Moduli spaces of quadratic differentials. *J. Analyse Math.*, 55:117–171, 1990.

[58] André Weil. The field of definition of a variety. *Amer. J. Math.*, 78:509–524, 1956.

Appendix: Program codes

The following is the implimentation of Algorithms in Section 6.1 that was used to obtain the calculation results in Section 6.2. The program codes are powered by Python (https://www.python.org/). Files including program codes and calculation results have been made publicly available at GitHubi (https://github.com/ShunKumagai/origami). The program codes were originally written by the author, and some of it was arranged with the help of the staff in Cyber Science Center(https://www.cc.tohoku.ac.jp/), Tohoku University.

Program1 (Algorithm 6.1.1 and 6.1.2): classification of all patterns of origamis in equivalence classes.

Before running the program, input the degree d (line 18) and remove the line breaks at '#partition data' (line 29-43).

Program1.py

```
1 #coding: UTF-8
2 import time
3 t1=time.time()
4 t0=time.time()
5
6 import itertools as it
7 import numpy as np
8 import math
9 import pickle
10 import copy
11 from functools import reduce
12 import multiprocessing
13 from multiprocessing import Pool
14 from concurrent.futures import ProcessPoolExecutor
15 from concurrent.futures import ThreadPoolExecutor
16
17 #input
18 d=3
19
20 #partition data: remove line breaks for d=>5
21 N0=[[[0]],[[1]]]
22 #d=2
23 N0.append([ [[2]],[[1,1]] ])
```

```
#d=3
N0.append([ [[3]],[[1,2]],[[1,1,1]] ])
#d=4
N0.append([[[4]],[[1,3],[2,2]],[[1,1,2]],[[1,1,1,1]]])
#d=5
N0.append([[[5]],[[1,4],[2,3]],[[1,1,3],[1,2,2]],[[1,1,1,2]],
  [[1,1,1,1,1]]])
#d=6
N0.append([[[6]],[[1,5],[2,4],[3,3]],[[1,1,4],[1,2,3],[2,2,2]],
  [[1,1,1,3],[1,1,2,2]],[[1,1,1,1,2]],[[1,1,1,1,1,1]]])
#d=7
N0.append([[[7]],[[1,6],[2,5],[3,4]],[[1,1,5],[1,2,4],[1,3,3],
  [2,2,3]],[[1,1,1,4],[1,1,2,3],[1,2,2,2]],[[1,1,1,1,3],
  [1,1,1,2,2]],[[1,1,1,1,1,2]],[[1,1,1,1,1,1,1]]])
 #d=8
 N0.append([[[8]],[[1,7],[2,6],[3,5],[4,4]],[[1,1,6],[1,2,5],
   [1,3,4],[2,2,4],[2,3,3]],[[1,1,1,5],[1,1,2,4],[1,1,3,3],
   [1,2,2,3]],[[1,1,1,1,4],[1,1,1,2,3],[1,1,2,2,2]],
   [[1,1,1,1,1,3],[1,1,1,1,2,2]],[[1,1,1,1,1,1,2]],
   [[1,1,1,1,1,1,1,1]]])

np.set_printoptions(threshold=np.inf)#Do not always omit print of
  numpy matrices


def indices(A):
    return range(len(A))
def M(x):
    return np.identity(len(x), dtype=int)[:, x]
def inv(A):
    return np.swapaxes(A, -1, -2)
def iM(x):
    return inv(M(x))
def get_time():
    t=time.time()
    dt=(t-t0)
    print("total time:",dt)
def finish():
    t=time.time()
    dt=(t-t0)
    print("total time:",dt)
    exit()

def cycle(v):
    rest=[i for i in indices(v)]
    cv=[]
    while len(rest)!=0:
        cvi=[rest[0]]
        rest.remove(rest[0])
        j=0
        while True:
```

```
            cvi.append(v[cvi[j]])
            try:
                rest.remove(v[cvi[j]])
            except ValueError:
                pass
            j=j+1
            if cvi[j]==cvi[0]:
                cvi.pop(j)
                break
        cv.append(cvi)
    return cv

def icycle(c,L="no data"):
    if L=="no data":L=sum([len(c[i]) for i in indices(c)])
    v=[i for i in range(L)]
    for i in indices(c):
        for j in range(len(c[i])-1):
            v[c[i][j]]=c[i][j+1]
        v[c[i][len(c[i])-1]]=c[i][0]
    return v

def Sym(d):
    a = np.identity(d, dtype='i')
    S = np.array([a[np.array(idx)] for idx in it.permutations(
  range(d))])
    return S

def represent(X,d):
    p=len(X)
    N=np.asarray([len(x) for x in X])
    p0=sum(N)
    Y=[]
    for i in range(p):
        for j in range(N[i]):
            Xij=[ np.asarray([0]) if X[i][j][k]==1 else np.
  asarray([s+1 for s in range(X[i][j][k]-1)]+[0]) for k in range(
  i+1)]
            Dij=[0]+[len(Xij[k]) for k in range(i)]
            Eij=np.asarray([sum(Dij[0:k+1]) for k in range(i+1)])
            Yij=np.concatenate([Xij[k]+Eij[k] for k in range(i+1)
  ])
            Y.append(Yij)
    return np.asarray(Y)

def Sign(d):
    return np.asarray([[(t//(2**s))%2 for s in range(d)] for t in
   range(2**d)])
def vinv(v):
    return M(v).dot(e)
def conjugate(sigma,x,invsigma=None):
    if invsigma is None:
```

```
         return np.array([sigma[ x[vinv(sigma)[j]] ] for j in range(d)])
     else:
         return np.array([sigma[ x[invsigma[j]] ] for j in range(d)])

 def ytosign(j,sign):#positive if sign=0 or ‘False’, negative if sign=1 or ‘True’
     return Vd[j] if sign == 1 else iVd[j]
 def xtosign(i,sign):#positive if sign=0 or ‘False’, negative if sign=1 or ‘True’
     return Xrep[i] if sign == 1 else iXrep[i]
 def vtosign(v,sign):
     return v if sign == 1 else vinv(v)
 def vtosigns(v,signs):
     return np.array([vtosign(v,not signs[m])[m] for m in Id])
 def xi(y,e):
     return np.array([e[m]!=e[vtosign(y,not e[m])[m]] for m in Id])

#Algorithm 6.1.1
 def isom_sub(i, j, k):
     ret=[]
     x=Xrep[i]
     y=Vd[j]
     invy=vinv(y)
     yi=iVd[j]
     eps=Signd[k]
     for nd in range(lSignd):
         delta=Signd[nd]
         vtosign_x=[vtosign(x,not delta[m])[m] for m in Id]
         if np.any(delta!=np.array([delta[x[m]] for m in Id])):
             continue
         for ns in range(lVd):
             sigma=Vd[ns]
             isigma=iVd[ns]
             invsigma = vinv(sigma)
             if np.any(conjugate(sigma,vtosign_x,invsigma)!=x):
                 continue
             Yeesd=np.array([conjugate(sigma,[invy[m] if (eps[m]+Signd[n][sigma[m]]+delta[m])%2 else y[m] for m in Id], invsigma) for n in indices(Signd)])

             NYeesd=[NVd[np.all(Vd==Yeesd[n],axis=1)] for n in NSignd]
             exNSignd=NSignd[np.array([len(NYeesd[n])!=0 for n in NSignd])]
             eta=np.array([np.all(1-xi(Yeesd[n],[(delta[isigma[m]]+eps[isigma[m]]+Signd[n][m])%2 for m in Id])) for n in exNSignd])
             trueNSignd=exNSignd[eta]
```

```
            if len(trueNSignd) > 0:
                ret.extend([[i,NYeesd[n][0],n] for n in trueNSignd])
    return ret

#Algorithm 6.1.2
def classify(i):
    rest=np.concatenate([[[j,k]for k in NSignd]for j in NVd])
    NYE0i=[]
    while len(rest)>0:
        ye=[rest[0][0],rest[0][1]]
        isom_pre2 = isom_sub(i, ye[0], ye[1])
        Isom=np.unique(isom_pre2,axis=0)
        NYE0i.append(Isom)
        rest=rest[[np.all(np.any(ye1!=Isom[:,1:],axis=1)) for ye1 in rest]]
    return NYE0i

Signd=Sign(d)
NSignd=np.arange(len(Signd))

S=Sym(d)
iS = inv(S)

e=np.arange(d)#[0,1,2,...,d-1]

Id=e
Vd=S.dot(e)
iVd=iS.dot(e)
NVd=np.arange(len(Vd))

Xrep=represent(N0[d],d)
iXrep=np.asarray([vinv(x) for x in Xrep])
X1=np.concatenate([NVd[np.all(Vd==x,axis=1)] for x in Xrep])
NX1=np.arange(len(X1))


lVd=len(Vd)
lSignd=len(Signd)

#multiprocessing
n_process=1
n_thread=1

if __name__ == "__main__":
    print(multiprocessing.cpu_count())
    p = Pool(n_process)
    NYE0 = p.map(classify, NX1)
    p.close()
    t=time.time()
    dt=(t-t0)
```

```
    NYE1=[np.array([[i,nye0[0][1],nye0[0][2]] for nye0 in NYE0[i]]) for i in NX1]
    YE1=[np.array([[Xrep[i],Vd[nye0[0][1]],Signd[nye0[0][2]]] for nye0 in NYE0[i]]) for i in NX1]
    lYE1=[len(NYE1[i]) for i in NX1]
    CNYE0=np.concatenate(NYE0)
    NCO=len(CNYE0)
    CNYE1=np.concatenate(NYE1)
    CNNYE1=np.array([i for i in indices(CNYE0)])
    CYE0=[[np.array([Xrep[a[0]],Vd[a[1]],Signd[a[2]]]) for a in c] for c in CNYE0]
    t = time.localtime()
    fname = str(t.tm_mon)+str(t.tm_mday)+str(t.tm_hour)+str(t.tm_min)
    with open('data_d={0}.txt'.format(d), 'w') as f:
        print('#d={0}'.format(d), file=f)
        print('import numpy as np'.format(d), file=f)
        print('NYE0=',NYE0, file=f)
        t=time.time()
        dt=(t-t0)
        print("#total time:",dt,file=f)
        finish()
```

Example of output of Program1.py (input:d=3) is as follows.

data_d=3.txt

```
#d=3
import numpy as np
NYE0= [[np.array([[0, 0, 0],[0, 0, 1],[0, 0, 2],[0, 0, 3],[0, 0, 4],[0, 0, 5],[0, 0, 6],[0, 0, 7]]),
 np.array([[0, 1, 0],[0, 1, 1],[0, 1, 6],[0, 1, 7],[0, 2, 0],[0, 2, 3],[0, 2, 4],[0, 2, 7],[0, 5, 0],[0, 5, 2],[0, 5, 5],[0, 5, 7]]),
 np.array([[0, 1, 2],[0, 1, 3],[0, 1, 4],[0, 1, 5],[0, 2, 1],[0, 2, 2],[0, 2, 5],[0, 2, 6],[0, 5, 1],[0, 5, 3],[0, 5, 4],[0, 5, 6]]),
 np.array([[0, 3, 0],[0, 4, 7]]),
 np.array([[0, 3, 1],[0, 3, 2],[0, 3, 4],[0, 4, 3],[0, 4, 5],[0, 4, 6]]),
 np.array([[0, 3, 3],[0, 3, 5],[0, 3, 6],[0, 4, 1],[0, 4, 2],[0, 4, 4]]),
 np.array([[0, 3, 7],[0, 4, 0]])],
 [np.array([[1, 0, 0],[1, 0, 1],[1, 0, 2],[1, 0, 3],[1, 0, 4],[1, 0, 5],[1, 0, 6],[1, 0, 7]]),
 np.array([[1, 1, 0],[1, 1, 1],[1, 1, 6],[1, 1, 7]]),
 np.array([[1, 1, 2],[1, 1, 3],[1, 1, 4],[1, 1, 5]]),
 np.array([[1, 2, 0],[1, 2, 1],[1, 2, 2],[1, 2, 3],[1, 2, 4],[1, 2, 5],[1, 2, 6],[1, 2, 7],[1, 5, 0],[1, 5, 1],[1, 5, 2],[1, 5, 3],[1, 5, 4],[1, 5, 5],[1, 5, 6],[1, 5, 7]]),
```

```
 np.array([[1, 3, 0],[1, 3, 1],[1, 3, 6],[1, 3, 7],[1, 4, 0],[1, 4, 1],[1, 4, 6],[1, 4, 7]]),
 np.array([[1, 3, 2],[1, 3, 3],[1, 3, 4],[1, 3, 5],[1, 4, 2],[1, 4, 3],[1, 4, 4],[1, 4, 5]])], [np.array([[2, 0, 0],[2, 0, 1],[2, 0, 2],[2, 0, 3],[2, 0, 4],[2, 0, 5],[2, 0, 6],[2, 0, 7]]),
 np.array([[2, 1, 0],[2, 1, 1],[2, 1, 2],[2, 1, 3],[2, 1, 4],[2, 1, 5],[2, 1, 6],[2, 1, 7],[2, 2, 0],[2, 2, 1],[2, 2, 2],[2, 2, 3],[2, 2, 4],[2, 2, 5],[2, 2, 6],[2, 2, 7],[2, 5, 0],[2, 5, 1],[2, 5, 2],[2, 5, 3],[2, 5, 4],[2, 5, 5],[2, 5, 6],[2, 5, 7]]),
 np.array([[2, 3, 0],[2, 3, 1],[2, 3, 2],[2, 3, 3],[2, 3, 4],[2, 3, 5],[2, 3, 6],[2, 3, 7],[2, 4, 0],[2, 4, 1],[2, 4, 2],[2, 4, 3],[2, 4, 4],[2, 4, 5],[2, 4, 6],[2, 4, 7]])]]
```

Program2 (Algorithm 6.1.3, 6.1.4, 6.1.5, and 6.1.6): Calculation of the action of $PSL(2,\mathbb{Z})$ on $\tilde{\Omega}_d$ and its orbit decompoisition. Finally, it output the data of Galois invariants of origami.

Before running the Program2, do the following: Input the degree d (line 13). Remove the line breaks at '#partition data' (line 26-41). Replace every string "array" in file data_d=*.txt with "asarray", change the filename extension to ".py", and place it in the same directory as the program.

Program2.py

```
1 import time
2 t1=time.time()
3 t0=time.time()
4
5 import itertools as it
6 import numpy as np
7 import math
8 import pickle
9 import copy
10 from functools import reduce
11
12 #input
13 d=3
14 import importlib
15 from importlib import import_module
16 data = import_module( 'data_d={}'.format(d))
17
18 #partition data: remove line breaks for d=>5
19 N0=[[[0]],[[1]]]
20 #d=2
21 N0.append([ [[2]],[[1,1]] ])
22 #d=3
23 N0.append([ [[3]],[[1,2]],[[1,1,1]] ])
```

```
#d=4
N0.append([[[4]],[[1,3],[2,2]],[[1,1,2]],[[1,1,1,1]]])
#d=5
N0.append([[[5]],[[1,4],[2,3]],[[1,1,3],[1,2,2]],[[1,1,1,2]],
  [[1,1,1,1,1]]])
#d=6
N0.append([[[6]],[[1,5],[2,4],[3,3]],[[1,1,4],[1,2,3],[2,2,2]],
  [[1,1,1,3],[1,1,2,2]],[[1,1,1,1,2]],[[1,1,1,1,1,1]]])
#d=7
N0.append([[[7]],[[1,6],[2,5],[3,4]],[[1,1,5],[1,2,4],[1,3,3],
  [2,2,3]],[[1,1,1,4],[1,1,2,3],[1,2,2,2]],[[1,1,1,1,3],
  [1,1,1,2,2]],[[1,1,1,1,1,2]],[[1,1,1,1,1,1,1]]])
 #d=8
 N0.append([[[8]],[[1,7],[2,6],[3,5],[4,4]],[[1,1,6],[1,2,5],
   [1,3,4],[2,2,4],[2,3,3]],[[1,1,1,5],[1,1,2,4],[1,1,3,3],
   [1,2,2,3]],[[1,1,1,1,4],[1,1,1,2,3],[1,1,2,2,2]],
   [[1,1,1,1,1,3],[1,1,1,1,2,2]],[[1,1,1,1,1,1,2]],
   [[1,1,1,1,1,1,1,1]]])

np.set_printoptions(threshold=np.inf)#Do not always omit print of numpy matrices

def lcm_base(x, y):
    return (x * y) // math.gcd(x, y)
def lcm(*numbers):
    return reduce(lcm_base, numbers, 1)
def lcm_list(numbers):
    return reduce(lcm_base, numbers, 1)
def indices(A):
    return range(len(A))
def Cdelta(a,b):
    D=1 if a==b else 0
    return D
def M(x):
    l=len(x)
    return np.asarray([[Cdelta(j,x[i])for i in range(l)] for j in range(l)])
def inv(A):
    l=len(A)
    return np.asarray([[A[i][j] for i in range(l)] for j in range(l)])
def iM(x):
    return inv(M(x))
def tttime():
    t=time.time()
    dt=(t-t0)
    print("total time:",dt)
def finish():
    t=time.time()
    dt=(t-t0)
    print("total time:",dt)
```

```
    exit()

def cycle(v):
    rest=[i for i in indices(v)]
    cv=[]
    while len(rest)!=0:
        cvi=[rest[0]]
        rest.remove(rest[0])
        j=0
        while True:
            cvi.append(v[cvi[j]])
            try:
                rest.remove(v[cvi[j]])
            except ValueError:
                pass
            j=j+1
            if cvi[j]==cvi[0]:
                cvi.pop(j)
                break
        cv.append(cvi)
    return cv
def icycle(c,L="no data"):
    if L=="no data":L=sum([len(c[i]) for i in indices(c)])
    v=[i for i in range(L)]
    for i in indices(c):
        for j in range(len(c[i])-1):
            v[c[i][j]]=c[i][j+1]
        v[c[i][len(c[i])-1]]=c[i][0]
    return v

def Sym(d):
    S=np.asarray([[[Cdelta(i,s[j])for i in range(d)] for j in range(d)] for s in list(it.permutations(list(range(d))))])
    return S
def represent(X,d):
    p=len(X)
    N=np.asarray([len(x) for x in X])
    p0=sum(N)
    Y=[]
    for i in range(p):
        for j in range(N[i]):
            #Xij[k]=[0] if X[i][j][k]==1 else np.concatenate([s+1 for s in range(X[i][j][k]-1)],[0])
            Xij=[ np.asarray([0]) if X[i][j][k]==1 else np.asarray([s+1 for s in range(X[i][j][k]-1)]+[0]) for k in range(i+1)]
            Dij=[0]+[len(Xij[k]) for k in range(i)]
            Eij=np.asarray([sum(Dij[0:k+1]) for k in range(i+1)])
            Yij=np.concatenate([Xij[k]+Eij[k] for k in range(i+1)])
            Y.append(Yij)
    return np.asarray(Y)
```

```

def Sign(d):
    return np.asarray([[(t//(2**s))%2 for s in range(d)] for t in range(2**d)])

Signd=Sign(d)
NSignd=np.arange(len(Signd))
e=np.arange(d)#[0,1,2,...,d-1]
S=Sym(d)
iS=np.asarray([inv(A) for A in S])


def vinv(v):
    return M(v).dot(e)

Id=e
Vd=S.dot(e)
iVd=iS.dot(e)
NVd=np.arange(len(Vd))

Xrep=represent(N0[d],d)
iXrep=np.asarray([vinv(x) for x in Xrep])
X1=np.concatenate([NVd[np.all(Vd==x,axis=1)] for x in Xrep])
NX1=np.arange(len(X1))

def conjugate(v,x):
    return ([v[ x[vinv(v)[j]] ] for j in range(d)])
Xclass=[[conjugate(v,x) for v in Vd] for x in Xrep]

def ytosign(j,sign):#sign=0,は正、falsesign=1,は負として処理true
    return Vd[j] if sign == 1 else iVd[j]
def xtosign(i,sign):#sign=0,は正、falsesign=1,は負として処理true
    return Xrep[i] if sign == 1 else iXrep[i]
def vtosign(v,sign):
    return v if sign == 1 else vinv(v)

def bx(x,y,eps,j):
    pm=j//d
    i=j%d
    bx=[pm, vtosign(x,1-pm)[i]]
    return d*bx[0]+bx[1]
def ibx(x,y,eps,j):
    pm=j//d
    i=j%d
    ibx=[pm, vtosign(x,pm)[i]]
    return d*ibx[0]+ibx[1]
def by(x,y,eps,j):
    pm=j//d
    i=j%d
    by=[(pm+eps[i]+eps[vtosign(y,eps[i]==pm)[i]])%2,vtosign(y,eps[i]==pm)[i]]
```

```
    return d*by[0]+by[1]
def iby(x,y,eps,j):
    pm=j//d
    i=j%d
    iby=[(pm+eps[i]+eps[vtosign(y,eps[i]!=pm)[i]])%2,vtosign(y,eps[i]!=pm)[i]]
    return d*iby[0]+iby[1]

NYE=np.concatenate([[[j,k]for k in NSignd]for j in NVd])
YE=np.array([[ Vd[nye[0]],Signd[nye[1]] ] for nye in NYE])


NYE0=data.NYE0
NYE1=[np.array([[i,nye0[0][1],nye0[0][2]] for nye0 in NYE0[i]]) for i in NX1]
YE1=[np.array([[Xrep[i],Vd[nye0[0][1]],Signd[nye0[0][2]]] for nye0 in NYE0[i]]) for i in NX1]

def restore(nxye):
    return [cycle(Xrep[nxye[0]]),cycle(Vd[nxye[1]]),Signd[nxye[2]]]

lYE1=[len(NYE1[i]) for i in NX1]
CNYE0=np.concatenate(NYE0)
NCO=len(CNYE0)
CNYE1=np.concatenate(NYE1)
CNNYE1=np.array([i for i in indices(CNYE0)])
CYE0=[[np.array([Xrep[a[0]],Vd[a[1]],Signd[a[2]]]) for a in c] for c in CNYE0]
CYE1=[c[0] for c in CYE0]


def Orbit(x,y):
    if len(x)!=len(y):print("error:Orbitlength")
    n=len(x)
    cx=cycle(x)
    Ncx=np.array(indices(cx))
    cy=cycle(y)
    Ncy=np.array(indices(cy))
    rest=np.array(range(n))
    decomp=[]
    while len(rest)!=0:
        i=rest[0]
        orbi=[i]
        resti=[i]
        donei=[]
        while len(resti)!=0:
            j=resti[0]
            cxj=cx[Ncx[([np.any(np.array(cx[k])==j) for k in indices(cx)])][0]]
            cyj=cy[Ncy[([np.any(np.array(cy[k])==j) for k in
```

```
indices(cy)])][0]]
            orbi=np.unique(np.concatenate([orbi,cxj,cyj]))
            resti=copy.deepcopy(orbi)
            donei.append(j)
            for l in donei:
                resti=resti[resti!=l]
            for l in orbi:
                rest=rest[rest!=l]
        decomp.append(orbi)
    return decomp

#Algorithm 6.1.3
print("start")
def PermT():
    permT=[[0 for j in NYE1[i]]for i in NX1]
    for i in NX1:
        x=Xrep[i]
        ix=iXrep[i]
        for j in indices(NYE1[i]):
            nye1=NYE1[i][j]#=[i,j,kの形], i=nye[0]
            y=Vd[nye1[1]]
            eps=Signd[nye1[2]]
            cyT=[]
            epsT=[0 for i in Id]
            rest=copy.deepcopy(Id)
            while len(rest)!=0:
                a1=rest[0]
                a2=rest[0]
                a0=rest[0]
                cyT1=[a0]
                eps_1=0
                while True:
                    rest=rest[rest!=a2]#remove a1 from rest
                    a2_=vtosign(y,not eps_1!=eps[vtosign(x,not eps_1)[a1]])[vtosign(x,not eps_1)[a1]]
                    eps_2=((eps_1!=eps[vtosign(x,not eps_1)[a1]])!=eps[a2_])
                    if eps_2==0:a2=a2_
                    else:a2=ix[a2_]
                    if a2==a0:break
                    cyT1=cyT1+[a2]
                    epsT[a2]=eps_2
                    eps_1=eps_2
                    a1=a2_
                cyT=cyT+[cyT1]
            epsT=np.array(epsT).astype(np.int)
            yT=icycle(cyT,d)
            permT[i][j]=np.concatenate([[i],NYE[np.all(np.all(YE==[yT,epsT],axis=2),axis=1)][0]])
    return permT

```

```
#Algorithm 6.1.5
def PermS():
    debugflag=False
    permS=[[0 for j in NYE1[i]]for i in NX1]
    for i in NX1:
        x=Xrep[i]
        ix=iXrep[i]
        for j in indices(NYE1[i]):
            nye1=NYE1[i][j]#=[i,j,k], i=nye[0]
            y=Vd[nye1[1]]
            eps=Signd[nye1[2]]

            if debugflag:print(x,y,eps)

            #Algorithm 6.1.4
            cxS=[]
            deltaS=[0 for i in Id]
            rest=copy.deepcopy(Id)
            while len(rest)!=0:#make cycle representation
                a1=rest[0]
                a2=rest[0]
                a0=rest[0]
                cxS1=[a0]
                delta1=0
                while True:#cycle starting from a0
                    rest=rest[rest!=a2]#remove a2 from rest
                    a2=vtosign(y,not eps[a1]!=delta1)[a1]
                    delta2=(delta1!=eps[a1])!=eps[a2]
                    if a2==a0:break
                    cxS1=cxS1+[a2]
                    deltaS[a2]=delta2
                    delta1=delta2
                    a1=a2
                cxS=cxS+[cxS1]
            xS_=icycle(cxS,d)

            cyS=[]
            epsS_=[0 for i in Id]
            rest=copy.deepcopy(Id)
            while len(rest)!=0:#make cycle representation
                b1=rest[0]
                b2=rest[0]
                b0=rest[0]
                cyS1=[b0]
                epsS_1=0
                while True:#cycle starting from b0
                    rest=rest[rest!=b2]#remove b2 from rest
                    b2=vtosign(x,not not epsS_1!=deltaS[b1])[b1]
                    epsS_2=(epsS_1!=deltaS[b1])!=deltaS[b2]
                    if debugflag:print(b2,epsS_2)
                    if b2==b0:
```

```
                        break
                    cyS1=cyS1+[b2]
                    epsS_[b2]=epsS_2
                    epsS_1=epsS_2
                    b1=b2
                cyS=cyS+[cyS1]
            epsS_=np.array(epsS_).astype(np.int)
            yS_=icycle(cyS,d)
            Xnum=NX1[np.any(np.all(Xclass==np.array(xS_),axis=2),axis=1)][0]#xS belongs to Xrep[Xnum]
            conj=iVd[np.all(Xclass==np.array(xS_),axis=2)[Xnum]][0]#conjugator
            if np.any(xS!=conjugate(conj,xS_)):
                print("conjugate error")
                finish()
            yS=conjugate(conj,yS_)
            epsS=[epsS_[vinv(conj)[i]] for i in Id]
            permS[i][j]=np.concatenate([[Xnum],NYE[np.all(np.all(YE==[yS,epsS],axis=2),axis=1)][0]])
    return permS


permT=np.concatenate(PermT())
print("PermT:done")
tttime()
permS=np.concatenate(PermS())
print("PermS:done")
tttime()
permT1=[CNNYE1[([np.any(np.all(CNYE0[j]==permT[i],axis=1)) for j in indices(CNYE0)])][0] for i in CNNYE1]
permS1=[CNNYE1[([np.any(np.all(CNYE0[j]==permS[i],axis=1)) for j in indices(CNYE0)])][0] for i in CNNYE1]
permTS1=[permT1[permS1[i]] for i in CNNYE1]

print("")

CpermT=cycle(permT1)
CpermS=cycle(permS1)
CpermTS=cycle(permTS1)

#Algorithm 6.1.6
Orb=Orbit(permT1,permS1)

#Output result
def output():
    with open('result_d={0}.txt'.format(d), 'w') as f:
        print(" ")
        print("output")
        print("  d =",d)
        print("#d =",d, file=f)
        print(" ")
```

```
    Num=1
    for orb in Orb:
        rep=orb[0]
        if np.any(np.any(CNYE0[rep]==[-1,-1,0],axis=1)) or np.any(np.any(CNYE0[rep]==[-1,-1,len(Signd)-1],axis=1)):Abelian=True
        else:Abelian=False
        Nxye=CNYE1[rep]
        x=Xrep[Nxye[0]]
        y=Vd[Nxye[1]]
        eps=Signd[Nxye[2]]
        decomp=Orbit(x,y)
        if len(decomp)!=1:
            continue
        if Abelian:
            z=[vinv(y)[vinv(x)[y[x[i]]]] for i in Id]
            vl=np.sort([4*len(c) for c in cycle(z)])#valency list
            Nv=len(vl)#number of vertices
        else:
            bz=[iby(x,y,eps,ibx(x,y,eps,by(x,y,eps,bx(x,y,eps,j)))) for j in range(2*d)]
            cbz=cycle(bz)
            vld=[]#valency list of double-paired vertices
            vlr=[]#valency list of ramified vertices
            for c in cbz:
                bxby0=by(x,y,eps,bx(x,y,eps,c[0]))
                cbz0=[c1 for c1 in cbz if np.any(np.array(c1)==c[0])][0]
                if np.any(cbz0==bxby0%d+(1-bxby0//d)*d):#ramified vertex
                    vlr.append(len(c)*2)
                else:#double-paired vertex
                    vld.append(len(c)*4)
            vldr=[np.sort(vld)[2*i] for i in range(len(vld)//2)]
            vl=np.sort(vlr+vldr)#valency list
            Nv=len(vl)#number of vertices
        CT=[c for c in CpermT if np.any(orb==c[0])]
        CS=[c for c in CpermS if np.any(orb==c[0])]
        CTS=[c for c in CpermTS if np.any(orb==c[0])]
        genus=1-(len(orb)-(3*len(orb)-len([c for c in CS if len(c)==1]))/2+len(CT)+len(CTS))/2
        WL=lcm(*[len(c) for c in CT])
        print("  Component No.",Num, file=f)
        print("  representatives: ",orb, file=f)
        print("  index of VG =",len(orb), file=f)
        print("  base: (x,y,eps) = (",cycle(x),cycle(y),eps,")", file=f)
        print("  surface type= (",int((d-Nv)/2+1),",",Nv,")", file=f)
```

```
            print("  valency list= ",vl , file=f)
            print("  Abelian:",Abelian, file=f)
            print("  stratum:","A_" if Abelian else "Q_",int((d-Nv)/2+1),[int(v/2-2) for v in vl],   file=f)
            print("  T =",CT, file=f)
            print("  widths list of T =",[len(c) for c in CT],len(CT), file=f)
            print("  S =",CS, file=f)
            print("  widths list of S =",[len(c) for c in CS],len(CS), file=f)
            print("  TS =",CTS, file=f)
            print("  widths list of TS =",[len(c) for c in CTS],len(CTS), file=f)
            print("  genus =",math.floor(genus), file=f)
            print("  Wolfarht level =",WL, file=f)
            #list of origamis in component
            for i in orb:
                print("    representative No.",i, file=f)
                Nxyei=CNYE1[i]
                xi=Xrep[Nxyei[0]]
                yi=Vd[Nxyei[1]]
                epsi=Signd[Nxyei[2]]
                print("    (x,y,eps) = (",cycle(xi),cycle(yi),epsi,")", file=f)

            print(" ", file=f)
            print(" ", file=f)
            print(" ", file=f)
            Num=Num+1
    print("end")

print("Output")
output()
finish()
```

Example of output of Program2.py (input:d=3) is as follows.

result_d=3.txt

```
#d = 3
  Component No. 1
  representatives:  [ 0  3  6 15]
  index of VG = 4
  base: (x,y,eps) = ( [[0, 1, 2]] [[0], [1], [2]] [0 0 0] )
  surface type= ( 1 , 3 )
  Abelian: True
  T = [[0, 3, 6], [15]]
  widths list = [3, 1] 2
  S = [[0, 15], [3, 6]]
  widths list = [2, 2] 2
```

```
TS = [[0, 15, 3], [6]]
widths list = [3, 1] 2
genus = 0
Wolfarht level = 3
  representative No. 0
  (x,y,eps) = ( [[0, 1, 2]] [[0], [1], [2]] [0 0 0] )
  representative No. 3
  (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [0 0 0] )
  representative No. 6
  (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [1 1 1] )
  representative No. 15
  (x,y,eps) = ( [[0], [1], [2]] [[0, 1, 2]] [0 0 0] )



Component No. 2
representatives:  [ 1 10 11]
index of VG = 3
base: (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 0 0] )
surface type= ( 2 , 1 )
Abelian: True
T = [[1], [10, 11]]
widths list = [1, 2] 2
S = [[1, 11], [10]]
widths list = [2, 1] 2
TS = [[1, 10, 11]]
widths list = [3] 1
genus = 0
Wolfarht level = 2
  representative No. 1
  (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 0 0] )
  representative No. 10
  (x,y,eps) = ( [[0], [1, 2]] [[0, 1], [2]] [0 0 0] )
  representative No. 11
  (x,y,eps) = ( [[0], [1, 2]] [[0, 1, 2]] [0 0 0] )



Component No. 3
representatives:  [ 2  4  5 12]
index of VG = 4
base: (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 1 0] )
surface type= ( 2 , 1 )
Abelian: False
T = [[2, 4, 5], [12]]
widths list = [3, 1] 2
S = [[2, 12], [4, 5]]
widths list = [2, 2] 2
TS = [[2, 12, 4], [5]]
widths list = [3, 1] 2
genus = 0
```

```
Wolfarht level = 3
  representative No. 2
  (x,y,eps) = ( [[0, 1, 2]] [[0], [1, 2]] [0 1 0] )
  representative No. 4
  (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [1 0 0] )
  representative No. 5
  (x,y,eps) = ( [[0, 1, 2]] [[0, 1, 2]] [1 1 0] )
  representative No. 12
  (x,y,eps) = ( [[0], [1, 2]] [[0, 1, 2]] [0 1 0] )
```