A Geometric Study on Ramanujan's Modular Equations and Hecke Groups

A Geometric Study on Ramanujan's Modular Equations and Hecke Groups

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Abstract

Srinivasa Ramanujan recorded many remarkable formulae for the solutions to generalized modular equations without proofs. Inspired by the work of Ramanujan, many people have studied generalized modular equations and numerous formulae found by Ramanujan. Many decades later, proofs of those formulae were provided by making use of highly nontrivial identities for theta series and hypergeometric functions. These formulae known as modular equations can be transformed into polynomial equations. There is an intimate relation between Hecke groups and generalized modular equations. Based on the relation, we offer a geometric approach to the proof of those formulae. We emphasize that our approach does not need any knowledge about the identities for Jacobi's theta functions and hypergeometric functions. Without prior knowledge about Ramanujan's formulae, one can derive those formulae through our approach. We prove that the solutions to generalized modular equations satisfy polynomial equations. There is no developed theory about how to find the degrees of those polynomials explicitly. We determine the degrees of those polynomials explicitly in terms of the indices of Hecke subgroups. In this thesis, we also study the relation between Hecke groups and modular equations in Ramanujan's theories of signatures 2, 3, and 4. Furthermore, we present some applications of our results by deriving geometrically some known modular equations.

To the Memory of My Father

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Symbol	Description
$\mathbb{C}$	the complex plane
$\widehat{\mathbb{C}}$	the Riemann sphere
$\mathbb{D}$	the open unit disc $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathbb{H}$	the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$
$\mathbb{H}^*$	union of the upper half-plane and the set of cusps of a Fuchsian group Γ
$\partial\mathbb{H}$	boundary of the upper half-plane $\mathbb{H}$
$\Gamma_0(m)$	congruence subgroup
$\Gamma(m)$	principal congruence subgroup of level m
$\Lambda(\Gamma)$	the limit set of the Fuchsian group Γ
Γ_τ	the stabilizer subgroup of Γ with respect to τ
H_k	the Hecke group
$\mathcal{K}(z)$	the complete elliptic integral of the first kind
$\mathcal{E}(z)$	the complete elliptic integral of the second kind
$\varphi_{t,L}(r)$	the modular function of degree $\frac{1}{L}$
$\mathbb{C}(x)$	the field of rational functions of x with coefficients in $\mathbb{C}$
$\mathbb{C}[x]$	the $\mathbb{C}$-algebra of polynomials in x
$\operatorname{Aut}(\mathbb{H})$	the group of analytic automorphisms of $\mathbb{H}$
G_q	covering group of the canonical projection $\pi_q : \mathbb{H} \to \Gamma\backslash\mathbb{H}$
$\lvert\Gamma_1 : \Gamma_2\rvert$	index of the subgroup Γ_2 in Γ_1
$\Gamma\backslash\mathbb{H}$	quotient Riemann surface
$\operatorname{Area}(\Gamma\backslash\mathbb{H})$	hyperbolic area of the quotient surface $\Gamma\backslash\mathbb{H}$
$\hat{X}$	compactification of the quotient Riemann surface $X = \Gamma\backslash\mathbb{H}$
$\Psi(N)$	the Dedekind psi function of $N \in \mathbb{N}$

Chapter 1

Introduction

For a given integer $p \geq 2$ and $t \in (0, \frac{1}{2}]$, Srinivasa Ramanujan, an Indian mathematical genius, extensively studied the equation

$$\frac{{}_2F_1(t, 1-t; 1; 1-\beta)}{{}_2F_1(t, 1-t; 1; \beta)} = p\,\frac{{}_2F_1(t, 1-t; 1; 1-\alpha)}{{}_2F_1(t, 1-t; 1; \alpha)}, \tag{1.1}$$

which is known as the generalized modular equation of degree p and signature $\frac{1}{t}$. Here, ${}_2F_1(a, b; c; z)$ denotes the Gaussian hypergeometric function whose definition will be given in Chapter 2. Ramanujan left many remarkable formulae known as modular equations describing relations between α and β in his unpublished notebooks but he did not record any proof of those formulae (see [16] and [63]). There were no developed theories related to Ramanujan's modular equations before the 1980s. Some mathematicians, for example, B. C. Berndt, S. Bhargava, J. M. Borwein, P. B. Borwein, F. G. Garvan developed and organized the theories and tried to give the proofs of many identities recorded by S. Ramanujan (see [15], [16], [17], [19], [21], [23], [75]). Also, G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen and others have investigated the theory of Ramanujan's modular equations from different perspective (see, e.g., [7] and [10]).

The case when $\frac{1}{t} = 2$ corresponds to the classical modular equation. Indeed, the complete elliptic integral of the first kind is described by

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}} = \frac{\pi}{2}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; r^2)$$

and the function

$$\mu(r) = \frac{\pi}{2} \cdot \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)} = \frac{\pi}{2} \cdot \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-r^2)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; r^2)}$$

is known to be the modulus of the Grötzsch ring

$$\{z \in \mathbb{C} : |z| < 1\} \setminus [0, r]$$

for $0 < r < 1$. The function $\mu(r)$ plays an important role in the theory of plane quasiconformal mappings (see, for instance, [10] or [49]). The generalized modular equation (1.1) for $\frac{1}{t} = 2$ now takes the form $\mu(s) = p\mu(r)$ with $\alpha = r^2$ and $\beta = s^2$. When $p = 2$, the solution to this modular equation is given by

$$s = \frac{1-\sqrt{1-r^2}}{1+\sqrt{1-r^2}} = \left(\frac{1-\sqrt{1-r^2}}{r}\right)^2 \tag{1.2}$$

(see [49, (2.4) on p. 60] or [10, (5.4)]). Mathematicians of nineteenth century studied the classical case deeply. For example, R. Russell studied the classical case systematically and obtained many modular equations which are known as Russel-type modular equations (see [65], [66]). Jacobi also studied this case and found modular equations for $p = 3$ and 5. The following equation is Jacobi's modular equation of degree $p = 3$:

$$y^4 + 2x^3y^3 - 2xy - x^4 = 0, \tag{1.3}$$

where $x = \alpha^{\frac{1}{8}}$ and $y = \beta^{\frac{1}{8}}$. Influenced by Jacobi, Sohnke found modular equations of degrees $p = 7, 11, 13, 17, 19$. Those modular equations are known as Jacobi-Sohnke equations (see [35, p. 495], [61]). Schläfli tried to find simpler forms of modular equations using the modular functions

$$g(\tau) = \left(\frac{2^4}{\alpha(1-\alpha)}\right)^{\frac{1}{24}} \tag{1.4}$$

and

$$g(p\tau) = \left(\frac{2^4}{\beta(1-\beta)}\right)^{\frac{1}{24}}. \tag{1.5}$$

For instance, the following equation is modular equation of degree $p = 3$

$$\left(\frac{x}{y}\right)^6 + \left(\frac{y}{x}\right)^6 = x^3y^3 - \frac{8}{x^3y^3}, \tag{1.6}$$

where $x = g(\tau)$ and $y = g(3\tau)$. Schläfli found modular equations of prime degrees $p = 3, 5, 7, 11, 13, 17, 19$, and of composite degree $p = 9$. Those equations are known as Schläfli modular equations (see [68]). In the classical case, Weber also found modular equations of prime degrees $p = 3, 5, 7, 11, 13, 17, 19, 23, 31, 47, 71$, and of composite degree $p = 15$. He developed function-theoretic technique to find those modular equations (see [77]).

Ramanujan mainly considered the cases when the signature $\frac{1}{t} \in \{3, 4, 6\}$. See [14], [15], [16], [18], [21] and [62] for his work in relation with the modular equations. Berndt, Bhargava and Garvan [19] derived the modular equations obtained by Ramanujan (see also [7]). Their derivations make use of highly nontrivial identities for Jacobi's theta functions and hypergeometric functions in addition to a number of ingenious ideas. For example, they gave rigorous proofs for the following results.

Theorem 1.1 ([19, Theorem 7.1])**.** *When $p = 2$ and $\frac{1}{t} = 3$, the solutions α and β to the equation* (1.1) *are related by*

$$(\alpha\beta)^{1/3} + \left\{(1-\alpha)(1-\beta)\right\}^{1/3} = 1. \tag{1.7}$$

Theorem 1.2 ([19, Lemma 7.4])**.** *When $p = 3$ and $\frac{1}{t} = 3$, the solutions α and β to the equation* (1.1) *are related by*

$$(1-\alpha)^{1/3} = \frac{1-\beta^{1/3}}{1+2\beta^{1/3}}. \tag{1.8}$$

We note that the above relations can be transformed to polynomial equations. For instance, (1.7) may be transformed to

$$(2\alpha-1)^3\beta^3 - 3\alpha(4\alpha^2 - 13\alpha + 10)\beta^2 + 3\alpha(2\alpha^2 - 10\alpha + 9)\beta - \alpha^3 = 0. \tag{1.9}$$

In particular, we observe that there are at most three values of β satisfying the modular equation (1.7) for each α. We can say that α and β satisfy a polynomial equation of degree 3 in this case. It is rather surprising that α and β are related algebraically, because the hypergeometric function is transcendental for the corresponding parameters. For instance, we do not have a complete answer to the question for which p and t the solutions to the generalized modular equation (1.1) are algebraic. In this thesis, we propose a geometric approach to this problem. In particular, the geometric observation suggests that it is more natural to look at $q = \frac{1}{1-2t}$ rather than the signature $\frac{1}{t}$. We will call q the *order* of the modular equation (1.1). For instance, the signatures $\frac{1}{t} = 2, 3, 4, 6$ correspond to $q = \infty, 3, 2, 3/2$, respectively. Though our approach does not cover the case $\frac{1}{t} = 6$, it may allow us to approach other cases when q are integers > 3.

In Chapter 2, we introduce the background materials on the group $\mathrm{SL}(2, \mathbb{R})$, Fuchsian groups, quotient spaces, Hecke groups, hypergeometric functions and modular equations. For a subgroup K of the Hecke group H_k of finite index, we discuss the construction of the fundamental domain for K and the geometric invariants of K. We present some well-known results, which will be used in other chapters, without proofs.

In Chapter 3, we investigate some known results on automorphic functions and on the space of automorphic forms. We show that

$$\alpha(\tau) = \pi_q(\tau) \quad \text{and} \quad \beta(\tau) = \pi_q(M_p\tau) = \alpha(p\tau)$$

are automorphic functions on G_q and $M_p^{-1}G_qM_p$, respectively, where G_q is the covering group of the canonical projection $\pi_q : \mathbb{H} \to G_q\backslash\mathbb{H}$, $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\tau \in \mathbb{H}$. We reprove two results of Y. Yang [79, Theorems 4, 9].

Chapter 4 is devoted to prove that the solutions (α, β) to the generalized modular equation (1.1) in the quotient Riemann surface $G_q\backslash\mathbb{H}$ satisfy the polynomial equation $P(\alpha, \beta) = 0$ (see Theorem 4.1). First, we construct the covering group G_q of the canonical projection $\pi_q : \mathbb{H} \to G_q\backslash\mathbb{H}$. The polynomial $P(\alpha, \beta)$ is irreducible of degree n in each of α and β, where n is the index of the subgroup

$$K = G_q \cap (M_p^{-1}G_qM_p)$$

in G_q. The quotient Riemann surface $Z = K\backslash\mathbb{H}$ will be used as a parameter space of the solutions (α, β) to (1.1). If $\varphi, \psi : Z \to X = G_q\backslash\mathbb{H}$ satisfy the relations

$$\varphi(\rho(\tau)) = \pi_q(\tau) \quad \text{and} \quad \psi(\rho(\tau)) = \pi_q(p\tau),$$

where ρ is the canonical projection $\mathbb{H} \to Z = K\backslash\mathbb{H}$, then the solutions are given by

$$\alpha = \varphi(z) \quad \text{and} \quad \beta = \psi(z)$$

for $z \in Z$. The maps φ and ψ extend to the compactifications $\hat{Z}$ to $\hat{X} = \widehat{\mathbb{C}}$ as n-sheeted branched (analytic) covering maps. This discussion can also be found in [4].

In Chapter 5, we study the relation between Hecke groups and the modular equations in Ramanujan's theories of signatures 2, 3, and 4. As far as we know that there is no developed theory about how to determine the degree of the polynomial $P(\alpha,\beta)$ when the modulus β has degree p over the modulus α in the theory of signature $\frac{1}{t}$. We determine the degree in each of α and β of the polynomial $P(\alpha,\beta)$ explicitly in terms of the indices of Hecke subgroups. We establish some mutually equivalent statements related to Hecke subgroups and modular equations, and prove that $(1-\beta, 1-\alpha)$ is also a solution to the generalized modular equation (1.1) and $P(1-\beta, 1-\alpha)=0$. The contents of this chapter were already discussed in [3].

Chapter 6 deals with the modular equations in the theory of signature 2. Parts of this chapter were presented in [4]. In the case of signature $\frac{1}{t}=2$, the covering group G_∞ is the principal congruence subgroup $\Gamma(2)$. We consider the cases $p=2$ and $p=3$. We construct the fundamental domains for the subgroups

$$G_\infty \cap (M_2^{-1} G_\infty M_2) \quad \text{and} \quad G_\infty \cap (M_3^{-1} G_\infty M_3).$$

We find the side-pairing transformations as the generators of the subgroup

$$K = G_\infty \cap (M_p^{-1} G_\infty M_p)$$

so that $M_p K M_p^{-1} \in G_\infty$. Applying the results of the preceding chapters, we derive geometrically the following modular equations

$$\beta = \left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2 \tag{1.10}$$

and

$$(\alpha\beta)^{1/4} + \left\{(1-\alpha)(1-\beta)\right\}^{1/4} = 1 \tag{1.11}$$

corresponding to the cases $p=2$ and $p=3$, respectively. Equations (1.10) and (1.11) can be transformed into the following polynomial equations

$$P(x,y) = x^2y^2 - 2(x^2-8x+8)y + x^2$$

and

$$P(x,y) = y^4 + 2x^3y^3 - 2xy - x^4,$$

respectively.

In Chapter 7, we consider the modular equations in the theory of signature $\frac{1}{t} = 3$. For the case of signature 3, the covering group G_3 of the canonical map $\pi_3 : \mathbb{H} \to G_3 \backslash \mathbb{H}$ is generated by

$$\begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

We construct the fundamental domains for the subgroups

$$K = G_3 \cap (M_p^{-1} G_3 M_p),$$

where $p = 2$, 3, 5 and find the generators of the subgroup K so that $M_p K M_p^{-1} \in G_3$. In this chapter, we geometrically deduce the modular equations (1.7) and (1.8), which correspond to the cases $p = 2$ and $p = 3$, respectively. Parts of this chapter can also be found in [4]. Since the quotient Riemann surface $K\backslash\mathbb{H}$, where

$$K = G_3 \cap (M_5^{-1} G_3 M_5),$$

is a genus one surface, we are not able to apply our approach to derive the corresponding modular equation.

In this thesis, we treat only the case when the Riemann surface $Z = K\backslash\mathbb{H}$, where $K = G_q \cap (M_p^{-1} G_q M_p)$, is planar. When Z is non-planar, it is technically difficult to find an explicit form of the polynomial $P(x, y)$. Let $\hat{Z}$ be the compactification of Z. Since the parametrizations

$$\varphi, \psi : \hat{Z} \to \widehat{\mathbb{C}}$$

are n-sheeted covering maps with critical values contained in $\{0, 1, \infty\}$ if

$$|G_q : K| = n,$$

Belyĭ's theorem implies that the compact Riemann surface $\hat{Z}$ is an algebraic curve defined over $\overline{\mathbb{Q}}$ (see [40]). Therefore, in principle, we could determine the surface Z and maps φ, ψ by the combinatorial information about the coverings. We hope to give further examples when Z is non-planar in the future work.

Chapter 2

Preliminaries

This chapter presents some basic notions related to the group $\mathrm{SL}(2,\mathbb{R})$, Fuchsian groups, quotient spaces, Hecke groups, hypergeometric functions and modular equations. The construction of fundamental domains for Hecke subgroups and their geometric invariants are also discussed in this chapter. We state some well-known results, which are relevant in subsequent chapters, without proofs.

2.1 The Group $\mathrm{SL}(2,\mathbb{R})$

Let $\mathbb{H}$ denote the upper half-plane $\{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$. The group $\mathrm{SL}(2,\mathbb{R})$ is defined by

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}$$

and it is the group of orientation-preserving isometries of the upper half-plane $\mathbb{H}$. Let I_2 denote the 2×2 identity matrix, then $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}$ (see [71, Chapter VII]). The group $\mathrm{PSL}(2,\mathbb{R})$ acts on the upper half-plane $\mathbb{H}$ as follows:

$$\tau \mapsto \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d},\ \text{for}\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}),\ \tau\in\mathbb{H}.$$

All transformations of $\mathrm{PSL}(2,\mathbb{R})$ are conformal. The group of automorphisms of the upper half-plane $\mathbb{H}$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ and is given by

$$\mathrm{Aut}(\mathbb{H}) = \left\{ \tau \mapsto \frac{a\tau+b}{c\tau+d} : a,b,c,d\in\mathbb{R} \text{ and } ad-bc\neq 0 \right\}.$$

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. The group of automorphisms of the unit disc $\mathbb{D}$ is given as follows:

$$\mathrm{Aut}(\mathbb{D}) = \left\{ \tau \mapsto e^{i\theta} \frac{\tau - \omega}{1 - \overline{\omega}\tau} : \theta \in \mathbb{R} \text{ and } \omega \in \mathbb{D} \right\},$$

which is also isomorphic to $\mathrm{PSL}(2, \mathbb{R})$.

The boundary of the upper half-plane $\mathbb{H}$ is $\mathbb{R} \cup \infty$. Semicircles orthogonal to the real axis and vertical lines are called *geodesics*. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and let $\mathrm{tr}(\gamma)$ denote the trace of γ, then the element γ is said to be *elliptic*, *parabolic* and *hyperbolic* when $|\,\mathrm{tr}(\gamma)| < 2$, $|\,\mathrm{tr}(\gamma)| = 2$ and $|\,\mathrm{tr}(\gamma)| > 2$, respectively. If $\gamma \in \mathrm{SL}(2, \mathbb{R})$ and $\gamma \neq \pm I_2$, then

(i) γ has only one fixed point on $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ if and only if γ is parabolic,

(ii) γ has only one fixed point τ in $\mathbb{H}$ (and the other fixed point is $\overline{\tau}$) if and only if γ is elliptic,

(iii) γ has two fixed points on $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ if and only if γ is hyperbolic,

see [72, Proposition 1.13].

For $D \subseteq \mathbb{H}$ and $z = x + iy \in \mathbb{H}$, let $\mathrm{Area}(D)$ denote the hyperbolic area given by

$$\mathrm{Area}(D) = \int_D \frac{dx\,dy}{y^2},$$

provided the integral exists. Then, $\mathrm{Area}(\gamma(D)) = \mathrm{Area}(D)$ for all $\gamma \in \mathrm{PSL}(2, \mathbb{R})$, that is, $\mathrm{Area}(D)$ is invariant under $\mathrm{PSL}(2, \mathbb{R})$. If Δ is a hyperbolic triangle, then $\mathrm{Area}(\Delta)$ depends on the angles of Δ by the following theorem (see [42, p. 13]).

Theorem 2.1 (Gauss-Bonnet)**.** *For a hyperbolic triangle Δ with angles θ_1, θ_2 and θ_3,*

$$\mathrm{Area}(\Delta) = \pi - (\theta_1 + \theta_2 + \theta_3).$$

2.1.1 The Modular Group $\mathrm{SL}(2, \mathbb{Z})$

The *modular group* $\mathrm{SL}(2, \mathbb{Z})$ is defined by

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I_2\}$. The generators of $\mathrm{SL}(2,\mathbb{Z})$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If m is a positive integer, then the *principal congruence subgroup* $\Gamma(m)$ is defined as

$$\Gamma(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : a \equiv d \equiv 1 \,(\mathrm{mod}\ m)\,\mathrm{and}\, b \equiv c \equiv 0 \,(\mathrm{mod}\ m) \right\},$$

and the *congruence subgroup* $\Gamma_0(m)$ is defined as

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) : c \equiv 0 \,(\mathrm{mod}\ m) \right\},$$

see [30, Chapter 1] and [71, Chapter VII] for more details on modular group.

2.2 Fuchsian Groups

A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, i.e., it is a group of orientation-preserving isometries of the upper half-plane $\mathbb{H}$.

The upper half-plane $\mathbb{H}$ or the unit disc $\mathbb{D}$ is invariant under a Fuchsian group Γ. The group Γ acts properly discontinuously on $\mathbb{H}$ or $\mathbb{D}$. The definition of properly discontinuous action will be given in Section 2.3. Note that Γ acts properly discontinuously on any conformal image of $\mathbb{D}$ (see [13, p. 121]). For any Fuchsian group Γ, let $\Lambda(\Gamma)$ denote the limit set of Γ. Then, $\Lambda(\Gamma) \subseteq \mathbb{R} \cup \{\infty\}$ for the upper half-plane model or $\Lambda(\Gamma) \subseteq \partial\mathbb{D}$ for the unit disc model. Let $\mathbf{D}$ be any conformal image of the unit disc $\mathbb{D}$ and let $\partial\mathbf{D}$ denote the boundary of $\mathbf{D}$. Then, a Fuchsian group Γ is of

(i) the *first kind* if $\Lambda(\Gamma) = \partial\mathbf{D}$,

(ii) the *second kind* if $\Lambda(\Gamma)$ is a proper subset of $\partial\mathbf{D}$,

see [13, p. 188] or [42, p. 67] for details.

Let us consider a Fuchsian group Γ, then a point $\tau \in \mathbb{H}$ is called an *elliptic point* of Γ if $\gamma(\tau) = \tau$ for an elliptic element $\gamma \in \Gamma$. The set of elliptic points of a

Fuchsian group Γ does not have a limit point in $\mathbb{H}$. For an elliptic point τ of Γ,

$$\Gamma_\tau := \{\gamma \in \Gamma : \gamma(\tau) = \tau\}$$

is called the *isotropy subgroup* or *stabilizer subgroup* of Γ with respect to τ. The subgroup Γ_τ is a cyclic group of finite order. Also, we call a point x of $\mathbb{R} \cup \{\infty\}$ a *cusp* of Γ if $\sigma(x) = x$ for a parabolic element $\sigma \in \Gamma$. One can prove the following result (see, e.g., [42, Theorem 2.2.3]).

Theorem 2.2. *Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Then, the following statements hold:*

(i) when Γ is parabolic or hyperbolic cyclic, it is a Fuchsian group,

(ii) when Γ is elliptic cyclic, then Γ is a Fuchsian group if and only if it is finite.

If a Fuchsian group Γ has cusps, then the quotient Riemann surface $\Gamma\backslash\mathbb{H}$ is not compact. Let $\mathbb{H}^*$ denote the union of the upper half-plane and the set of cusps of Γ. Since Γ acts on $\mathbb{H}^*$, the quotient surface $\Gamma\backslash\mathbb{H}^*$ is a compact Riemann surface if Γ has finite number of cusps. If Γ has no cusps, then $\mathbb{H}^* = \mathbb{H}$ and the quotient surface $\Gamma\backslash\mathbb{H}$ is compact (see [72, Section 1.3]).

For a Fuchsian group Γ, consider a subset F of $\mathbb{H}$. The subset F is called a *fundamental domain* for Γ if the following conditions (see [72, p. 15]) are satisfied:

(i) F is open and connected,

(ii) all points of F are Γ-inequivalent,

(iii) each point of $\mathbb{H}$ is Γ-equivalent to a point of the closure of F.

There is a fundamental domain for every Fuchsian group Γ. Fundamental domain for a given Fuchsian group can be constructed in different ways. If the area of a fundamental domain for Γ is finite, then it is invariant under Γ.

Theorem 2.3 ([42, Theorem 3.1.1])**.** *For a given Fuchsian group Γ, assume that F_1 and F_2 are two fundamental domains for Γ with $\mathrm{Area}(F_1) < \infty$. If the hyperbolic areas of the boundaries ∂F_1 and ∂F_2 are zero, then $\mathrm{Area}(F_1) = \mathrm{Area}(F_2)$.*

2.3 Quotient Spaces

Assume that $\mathcal{M}$ is a topological space and Γ is a topological group. For $x \in \mathcal{M}$ and $\gamma \in \Gamma$, the mapping $x \mapsto \gamma \cdot x$ is a homeomorphism of $\mathcal{M}$ onto itself. For every $x \in \mathcal{M}$, the Γ-orbit of x is given by

$$\Gamma x = \big\{\gamma \cdot x : \gamma \in \Gamma\big\}.$$

If E is a compact subset of $\mathcal{M}$, then Γ acts *properly discontinuously* on $\mathcal{M}$ if and only if

$$\gamma(E) \cap E \neq \emptyset$$

for finitely many $\gamma \in \Gamma$ (see [13, p. 94]). The set of all Γ-orbits of points on $\mathcal{M}$ is denoted by $\Gamma\backslash\mathcal{M}$. Suppose $\pi : \mathcal{M} \to \Gamma\backslash\mathcal{M}$ is the canonical projection defined by $\pi(x) = \Gamma x$. If $Y \subset \Gamma\backslash\mathcal{M}$, then Y is called open if $\pi^{-1}(Y)$ is open in $\mathcal{M}$. It is well-known that π defines a topology on $\Gamma\backslash\mathcal{M}$. This topology is called the *quotient topology* (see [72, Chapter 1]). The following theorem is a very useful tool to construct Riemann surfaces by making the quotient space through properly discontinuous group action.

Theorem 2.4 ([13, Theorem 6.2.1])**.** *Suppose* $\mathbf{C}$ *is a subdomain of the Riemann sphere* $\widehat{\mathbb{C}}$ *and* $\mathbf{C}$ *is invariant under a group* Γ *of Möbius transformations. If* Γ *acts properly discontinuously on* $\mathbf{C}$*, then* $\Gamma\backslash\mathbf{C}$ *is a Riemann surface.*

Let a Fuchsian group Γ act properly discontinuously on the upper half-plane $\mathbb{H}$. We call Γ *geometrically finite* if the fundamental domain F for Γ has finitely many sides. The vertices of F are isolated (see [42, Chapter 4]).

Theorem 2.5 (Siegel's Theorem)**.** *Let* $\mathrm{Area}(\Gamma\backslash\mathbb{H})$ *denote the hyperbolic area of the quotient surface* $\Gamma\backslash\mathbb{H}$*. If* $\mathrm{Area}(\Gamma\backslash\mathbb{H}) < \infty$*, then* Γ *is geometrically finite.*

Consider the canonical projection $\pi : \mathbb{H} \to \Gamma\backslash\mathbb{H}$. The map π is open and continuous. Let π_F denote the restriction of the map π to the fundamental domain F for Γ, then the congruent points, i.e., the Γ-equivalent points of F are identified by the map

$$\pi_F : F \to \Gamma\backslash F.$$

Note that, for $\tau \in \mathbb{H}$, the orbits $\Gamma\tau$ and the sets $F \cap \Gamma\tau$ are the elements of $\Gamma\backslash\mathbb{H}$ and $\Gamma\backslash F$, respectively. Thus,

$$\pi_F(\tau) = F \cap \Gamma\tau.$$

Let us consider the maps

$$g : F \to \mathbb{H} \quad \text{and} \quad h : F \cap \Gamma\tau \to \Gamma\tau$$

such that

$$h \circ \pi_F = \pi \circ g.$$

Thus, we have the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "\pi_F"'] & \mathbb{H} \arrow[d, "\pi"] \\ \Gamma\backslash F \arrow[r, "h"'] & \Gamma\backslash\mathbb{H} \end{tikzcd}$$

(see [13, Chapter 9]).

Theorem 2.6 ([13, Proposition 9.2.2]). *Consider the maps π_F, g, and h defined above. Then,*

(i) the map π_F is surjective and continuous,

(ii) the map g is injective and continuous,

(iii) the map h is bijective and continuous.

The fundamental domain F for Γ is called *locally finite* if and only if the set

$$E \cap \Gamma\backslash F$$

is finite for each compact subset E of $\mathbb{H}$.

By the definition of fundamental domain, the Γ-equivalent points of F are located at the boundary ∂F. Thus, the sides of F are identified by the action of Γ on F and we obtain an oriented surface $\Gamma\backslash F$. The surface $\Gamma\backslash F$ has (possibly) some marked points, which are the elliptic points of finite orders and cusps. By the following theorem, the quotient surface $\Gamma\backslash\mathbb{H}$ is homeomorphic to $\Gamma\backslash F$.

Theorem 2.7 ([13, Theorem 9.2.4]). *Let F be the fundamental domain for a Fuchsian group Γ. Then, F is locally finite if and only if the map $h : \Gamma\backslash F \to \Gamma\backslash\mathbb{H}$ is a homeomorphism.*

The quotient surface $\Gamma\backslash\mathbb{H}$ is known as an *orbifold*. The hyperbolic area, $\text{Area}(\Gamma\backslash\mathbb{H})$, of the quotient surface $\Gamma\backslash\mathbb{H}$ is equal to the hyperbolic area, $\text{Area}(\Gamma\backslash F)$, of $\Gamma\backslash F$. Let $m_1, \ldots, m_s$ be the orders of the vertices of the fundamental domain F for Γ. For each $\nu = 1, \ldots, s$, the order, m_ν, is finite if the corresponding vertex is an elliptic point and $m_\nu = \infty$ if the corresponding vertex is a cusp. The number of cusps and genus determine the topological configuration of the surface. If g is the genus of the quotient surface $\Gamma\backslash\mathbb{H}$, then $(g; m_1, \ldots, m_s)$ is known as the *signature* of Γ.

Theorem 2.8 ([42, Theorem 4.3.1]). *If* $(g; m_1, \ldots, m_s)$ *is the signature of a Fuchsian group* Γ*, then*

$$\mathrm{Area}(\Gamma\backslash\mathbb{H}) = 2\pi\left((2g-2) + \sum_{\nu=1}^{s}\left(1 - \frac{1}{m_\nu}\right)\right).$$

Assume that for $\nu = 1, \ldots, m$, A_ν are the generators of the elliptic isotropy subgroups of orders m_ν, and for $\mu = 1, \ldots, n$, B_μ are the generators of the cuspidal isotropy subgroups of the Fuchsian group Γ. If the compact Riemann surface $X = \Gamma\backslash\mathbb{H}^*$ has genus zero, then the Fuchsian group Γ can be expressed by the following form:

$$\Gamma = \langle A_1, \ldots, A_m, B_1, \ldots, B_n : A_\nu^{m_\nu} = 1 \text{ for } \nu = 1, \ldots, m \text{ and } A_1 \cdots A_m B_1 \cdots B_n = 1\rangle.$$

The Fuchsian group Γ can be expressed as the free product of A_ν, i.e., its elliptic isotropy subgroups if Γ has a unique cusp. These facts are well-known (see, e.g., [13], [42] and [74]).

2.4 Hecke Groups

For an integer $k \geq 3$, the Hecke group H_k, introduced by E. Hecke [36], is defined as the discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ generated by the two elements $\pm S$ and $\pm T_k$, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_k = \begin{pmatrix} 1 & \lambda_k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda_k = 2\cos\frac{\pi}{k}.$$

See [26] for details about the Hecke groups. Here and in what follows, we often identify a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the Möbius transformation

$$\tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d}$$

so that S and T_k are regarded as elements of $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$.

Let $H_k = H(\lambda_k)$, then $H_3 = H(1)$ is the classical modular group $\mathrm{PSL}(2,\mathbb{Z})$, $H_4 = H(\sqrt{2})$, $H_5 = H\left(\frac{1+\sqrt{5}}{2}\right)$ and $H_6 = H(\sqrt{3})$. For $k > 3$, the set of cusps of H_k is a subset of $\mathbb{Q}[\lambda_k] \cup \{\infty\}$. If we put

$$U_k = T_k S = \begin{pmatrix} \lambda_k & -1 \\ 1 & 0 \end{pmatrix},$$

then U_k is an elliptic element of order k with fixed point at $e^{\pi i/k}$. Let the set F_k be defined by

$$F_k = \left\{\tau \in \mathbb{H} : |\mathrm{Re}\,\tau| \leq \cos\frac{\pi}{k}, |\tau| \geq 1\right\}.$$

Then F_k is a fundamental domain for H_k with side pairing transformations S and T_k. Note that S is an elliptic element of order 2 with fixed point at $\tau = i$. In particular, we see that H_k is a triangle group of signature $(2, k, \infty)$ for $3 \leq k \leq \infty$ (see [13, p. 293]).

Let us restrict ourselves on the case when $k = 2q$ is an even number with $q \geq 2$. Then

$$\hat{F}_q = F_{2q} \cup S(F_{2q})$$

is a fundamental domain for the (normal) subgroup G of H_{2q} of index 2 generated by

$$T_{2q} = \begin{pmatrix} 1 & \lambda_{2q} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W_{2q} = S^{-1}T_{2q}^{-1}S = \begin{pmatrix} 1 & 0 \\ \lambda_{2q} & 1 \end{pmatrix}$$

(see Figure 2.1).

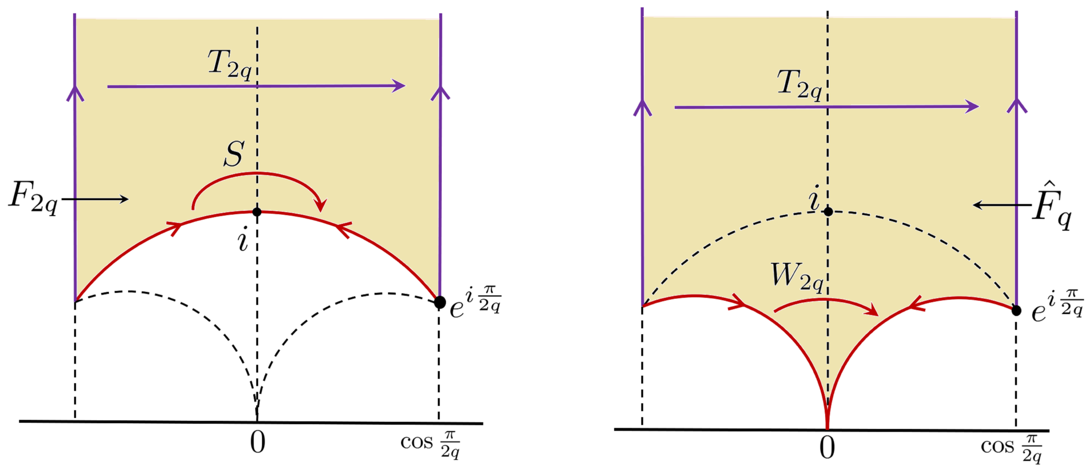


Figure 2.1: Fundamental domains for $H_{2q} = \langle T_{2q}, S\rangle$ and $G = \langle T_{2q}, W_{2q}\rangle$

Let $\mathbb{H}^*$ denote the union of the upper half-plane $\mathbb{H}$ and the set of cusps of H_k. Let Δ be the hyperbolic triangle $(2, k, \infty)$ with vertices at i, $e^{i\frac{\pi}{k}}$ and ∞. If H_k^* is the group generated by the reflections across the sides of the triangle Δ, then $H_k \triangleleft H_k^*$, $|H_k^* : H_k| = 2$ and all transformations of H_k is orientation preserving. Now, let

$$V_k = ST_k^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -\lambda_k \end{pmatrix}.$$

Then, we can consider S and V_k as the independent generators of H_k. In this case, one can show that the interior of F_k defined by

$$F_k = \left\{ \tau \in \mathbb{H} : 0 \leq \operatorname{Re}\tau \leq \cos\frac{\pi}{k}, |2\tau\cos\frac{\pi}{k} - 1| \geq 1 \right\} \tag{2.1}$$

is a fundamental domain for $H_k = \langle S, V_k \rangle$. Note that V_k is a rotation about $e^{i\frac{\pi}{k}}$ of an angle $\frac{2\pi}{k}$ and S is a rotation about i of an angle π (see Figure 2.2).

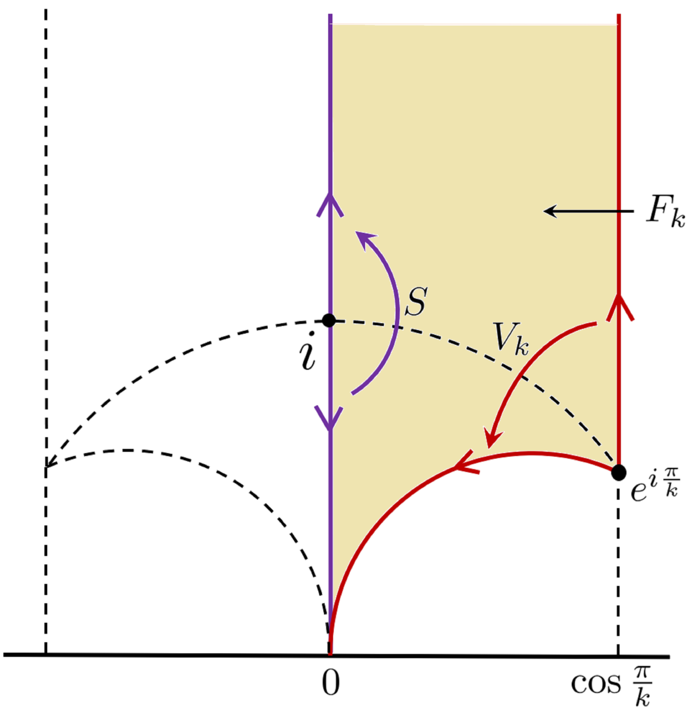


Figure 2.2: Fundamental domain for the Hecke group $H_k = \langle S, V_k \rangle$

There are k-tiles (H_k-translates of F_k) which are joined at the elliptic point $e^{i\frac{\pi}{k}}$. There are three tiles $S(F_k)$, $U_k(F_k)$, and $V_k(F_k)$ adjacent to the tile F_k. In general, if $A \in H_k$, then the tiles $AS(F_k)$, $AU_k(F_k)$, and $AV_k(F_k)$ are adjacent to the tile $A(F_k)$ (see [38]).

2.5 Construction of Fundamental Domain for a Hecke Subgroup

In this section, we mainly discuss some useful results related to the construction of fundamental domains for a subgroup of the Hecke group H_k of finite index. These results and related discussions can be found in [44], [45], and [46].

There are the following two main methods:

(1) Dirichlet's polygon construction (see [48]),

(2) Ford's isometric circle method (see [33])

to construct fundamental domain for a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. To construct a hyperbolically convex fundamental domain by these two methods, one needs to know almost all elements of the group considered. The most useful and convenient method to construct fundamental domain for a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ is to use the right coset decomposition (see [82], [83]). R. S. Kulkarni [44] considered the modular group $\mathrm{PSL}(2,\mathbb{Z})$ which corresponds to the case $k = 3$ and M. L. Lang [46] considered the Hecke group H_k for prime $k > 3$ to show that each subgroup of H_k of finite index has a fundamental domain which is a special polygon.

Suppose that K is a Hecke subgroup which has index $n < \infty$ in H_k, i.e.,

$$|H_k : K| = n.$$

Then, we can express H_k as a disjoint union of n cosets of K as follows

$$H_k = \bigcup_{j=1}^{n} \gamma_j K,$$

where $\gamma_j \in H_k$. Let F be the fundamental domain for the subgroup K. Then, we have

$$F = \bigcup_{j=1}^{n} \gamma_j^{-1} F_k,$$

where F_k is the fundamental domain for H_k. For any $\tau \in \mathbb{H}$, there exists an element $\gamma \in H_k$ such that $\gamma(\tau) \in F_k$. We can find an element $\delta \in K$ so that

$$\gamma = \gamma_j \delta \implies \delta = \gamma_j^{-1} \gamma,$$

for some j. Thus,

$$\delta(\tau) = \gamma_j^{-1}\gamma(\tau) \in \gamma_j^{-1}F_k.$$

As $\gamma_j^{-1}F_k \subset F$, we deduce that $\delta(\tau) \in F$. One can easily show that all interior points of F are K-inequivalents which implies that F is a fundamental domain for the subgroup K. Note that there are many options to choose $\gamma_j \in H_k$. We choose γ_j so that F is simply connected.

Recall that H_k^* is the group generated by the reflections across the sides of the triangle Δ, where Δ is the hyperbolic triangle $(2,\, k,\, \infty)$ with vertices at i, $e^{i\frac{\pi}{k}}$ and ∞. Let $\mathcal{T}$ denote the tessellation or tiling of $\mathbb{H}$ constructed by the H_k^*-translates of Δ. We call the H_k^*-translates of $e^{i\frac{\pi}{k}}$, i and ∞, respectively, *odd vertices*, *even vertices*, and *cusps* of $\mathcal{T}$. We denote by (τ_1, τ_2) the geodesic joining τ_1 to τ_2. The H_k^*-translates of the geodesic joining $e^{i\frac{\pi}{k}}$ to ∞ (resp. joining i to ∞) are known as *odd edges* (resp. *even edges*) of $\mathcal{T}$. The H_k^*-translates of the geodesics joining i to $e^{i\frac{\pi}{k}}$ are known as *f-edges* of $\mathcal{T}$. The geodesic joining 0 to ∞, i.e., $(0, \infty)$ comprises two even edges and the *even lines* of $\mathcal{T}$ are the H_k^*-translates of $(0, \infty)$ (see [44], [45], [46]). The even lines of $\mathcal{T}$ always intersect at the boundary of $\mathbb{H}$ and we obtain a tessellation, which we denote by $\mathcal{T}^*$, of $\mathbb{H}$ into ideal k-gons by the even lines. Consider the canonical projection $\pi : \mathbb{H} \to K\backslash\mathbb{H}$ and the even line $(0, \infty)$. Let us denote the tile by T whose boundary contains the projection of the even line $(0, \infty)$ by π, i.e., $\pi(0, \infty)$. The tile T is developed to $\mathbb{H}$ such that $\pi(0, \infty)$ is developed to $(0, \infty)$. Next, the tiles adjacent to T are developed. In this manner, all tiles of $K\backslash\mathbb{H}$ are developed one by one. For $K < H_k$, if s is a cusp of $K\backslash\mathbb{H}$, then the number of even lines in $K\backslash\mathbb{H}$ meeting at s determines the width of the cusp s. If a finite number of k-gons form a convex hyperbolic polygon which belongs to the tessellation $\mathcal{T}^*$, then the polygon is called an *ideal polygon*. If the set of odd vertices is regarded as the set of vertices of a tree, then the f-edges construct a k-regular tree. A unique vertex, say v_F, of the k-regular tree is contained in every k-gon F. The distance between the vertex at i and v_F is known as the *depth* of F denoted by $d(F)$. Note that the distance of the vertices adjacent to the vertex at i is 1.

For $\lambda_k = 2\cos\frac{\pi}{k}$, the set of cusps of the Hecke group H_k is a subset of $\mathbb{Q}[\lambda_k] \cup \{\infty\}$. A cusp $s = \frac{a}{c}$ is said to be in *reduced form* if

(i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_k$ for some $\frac{b}{d}$,

(ii) $c \geq 0$.

Lemma 2.9 ([45, Lemma 3.2])**.** *Let $\{x_1, x_2, \ldots, x_k\}$, where $x_1 < \cdots < x_k$, be the set of cusps of an ideal k-gon F (if ∞ is a cusp, then $x_k = \infty = \frac{1}{0}$ or $x_1 = -\infty = \frac{-1}{0}$ depending on F lies in the right half-plane or in the left half-plane). If $x_j = \frac{a_j}{c_j}$ is in reduced form and $\frac{a_0}{c_0} = \frac{-a_k}{-c_k}$, then*

$$a_j = \lambda_k a_{j-1} - a_{j-2}$$

and

$$c_j = \lambda_k c_{j-1} - c_{j-2}$$

for $2 \leq j \leq k-1$.

If the both end points of a geodesic belong to the boundary of $\mathbb{H}$, i.e., $\mathbb{R} \cup \{\infty\}$, then the geodesic is called a *complete geodesic*. We regard ∞ as $\frac{1}{0}$, and a whole number m as $\frac{m}{1}$.

Consider the fundamental domain F_k for the Hecke group H_k, where F_k is the hyperbolic triangle with vertices at ∞, 0 and $e^{i\frac{\pi}{k}}$. Then, a *special triangle* is the H_k-translate of F_k. For $1 \leq l < k$ and $l|k$, let

$$D_l = F_k \cup V_k(F_k) \cup V_k^2(F_k) \cup \cdots \cup V_k^{l-1}(F_k).$$

Then, l copies of the special triangle F_k (all of them meet at $e^{i\frac{\pi}{k}}$) form D_l. We call the H_k-translates of D_l the *l-clusters*. Each l-cluster has one odd vertex and $l+1$ cusps known as *free vertices*. The boundary ∂D_l consists of l even lines and two odd edges. The two odd edges make an angle $\frac{2l\pi}{k}$ at the odd vertex. If the set of cusps of the depth 1 k-gon in the right half-plane is given by

$$\{s_1 = 0 = \frac{0}{1}, s_2, \ldots, s_{k-1}, s_k = \infty = \frac{1}{0}\}, \tag{2.2}$$

then the set of free vertices of the l-cluster D_l is given by

$$\{s_1 = 0 = \frac{0}{1}, s_2, \ldots, s_l, s_k = \infty = \frac{1}{0}\}, \tag{2.3}$$

where s_j's are arranged in ascending order (see [45]).

Let F_0 be an ideal polygon and let F be a convex hyperbolic polygon. Assume that F is the union of F_0 and a particular number of special triangles attached externally to F_0. Let us denote by $F_K = (F, \mathcal{A}_K)$ a convex hyperbolic polygon F along with a set of side-pairing transformations $\mathcal{A}_K$. We call F_K a special polygon if the following rules are satisfied.

(A) We consider 0 and ∞ as two of the vertices of the polygon F.

(B) Two odd edges are always paired with each other. If e_1 and e_2 are two odd edges which are paired, then the internal angle between them is $\frac{2\pi}{k}$ and they are considered as sides of F. In this case, e_1 and e_2 are known as odd sides.

(C) If e_1 and e_2 are two even edges in ∂F which form an even line, then either

  (i) e_1 and e_2 make a free side of F; a different free side of F is paired with this free side, or

  (ii) the edges e_1 and e_2 are paired, in that case the both edges are regarded as sides of F; they are known as even sides of F.

For a special polygon $F_K = (F, \mathcal{A}_K)$, a cusp in F is known as *free vertex* of F_K. Consider a finite sequence

$$\{-\infty = \frac{-1}{0} = x_{-1},\, x_0,\, x_1,\, \ldots,\, x_m,\, x_{m+1} = \infty = \frac{1}{0}\} \tag{2.4}$$

of numbers in ascending order such that

(i) for $0 \leq j \leq m$, $x_j \in \mathbb{Q}[\lambda_k]$ and $x_j = 0$ for some j,

(ii) each $x_j = \frac{a_j}{c_j}$ is in reduced form,

(iii) if $a_{j+1}c_j - a_jc_{j+1} \neq 1$, there exists $\gamma \in H_k$ and an l-cluster D_l, where $1 < l < k$ and $l|k$, so that $\gamma(s_l) = x_j$, $\gamma(s_k) = x_{j+1}$, where s_l and s_k are given in (2.3),

(iv) if $a_{j+1}c_j - a_jc_{j+1} = 1$, then $\{x_j,\, x_{j+1}\}$ is known as ordinary interval and the geodesic $(x_j,\, x_{j+1})$ is an even line.

The sequence in (2.4) is called *Hecke-Farey sequence*. If a Hecke-Farey sequence is equipped with an additional structure of side-pairings between adjacent points x_j's, then it is called a *Hecke-Farey symbol*. For details on Hecke-Farey symbol and side-pairings, see [45] and [46].

Proposition 2.10 ([45, Proposition 4.4])**.** *The set of Hecke-Farey symbols and the set of special polygons are related by a one to one correspondence.*

Suppose F_1 and F_2 are two special polygons. We call F_1 and F_2 are equivalent if one can be constructed from other using elementary cut operations (see [46, p. 588]) and we write $F_1 \sim F_2$.

Theorem 2.11 ([46, Theorem 11]). *Consider a subgroup K of the Hecke group H_k and two special polygons F_1 and F_2. Then, F_1 and F_2 generate the same subgroup K if and only if $F_1 \sim F_2$.*

The following theorem is related to the fundamental domain for the subgroup K of H_k.

Theorem 2.12 ([45, Theorem 5.1]). *Let $F_K = (F, \mathcal{A}_K)$ be a special polygon. Then, there exists a subgroup K of H_k generated independently by the set $\mathcal{A}_K$ of side-pairing transformations so that the hyperbolic polygon F is a fundamental domain for K.*

The special polygon $F_K = (F, \mathcal{A}_K)$ is known as the *admissible fundamental domain* for the subgroup K of H_k.

Theorem 2.13 ([45, Theorem 5.2]). *Every subgroup K of the Hecke group H_k of finite index has an admissible fundamental domain $F_K = (F, \mathcal{A}_K)$.*

In [38], B. Ibrahimou and O. Yayenie extended the results of Kulkarni [44] and Lang [46] to show that for any $k \geq 3$ every subgroup of finite index of the Hecke modular group H_k has fundamental domain which is a special polygon and they call it *convex standard fundamental domain* for a Hecke subgroup K.

Proposition 2.14 ([38, Proposition 2.2]). *Let $K < H_k$ and $|H_k : K| = n < \infty$. Let the set $\mathcal{S}$ contain a finite number of inequivalent elements of H_k modulo K. If*

$$F = \bigcup_{A \in \mathcal{S}} A(F_k)$$

is connected and if any tile adjacent to F is equivalent to a tile $A(F_k) \subset F$ modulo K, then $|\mathcal{S}| = n$, i.e.,

$$H_k = K \cdot \mathcal{S}.$$

The following theorem is proved in [38] using algorithmic approach.

Theorem 2.15 ([38, Theorem 3.1]). *Let the index of the subgroup K be $n < \infty$ in the Hecke group H_k. Then, there exist $r < \infty$ elements $A_1, A_2, \ldots, A_r \in H_k$ and r disjoint sets*

$$\mathcal{S}_j := \left\{ A_j,\, A_j V_k,\, \ldots,\, A_j (V_k)^{m_j - 1} \right\}$$

for $j = 1, 2, \ldots, r$ so that

1. $n = m_1 + m_2 + \cdots + m_r$, *where each* $m_j | k$,

2. $H_k = K \cdot \Sigma$, *where* $\Sigma = \bigcup_{j=1}^{r} \mathcal{S}_j$,

3. $F = \bigcup_{A \in \Sigma} A(F_k)$ *is a convex standard fundamental domain for* K.

2.6 Geometric Invariants of a Hecke Subgroup

Let K be a subgroup of the Hecke group H_k of index $n < \infty$. Let $F_K = (F, \mathcal{A}_K)$ be an admissible fundamental domain for K, that is, $F_K = (F, \mathcal{A}_K)$ is a special polygon related to K. One can determine the geometric invariants of the quotient Riemann surface by studying the special polygon $F_K = (F, \mathcal{A}_K)$. From the subgroup relation $K < H_k$, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{H} \arrow[d] \arrow[dr] & \\ K\backslash\mathbb{H} \arrow[r] & H_k\backslash\mathbb{H}. \end{tikzcd}$$

The following invariants are the geometric invariants (see [44, Section 7] and [45, Section 6]) of the subgroup K:

(i) the number, say ν_2, of elliptic or branch points of order 2 of $\mathbb{H} \to K\backslash\mathbb{H}$,

(ii) the number, say ν_k, of elliptic or branch points of order k of $\mathbb{H} \to K\backslash\mathbb{H}$,

(iii) the degree, $n = |H_k : K|$, of the branched or ramified covering

$$K\backslash\mathbb{H} \to H_k\backslash\mathbb{H},$$

which is the number of special triangles in the special polygon $F_K = (F, \mathcal{A}_K)$,

(iv) the number, say ν_l, of inequivalent classes of order l elliptic elements which are conjugates of $V_k^{\frac{k}{l}}$, where $1 < l < k$ and $l | k$,

(v) the number, say ν_∞, of inequivalent cusps of K, that is, the number of punctures of the quotient Riemann surface $K\backslash\mathbb{H}$,

(vi) the width of the j-th inequivalent cusp, say $w(s_j)$, $j = 1, 2, \ldots, \nu_\infty$,

(vii) the genus, g, of the quotient Riemann surface $K\backslash\mathbb{H}$,

(viii) the rank, say r, of the fundamental group of $K\backslash\mathbb{H}$, i.e., $\pi_1(K\backslash\mathbb{H})$.

One can describe (i), (ii), (iii), and (viii) in the context of group theory as follows:

- ν_2 = the number of inequivalent classes of order 2 elements which are conjugates of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or the number of conjugacy classes of order 2 elements in the independent set of generators of K,
- ν_k = the number of inequivalent classes of order k elements which are conjugates of $V_k = \begin{pmatrix} 0 & -1 \\ 1 & -\lambda_k \end{pmatrix}$, or the number of conjugacy classes of order k elements in the independent set of generators of K,
- n = the index of the subgroup K in the Hecke group H_k,
- r = the number of cycles of the permutation representation of T_k on the set of cosets H_k/K, or the rank of the free factors of K, or the number of infinite order generators in the independent set of generators of K.

Note that one can obtain r from the side-pairing transformations of the special polygon $F_K = (F, \mathcal{A}_K)$. If $2 \leq l_j \leq k$ and $l_j|k$, then the genus of the quotient Riemann surface $K\backslash\mathbb{H}$ is given by the Riemann Hurwitz formula as follows:

$$2g - 2 = n\left(\frac{k-2}{2k}\right) - \frac{\nu_2}{2} - \sum_{\substack{l_j|k \\ 2\leq l_j \leq k}} \nu_{l_j}\left(1 - \frac{1}{l_j}\right) - \nu_\infty. \tag{2.5}$$

Also, g, r and ν_∞ are related by

$$2g = r - \nu_\infty + 1. \tag{2.6}$$

2.7 Hypergeometric Functions

For complex numbers a, b, c with $c \neq 0, -1, -2, \ldots$, and nonnegative integer n, the *Gaussian hypergeometric function*, ${}_2F_1(a, b; c; z)$, is defined as

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad z \in \mathbb{D},$$

where $(a)_n$ is the Pochhammer symbol or shifted factorial function given by

$$(a)_n = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\cdots(a+n-1), & \text{if } n \geq 1. \end{cases}$$

By Euler's integral representation formula, we know that ${}_2F_1(a, b; c; z)$ analytically extends to the slit domain $\mathbb{C} \setminus [1, +\infty)$. For more details, see Chapter II of [11] and Chapter XIV of [78]. In many branches of Mathematics and Physics, the hypergeometric function ${}_2F_1(a, b; c; z)$ has various applications and many special functions can be derived as limiting values of ${}_2F_1(a, b; c; z)$ (see [10], [53], [70], [84]).

Let us denote the *complete elliptic integral* of the first and the second kind, respectively, by $\mathcal{K}(z)$ and $\mathcal{E}(z)$. Then $\mathcal{K}(z)$ and $\mathcal{E}(z)$ are defined by

$$\mathcal{K}(z) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-zu^2)}} \tag{2.7}$$

and

$$\mathcal{E}(z) = \int_0^1 \sqrt{\frac{1-zu^2}{1-u^2}}du, \tag{2.8}$$

respectively. For $t \in (0, 1)$, let $\mathcal{K}_t(z)$ and $\mathcal{E}_t(z)$ denote the *generalized complete elliptic integrals* of the first and the second kind, respectively. Then $\mathcal{K}_t(z)$ and $\mathcal{E}_t(z)$ are defined, respectively, by

$$\mathcal{K}_t(z) = \sin(\pi t) \int_0^1 \frac{u^{1-2t}}{(1-u^2)^{1-t}(1-zu^2)^t}du \tag{2.9}$$

and

$$\mathcal{E}_t(z) = \sin(\pi t) \int_0^1 \left(\frac{1-zu^2}{1-u^2}\right)^{1-t} u^{1-2t}du. \tag{2.10}$$

Also, $\mathcal{K}(z)$ and $\mathcal{E}(z)$ can be defined in terms of the Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$, respectively, by

$$\mathcal{K}(z) = \frac{\pi}{2}\, {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; z\Big) \tag{2.11}$$

and

$$\mathcal{E}(z) = \frac{\pi}{2}\, {}_2F_1\Big(-\frac{1}{2}, \frac{1}{2}; 1; z\Big). \tag{2.12}$$

In terms of the Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$, $\mathcal{K}_t(z)$ and $\mathcal{E}_t(z)$

are defined , respectively, by

$$\mathcal{K}_t(z) = \frac{\pi}{2}\,{}_2F_1(t, 1-t; 1; z) \tag{2.13}$$

and

$$\mathcal{E}_t(z) = \frac{\pi}{2}\,{}_2F_1(t-1, 1-t; 1; z). \tag{2.14}$$

For $z \in \mathbb{C} \setminus [1, +\infty)$, $\mathcal{K}_t(z)$ and $\mathcal{E}_t(z)$ are (single-valued) analytic functions. For details, see [7], [8], [19], and [21].

If we set $c = a+b$, then the hypergeometric function ${}_2F_1(a, b; a+b; z)$ is known to be *zero-balanced*. The above discussed complete elliptic integral of first kind is a special case of zero-balanced hypergeometric function. The zero-balanced hypergeometric function ${}_2F_1(a, b; a+b; z)$ has many important and nice transformation identities. These identities and the inequalities involving zero-balanced hypergeometric functions are intensely studied by many mathematicians (see [6], [55], [56], [57], [76], [80], [81]). For example, the following identities are known as Landen transformations

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4x}{(1+x)^2}\right) = (1+x)\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \tag{2.15}$$

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1-x}{1+x}\right)^2\right) = \frac{(1+x)}{2}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right), \tag{2.16}$$

for $x \in (0, 1)$ (see [5], [47]) and the Ramanujan's beautiful cubic transformation identities are given by

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right), \tag{2.17}$$

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-x}{1+2x}\right)^3\right) = \frac{1+2x}{3}\ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3\right), \tag{2.18}$$

for $x \in (0, 1)$ (see [19], [22], [23]). For more transformations of hypergeometric functions, see [1], [11], and [78].

We now construct a connection between the Schwarz triangle function and the Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$. Since

$$w_1 = {}_2F_1(a, b; c; z)$$

and

$$w_2 = {}_2F_1(a, b; a + b + 1 - c; 1 - z)$$

are two linearly independent solutions of the following hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + \{c - (a + b + 1)z\}\frac{dw}{dz} - abw = 0, \tag{2.19}$$

it is a well-known fact that the Schwarz triangle function defined by

$$S(z) = i\,\frac{{}_2F_1(a, b; a + b + 1 - c; 1 - z)}{{}_2F_1(a, b; c; z)}$$

maps the upper half-plane $\mathbb{H}$ conformally onto a curvilinear triangle Δ_t, which has interior angles $(1-c)\pi$, $(c-a-b)\pi$ and $(b-a)\pi$ at the vertices $S(0)$, $S(1)$ and $S(\infty)$, respectively. For details, we recommend the readers to go through Chapter V, Section 7 of [52]. For $t \in (0, \frac{1}{2}]$, let $a = t$, $b = 1 - a = 1 - t$ and $c = 1$, then $S(z)$ can be expressed as

$$S(z) = f_t(z) = i\,\frac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)}. \tag{2.20}$$

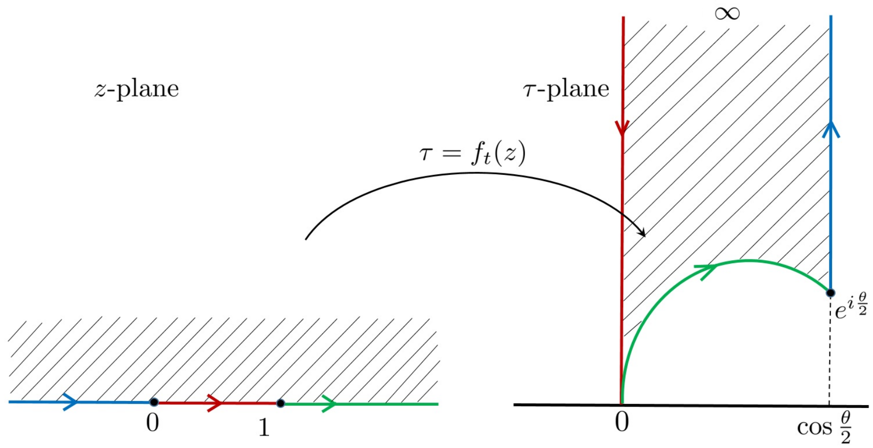


Figure 2.3: Mapping of the upper half-plane $\mathbb{H}$ onto Δ_t by f_t

If $\theta_1 = \frac{\pi}{m_1}$, $\theta_2 = \frac{\pi}{m_2}$ and $\theta_3 = \frac{\pi}{m_3}$, then a curvilinear triangle with angles θ_j (for $j = 1,\ 2,\ 3$) can be continued as a single-valued function across the sides of the triangle if and only if m_j is an integer greater than 1 including ∞ (see [67, p. 416]). Therefore,

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1. \tag{2.21}$$

The following lemma is related to the above facts.

Lemma 2.16 ([8], Lemma 4.1)**.** *For $t \in (0, \frac{1}{2}]$, if*

$$f_t(z) = i\, \frac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)},$$

then the upper half-plane $\mathbb{H}$ is mapped by f_t onto the hyperbolic triangle Δ_t given by

$$\Delta_t = \left\{ \tau \in \mathbb{H} : 0 < Re\,\tau < \cos\frac{\theta}{2}, \left| 2\tau \cos\frac{\theta}{2} - 1 \right| > 1 \right\},$$

where $\theta = (1-2t)\pi$. The interior angles of Δ_t are 0, 0 and $\theta = (1-2t)\pi$ at the vertices $f_t(0) = \infty$, $f_t(1) = 0$ and $f_t(\infty) = e^{i\frac{\theta}{2}}$, respectively.

2.8 Modular Equations

Let Γ be a finite index subgroup of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. If $f(\tau)$ is an automorphic function on Γ, i.e.,

$$f(\gamma \cdot \tau) = f\Big(\frac{a\tau + b}{c\tau + d}\Big) = f(\tau), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \tau \in \mathbb{H},$$

then $f(p\tau)$ is an automorphic function on $M_p^{-1}\Gamma M_p$, where $p > 1$ is an integer and $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (see Chapter 3). The functions $f(\tau)$ and $f(p\tau)$ are related by an algebraic equation, which is called a *modular equation* for $f(\tau)$ of *degree* p (see [61]).

For $t \in (0, \frac{1}{2}]$, $r \in (0, 1)$ and an integer $p > 1$, the *generalized modular equation* is given by

$$\frac{{}_2F_1(t, 1-t; 1; 1-s^2)}{{}_2F_1(t, 1-t; 1; s^2)} = p\, \frac{{}_2F_1(t, 1-t; 1; 1-r^2)}{{}_2F_1(t, 1-t; 1; r^2)}, \tag{2.22}$$

where $\frac{1}{t}$ is called the *signature of the modular equation*. Let $\alpha = r^2$ and $\beta = s^2$, then the modulus β has degree p over the modulus α. The *multiplier* m is given by

$$m = \frac{{}_2F_1(t, 1-t; 1; \alpha)}{{}_2F_1(t, 1-t; 1; \beta)}.$$

A modular equation of degree p in the theory of signature $\frac{1}{t}$ is an explicit relation between α and β induced by (2.22), where $\alpha = r^2$ and $\beta = s^2$ (see [19]). Let us define μ_t by

$$\mu_t(r) \equiv \frac{\pi}{2\sin(t\pi)} \frac{{}_2F_1(t, 1-t; 1; 1-r^2)}{{}_2F_1(t, 1-t; 1; r^2)},$$

then $\mu_t : (0,1) \to (0,\infty)$ is a decreasing homeomorphism. Thus, (2.22) becomes

$$\mu_t(s) = p\,\mu_t(r),$$

and for $p = \frac{1}{L}$, its solution is

$$s = \varphi_{t,L}(r) \equiv \mu_t^{-1}\Big(\frac{1}{L}\mu_t(r)\Big).$$

Here, $\varphi_{t,L}(r)$ is called the *modular function* of degree $p = \frac{1}{L}$ in the Ramanujan's theory of signature $\frac{1}{t}$. For $t = \frac{1}{2}$, in the context of quasiconformal theory, μ_t and $\varphi_{t,L}(r)$ are known as *Grötzsch ring function* and *Hersch-Pfluger distortion function*, respectively (see [7], [9], [58], [59], [60]).

In the theory of signature $\frac{1}{t} = 2$, when $p = 3$, 5, 7 and 11, α and β are related by the following modular equations:

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} = 1,$$

$$(\alpha\beta)^{\frac{1}{2}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{2}} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}} = 1,$$

$$(\alpha\beta)^{\frac{1}{8}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{8}} = 1,$$

and

$$(\alpha\beta)^{\frac{1}{4}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}} = 1,$$

respectively (see Entries 5(ii), 13(i), 19(i) of Chapter 19 and Entry 7(i) of Chapter 20 in [16]).

In the theory of signature $\frac{1}{t} = 3$, when $p = 2$, 5, and 11, α and β are related by the following modular equations:

$$(\alpha\beta)^{\frac{1}{3}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{3}} = 1,$$

$$(\alpha\beta)^{\frac{1}{3}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{3}} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}} = 1,$$

and

$$\begin{aligned}(\alpha\beta)^{\frac{1}{3}}+&\{(1-\alpha)(1-\beta)\}^{\frac{1}{3}}+6\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\\&+3\sqrt{3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}\Big\{(\alpha\beta)^{\frac{1}{6}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\Big\}=1,\end{aligned}$$

respectively (see Theorems 7.1, 7.6, and 7.8 in [19]).

In the theory of signature $\frac{1}{t}=4$, when β has degree 3, 5, 7, and 11, the corresponding modular equations are

$$(\alpha\beta)^{\frac{1}{2}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{2}}+4\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{4}}=1,$$

$$(\alpha\beta)^{\frac{1}{2}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{2}}+8\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\Big\{(\alpha\beta)^{\frac{1}{6}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\Big\}=1,$$

$$\begin{aligned}(\alpha\beta)^{\frac{1}{2}}+&\{(1-\alpha)(1-\beta)\}^{\frac{1}{2}}+20\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\\&+8\sqrt{2}\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{8}}\Big\{(\alpha\beta)^{\frac{1}{4}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\Big\}=1,\end{aligned}$$

and

$$\begin{aligned}(\alpha\beta)^{\frac{1}{2}}+&\{(1-\alpha)(1-\beta)\}^{\frac{1}{2}}+68\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{4}}\\&+16\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{12}}\Big\{(\alpha\beta)^{\frac{1}{3}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{3}}\Big\}\\&+48\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\Big\{(\alpha\beta)^{\frac{1}{6}}+\{(1-\alpha)(1-\beta)\}^{\frac{1}{6}}\Big\}=1,\end{aligned}$$

respectively (see Theorems 10.1, 10.2, 10.3, and 10.4 in [19]).

We observe that in the theory of signature 3, we have the symmetric modular equation in α and β if p is prime and $p \equiv 2 \pmod 3$ (see [29, p. 43]). In the theories of signatures 2 and 4, we have symmetric modular equations in α and β when p is an odd prime.

Chapter 3

Automorphic Functions and Space of Automorphic Forms

In this chapter, we mainly investigate some known results on automorphic functions and on the space of automorphic forms.

3.1 Automorphic Functions

Let Γ be a Fuchsian group of the first kind which leaves the upper half-plane $\mathbb{H}$ or the unit disc $\mathbb{D}$ invariant. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$, then a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is said to be an *automorphic form of weight k* if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

If $k = 0$, then

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

and the meromorphic function $f : \mathbb{H} \to \Gamma\backslash\mathbb{H}^*$ is called an *automorphic function.* When the genus of the quotient Riemann surface $\Gamma\backslash\mathbb{H}^*$ is zero, an automorphic function is called a *Hauptmodul.* If an automorphic function has no poles, then it is constant according to the consequence of maximum modulus principle. For details, we refer the reader to [24], [30], [39], and [51].

Assume that $z(\tau)$ is the Hauptmodul of the quotient space $\Gamma\backslash\mathbb{H}^*$ which has genus 0. Let τ_1, τ_2 and τ_3 be the elliptic points of orders m_1, m_2 and m_3, respectively, of $\Gamma\backslash\mathbb{H}^*$ such that

$$z(\tau_1) = 0, \quad z(\tau_2) = 1, \quad z(\tau_3) = \infty.$$

Since $z(\tau)$ takes on values $0, 1$ and ∞ at the elliptic points, $z(\tau)$ satisfies the following hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + \{c - (a+b+1)z\}\frac{dw}{dz} - abw = 0. \tag{3.1}$$

If w_1 and w_2 are two linearly independent solutions of (3.1), then the quotient $\tau = S(z) = \dfrac{w_2(z)}{w_1(z)}$ satisfies the following Schwarzian equation

$$\{S, z\} = \frac{1-p_1^2}{2z^2} + \frac{1-p_2^2}{2(1-z)^2} + \frac{1-p_1^2-p_2^2+p_3^2}{2z(1-z)}, \tag{3.2}$$

where $p_1 = \frac{1}{m_1}$, $p_2 = \frac{1}{m_2}$, $p_3 = \frac{1}{m_3}$ are called the *accessory parameters* and $\{S, z\}$ is the *Schwarzian derivative* of $S(z)$ defined by

$$\{S, z\} = \Big(\frac{S''(z)}{S'(z)}\Big)' - \frac{1}{2}\Big(\frac{S''(z)}{S'(z)}\Big)^2.$$

For details, we recommend the readers to go through Chapter 5, Section 7 of [52]. The following theorem is related to these facts.

Theorem 3.1 ([12, Theorem 6.2])**.** *If $c \neq 1$, then the functions*

$${}_2F_1(a, b; c; z) \quad \textit{and} \quad z^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; z)$$

are two linearly independent solutions of the hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + \{c - (a+b+1)z\}\frac{dw}{dz} - abw = 0.$$

The function

$$\tau = S(z) = \frac{z^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; z)}{{}_2F_1(a, b; c; z)}$$

maps the upper half z-plane $\mathbb{H}$ conformally onto the interior of the curvilinear triangle $[P, Q, R]$ in the τ-plane and establishes a homeomorphism between the boundary of $\mathbb{H}$, i.e., $\mathbb{R} \cup \infty$ and the boundary of $[P, Q, R]$. The vertices of the *triangle are given in terms of Euler's gamma function as*

$$\begin{aligned}
P &= S(0) = 0,\\
Q &= S(1) = \frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)},\\
R &= S(\infty) = e^{\pi i(1-c)}\frac{\Gamma(a)\Gamma(c-b)\Gamma(2-b)}{\Gamma(c)\Gamma(a-c+1)\Gamma(1-b)}.
\end{aligned}$$

The interior angles at the vertices P, Q *and* R *are* $(1-c)\pi$, $(c-a-b)\pi$ *and* $(b-a)\pi$, *respectively.*

Let G_q denote the group generated by

$$T_{2q} = \begin{pmatrix} 1 & \lambda_{2q} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W_{2q} = \begin{pmatrix} 1 & 0 \\ \lambda_{2q} & 1 \end{pmatrix},$$

where $\lambda_{2q} = 2\cos\frac{\pi}{2q}$. The following lemma shows that $\alpha(\tau) = \pi_q(\tau)$ and $\beta(\tau) = \pi_q(M_p\tau) = \alpha(p\tau)$ are automorphic functions on G_q and $M_p^{-1}G_qM_p$, respectively, for the canonical projection $\pi_q : \mathbb{H} \to G_q\backslash\mathbb{H}$, $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\tau \in \mathbb{H}$.

Lemma 3.2. *Let* p *be an integer* > 1 *and* $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. *Let* $\pi_q : \mathbb{H} \to G_q\backslash\mathbb{H}$. *If* $\alpha(\tau) = \pi_q(\tau)$ *and* $\beta(\tau) = \pi_q(M_p\tau) = \alpha(p\tau)$ *for* $\tau \in \mathbb{H}$, *then* $\alpha(\tau)$ *and* $\beta(\tau)$ *are automorphic functions on* G_q *and* $G_q^{M_p} = M_p^{-1}G_qM_p$, *respectively.*

Proof. If T'_{2q} and W'_{2q} are the generators of $G_q^{M_p}$, then

$$T'_{2q} = M_p^{-1}T_{2q}M_p = \begin{pmatrix} 1 & \frac{\lambda_{2q}}{p} \\ 0 & 1 \end{pmatrix}$$

and

$$W'_{2q} = M_p^{-1}W_{2q}M_p = \begin{pmatrix} 1 & 0 \\ p\lambda_{2q} & 1 \end{pmatrix}.$$

Assume that

$$\tau(\alpha) = i\,\frac{{}_2F_1(t, 1-t; 1; 1-\alpha)}{{}_2F_1(t, 1-t; 1; \alpha)},$$

then

$$\tau(1-\alpha) = i\,\frac{{}_2F_1(t, 1-t; 1; \alpha)}{{}_2F_1(t, 1-t; 1; 1-\alpha)} = -\frac{1}{\tau(\alpha)}.$$

Thus, we have

$$\alpha(-\frac{1}{\tau}) = 1 - \alpha(\tau).$$

Let $q(\tau) = \exp\left(\frac{2\pi}{\lambda_{2q}} i\tau\right)$, then

$$q(\tau + \lambda_{2q}) = \exp\Big(\frac{2\pi}{\lambda_{2q}} i(\tau + \lambda_{2q})\Big) = q(\tau).$$

Hence, we deduce that

$$\alpha\big(T_{2q}\tau\big) = \alpha(\tau + \lambda_{2q}) = \alpha(\tau).$$

Also,

$$\begin{aligned}
\alpha\big(W_{2q}\tau\big) &= \alpha\left(\frac{\tau}{\lambda_{2q}\tau + 1}\right) \\
&= \alpha\left(\frac{1}{\lambda_{2q} + 1/\tau}\right) \\
&= 1 - \alpha\big(-(\lambda_{2q} + 1/\tau)\big) \\
&= 1 - \alpha\left(-\frac{1}{\tau}\right) = \alpha(\tau).
\end{aligned}$$

Hence, $\alpha(\tau)$ is an automorphic function on G_q.

Next, we have

$$\begin{aligned}
\beta\Big(T'_{2q}\tau\Big) &= \beta\Big(\tau + \frac{\lambda_{2q}}{p}\Big) \\
&= \alpha\left(p\Big(\tau + \frac{\lambda_{2q}}{p}\Big)\right) \\
&= \alpha(p\tau) = \beta(\tau)
\end{aligned}$$

and

$$\begin{aligned}
\beta\Big(W'_{2q}\tau\Big) &= \beta\left(\frac{\tau}{p\lambda_{2q}\tau + 1}\right) \\
&= \alpha\left(p\Big(\frac{\tau}{p\lambda_{2q}\tau + 1}\Big)\right) \\
&= \alpha\left(\frac{1}{\lambda_{2q} + \frac{1}{p\tau}}\right)
\end{aligned}$$

$$= 1 - \alpha\Bigg(- \Big(- \lambda_{2q} + \frac{1}{p\tau}\Big)\Bigg)$$
$$= 1 - \alpha\Big(- \frac{1}{p\tau}\Big) = \alpha(p\tau) = \beta(\tau).$$

Thus, $\beta(\tau)$ is an automorphic function on $G_q^{M_p}$. $\square$

3.2 Space of Automorphic Forms

Let F be the fundamental domain for the Fuchsian group Γ. If F is compact, then it has finitely many vertices. Let $P_1, \ldots, P_n$ be the vertices whose orders are $m_1, m_2, \ldots, m_n$, respectively. If the number of elliptic elements and cusps of Γ are r and l, respectively, then for definiteness we suppose that

$$2 \leq m_1 \leq m_2 \leq \ldots \leq m_r < \infty$$

and

$$m_{r+1} = \ldots = m_n = \infty,$$

where $r + l = n$.

Let X denote the quotient Riemann surface $\Gamma\backslash\mathbb{H}$ and let $\hat{X}$ denote its compactification with cusps, i.e., $\hat{X} = \Gamma\backslash\mathbb{H}^*$, where $\mathbb{H}^*$ is the union of the upper half-plane $\mathbb{H}$ and the set of cusps of the Fuchsian group Γ. Let the genus of $\hat{X}$ be g and let π be the canonical mapping of $\mathbb{H}^*$ onto $\Gamma\backslash\mathbb{H}^*$. If f is a conformal mapping of $\Gamma\backslash\mathbb{H}^*$ onto $\hat{X} \setminus \{P_{r+1}, \ldots, P_n\}$ such that $\tau \in \mathbb{H}$ is not a fixed point of an elliptic element of Γ when

$$f \circ \pi(\tau) \neq P_j, \quad j = 1, \ldots, r$$

and is a fixed point of an elliptic element of order m_j when

$$f \circ \pi(\tau) = P_j, \quad j = 1, \ldots, r,$$

then we say that Γ has signature $(g; m_1, \ldots, m_n)$. For more detailed discussion, reader may consult Section 2.1 of [20], Chapter 4 of [42], and Section 2 of [73]. Let us denote by S_k the space of automorphic forms of weight k with respect to Γ. The following theorem determines the dimension of the space of automorphic forms S_k.

Theorem 3.3 ([72, Theorem 2.23]). *Let* $(g; m_1, \ldots, m_r)$ *be the signature of a Fuchsian group* Γ *and let* $\dim S_k$ *denote the dimension of the space of automorphic forms* S_k. *Then, for an even integer* k,

$$\dim S_k = \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k = 0, \\ g, & \text{if } k = 2, \\ (k-1)(g-1) + \sum_{i=1}^{r} \left\lfloor \frac{k}{2}(1 - \frac{1}{m_i}) \right\rfloor, & \text{if } k \geq 4. \end{cases}$$

Now, we present the following two theorems from [79] related to the basis for the space of automorphic forms S_k.

Theorem 3.4 ([79, Theorem 4]). *Let* $X = \Gamma \backslash \mathbb{H}$ *be a quotient Riemann surface for the Fuchsian group* Γ *with signature* $(0; m_1, \ldots, m_r)$. *Assume that* $z(\tau)$ *is a Hauptmodul of* X *and* $\tau_1, \ldots, \tau_r$ *are elliptic points of orders* $m_1, \ldots, m_r$, *respectively. For an even integer* $k \geq 4$, *let*

$$e_i = \left\lfloor \frac{k}{2}\left(1 - \frac{1}{m_i}\right) \right\rfloor \quad \text{and} \quad n = \dim S_k = 1 + \sum_{i=1}^{r} e_i - k.$$

If $z(\tau_i) = z_i$ *for* $i = 1, \ldots, r$ *and*

$$w(\tau) = \frac{\big(z'(\tau)\big)^{k/2}}{\prod_{i=1, z_i \neq \infty}^{r} \big(z(\tau) - z_i\big)^{e_i}},$$

then the set

$$\left\{ w(\tau)\big(z(\tau)\big)^{\nu} : \nu = 0, \ldots, n-1 \right\}$$

is a basis for the space of automorphic forms of weight k *on* X.

Theorem 3.5 ([79, Theorem 9]). *Suppose that* Γ *is a triangle group with signature* $(0; m_1, m_2, m_3)$. *Let the Hauptmodul of the quotient space* $X = \Gamma \backslash \mathbb{H}$ *be* $z(\tau)$ *which takes on values* 0, 1 *and* ∞ *at the elliptic points of orders* m_1, m_2 *and* m_3, *respectively. For an even integer* $k \geq 4$, *let*

$$q_1 = \frac{k}{2}\left(1 - \frac{1}{m_1}\right) - \left\lfloor \frac{k}{2}\left(1 - \frac{1}{m_1}\right) \right\rfloor, \quad q_2 = \frac{k}{2}\left(1 - \frac{1}{m_2}\right) - \left\lfloor \frac{k}{2}\left(1 - \frac{1}{m_2}\right) \right\rfloor$$

and

$$l = \left\lfloor \frac{k}{2}\Big(1 - \frac{1}{m_1}\Big) \right\rfloor + \left\lfloor \frac{k}{2}\Big(1 - \frac{1}{m_2}\Big) \right\rfloor + \left\lfloor \frac{k}{2}\Big(1 - \frac{1}{m_3}\Big) \right\rfloor - k.$$

Then, the following set is a basis for the space of automorphic forms of weight k on X:

$$\left\{ z^{q_1}(1-z)^{q_2}\big(z(\tau)\big)^{\nu}\Big({}_2F_1(a,b;c;z) + Cz^{\frac{1}{m_1}}\ {}_2F_1(a',b';c';z)\Big)^k : \nu = 0,\dots,l \right\},$$

where $C \in \mathbb{C}\setminus\{0\}$; a,b,c and a',b',c' are given by

$$a = \frac{1}{2}\Big(1 - \frac{1}{m_1} - \frac{1}{m_2} + \frac{1}{m_3}\Big), \quad b = \frac{1}{2}\Big(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\Big), \quad c = 1 - \frac{1}{m_1} \tag{3.3}$$

and

$$a' = \frac{1}{2}\Big(1 + \frac{1}{m_1} - \frac{1}{m_2} + \frac{1}{m_3}\Big), \quad b' = \frac{1}{2}\Big(1 + \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\Big), \quad c' = 1 + \frac{1}{m_1}. \tag{3.4}$$

3.2.1 Proof of Theorem 3.4

For $\nu = 0,\dots,n-1$ and $e_i = \left\lfloor \frac{k}{2}\Big(1 - \frac{1}{m_i}\Big) \right\rfloor$, let

$$g_\nu(\tau) = w(\tau)\big(z(\tau)\big)^{\nu} = \frac{\big(z'(\tau)\big)^{k/2}\big(z(\tau)\big)^{\nu}}{\displaystyle\prod_{i=1, z_i\neq\infty}^{r} \big(z(\tau) - z_i\big)^{e_i}}. \tag{3.5}$$

Then,

$$\begin{aligned} g_\nu\Big(\frac{a\tau+b}{c\tau+d}\Big) &= \frac{\Big(z'\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)^{k/2}\Big(z\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)^{\nu}}{\displaystyle\prod_{i=1, z_i\neq\infty}^{r} \Big(z\Big(\frac{a\tau+b}{c\tau+d}\Big) - z_i\Big)^{e_i}} \\ &= \frac{(c\tau+d)^k\big(z'(\tau)\big)^{k/2}\big(z(\tau)\big)^{\nu}}{\displaystyle\prod_{i=1, z_i\neq\infty}^{r} \big(z(\tau) - z_i\big)^{e_i}} \\ &= (c\tau+d)^k g_\nu(\tau). \end{aligned}$$

Thus, g_ν is an automorphic form of weight k, i.e., $g_\nu(\tau) = w(\tau)\big(z(\tau)\big)^{\nu} \in S_k$.

Assume that the Hauptmodul $z(\tau)$ does not have any pole at the elliptic points τ_i, i.e., $z_i \neq \infty$ for $i = 1, \ldots, r$. Since τ_i is an elliptic point of order m_i, we have, in a neighbourhood of $\tau = \tau_i$,

$$\begin{aligned} z(\tau) - z_i &= a_i(\tau - \tau_i)^{m_i} + O\big((\tau - \tau_i)^{m_i+1}\big) \\ &= (\tau - \tau_i)^{m_i} z^*(\tau), \end{aligned}$$

where $a_i \in \mathbb{C} \setminus \{0\}$, $z^*(\tau_i) \neq 0$ and $z^*(\tau)$ is analytic in a neighbourhood of $\tau = \tau_i$. Therefore, a single-valued analytic m_i-th root can be defined in a neighbourhood of $\tau = \tau_i$. This can be done at all points equivalent to τ_i. Since $(z - z_i)$ is non-zero and analytic on the other part of the upper half-plane $\mathbb{H}$, its m_i-th root is analytic at each point of the remainder of $\mathbb{H}$. As $(z - z_i)^{m_i}$ is locally single-valued (locally analytic) at each point of the upper half-plane $\mathbb{H}$, which is simply connected, so it follows from monodromy theorem that a single-valued and analytic m_i-th root of $(z - z_i)$ can be defined on the whole upper half-plane $\mathbb{H}$.

The Hauptmodul $\big(z(\tau) - z_i\big)$ has a zero of order m_i at $\tau = \tau_i$. Hence $\big(z'(\tau)\big)^{k/2}$ has a zero of order $\frac{k}{2}(m_i - 1)$ and $\prod_{i=1, z_i \neq \infty}^{r} \big(z(\tau) - z_i\big)^{e_i}$ has a zero of order $m_i \left\lfloor \frac{k}{2}\left(1 - \frac{1}{m_i}\right)\right\rfloor$ at $\tau = \tau_i$. Therefore, from (3.5) we observe that g_ν has a simple zero at $\tau = \tau_i$ and is holomorphic on the upper half-plane $\mathbb{H}$.

Now, let us assume that the Hauptmodul $z(\tau)$ has a pole at one of the elliptic points τ_i for $i = 1, \ldots, r$. Without loss of generality, we choose τ_1 such that $z(\tau_1) = z_1 = \infty$. Since $z(\tau)$ is a Hauptmodul and τ_1 is an elliptic point of order m_1, it follows that

$$z(\tau) = \frac{b_1}{(\tau - \tau_1)^{m_1}} + O\big((\tau - \tau_1)^{1-m_1}\big), \quad b_1 \in \mathbb{C} \setminus \{0\}$$

and

$$z'(\tau) = -\frac{b_1 m_1}{(\tau - \tau_1)^{m_1+1}} + O\big((\tau - \tau_1)^{-m_1}\big).$$

In this case, from (3.5) we have

$$g_\nu(\tau) = \frac{\big(z'(\tau)\big)^{k/2}\big(z(\tau)\big)^{\nu}}{\prod_{i=2, z_i \neq \infty}^{r} \big(z(\tau) - z_i\big)^{e_i}}. \tag{3.6}$$

The Hauptmodul $z(\tau)$ has a pole of order m_1 at $\tau = \tau_1$. Thus, $\big(z'(\tau)\big)^{k/2}$ has a pole of order $\frac{k}{2}(m_1+1)$ and $\prod_{i=2, z_i\neq\infty}^{r} \big(z(\tau)-z_i\big)^{e_i}$ has a pole of order $m_1 \sum_{i=2}^{r} \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_i}\Big)\Big\rfloor$ at $\tau = \tau_1$. Since $\nu = 0, \ldots, \sum_{i=1}^{r} e_i - k$, so $\big(z(\tau)\big)^{\nu}$ has a pole of order at most $m_1\Big(\sum_{i=1}^{r} \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_i}\Big)\Big\rfloor - k\Big)$ at $\tau = \tau_1$. As a result, if $\mathrm{ord}(g_\nu, \tau)$ denote the order of the function g_ν in (3.6), then for $\tau = \tau_1$, we have

$$\begin{aligned}
-\mathrm{ord}(g_\nu, \tau_1) &\leq \frac{k}{2}(m_1+1) - m_1 \sum_{i=2}^{r} \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_i}\Big)\Big\rfloor \\
&\quad + m_1\Big(\sum_{i=1}^{r} \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_i}\Big)\Big\rfloor - k\Big) \\
&= -\frac{k}{2}\Big(m_1 - 1\Big) + m_1 \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_1}\Big)\Big\rfloor \\
&\leq 0.
\end{aligned}$$

Consequently, $\mathrm{ord}(g_\nu, \tau_1) \geq 0$ and g_ν is holomorphic on the upper half-plane $\mathbb{H}$.

Finally, assume that $z_i \neq \infty$ for $i = 1, \ldots, r$. Let the Hauptmodul $z(\tau)$ have the value ∞ at the point $\tau = \tau_0$. It follows that $z(\tau)$ has a simple pole at τ_0. Therefore, we have

$$z(\tau) = \frac{a_0}{(\tau - \tau_0)} + O(1), \quad a_0 \in \mathbb{C} \setminus \{0\}$$

and

$$z'(\tau) = -\frac{a_0}{(\tau-\tau_0)^2} + O(1).$$

At the point $\tau = \tau_0$, $\big(z'(\tau)\big)^{k/2}$ has a pole of order k, $\prod_{i=1, z_i\neq\infty}^{r} \big(z(\tau)-z_i\big)^{e_i}$ has a pole of order $\sum_{i=1}^{r} \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_i}\Big)\Big\rfloor$ and z^{ν} has a pole of order at most $\sum_{i=1}^{r} \Big\lfloor \frac{k}{2}\Big(1-\frac{1}{m_i}\Big)\Big\rfloor - k$. From (3.5), it follows easily that

$$\mathrm{ord}(g_\nu, \tau_0) \geq 0.$$

Thus, g_ν is holomorphic on $\mathbb{H}$ in this case also. This completes the proof.

3.2.2 Proof of Theorem 3.5

If, for $0 < c < 1$,

$$w_1 = {}_2F_1(a, b; c; z)$$

and

$$w_2 = z^{1-c}{}_2F_1(a', b'; c'; z)$$

are two linearly independent solutions of the hypergeometric differential equation (3.1), then

$$a' = a - c + 1, \quad b' = b - c + 1, \quad c' = 2 - c. \tag{3.7}$$

The relations between a, b, c and the accessory parameters are given as follows

$$p_1 = 1 - c, \quad p_2 = c - b - a, \quad p_3 = a - b$$

or,

$$a = \frac{1}{2}(1 - p_1 - p_2 + p_3), \quad b = \frac{1}{2}(1 - p_1 - p_2 - p_3), \quad c = 1 - p_1. \tag{3.8}$$

The relations (3.3) and (3.4) follow easily from (3.7) and (3.8).

The quotient $\tau = S(z) = \dfrac{w_2(z)}{w_1(z)}$ provides conformal representation of the upper half z-plane $\mathbb{H}$ onto the interior of a triangle with vertices corresponding to τ_1, τ_2 and τ_3 in τ-plane and forms a homeomorphism between $\mathbb{R} \cup \infty$ and the boundary of the triangle. Since m_1, m_2, m_3 are positive integers greater than 1 and

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1,$$

the inverse $z(\tau)$ of $S(z)$ is single-valued. For $z : \Gamma\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{P}^1$, up to a Möbius transformation, by the quotient $S(z) = \dfrac{w_2(z)}{w_1(z)}$ of two hypergeometric functions in z, each $z \in \mathbb{P}^1$ can be associated to a representative of its corresponding Γ-orbit in $\mathbb{H}$.

Now let us choose a representative $\tau_1 \in \mathbb{H}$ of elliptic points of order m_1. If $A \in \Gamma$ is the generator of the stabilizer subgroup for the elliptic point τ_1, then we have

$$\frac{A\tau - \tau_1}{A\tau - \overline{\tau}_1} = e^{\frac{2\pi i}{m_1}} \frac{\tau - \tau_1}{\tau - \overline{\tau}_1}. \tag{3.9}$$

Also, for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ we have

$$\tau = \frac{\alpha w_2 + \beta w_1}{\gamma w_2 + \delta w_1}. \tag{3.10}$$

The Möbius transformation

$$\tau \to \frac{\tau - \tau_1}{\tau - \overline{\tau}_1}$$

maps the two sides (hyperbolic lines) of the triangle through τ_1 to straight lines through the origin. For $|z| < 1$, the point on $\Gamma\backslash\mathbb{H}$ is the Γ-orbit of τ near the elliptic point τ_1 such that

$$\frac{\tau - \tau_1}{\tau - \overline{\tau}_1} = k_0 \frac{w_2}{w_1} \tag{3.11}$$

for a nonzero constant k_0. Also, as $z(\tau_2) = 1$, the value of k_0 can be determined from (3.11) when $\tau \to \tau_2$ using the following formula of Gauss

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

From (3.11) we have

$$\tau = \frac{\tau_1 w_1 - k_0 \overline{\tau}_1 w_2}{w_1 - K w_2}. \tag{3.12}$$

Differentiating both sides of (3.12) with respect to z, it follows immediately

$$\frac{d\tau}{dz} = k_0(\tau_1 - \overline{\tau}_1)\frac{w_1 w_2' - w_1' w_2}{(w_1 - K w_2)^2} = k_0(\tau_1 - \overline{\tau}_1)\frac{W}{(w_1 - k_0 w_2)^2}, \tag{3.13}$$

where $w_1' = \dfrac{dw_1}{dz}$, $w_2' = \dfrac{dw_2}{dz}$ and W is the Wronskian of w_1 and w_2 given by

$$W = w_1 w_2' - w_1' w_2.$$

The hypergeometric differential equation (3.1) can be written as

$$\frac{d^2 w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0, \tag{3.14}$$

where

$$P(z) = \frac{c - (a+b+1)z}{z(1-z)}$$

and

$$Q(z) = \frac{ab}{z(z-1)}.$$

Since w_1 and w_2 are two linearly independent solutions of (3.14), we have

$$W(z) = W(z_0)\exp\Big(-\int_{z_0}^{z} P(z)dz\Big), \tag{3.15}$$

where w_1 and w_2 are regular at the point $z = z_0$ and $W(z_0) \neq 0$. From (3.15), it can be shown immediately that

$$W(z) = C_0 z^{-c}(1-z)^{c-a-b-1} \tag{3.16}$$

for some constant $C_0 \in \mathbb{C} \setminus \{0\}$ depending on the point z_0. Consequently, from (3.13) and (3.16) we have

$$\frac{d\tau}{dz} = C_0 k_0(\tau_1 - \overline{\tau}_1)\frac{z^{-c}(1-z)^{c-a-b-1}}{(w_1 - k_0 w_2)^2}. \tag{3.17}$$

Thus,

$$z'(\tau) = \kappa z^{(1-\frac{1}{m_1})}(1-z)^{(1-\frac{1}{m_2})}\Big({}_2F_1(a,b;c;z) + Cz^{\frac{1}{m_1}}\ {}_2F_1(a',b';c';z)\Big)^2, \tag{3.18}$$

where $z'(\tau) = \dfrac{dz(\tau)}{d\tau}$, $C = -k_0$ and $\kappa = \dfrac{1}{C_0 k_0(\tau_1 - \overline{\tau}_1)}$.

Since $z(\tau)$ is a Hauptmodul on $\Gamma\backslash\mathbb{H}$, $z'(\tau)$ is an automorphic form of weight 2 on $\Gamma\backslash\mathbb{H}$. Therefore, $\big(z'(\tau)\big)^{k/2}$ is an automorphic form of weight k on $\Gamma\backslash\mathbb{H}$. Thus, Theorem 3.5 follows from Theorem 3.4.

Lemma 3.6 ([43, Lemma 4.2]). *The constant k_0 in (3.11) can be expressed in terms of Euler's gamma function as*

$$k_0 = \left(\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\right)^{\frac{1}{2}}\frac{\Gamma(c)}{\Gamma(2-c)}.$$

Proof. Since $z(\tau_2) = 1$, and

$$\begin{aligned}{}_2F_1(a,b;c;1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\\ {}_2F_1(a-c+1,b-c+1;2-c;1) &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)},\end{aligned}$$

so, when $\tau \to \tau_2$, from (3.11) we have

$$k_0 = \frac{\tau_2 - \tau_1}{\tau_2 - \overline{\tau}_1}\frac{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}. \tag{3.19}$$

If $d(\tau_1,\tau_2)$ denote the hyperbolic distance between τ_1 and τ_2, then using (75.1) of [28], we have

$$\tanh^2\frac{d(\tau_1,\tau_2)}{2}=\frac{\cos\left\{\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}\right)\pi\right\}\cos\left\{\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}-\frac{1}{m_3}\right)\pi\right\}}{\cos\left\{\frac{1}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}+\frac{1}{m_3}\right)\pi\right\}\cos\left\{\frac{1}{2}\left(-\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}\right)\pi\right\}}.$$

From (3.3), it follows

$$\tanh^2\frac{d(\tau_1,\tau_2)}{2}=\frac{\sin\pi a\sin\pi b}{\sin\pi(c-a)\sin\pi(c-b)}.$$

For $a\in\mathbb{Z}$, we have $\Gamma(a)\Gamma(1-a)=\dfrac{\pi}{\sin\pi a}$, which implies

$$\tanh^2\frac{d(\tau_1,\tau_2)}{2}=\frac{\Gamma(c-a)\Gamma(a-c+1)\Gamma(c-b)\Gamma(b-c+1)}{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}. \tag{3.20}$$

Also, from (85.3) of [28], we have

$$\tanh^2\frac{d(\tau_1,\tau_2)}{2}=\left(\frac{\tau_2-\tau_1}{\tau_2-\overline{\tau}_1}\right)^2. \tag{3.21}$$

Thus, from (3.19), (3.20) and (3.21), we have the required expression for k_0. □

Chapter 4

Generalized Modular Equations

We first recall that the generalized modular equation of degree p and signature $\frac{1}{t}$ is given by

$$\frac{{}_2F_1(t, 1-t; 1; 1-\beta)}{{}_2F_1(t, 1-t; 1; \beta)} = p\,\frac{{}_2F_1(t, 1-t; 1; 1-\alpha)}{{}_2F_1(t, 1-t; 1; \alpha)}, \tag{4.1}$$

where $t \in (0, \frac{1}{2}]$ and $p > 1$ is an integer. Let $G_q = \langle T, W \rangle$ be the Fuchsian group generated by

$$T\tau = \tau + \lambda \quad \text{and} \quad W\tau = \frac{\tau}{1 + \lambda\tau},$$

where $\lambda = 2\cos\frac{\pi}{2q}$ and let M_p be the Möbius transformation defined by $M_p\tau = p\tau$. To speak about the solutions to (4.1), we have to clarify the range of the solutions. Originally, the range of α and β was the interval $[0, 1]$. However, in our setting, it is natural to choose the quotient Riemann surface $X = G_q\backslash\mathbb{H}$ as the range of the solutions (α, β) to the equation (4.1). Here, we take $X = \widehat{\mathbb{C}} \setminus \{0, 1\}$ for finite q and $X = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ for $q = \infty$. For the above assumptions, we have the following result.

Theorem 4.1. *Let n, p and q be integers with n, p, $q \geq 2$ (possibly $q = \infty$) and set*

$$t = \frac{q-1}{2q}.$$

Then, the solutions (α, β) to the equation (4.1) in $X = G_q\backslash\mathbb{H}$ satisfy the equation $P(\alpha, \beta) = 0$ for an irreducible polynomial $P(x, y)$ of degree n if and only if $G_q \cap G_q^{M_p}$ is a subgroup of G_q of index n, where $G_q^{M_p} = M_p^{-1} G_q M_p$.

As we will see in the proof, the solutions α and β are parametrized as $\alpha = \varphi(z)$ and $\beta = \psi(z)$ on the Riemann surface $Z = (G_q \cap G_q^{M_p})\backslash\mathbb{H}$ and they satisfy the polynomial equation $P(\alpha,\beta)=0$ of degree n. More precisely, $P(x,y)$ has the forms

$$\begin{aligned} P(x,y) &= a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y + a_n(x) \\ &= b_0(y)x^n + b_1(y)x^{n-1} + \cdots + b_{n-1}(y)x + b_n(y), \end{aligned}$$

where

$$a_j(x) \in \mathbb{C}[x],\ b_j(y) \in \mathbb{C}[y]\ \ (j = 0, 1, \ldots, n),$$

such that $P(x,y)$ is irreducible as an element of $\mathbb{C}(x)[y]$ and $\mathbb{C}(y)[x]$, respectively. Recall that $\mathbb{C}[x]$ and $\mathbb{C}(x)$ stand for the $\mathbb{C}$-algebra of polynomials in x and the field of rational functions of x with coefficients in $\mathbb{C}$, respectively.

4.1 Construction of Covering Group

Let $0 < t \le 1/2$. Then it is classically known [27] that the function

$$\tau = f_t(z) = i \cdot \frac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)}$$

maps the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto the domain

$$\Delta_t = \Big\{\tau \in \mathbb{H} : 0 < \operatorname{Re}\tau < \sin \pi t, |2\tau \sin \pi t - 1| > 1\Big\}$$

and that Δ_t is the hyperbolic triangle with vertices at 0, ∞ and $ie^{-\pi t i}$ of interior angles 0, 0 and $(1-2t)\pi$, respectively (see also [8, Lemma 4.1]). Note also that f_t extends homeomorphically to the boundary and

$$f_t(0) = \infty, \quad f_t(1) = 0, \quad f_t(\infty) = ie^{-\pi t i} = e^{i\frac{(1-2t)\pi}{2}}. \tag{4.2}$$

Suppose now that $q = \frac{1}{1-2t}$ is an integer or ∞. Let

$$\pi_q : \Delta_t \to \mathbb{H}$$

be the inverse map, $f_t^{-1}(\tau)$, of $\tau = f_t(z)$. Then, by repeated applications of the Schwarz reflection principle, π_q extends to a holomorphic map from $\mathbb{H}$ into $\widehat{\mathbb{C}} \setminus \{0, 1\}$. Here, we note that π_q is locally q to 1 at the point $ie^{-\pi t i}$. By construction, the covering group

$$G_q = \Big\{\gamma \in \operatorname{Aut}(\mathbb{H}) : \pi_q \circ \gamma = \pi_q\Big\}$$

is the triangle group of signature (q, ∞, ∞) arising from the hyperbolic triangle $\Delta_t,\ q = 1/(1-2t)$. Here, the group $\mathrm{Aut}(\mathbb{H})$ of analytic automorphisms of $\mathbb{H}$ is identified with

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}.$$

By its form, the function f_t satisfies the relation

$$f_t(1-z) = -\frac{1}{f_t(z)}.$$

Since $\pi_q = f_t^{-1}$ on Δ_t, the following result follows.

Lemma 4.2. *The covering map π_q satisfies the functional equation*

$$\pi_q(\tau) + \pi_q(-\frac{1}{\tau}) = 1, \quad \tau \in \mathbb{H}.$$

Recall that $\hat{F}_q = F_{2q} \cup S(F_{2q})$ is a fundamental domain for the (normal) subgroup G of H_{2q} of index 2 generated by

$$T_{2q} = \begin{pmatrix} 1 & \lambda_{2q} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W_{2q} = \begin{pmatrix} 1 & 0 \\ \lambda_{2q} & 1 \end{pmatrix}.$$

To adapt with our aim, we modify the fundamental domain as follows. For $t = (q-1)/(2q)$, let

$$\tilde{F}_q = \Delta_t \cup \Delta_t',$$

where Δ_t' is the reflection of Δ_t across the line $\mathrm{Re}\,\tau = \sin \pi t$. Then, $\tilde{F}_q$ serves as a fundamental domain for G, which is the same as G_q, the above-defined covering group of π_q. (This appears in Example 1.3 of [26].) Note that the element

$$V_q := T_{2q} W_{2q}^{-1} = U_{2q}^2 = \begin{pmatrix} \lambda_{2q}^2 - 1 & -\lambda_{2q} \\ \lambda_{2q} & -1 \end{pmatrix} = \begin{pmatrix} \lambda_q + 1 & -\lambda_{2q} \\ \lambda_{2q} & -1 \end{pmatrix}$$

is elliptic of order q and that G_q is generated by T_{2q} and V_q (see Figure 4.1).

Here, we note the following property.

Lemma 4.3. *Let p, q be integers with $p, q \geq 2$ (where q may be ∞) and let*

$$K = G_q \cap G_q^{M_p},$$

where $M_p \tau = p\tau$. Then,

$$|G_q : K| = |G_q^{M_p} : K|.$$

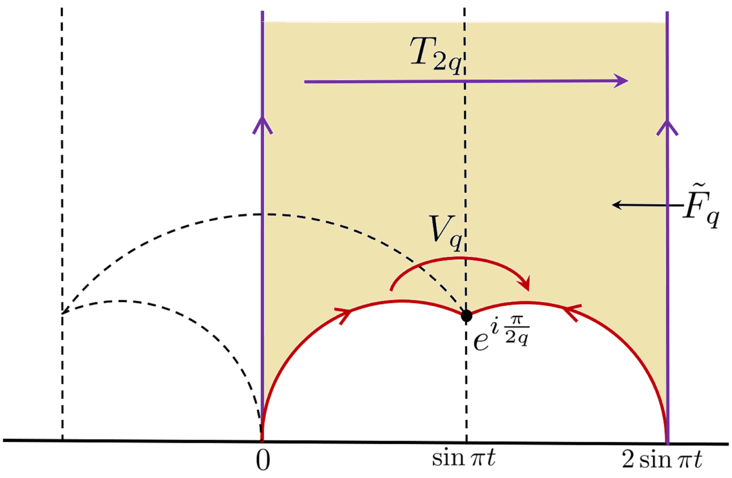


Figure 4.1: Fundamental domain for $G_q = \langle T_{2q}, V_q \rangle$

Proof. If both of the indices are infinite, there is nothing to show. Assume that

$$n = |G_q : K| < \infty.$$

Then the hyperbolic area $\mathrm{Area}(Z)$ of the Riemann orbifold $Z = K\backslash\mathbb{H}$ is computed as (see [13, p. 150] or [42, p. 13])

$$\begin{aligned} \mathrm{Area}(Z) &= n\,\mathrm{Area}(G_q\backslash\mathbb{H}) \\ &= n\,\mathrm{Area}(\tilde{F}_q) \\ &= 2n(\pi - \pi/q). \end{aligned}$$

Since $G_q^{M_p}\backslash\mathbb{H}$ has the same hyperbolic area as $G_q\backslash\mathbb{H}$, we have

$$\begin{aligned} |G_q^{M_p} : K| &= \frac{\mathrm{Area}(K\backslash\mathbb{H})}{\mathrm{Area}(G_q^{M_p}\backslash\mathbb{H})} \\ &= \frac{\mathrm{Area}(Z)}{\mathrm{Area}(G_q\backslash\mathbb{H})} \\ &= n \end{aligned}$$

as required. When $|G_q^{M_p} : K| < \infty$, the same argument works for the proof. $\square$

Remark 4.1. *In view of the proof, one can also show that the formula*

$$|G : G \cap G^M| = |G^M : G \cap G^M|$$

holds for a cofinite Fuchsian group G; namely, if the surface $G\backslash\mathbb{H}$ has finite hyper-

bolic area. However, this formula is not true in general. For instance, Jørgensen, Marden and Pommerenke [41] constructed a Fuchsian group Γ and its subgroup G of index 2 such that $G^M = M^{-1}GM$ is a proper subgroup of G for some $M \in \Gamma$ so that $G \cap G^M = G^M$. In particular,

$$|G : G \cap G^M| > |G^M : G \cap G^M| = 1.$$

The index $|G_q : K|$ can be computed explicitly when $q = 2$, 3, and ∞ (see Chapter 5).

4.2 Criteria for Finiteness

Let Γ be a Fuchsian group acting on the upper half-plane $\mathbb{H}$ and X be the quotient Riemann surface $\Gamma\backslash\mathbb{H}$. We denote by π the canonical projection $\pi : \mathbb{H} \to X$. When $\gamma \in \Gamma$ is a parabolic element with fixed point at $\tau_0 \in \partial\mathbb{H}$, it is known that $\pi(\tau)$ tends to a puncture, say P, of X as $\tau \to \tau_0$ nontangentially. As a convention, we will write $\pi(\tau_0) = P$ in the sequel. A point $\tau_0 \in \mathbb{H}$ is called a fixed point of an element γ of $\mathrm{PSL}(2, \mathbb{R})$ if $\gamma\tau_0 = \tau_0$ and the set of fixed points of γ in $\mathbb{H}$ is denoted by $\mathrm{Fix}(\gamma)$. A non-identity element γ is called elliptic if $\mathrm{Fix}(\gamma) \neq \emptyset$. Let $M \in \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{H})$ such that $M \notin \Gamma$. Then we consider the number (possibly ∞) defined by

$$N_{X_0}(M, \Gamma) = \sup_{x \in X_0} \#\{\pi(M\tau) : \tau \in \pi^{-1}(x)\} \tag{4.3}$$

for $X_0 \subset X$.

The next result is our main lemma and it gives a criterion for finiteness of $N_{X_0}(M, \Gamma)$.

Lemma 4.4. *Let Γ be a Fuchsian group and $M \in \mathrm{PSL}(2, \mathbb{R}) \setminus \Gamma$. Suppose that X_0 is an uncountable subset of the quotient Riemann surface $X = \Gamma\backslash\mathbb{H}$. Then*

$$N_{X_0}(M, \Gamma) < \infty$$

if and only if

$$\Gamma^M := M^{-1}\Gamma M$$

is commensurable with Γ in the sense that $|\Gamma : G| < \infty$ for $G = \Gamma \cap \Gamma^M$. Moreover, *in this case,*

$$N_{X_0}(M, \Gamma) = |\Gamma : G|.$$

Proof. The "if" part is almost trivial. Let

$$N := |\Gamma : G| < \infty.$$

Then we take

$$\gamma_1, \ldots, \gamma_N \in \Gamma$$

so that

$$\Gamma = G\gamma_1 \cup \cdots \cup G\gamma_N.$$

For $x \in X$ and $\tau_0 \in \pi^{-1}(x)$, we observe

$$\begin{aligned} M(\pi^{-1}(x)) &= M\Gamma\tau_0 \\ &= \bigcup_{j=1}^{N} MG\gamma_j\tau_0 \\ &= \bigcup_{j=1}^{N} MGM^{-1} \cdot M\gamma_j\tau_0. \end{aligned}$$

Since $MGM^{-1} \subset \Gamma$, the set $MGM^{-1} \cdot M\gamma_j\tau_0$ is projected to the point $\pi(M\gamma_j\tau_0)$ by π. Therefore,

$$\#(\pi(M(\pi^{-1}(x)))) \leq N.$$

Hence, we conclude that

$$N_{X_0}(M, \Gamma) \leq |\Gamma : G|. \tag{4.4}$$

To show the "only if" part, we assume that $N := N_{X_0}(M, \Gamma)$ is finite. Let

$$E_1 = \bigcup_{\gamma,\delta} \mathrm{Fix}(\gamma^{-1}\delta),$$

where γ and δ range over $\gamma \in \Gamma$ and $\delta \in \Gamma^M$ with $\gamma \neq \delta$. Since each $\mathrm{Fix}(\gamma^{-1}\delta)$ contains at most one point, the set E_1 is at most countable. Moreover, the set

$$E = \Gamma \cdot E_1 = \{\gamma\tau : \gamma \in \Gamma,\, \tau \in E_1\}$$

is also at most countable. Take a point τ_0 from the uncountable subset $\pi^{-1}(X_0) \setminus E$ of $\mathbb{H}$ and fix it. We regard $G \backslash \Gamma$ as the set of right cosets $\{G\gamma : \gamma \in \Gamma\}$. As we saw above, each set $G\gamma \cdot \tau_0$ projects to one point $\pi(M\gamma\tau_0)$ under the mapping

$\pi \circ M : \mathbb{H} \to X$. We now show that the mapping

$$\phi : G\backslash\Gamma \to X$$

defined by

$$\phi(G\gamma) = \pi(M\gamma\tau_0)$$

is injective. To this end, we suppose that

$$\phi(G\gamma_1) = \phi(G\gamma_2)$$

for some $\gamma_1, \gamma_2 \in \Gamma$. Then

$$M\gamma_2\tau_0 = \gamma M\gamma_1\tau_0$$

for some $\gamma \in \Gamma$. It says that $\gamma_1\tau_0$ is a fixed point of $(\gamma_2\gamma_1^{-1})^{-1}\delta$, where

$$\delta = M^{-1}\gamma M \in \Gamma^M.$$

Since $\gamma_1\tau_0 \notin E$, the element $(\gamma_2\gamma_1^{-1})^{-1}\delta$ must be the identity, which implies

$$\gamma_2\gamma_1^{-1} = \delta \in \Gamma \cap \Gamma^M = G.$$

Hence $G\gamma_1 = G\gamma_2$.

We have seen that

$$\phi : G\backslash\Gamma \to X$$

is injective. On the other hand, the image

$$\phi(G\backslash\Gamma) = \pi(M\Gamma\tau_0)$$

consists of at most N points. Therefore, we obtain

$$|\Gamma : G| = \#(G\backslash\Gamma) \leq N.$$

Combining with (4.4), we obtain

$$|\Gamma : G| = N_{X_0}(M, \Gamma)$$

as required. □

In particular, we see that $N_{X_0}(M, \Gamma)$ does not depend on the uncountable set

$X_0 \subset X$. Thus we denote by $N(M,\Gamma)$ the common number $N_{X_0}(M,\Gamma)$.

4.3 Proof of Theorem 4.1

Recall that

$$t = \frac{q-1}{2q} \quad \text{and} \quad M_p\tau = p\tau.$$

In this proof, we will use the notation introduced in Section 4.1. The generalized modular equation (4.1) may be expressed by

$$f_t(\beta) = pf_t(\alpha),$$

where

$$f_t(z) = i \cdot \frac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)}.$$

Therefore, α and β in $\widehat{\mathbb{C}} \setminus \{0,1\}$ satisfy (4.1) if and only if

$$\alpha = \pi_q(\tau) \quad \text{and} \quad \beta = \pi_q(p\tau)$$

for some $\tau \in \mathbb{H}$.

Proof of Theorem 4.1. First we assume that α and β in (4.1) satisfy the algebraic equation $P(\alpha,\beta) = 0$, where

$$P(x,y) = \sum_{j=0}^{n} a_j(x)y^j, \quad a_j(x) \in \mathbb{C}[x] \ (j = 0, 1, \ldots, n),$$

is an irreducible polynomial of degree n in $\mathbb{C}(x)[y]$. Let X_0 be the set of those points $x \in X$ for which $a_k(x) \neq 0$ for some k. Then for a fixed $x_0 \in X_0$, the algebraic equation $P(x_0, y) = 0$ in y has at most n solutions. Thus we conclude that

$$N_{X_0}(M_p, G_q) \leq n.$$

Lemma 4.4 now implies that

$$|G_q : G_q \cap G_q^{M_p}| \leq n.$$

By the irreducibility of $P(x,y)$, we see that the equality holds.

Conversely, we assume that the equality $|G_q : K| = n$ holds, where

$$K = G_q \cap G_q^{M_p}.$$

Note that $|G_q^{M_p} : K| = n$ by Lemma 4.3. We denote by Z the quotient Riemann surface $K\backslash\mathbb{H}$. Let $\rho : \mathbb{H} \to Z$ be the canonical projection and let

$$\varphi : Z \to X \quad \text{and} \quad \psi : Z \to X$$

be the induced mappings satisfying the relations

$$\pi_q = \varphi \circ \rho \quad \text{and} \quad \pi_q \circ M_p = \psi \circ \rho,$$

respectively. Thus we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{H} \arrow[r, "M_p"] \arrow[dl, "\pi_q"'] \arrow[d, "\rho"] & \mathbb{H} \arrow[d, "\pi_q"] \\
X & Z \arrow[l, "\varphi"] \arrow[r, "\psi"'] & X.
\end{tikzcd}$$

Note that the solution (α, β) to the modular equation (4.1) is now parametrized by

$$\alpha = \varphi(z) \quad \text{and} \quad \beta = \psi(z)$$

for $z \in Z$. We denote by $\hat{X}$ and $\hat{Z}$ the compact Riemann surfaces obtained by filling in the punctures of X and Z, respectively. Note that $\widehat{\mathbb{C}}$ can be taken as $\hat{X}$. Then φ and ψ extend to n-sheeted branched covering maps of $\hat{Z}$ onto $\hat{X} = \widehat{\mathbb{C}}$. In particular, ψ may be regarded as a meromorphic function on $\hat{Z}$.

Let X_0 denote X minus the set of critical values of

$$\varphi : Z \to X;$$

indeed $X_0 = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ in this case. Similarly, X_1 is defined for

$$\psi : Z \to X.$$

For a point $x_0 \in X_0$, we choose a small disk $U = U(x_0)$ with $x_0 \in U \subset X_0$. Let s_j be the elementary symmetric functions of

$$\psi \circ \eta_1, \ldots, \psi \circ \eta_n$$

of degree j, where $\eta_1, \ldots, \eta_n$ are local inverses of φ of U. Then for each $j = 1, 2, \ldots, n$, all

$$s_j : U(x_0) \to X$$

piece together to one function and it extends meromorphically to $\hat{X} = \widehat{\mathbb{C}}$ and ψ satisfies the equation

$$\psi^n - s_1 \circ \varphi \cdot \psi^{n-1} + \cdots + (-1)^{n-1} s_{n-1} \circ \varphi \cdot \psi + (-1)^n s_n \circ \varphi = 0$$

(see [34, Theorem 8.3] for details). Note that each $s_j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function. By writing

$$(-1)^j s_j(x) = \frac{a_j(x)}{a_0(x)}$$

for polynomials

$$a_0(x), \ldots, a_n(x) \in \mathbb{C}[x]$$

without non-trivial common factor, we define

$$P(x, y) = a_0(x) y^n + a_1(x) y^{n-1} + \cdots + a_{n-1}(x) y + a_n(x).$$

Then,

$$P(\alpha, \beta) = P\big(\varphi(z), \psi(z)\big) = 0$$

for $z \in Z$.

Conversely, suppose that $x_0 \in X_0$ with $a_0(x_0) \neq 0$ and $y_0 \in X_1$ satisfy

$$P(x_0, y_0) = 0.$$

Let

$$\varphi^{-1}(x_0) = \{z_1, \ldots, z_n\}$$

and let $\sigma_1, \ldots, \sigma_n$ be the elementary symmetric functions of $\psi(z_1), \ldots, \psi(z_n)$. Note that $\sigma_j = s_j(x_0)$ for $j = 1, \ldots, n$. Then,

$$\begin{aligned} a_0(x_0)\big(y_0 - \psi(z_1)\big) \cdots \big(y_0 - \psi(z_n)\big) &= a_0(x_0)\big(y_0^n - \sigma_1 y_0^{n-1} + \cdots + (-1)^n \sigma_n\big) \\ &= \sum_{j=0}^{n} a_j(x_0) y_0^{n-j} \\ &= P(x_0, y_0) \\ &= 0. \end{aligned}$$

Hence, $y_0 = \psi(z_k)$ for some k. Since

$$x_0 = \varphi(z_k) \quad \text{and} \quad y_0 = \psi(z_k),$$

we have shown that the converse is true. We next show that the polynomial $P(x, y)$ is irreducible in y with coefficients in $\mathbb{C}[x]$. Suppose, on the contrary, that $P(x, y)$ reduces to the product $P_1(x, y)P_2(x, y)$ of nonconstant polynomials P_1 and P_2. Let

$$Y_l = \{(x, y) \in X_0 \times X_1 : P_l(x, y) = 0\}$$

for $l = 1$, 2. We claim that $Y_1 \cap Y_2 = \emptyset$. Indeed, if $(x_0, y_0) \in Y_1 \cap Y_2$, then the polynomial

$$P(x_0, y) = P_1(x_0, y)P_2(x_0, y)$$

has a multiple zero at $y = y_0$, which is impossible because the set $\psi^{-1}(y_0)$ consists of n points for $y_0 \in X_1$. Since

$$\varphi \times \psi : Z_0 \to X_0 \times X_1$$

is continuous, where

$$Z_0 = \varphi^{-1}(X_0) \cap \psi^{-1}(X_1)$$

and Z_0 is connected, the image $(\varphi \times \psi)(Z_0)$ is contained in either Y_1 or Y_2. But, this is impossible. Thus, the claim has been shown.

Finally, consider the polynomial $P(x, y)$ in $\mathbb{C}(y)[x]$. Then, $P(x, y)$ is a polynomial in x with coefficients in $\mathbb{C}[y]$ and we may write

$$P(x, y) = b_0(y)x^m + b_1(y)x^{m-1} + \cdots + b_{m-1}(y)x + b_m(y), \quad b_j[y] \in \mathbb{C}[y],$$

where

$$m = \max_{0 \le j \le n} \deg a_j(x).$$

Note that $b_0 \neq 0$. Since $\psi : Z \to X$ is n-sheeted, as in the case of $\varphi : Z \to X$, we can see that $m = n$ and $P(x, y)$ is irreducible in $\mathbb{C}(y)[x]$. $\square$

Let us summarize the above observations for later use. Suppose that

$$K = G_q \cap G_q^{M_p}$$

is a subgroup of G_q of finite index n. The intermediate Riemann surface $Z = K\backslash\mathbb{H}$ may be used as a parameter space of the solutions (α, β) to the generalized modular equation (4.1). Indeed, the solutions are given by

$$\alpha = \varphi(z) \quad \text{and} \quad \beta = \psi(z)$$

for $z \in Z$, where

$$\varphi,\, \psi : Z \to X = G_q \backslash \mathbb{H}$$

are (possibly branched) covering maps satisfying the relations

$$\varphi\big(\rho(\tau)\big) = \pi_q(\tau) \quad \text{and} \quad \psi\big(\rho(\tau)\big) = \pi_q(p\tau)$$

for $\tau \in \mathbb{H}$. Note that φ and ψ extend to the compactifications $\hat{Z}$ to $\hat{X} = \widehat{\mathbb{C}}$ as n-sheeted branched (analytic) covering maps. The polynomial $P(x, y)$ whose zero set describes the solutions (α, β) can be computed as in the above proof.

4.4 Fricke Involution

Recall that the Möbius transformations S and M_p are given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

where $p > 1$ is an integer. The Fricke involution is defined by

$$SM_p : \tau \to -\frac{1}{p\tau},$$

which is also known as Atkin-Lehner involution. The involution SM_p swaps the cusps 0 and ∞ (see Figure 4.2). The following lemma helps us to compute ψ when we know about φ.

Lemma 4.5. *Under the assumption of Theorem 4.1, the Möbius transformation*

$$SM_p : \tau \to -\frac{1}{p\tau}$$

induces an analytic involution $\omega : Z \to Z$ which satisfies the relation

$$\rho \circ SM_p = \omega \circ \rho$$

and the functional equation

$$\psi = 1 - \varphi \circ \omega.$$

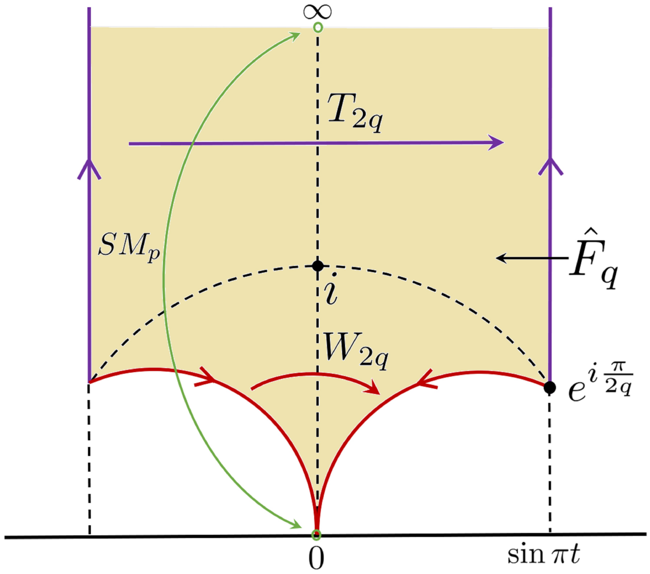


Figure 4.2: Fricke involution SM_p on the fundamental domain $\hat{F}_q$ for $G = \langle T_{2q}, W_{2q} \rangle$

Proof. Let

$$K = G_q \cap G_q^{M_p}.$$

We first note that the Möbius transformations S and M_p satisfy the relation

$$SM_p = M_p^{-1} S$$

and thus

$$M_p S M_p = S.$$

Recall that S normalizes G_q; namely,

$$G_q^S := S^{-1} G_q S = G_q.$$

Therefore,

$$\begin{aligned} K^{SM_p} &= G_q^{SM_p} \cap (G_q^{M_p})^{SM_p} \\ &= G_q^{M_p} \cap G_q^{M_p S M_p} \\ &= G_q^{M_p} \cap G_q^S \\ &= K, \end{aligned}$$

which means that $SM_p : \mathbb{H} \to \mathbb{H}$ descends to an automorphism $\omega : Z \to Z$ such that

$$\rho \circ SM_p = \omega \circ \rho.$$

Since $(SM_p)^2 = I$, we see that $\omega \circ \omega = \mathrm{id}$. We recall Lemma 4.2 which says that

$$\pi_q \circ S = 1 - \pi_q.$$

Then we compute

$$\begin{aligned}
\psi \circ \omega \circ \rho &= \psi \circ \rho \circ SM_p \\
&= (\pi_q \circ M_p) \circ SM_p \\
&= \pi_q \circ S \\
&= 1 - \pi_q \\
&= 1 - \varphi \circ \rho
\end{aligned}$$

and therefore we have

$$\psi \circ \omega = 1 - \varphi.$$

Since ω is an involution, we obtain the required relation. □

Remark 4.2. *Since ω comes from the normalizer of K, ω is indeed an automorphism of the Riemann* orbifold $K\backslash\mathbb{H}$. *In particular, ω maps a cone point of angle $2\pi/m$ to another (possibly the same) cone point of the same angle for an integer $m \geq 2$.*

Chapter 5

Hecke Groups and Ramanujan's Modular Equations

There is an intimate relation between the modular equations in Ramanujan's theories of signatures $\frac{1}{t} = 2, 3, 4$ and the Hecke groups. In this chapter, we study the relation between Hecke groups and the modular equations. There are different forms of modular equations for the same degree of β over α in the theory of signature $\frac{1}{t}$. For example,

$$(\alpha\beta)^{1/3} + \big\{(1-\alpha)(1-\beta)\big\}^{1/3} = 1, \tag{5.1}$$

$$\Big\{\frac{(1-\beta)^2}{1-\alpha}\Big\}^{\frac{1}{3}} - \Big(\frac{\beta^2}{\alpha}\Big)^{\frac{1}{3}} = m,$$

and

$$\Big(\frac{\alpha^2}{\beta}\Big)^{\frac{1}{3}} + \Big\{\frac{(1-\alpha)^2}{1-\beta}\Big\}^{\frac{1}{3}} = \frac{4}{m^4}$$

are the modular equations when the modulus β has degree 2 over the modulus α in the theory of signature 3 (see [19, Theorem 7.1]). Note that (5.1) can be transformed to the polynomial equation

$$(2\alpha-1)^3\beta^3 - 3\alpha(4\alpha^2-13\alpha+10)\beta^2 + 3\alpha(2\alpha^2-10\alpha+9)\beta - \alpha^3 = 0.$$

In Chapter 4, we offered a geometric approach to the proof of Ramanujan's identities for the solutions (α, β) to the generalized modular equation

$$\frac{{}_2F_1(t,1-t;1;1-\beta)}{{}_2F_1(t,1-t;1;\beta)} = p\,\frac{{}_2F_1(t,1-t;1;1-\alpha)}{{}_2F_1(t,1-t;1;\alpha)}. \tag{5.2}$$

By Theorem 4.1, the solution (α, β) satisfies a polynomial equation $P(\alpha, \beta) = 0$. In this chapter, we compute the degree in each of α and β of the polynomial $P(\alpha, \beta)$ explicitly based on the relation between the Hecke groups and modular equations. We establish some mutually equivalent statements related to Hecke subgroups and modular equations in Theorem 5.4, which is given based on Theorem 4.1, but the proof is different. We prove that if (α, β) is a solution to the generalized modular equation (5.2), then $(1-\beta, 1-\alpha)$ is also a solution to (5.2) and $P(1-\beta, 1-\alpha) = 0$. Note that by the degree n of the polynomial $P(\alpha, \beta)$, we mean that $P(\alpha, \beta)$ is a polynomial of degree n in each of α and β.

5.1 The Covering Group G_q and Even Hecke Subgroups

Recall that for an integer $k \geq 3$, the Hecke group H_k is defined as the discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ generated by the two elements $\pm S$ and $\pm T_k$, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_k = \begin{pmatrix} 1 & \lambda_k \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda_k = 2\cos\frac{\pi}{k}.$$

If $k = 2q$ is an even number with $q \geq 2$, then the Hecke group H_{2q} is isomorphic to $\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ and $G = \langle T_{2q}, W_{2q} \rangle$ is a (normal) subgroup of H_{2q} of index 2, where

$$T_{2q} := \begin{pmatrix} 1 & \lambda_{2q} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W_{2q} := S^{-1} T_{2q}^{-1} S = \begin{pmatrix} 1 & 0 \\ \lambda_{2q} & 1 \end{pmatrix}.$$

For $q = 2$ and $q = 3$, we have two important and interesting Hecke subgroups H_4 and H_6, respectively. The elements of H_4 and H_6 are completely known (see [54]). For these two cases, the Hecke group H_{2q} consists of the following two types of elements:

1. $\begin{pmatrix} a\lambda_{2q} & b \\ c & d\lambda_{2q} \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $ad\lambda_{2q}^2 - bc = 1$,
2. $\begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc\lambda_{2q}^2 = 1$,

which are known as the odd type and the even type, respectively (see [25] and [26]).

In Section 4.1, we have seen that the covering group G_q of π_q is generated by

$$T_{2q} := \begin{pmatrix} 1 & \lambda_{2q} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V_q := T_{2q}W_{2q}^{-1} = \begin{pmatrix} \lambda_q + 1 & -\lambda_{2q} \\ \lambda_{2q} & -1 \end{pmatrix}.$$

The group G_q is same as the group $G = \langle T_{2q}, W_{2q} \rangle$. Thus, we may conclude that the covering group G_q is the even type subgroup of the Hecke group H_{2q} and it can be represented by

$$G_q = \left\{ \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc\lambda_{2q}^2 = 1 \right\}.$$

Note that in [26], the group G_q is considered as even type for $q = 2$ and $q = 3$, i.e., for $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$. We will consider G_q also for the case $q = \infty$, i.e., for the case $\lambda_\infty = 2$. Since $q = \frac{1}{1-2t}$, the cases $q = 2$, 3, and ∞ correspond to Ramanujan's theories of signatures $\frac{1}{t} = 4$, 3, and 2, respectively.

Though we have discussed the construction of the covering group G_q in Section 4.1, we give the following lemma, which provides the reason for considering only the theories of signatures $\frac{1}{t} = 2$, 3, and 4.

Lemma 5.1. *If $\pi_q : \Delta_t \to \mathbb{H}$ is the inverse map of f_t, then π_q can be extended analytically to a single-valued function on $\mathbb{H}$ only for the theories of signatures 2, 3, and 4, i.e., for $q \in \{2, 3, \infty\}$ and the covering group of π_q is the even type subgroup of the Hecke group H_{2q}.*

Proof. Since

$$\tau = f_t(z) = i\,\frac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)}$$

maps the upper half z-plane to the curvilinear triangle Δ_t in the upper half τ-plane with internal angles 0, 0 and $\theta = (1-2t)\pi = \frac{\pi}{q}$, by Lemma 2.16, the condition (2.21) becomes

$$\frac{1}{q} < 1,$$

i.e., it depends only on the third fixed point $f_t(\infty) = e^{i\frac{\theta}{2}}$. As $\theta = (1-2t)\pi$, i.e., $q = \frac{1}{1-2t}$, we conclude that $q \in \mathbb{N} \cup \{\infty\} \setminus \{1\}$ only for $t = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. As $\pi_q : \Delta_t \to \mathbb{H}$ is the inverse map of f_t, we have

$$\pi_q(\infty) = 0, \quad \pi_q(0) = 1 \quad \text{and} \quad \pi_q(e^{i\frac{\theta}{2}}) = \infty.$$

The Riemann Mapping Theorem confirms the existence and uniqueness of the map π_q. By applying the Schwarz reflection principle repeatedly, we can extend $\pi_q(\tau)$ analytically to a single-valued function on $\mathbb{H}$ with the real axis as its natural boundary. The map π_q constructs an infinite cover of $\widehat{\mathbb{C}} \setminus \{0,1,\infty\}$ and π_q has branch points at 0, 1, and ∞. At the point $e^{i\frac{\theta}{2}}$, π_q is locally q to 1. The covering group of π_q is

$$\{\sigma \in \mathrm{PSL}(2,\mathbb{R}) : \pi_q \circ \sigma = \pi_q\}.$$

By the above construction, Δ_t is the fundamental half-domain for the covering group of π_q. Hence, we deduce that the covering group of π_q is the even type subgroup of the Hecke group H_{2q}. $\square$

Remark 5.1. *In fact, the quotient Riemann surface $G_q\backslash\mathbb{H}$ is a modular surface only for the Ramanujan's theories of signatures $\frac{1}{t} = 2$, 3, and 4. The surface corresponding to the theory of signature 6 is not a modular surface since $q = \frac{1}{1-2t} \notin \mathbb{N} \cup \{\infty\} \setminus \{1\}$ for $\frac{1}{t} = 6$ (see [50, Section 10]).*

For $\alpha \in (0,1)$ and $t \in (0, \frac{1}{2}]$, let

$$x = \frac{{}_2F_1(t, 1-t; 1; 1-\alpha)}{{}_2F_1(t, 1-t; 1; \alpha)}$$

and

$$q_t(\alpha) := \exp\Big(-\frac{\pi x}{\sin \pi t}\Big).$$

Then,

$$\begin{aligned}\exp\Big(-\frac{\pi x}{\sin \pi t}\Big) =& \alpha \exp\big(\psi(t) + \psi(1-t) + 2\gamma\big)\\ &\times \Bigg(1 + (2t^2 - 2t + 1)\alpha\\ &+ \Big(1 - \frac{7}{2}(t-t^2) + \frac{13}{4}(t-t^2)^2\Big)\alpha^2 + \cdots\Bigg),\end{aligned}$$

where ψ is the Euler digamma function (logarithmic derivative of the Gamma function), i.e., $\psi(\alpha) = \frac{d(\ln\Gamma(\alpha))}{d\alpha}$. For $t = \frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$, we have, respectively,

$$\begin{aligned}\exp\Big(2\psi(\frac{1}{2}) + 2\gamma\Big) &= \frac{1}{16},\\ \exp\Big(\psi(\frac{1}{3}) + \psi(\frac{2}{3}) + 2\gamma\Big) &= \frac{1}{27},\\ \exp\Big(\psi(\frac{1}{4}) + \psi(\frac{3}{4}) + 2\gamma\Big) &= \frac{1}{64},\end{aligned}$$

and

$$\exp\Big(\psi(\frac{1}{6})+\psi(\frac{5}{6})+2\gamma\Big)=\frac{1}{432},$$

that is, only for $t=\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}$ and $\frac{1}{6}$, the expression $\exp\big(\psi(t)+\psi(1-t)+2\gamma\big)$ takes on rational values. For other values of t, $\exp\big(\psi(t)+\psi(1-t)+2\gamma\big)$ is transcendental. For example, if $t=\frac{1}{5}$, then

$$\exp\Big(\psi(\frac{1}{5})+\psi(\frac{4}{5})+2\gamma\Big)=(\sqrt{5})^{-5}\Big(\frac{1+\sqrt{5}}{2}\Big)^{-\sqrt{5}}.$$

For this reason, the signature, $\frac{1}{t}$, takes on one of the values 2, 3, 4, and 6. For details, see Section 12 of [19]. Note that we will not use further ψ to denote the Euler digamma function.

The subgroup G_q has two cusps and one elliptic point for $t\in\Big\{\frac{1}{3},\ \frac{1}{4}\Big\}$ and has three cusps for $t=\frac{1}{2}$. Thus, the quotient Riemann surface $G_q\backslash\mathbb{H}$ is the two punctured Riemann sphere $\widehat{\mathbb{C}}\setminus\{0,1\}$ for $t\in\Big\{\frac{1}{3},\ \frac{1}{4}\Big\}$ and the thrice punctured Riemann sphere $\widehat{\mathbb{C}}\setminus\{0,1,\infty\}$ for $t=\frac{1}{2}$. The set of cusps of the Hecke group H_{2q} is $\mathbb{Q}[\lambda_{2q}]\cup\{\infty\}$. To compactify the quotient Riemann surface $G_q\backslash\mathbb{H}$, let $\mathbb{H}^*=\mathbb{H}\cup\mathbb{Q}[\lambda_{2q}]\cup\{\infty\}$. Then, $G_q\backslash\mathbb{H}^*$ is a compact Riemann surface. For all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in H_{2q}$ and $\tau\in\mathbb{H}$, the meromorphic function $g:\mathbb{H}\to H_{2q}\backslash\mathbb{H}^*$ is called an automorphic function if $g\Big(\dfrac{a\tau+b}{c\tau+d}\Big)=g(\tau)$ (see [24]).

5.2 Degree of the Polynomial $P(x,y)$

Let $\Psi(N)$ denote the Dedekind psi function given by

$$\Psi(N)=N\prod_{\substack{q|N\\ q\text{ prime}}}\Big(1+\frac{1}{q}\Big),\qquad N\in\mathbb{N}\tag{5.3}$$

(see [31, p. 123]). By the following theorem, one can determine the degree in each of α and β of the polynomial $P(\alpha,\beta)$ explicitly in Ramanujan's theories of signatures $\frac{1}{t}=2,\ 3,$ and 4.

Theorem 5.2. *For an integer $p>1$, suppose β has degree p over α in the theories of signatures $\frac{1}{t}=2,\ 3,$ and 4. Let $n\big(p,\frac{1}{t}\big)$ be the degree in each of α and β of the* *polynomial $P(\alpha, \beta)$ in Theorem 4.1, then*

$$n(p,2) = n(p,4) = \frac{1}{3}\Psi(2p) \quad \textit{and} \quad n(p,3) = \frac{1}{4}\Psi(3p).$$

Remark 5.2. *If p is an odd prime, then $n(p,2) = n(p,4) = p+1$. If $p \neq 3$ is a prime, then $n(p,3) = p+1$.*

H. H. Chan and W.-C. Liaw [29] studied modular equations in the theory of signature 3 based on the modular equations studied by R. Russell [65].

Theorem A ([29, Theorems 2.1, 3.1])**.** *If $p > 2$ is a prime, $u = (\alpha\beta)^{l/8}$ and $v = \{(1-\alpha)(1-\beta)\}^{l/8}$, where $(p+1)/8 = m/l$ in lowest terms, then (u, v) satisfies a polynomial equation $Q(u,v) = 0$, where $Q(x,y)$ is of degree m in each of x and y in the theory of signature 2. If $p > 3$ is a prime, $u = (\alpha\beta)^{l/6}$ and $v = \{(1-\alpha)(1-\beta)\}^{l/6}$, where $(p+1)/3 = m/l$ in lowest terms, then (u,v) satisfies a polynomial equation $Q(u,v) = 0$, where $Q(x,y)$ is of degree m in each of x and y in the theory of signature 3.*

Remark 5.3. *In the theory of signature 3, the degree n of the polynomial $P(\alpha, \beta)$ in Theorem 4.1 and the degree m of the polynomial $Q(u,v)$ in Theorem A are related as follows:*

(i) $n = 3m$ when $p \equiv 2 \,(mod\ 3)$,

(ii) $n = m$ when $p \equiv 1 \,(mod\ 3)$.

We compute the degree $n\big(p, \frac{1}{t}\big)$ for some small values of p and $\frac{1}{t} \in \big\{2, 3, 4\big\}$ in Table 5.1, which can also be used to calculate the index of the subgroup $G_q \cap G_q^{M_p}$ in G_q, i.e., $|G_q : G_q \cap G_q^{M_p}|$.

5.2.1 Proof of Theorem 5.2

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. For an integer $p > 1$, let $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, then the transformation group of order p (see Chapter VI of [69]), Γ_{M_p}, is given by

$$\Gamma_{M_p} := \Gamma \cap \Big(M_p^{-1}\Gamma M_p\Big),$$

which can be written as the group of Möbius transformations

$$\Gamma_{M_p} = \Big\{\gamma \in \Gamma : \ M_p \gamma M_p^{-1} \in \Gamma\Big\}.$$

p	$n(p,2)$ and $n(p,4)$	$n(p,3)$
2	2	3
3	4	3
4	4	6
5	6	6
6	8	9
7	8	8
8	8	12
9	12	9
10	12	18
11	12	12
12	16	18
13	14	14
14	16	24
15	24	18
16	16	24
17	18	18
18	24	27
19	20	20
20	24	36

Table 5.1: Values of $n\big(p, \frac{1}{t}\big)$ for some small values of p and $\frac{1}{t} \in \{2, 3, 4\}$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then $M_p \gamma M_p^{-1} = \begin{pmatrix} a & pb \\ \frac{c}{p} & d \end{pmatrix}$. Hence $M_p \gamma M_p^{-1} \in \Gamma$ only when $c \equiv 0 \pmod p$ and we have

$$\Gamma_{M_p} = \Gamma_0(p),$$

where

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod p \right\}.$$

The following lemma is a well-known result, e.g., see [69, p. 79] or Proposition 1.43 in [72].

Lemma 5.3. *For any positive integer N,*

$$\big|\Gamma : \Gamma_0(N)\big| = \Psi(N) = N \prod_{\substack{q|N \\ q \text{ prime}}} \Big(1 + \frac{1}{q}\Big).$$

Proof of Theorem 5.2. For $t \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$ and $\lambda_{2q} = 2\cos\frac{\pi}{2q}$, let

$$K = \left\{\gamma \in G_q : M_p\gamma M_p^{-1} \in G_q\right\},$$

where $p > 1$ is an integer, $q = 1/(1-2t)$ and $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

If

$$\gamma = \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} \in G_q,$$

then

$$M_p\gamma M_p^{-1} = \begin{pmatrix} a & p\,b\lambda_{2q} \\ \frac{c}{p}\,\lambda_{2q} & d \end{pmatrix}.$$

Therefore, $M_p\gamma M_p^{-1} \in G_q$ only when $c \equiv 0 \pmod{p}$ and we have

$$K = \left\{\begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} \in G_q : c \equiv 0 \pmod{p}\right\}. \tag{5.4}$$

Consequently,

$$G_q \cap \left(M_p^{-1}G_qM_p\right) = K \tag{5.5}$$

and

$$K < G_q < H_{2q}.$$

Let π_q and ρ denote the canonical projections $\mathbb{H} \to G_q\backslash\mathbb{H}$ and $\mathbb{H} \to K\backslash\mathbb{H}$, respectively. From the subgroup relation $K < G_q$, we have the branched covering map

$$\varphi : K\backslash\mathbb{H} \to G_q\backslash\mathbb{H}$$

and the following commutative diagram:

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\rho"'] \arrow[dr, "\pi_q"] & \\ K\backslash\mathbb{H} \arrow[r, "\varphi"'] & G_q\backslash\mathbb{H}. \end{tikzcd}$$

The degree of the branched covering $K\backslash\mathbb{H} \to G_q\backslash\mathbb{H}$ is $\left|G_q : K\right|$, which is the degree $n\left(p, \frac{1}{t}\right)$ in each of α and β of the polynomial $P(\alpha, \beta)$ by Theorem 4.1.

Also, for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, we have

$$\Gamma \cap \left(M_p^{-1}\Gamma M_p\right) = \Gamma_0(p),$$

and

$$\Gamma_0(\lambda_{2q}^2 p) < \Gamma_0(\lambda_{2q}^2) < \Gamma.$$

Let us consider the mapping

$$\Theta : G_q \to \Gamma_0(\lambda_{2q}^2)$$

defined by

$$\Theta(A) = M_{\lambda_{2q}}^{-1} A M_{\lambda_{2q}},$$

where $A = \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} \in G_q$ and $M_{\lambda_{2q}} = \begin{pmatrix} \lambda_{2q} & 0 \\ 0 & 1 \end{pmatrix}$. Then, we have

$$\Theta(G_q) = \Gamma_0(\lambda_{2q}^2) \quad \text{and} \quad \Theta(K) = \Gamma_0(\lambda_{2q}^2 p).$$

Therefore, $G_q \cong \Gamma_0(\lambda_{2q}^2)$, $K \cong \Gamma_0(\lambda_{2q}^2 p)$ and we have

$$\begin{aligned} \left|G_q : K\right| &= \left|\Gamma_0(\lambda_{2q}^2) : \Gamma_0(\lambda_{2q}^2 p)\right| \\ &= \frac{\left|\Gamma : \Gamma_0(\lambda_{2q}^2 p)\right|}{\left|\Gamma : \Gamma_0(\lambda_{2q}^2)\right|}. \end{aligned}$$

By Lemma 5.3,

$$\left|\Gamma : \Gamma_0(\lambda_{2q}^2 p)\right| = \Psi(\lambda_{2q}^2 p) \quad \text{and} \quad \left|\Gamma : \Gamma_0(\lambda_{2q}^2)\right| = \Psi(\lambda_{2q}^2).$$

Hence

$$n\Big(p, \frac{1}{t}\Big) = \left|G_q : K\right| = \frac{\Psi(\lambda_{2q}^2 p)}{\Psi(\lambda_{2q}^2)},$$

which implies

$$n(p,2) = \frac{1}{6}\Psi(4p), \quad n(p,3) = \frac{1}{4}\Psi(3p) \quad \text{and} \quad n(p,4) = \frac{1}{3}\Psi(2p).$$

By (5.3), it is easy to show that

$$\Psi(4p) = 2\Psi(2p).$$

Thus, $n(p,2) = n(p,4)$ as required. $\square$

5.3 The Modular Equation $P(\alpha, \beta) = 0$ and Hecke Subgroups

In this section, we first prove the following theorem, which is given based on Theorem 4.1 and is related to Hecke subgroups and the modular equations in Ramanujan's theories of signatures 2, 3, and 4.

Theorem 5.4. *For a given integer $p > 1$, suppose that β has degree p over α in the theories of signatures $\frac{1}{t} = 2$, 3, and 4. If $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, then*

(i) there exists a Hecke subgroup, say Γ_1, of finite index in H_{2q},

(ii) $\left(M_p^{-1}\Gamma_1 M_p\right) \cap \Gamma_1$ has finite index in H_{2q},

(iii) the degree of the branched covering

$$\Gamma_2 \backslash \mathbb{H} \to \Gamma_1 \backslash \mathbb{H}$$

is finite, where $\Gamma_2 = \left(M_p^{-1}\Gamma_1 M_p\right) \cap \Gamma_1$,

(iv) there is a polynomial equation $P(\alpha, \beta) = 0$, where the polynomial $P(\alpha, \beta)$ has degree $n = \left|\Gamma_1 : \Gamma_2\right|$ in each of α and β.

Remark 5.4. *In fact, the statements in Theorem 5.4 are mutually equivalent.*

We now recall some relevant facts from Chapter 4. If $f_t(z) = i\,\dfrac{{}_2F_1(t, 1-t; 1; 1-z)}{{}_2F_1(t, 1-t; 1; z)}$, then f_t maps conformally the upper half-plane $\mathbb{H}$ onto

$$\Delta_t = \left\{ \tau \in \mathbb{H} : 0 < Re\,\tau < \cos\frac{\theta}{2}, \left|2\tau\cos\frac{\theta}{2} - 1\right| > 1 \right\},$$

and the fundamental half-domain for G_q is Δ_t, where $t \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. The generalized modular equation (5.2) can be expressed as $f_t(\beta) = pf_t(\alpha)$, where p is an integer > 1. Thus, α and β in $\widehat{\mathbb{C}} \setminus \{0, 1\}$ satisfy (5.2) if and only if $\alpha = \pi_q(\tau)$ and $\beta(\tau) = \pi_q(p\tau)$ for $\tau \in \mathbb{H}$. Let $X = G_q \backslash \mathbb{H}$ and $Z = K \backslash \mathbb{H}$, where $K = G_q \cap G_q^{M_p}$. For the canonical projections $\pi_q : \mathbb{H} \to X$ and $\rho : \mathbb{H} \to Z$, consider the mappings $\varphi : Z \to X$ and $\psi : Z \to X$ such that $\pi_q = \varphi \circ \rho$ and $\pi_q \circ M_p = \psi \circ \rho$, i.e., the following diagrams commute:

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\rho"'] \arrow[dr, "\pi_q"] & \\ Z \arrow[r, "\varphi"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{H} \arrow[r, "M_p"] \arrow[d, "\rho"'] & \mathbb{H} \arrow[d, "\pi_q"] \\ Z \arrow[r, "\psi"'] & X. \end{tikzcd}$$

Thus, for $z \in Z$, the solution (α, β) to the generalized modular equation (5.2) is parametrized by $\alpha = \varphi(z)$ and $\beta = \psi(z)$. By the following theorem, $(1-\beta, 1-\alpha)$ is also a solution to (5.2).

Theorem 5.5. *If the solution (α, β) to the generalized modular equation (5.2) satisfies the equation $P(x, y) = 0$, then $(1-\beta, 1-\alpha)$ is also a solution to (5.2) and satisfies the equation $P(x, y) = 0$, where $P(x, y)$ is the polynomial in Theorem 4.1.*

If the moduli α and β are replaced by $1-\beta$ and $1-\alpha$, respectively and the multiplier m, defined by

$$m = \frac{{}_2F_1(t, 1-t; 1; \alpha)}{{}_2F_1(t, 1-t; 1; \beta)}, \tag{5.6}$$

is replaced by $\frac{p}{m}$, where p is the degree of β over α, then we have

$$\frac{p}{m} = \frac{{}_2F_1(t, 1-t; 1; 1-\beta)}{{}_2F_1(t, 1-t; 1; 1-\alpha)}. \tag{5.7}$$

From the equations (5.6) and (5.7), we have

$$p\,\frac{{}_2F_1(t, 1-t; 1; 1-\alpha)}{{}_2F_1(t, 1-t; 1; \alpha)} = \frac{{}_2F_1(t, 1-t; 1; 1-\beta)}{{}_2F_1(t, 1-t; 1; \beta)}.$$

Thus, the obtained modular equation has the same degree p (see [16, Entry 24(v), p. 216]).

5.3.1 Proofs of Theorem 5.4 and Theorem 5.5

Proof of Theorem 5.4. First, recall that the covering group of the map π_q is given by

$$G_q = \left\{ \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc\lambda_{2q}^2 = 1 \right\},$$

which is the even type subgroup of the Hecke group H_{2q}. It is well-known that the index of G_q in H_{2q} is 2 (see [26, p. 61]). Thus, $\Gamma_1 = G_q$ and (i) follows easily from this fact.

From (5.5), we have

$$\Gamma_2 = \Gamma_1 \cap \left(M_p^{-1}\Gamma_1 M_p\right) = K.$$

By virtue of the proof of Theorem 5.2, we have $\Gamma_2 \cong \Gamma_0(\lambda_{2q}^2 p)$ and hence Γ_2 is isomorphic to $\Gamma_0(4p)$, $\Gamma_0(3p)$ and $\Gamma_0(2p)$ for $t = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$, respectively. Each of $\Gamma_0(4p)$, $\Gamma_0(3p)$ and $\Gamma_0(2p)$ has finite index in $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. Therefore, Γ_2 has finite index in H_{2q}, which implies (ii).

If $X = \Gamma_1 \backslash \mathbb{H}$ and $Z = \Gamma_2 \backslash \mathbb{H}$, then the degree of the branched covering $Z \to X$ is equal to the index of Γ_2 in Γ_1. Since each of Γ_1 and Γ_2 has finite index in H_{2q}, the index of Γ_2 in Γ_1 is finite. Therefore, (iii) holds.

By Lemma 3.2, we deduce that $\alpha(\tau)$ and $\beta(\tau) = \alpha(p\tau)$ are automorphic functions on Γ_1 and $\Gamma_1' := M_p^{-1}\Gamma_1 M_p$, respectively. Recall that the quotient Riemann surface $X = \Gamma_1 \backslash \mathbb{H}$ is $\widehat{\mathbb{C}} \setminus \{0,1\}$ for $t \in \left\{\frac{1}{3}, \frac{1}{4}\right\}$ and $\widehat{\mathbb{C}} \setminus \{0,1,\infty\}$ for $t = \frac{1}{2}$. If $\widehat{X}$ is the compactification of X, then $\widehat{X}$ is the Riemann sphere $\widehat{\mathbb{C}}$. Thus, the field of automorphic functions for Γ_1 is $\mathbb{C}\big(\alpha(\tau)\big)$. If $X' = \Gamma_1' \backslash \mathbb{H}$ and $\widehat{X}'$ is the compactification of X', then $\widehat{X}' = \widehat{\mathbb{C}}$. The field of automorphic functions for Γ_1' is $\mathbb{C}\big(\beta(\tau)\big)$. Since $\Gamma_2 < \Gamma_1$ and $\Gamma_2 < \Gamma_1'$, both $\mathbb{C}\big(\alpha(\tau)\big)$ and $\mathbb{C}\big(\beta(\tau)\big)$ are subfields of the field of automorphic functions for $\Gamma_2 = \Gamma_1 \cap \Gamma_1'$, i.e., $\mathbb{C}\big(\alpha(\tau), \beta(\tau)\big)$. If $n = \big|\Gamma_1 : \Gamma_2\big|$, then $\varphi : Z \to X$ is a n-sheeted branched covering map. For any function $g \in \mathbb{C}\big(\alpha(\tau)\big)$, we have a function $f \in \mathbb{C}\big(\alpha(\tau), \beta(\tau)\big)$ by virtue of the pullback

$$\varphi^*(g) = g \circ \varphi = f,$$

where

$$\varphi^* : \mathbb{C}\big(\alpha(\tau)\big) \to \mathbb{C}\big(\alpha(\tau), \beta(\tau)\big)$$

is an algebraic field extension of degree n (see [34, Theorem 8.3]). Similarly, if ψ is the branched covering map $Z \to X'$, then ψ is also a n-sheeted covering map, since $|\Gamma_1' : \Gamma_2| = n$ by Lemma 4.3. Hence,

$$\psi^* : \mathbb{C}\big(\beta(\tau)\big) \to \mathbb{C}\big(\alpha(\tau), \beta(\tau)\big)$$

is an algebraic field extension of degree n. Consequently, there is a polynomial $P\big(\alpha(\tau), \beta(\tau)\big)$ which has degree n in each of $\alpha(\tau)$ and $\beta(\tau)$. The polynomial

$P\big(\alpha(\tau), \beta(\tau)\big)$ is determined up to a scalar factor so that

$$P\big(\alpha(\tau), \beta(\tau)\big) = 0,$$

which implies (iv) and completes the proof. $\square$

Proof of Theorem 5.5. Recall that the Hecke subgroup K is given by

$$K = \left\{ \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} \in G_q : c \equiv 0 \,(\mathrm{mod}\ p) \right\}.$$

Let $W_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$, then

$$W_p^{-1} \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} W_p = \begin{pmatrix} d & -\frac{c}{p}\lambda_{2q} \\ -pb\lambda_{2q} & a \end{pmatrix},$$

where $\begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} \in K$. Since $c \equiv 0 \,(\mathrm{mod}\ p)$, it follows that

$$W_p^{-1} \begin{pmatrix} a & b\lambda_{2q} \\ c\lambda_{2q} & d \end{pmatrix} W_p \in K.$$

Thus, K is normalized by W_p in $\mathrm{PSL}(2, \mathbb{R})$. The Möbius transformation

$$W_p\tau = -\frac{1}{p\tau}$$

induces an automorphism ω on $Z = K\backslash\mathbb{H}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "W_p"] \arrow[d, "\rho"'] & \mathbb{H} \arrow[d, "\rho"] \\ Z \arrow[r, "\omega"'] & Z. \end{tikzcd}$$

Moreover, by Lemma 4.5, $\omega : Z \to Z$ satisfies the following functional equations:

$$\varphi \circ \omega = 1 - \psi,$$
$$\psi \circ \omega = 1 - \varphi.$$

Hence, for $z \in Z$, we have

$$\varphi\big(\omega(z)\big) = 1 - \psi(z) = 1 - \beta$$

and

$$\psi\big(\omega(z)\big) = 1 - \varphi(z) = 1 - \alpha,$$

that is, ω interchanges α and $1-\beta$; β and $1-\alpha$. Thus, we deduce that $(1-\beta, 1-\alpha)$ is also a solution to (5.2) and $P(1-\beta, 1-\alpha) = 0$. $\square$

Chapter 6

Modular Equations in the Theory of Signature 2

In the theory of signature $\frac{1}{t} = 2$, the generalized modular equation is

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)} = p\,\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}. \tag{6.1}$$

Recall that the case of signature $\frac{1}{t} = 2$ corresponds to the classical modular equation. In this chapter, we consider the modular equations corresponding to the cases $p = 2$ and $p = 3$. Since $q = \frac{1}{1-2t}$, the case of signature $\frac{1}{t} = 2$ corresponds to the case $q = \infty$.

6.1 The Subgroup G_∞

As $q \to \infty$, the Hecke group H_{2q} converges to the group $H_\infty = \langle S, T_\infty \rangle$, which is a subgroup of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ of index 3, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

At the same time, the subgroup G_q converges to the principal congruence subgroup $\Gamma(2)$ of level 2, which will be denoted by G_∞ in this section. That is to say,

$$G_\infty = \{A \in \mathrm{SL}(2, \mathbb{Z}) : A \equiv I \bmod 2\}/\{\pm I\}.$$

The fundamental domain $\tilde{F}_\infty$ for G_∞ is described by $0 \leq \operatorname{Re}\tau \leq 2, |\tau - 1/2| \geq 1/2, |\tau - 3/2| \geq 1/2$ for $\tau \in \mathbb{H}$ (see Figure 6.1).

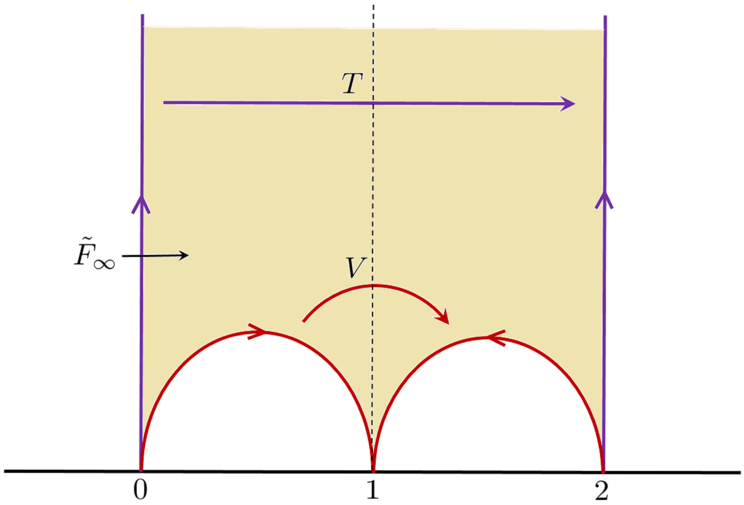


Figure 6.1: Fundamental domain for $G_\infty = \langle T, V \rangle$

A pair of generators of G_∞ are given by

$$T = T_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.$$

Note that V is a parabolic element with fixed point at $\tau = 1$ and that $V(0) = 2$. Also, let

$$W = -V^{-1}T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is well-known that $\Gamma(2) = G_\infty$ is a Fuchsian group uniformizing the thrice punctured sphere $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ (see §§3.4-5 in Chapter 7 of [2] and §§4.3 in Chapter 1 of [32], where the symbol λ is used for π_∞). The canonical projection

$$\pi_\infty : \mathbb{H} \to X$$

may be considered to be

$$\pi_\infty : \mathbb{H} \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$$

with

$$\pi_\infty(0) = 1, \quad \pi_\infty(\infty) = 0 \quad \text{and} \quad \pi_\infty(1) = \infty \tag{6.2}$$

by (4.2). Let $\rho : \mathbb{H} \to Z$, $\varphi, \psi : Z \to X$ be as in the proof of Theorem 4.1. Here is a simple but useful observation. The conjugation of a matrix by the Möbius transformation $M_p\tau = p\tau$ is computed by

$$M_p^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} M_p = \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix}.$$

Hence, an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of G_∞ is a member of $G_\infty^{M_p}$ precisely if

$$c \equiv 0 \pmod{2p}.$$

6.2 Case $p = 2$

In this section, we prove the following theorem corresponding to the case $p = 2$ in the theory of signature 2 (see [16, (24.12) on p. 213 and (24.21) on p. 215]).

Theorem 6.1. *In the theory of signature 2, suppose the modulus β has degree 2 over the modulus α. Then*

$$\alpha := \varphi(z) = 1 - z^2 \quad \textit{and} \quad \beta := \psi(z) = \frac{(z-1)^2}{(z+1)^2}$$

for $z \in \widehat{\mathbb{C}} \setminus \{0, \infty, 1, -1\}$. The modular equation is given by

$$\beta = \left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2. \tag{6.3}$$

Note that (6.3) is equivalent to (1.2) with $\alpha = r^2$ and $\beta = s^2$. The polynomial in Theorem 4.1 is given by

$$P(x, y) = x^2y^2 - 2(x^2 - 8x + 8)y + x^2.$$

In this case, the polynomial is not symmetric in x and y.

6.2.1 Construction of Fundamental Domain for $G_\infty \cap G_\infty^{M_2}$

Let $K = G_\infty \cap G_\infty^{M_2}$. Then the index of K in G_∞ is 2 (see Table 5.1). Let $F_K = (F, \mathcal{A}_K)$ denote the admissible fundamental domain for K, where F is a convex hyperbolic polygon and $\mathcal{A}_K$ is the set of side-pairing transformations which generate K. Then we choose F so that

$$F = \tilde{F}_\infty \cup V^{-1}(\tilde{F}_\infty).$$

If we identify the geodesic joining 0 to ∞ with the geodesic joining 2 to ∞ by A_1, the geodesic joining $\frac{2}{3}$ to 1 with the geodesic joining 2 to 1 by A_2 and the geodesic joining $\frac{2}{3}$ to $\frac{1}{2}$ with the geodesic joining 0 to $\frac{1}{2}$ by A_3, then

$$A_1 := T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$
$$A_2 := V^2 = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix},$$
$$A_3 := V^{-1}TV = \begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}$$

are side-paring transformations of the hyperbolic polygon $F = \tilde{F}_\infty \cup V^{-1}(\tilde{F}_\infty)$ (see Figure 6.2), that is, $\mathcal{A}_K = \{A_1,\ A_2,\ A_3\}$. Figure 6.3 shows the hyperbolic polygon

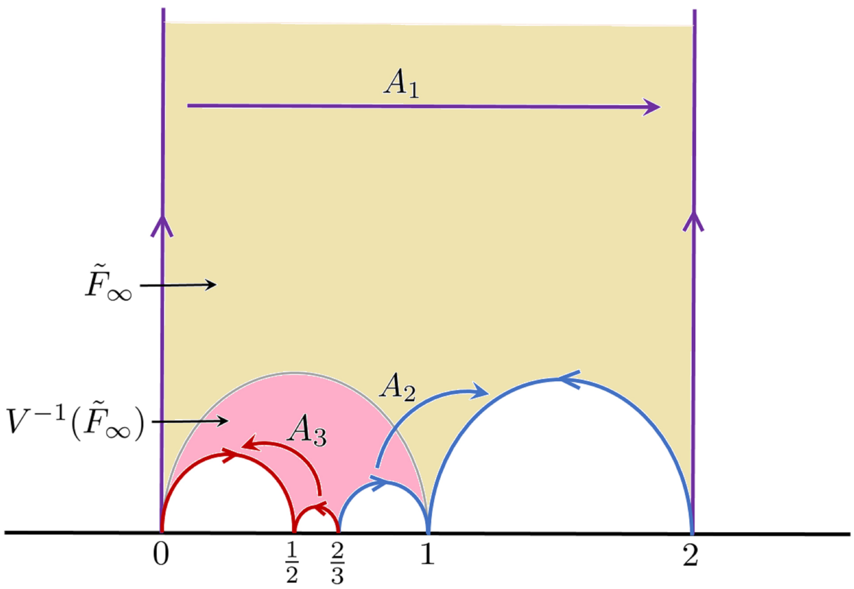


Figure 6.2: Fundamental domain for $K = G_\infty \cap G_\infty^{M_2}$

F in the Poincaré disc model, where $v_1,\ v_2,\ v_3$, and v_4 represent the inequivalent cusps. Therefore, the elements A_1, A_2, A_3 generate a subgroup of G_∞ of index 2. In view of the forms of A_j, we see that $A_j \in K$ for $j = 1, 2, 3$ and thus we have

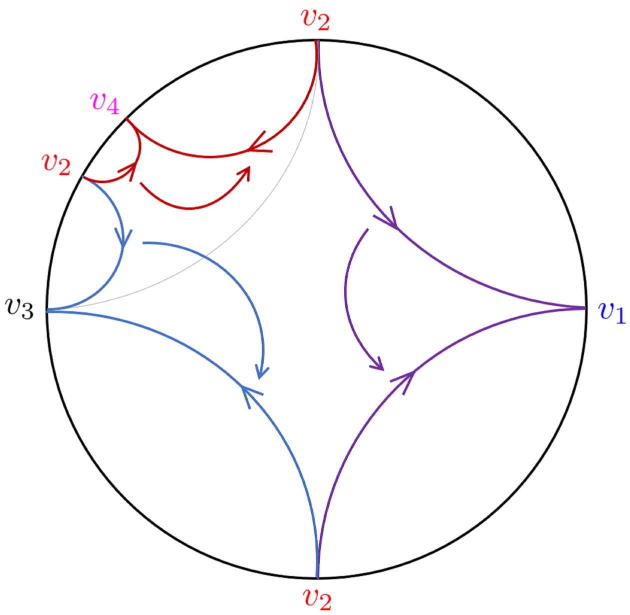


Figure 6.3: The hyperbolic polygon F for $K = G_\infty \cap G_\infty^{M_2}$ in the Poincaré disc model

$K = \langle A_1, A_2, A_3\rangle$. Moreover, since

$$\begin{aligned}\pi_\infty(M_2\tau) = \pi_\infty(M_2A_j\tau) &\Leftrightarrow M_2\tau \equiv M_2A_j\tau \pmod{G_\infty}\\ &\Leftrightarrow M_2A_j\tau = \sigma(M_2\tau) \text{ for } \sigma \in G_\infty\\ &\Leftrightarrow M_2A_jM_2^{-1} = \sigma \in G_\infty,\end{aligned}$$

we find the generators of the Fuchsian group K, i.e., A_1, A_2, A_3, such that

$$M_2A_jM_2^{-1} \in G_\infty$$

for $j = 1,\ 2,\ 3$ and $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to verify that

$$M_2A_1M_2^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = T^2 \in G_\infty,$$

$$M_2A_2M_2^{-1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} = -TV^{-1} \in G_\infty$$

and

$$M_2A_3M_2^{-1} = \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix} = V^{-1}V^{-1} \in G_\infty.$$

6.2.2 Proof of Theorem 6.1

It is easy to observe that the Fuchsian group $K = G_\infty \cap G_\infty^{M_2}$ has four inequivalent cusps. Therefore, the quotient Riemann surface $Z = K\backslash\mathbb{H}$ is conformally equivalent to a four-times punctured sphere. Recall that ρ is the canonical projection $\mathbb{H} \to Z$. Thus we can assume that $Z = \widehat{\mathbb{C}} \setminus \{0, \infty, 1, b\}$, where

$$\rho(0) = 0, \quad \rho(1) = \infty, \quad \rho(\infty) = 1 \quad \text{and} \quad \rho\left(\frac{1}{2}\right) = b.$$

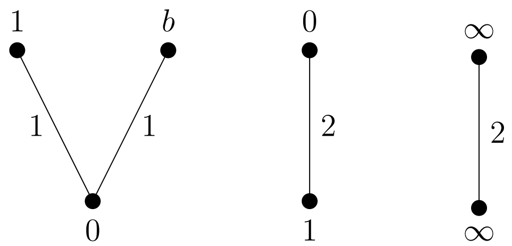
In view of (6.2), we have the conditions

$$\varphi(0) = 1, \quad \varphi(\infty) = \infty \quad \text{and} \quad \varphi(1) = \varphi(b) = 0.$$

Here we used the fact that $\frac{1}{2}$ is conjugate to ∞ under the action of G_∞. We note that the extension

$$\varphi : \hat{Z} = \widehat{\mathbb{C}} \to \hat{X} = \widehat{\mathbb{C}}$$

is a rational map of degree 2. The following figure shows the ramification data for the map $\varphi : \hat{Z} \to \hat{X}$.



Here, the ramification indices are expressed by the numbers attached to the lines. Since

$$\varphi^{-1}(1) = \{0\} \quad \text{and} \quad \varphi^{-1}(\infty) = \{\infty\},$$

we see that φ takes the values 0 and ∞ with multiplicity 2. Therefore, φ should have the form

$$\varphi(z) = cz^2 + 1$$

for a constant $c \neq 0$. Since $\varphi(1) = 0$, we conclude that $c = -1$ and $b = -1$. Hence,

$$\varphi(z) = 1 - z^2$$

under the above normalization.

Next we determine the form of $\psi : Z \to X$. By the relation

$$\pi_\infty(2\tau) = \psi\big(\rho(\tau)\big),$$

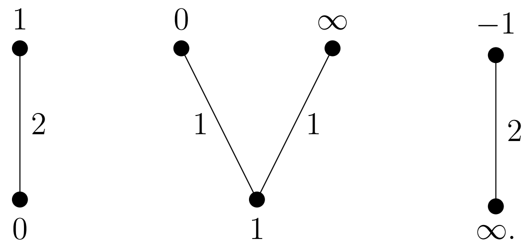
we have the necessary conditions

$$\begin{aligned} \psi(0) &= \pi_\infty(0) = 1, \\ \psi(\infty) &= \pi_\infty(2) = \pi_\infty(0) = 1, \\ \psi(1) &= \pi_\infty(\infty) = 0, \end{aligned}$$

and

$$\psi(-1) = \pi_\infty(1) = \infty.$$

As a result, we have the following ramification data:



In particular,

$$\psi^{-1}(0) = \{1\} \quad \text{and} \quad \psi^{-1}(\infty) = \{-1\}.$$

Since $\psi : \hat{Z} = \widehat{\mathbb{C}} \to \hat{X} = \widehat{\mathbb{C}}$ is a rational map of degree 2, ψ has the form

$$\psi(z) = \frac{c(z-1)^2}{(z+1)^2}$$

for a constant $c \neq 0$. Since $\psi(0) = 1$, we have $c = 1$. We note that $\psi(\infty) = 1$ is also satisfied. In this way, the solution (α, β) of the modular equation (4.1) with $t = 1/2$, $p = 2$ is parametrized by

$$\alpha = \varphi(z) = 1 - z^2 \quad \text{and} \quad \beta = \psi(z) = \frac{(z-1)^2}{(z+1)^2}.$$

Eliminating the variable z, we obtain the modular equation

$$\beta = \left(\frac{1 - \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha}} \right)^2.$$

6.3 Case $p = 3$

In this section, we prove the following theorem corresponding to the case $p = 3$ in the theory of signature 2 (see [16, Entry 5(ii), Chapter 19]).

Theorem 6.2. *In the theory of signature 2, suppose the modulus β has degree 3 over the modulus α. Then*

$$\alpha := \varphi(z) = \frac{z(z+2)^3}{(2z+1)^3}$$

and

$$\beta := \psi(z) = \frac{z^3(z+2)}{2z+1}$$

for $z \in \widehat{\mathbb{C}} \setminus \{0, \infty, 1, -\frac{1}{2}, -1, -2\}$. The modular equation is given by

$$(\alpha\beta)^{1/4} + \big\{(1-\alpha)(1-\beta)\big\}^{1/4} = 1. \tag{6.4}$$

Note that (6.4) is known as Legendre's modular equation (see (5.37) in [10] and (4.1.16) in [21]). Formula (6.4) may be transformed to the polynomial equation $P(\alpha, \beta) = 0$ as in Theorem 4.1, where

$$P(x, y) = y^4 + 2x^3y^3 - 2xy - x^4.$$

This is known as the modular equation of third order in Jacobi's form (see (3.42) in [64]).

6.3.1 Construction of Fundamental Domain for $G_\infty \cap G_\infty^{M_3}$

In this case, let $K = G_\infty \cap G_\infty^{M_3}$. Then the index of K in G_∞ is 4 (see Table 5.1). If $F_K = (F, \mathcal{A}_K)$ is the special polygon, i.e., the admissible fundamental domain for K, then we choose the hyperbolic polygon F such that

$$F = \tilde{F}_\infty \cup V(\tilde{F}_\infty) \cup V^{-1}(\tilde{F}_\infty) \cup W(\tilde{F}_\infty).$$

As a result, we obtain F given by the hyperbolic 10-gon with vertices at

$$0,\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{4}{3},\ \frac{3}{2},\ 2,\ \infty$$

in the counterclockwise order.

In this case, the elements

$$\begin{aligned}
A_1 &:= T, \\
A_2 &:= V^{-1}T^2V^{-1} = \begin{pmatrix} 5 & -8 \\ 12 & -19 \end{pmatrix}, \\
A_3 &:= V^{-1}T^{-1}V^{-1} = \begin{pmatrix} -7 & 10 \\ -12 & 17 \end{pmatrix}, \\
A_4 &:= V^{-3} = \begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix}, \\
A_5 &:= V^{-1}TV^{-1}T^{-1}V = \begin{pmatrix} -5 & 2 \\ -18 & 7 \end{pmatrix}
\end{aligned}$$

belong to the set, $\mathcal{A}_K$, of side-pairing transformations of F (see Figure 6.4). There-

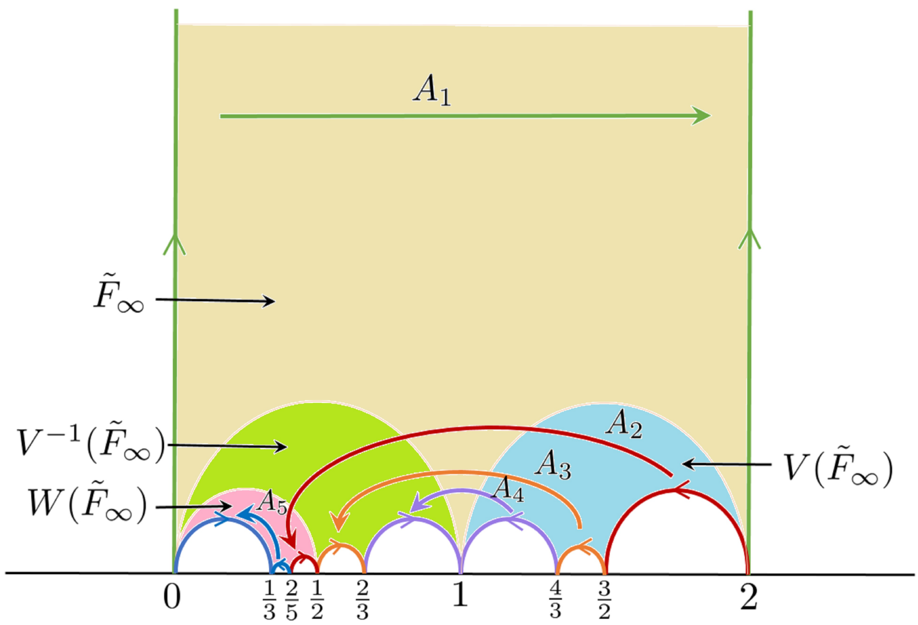


Figure 6.4: Fundamental domain for $K = G_\infty \cap G_\infty^{M_3}$

fore, $K = \langle A_1, A_2, A_3, A_4, A_5 \rangle$ is a subgroup of G_∞ for which $F_K = (F, \mathcal{A}_K)$ is the admissible fundamental domain. Figure 6.5 illustrates the hyperbolic polygon F in the Poincaré disc model, where v_1, v_2, v_3, v_4, v_5, and v_6 represent the inequivalent cusps. In view of the forms of A_j, we observe that $K \subset G_\infty \cap G_\infty^{M_3}$. On the other hand, the elements V, V^{-1}, and W are not contained in $G_\infty^{M_3}$, therefore $K = G_\infty \cap G_\infty^{M_3}$.

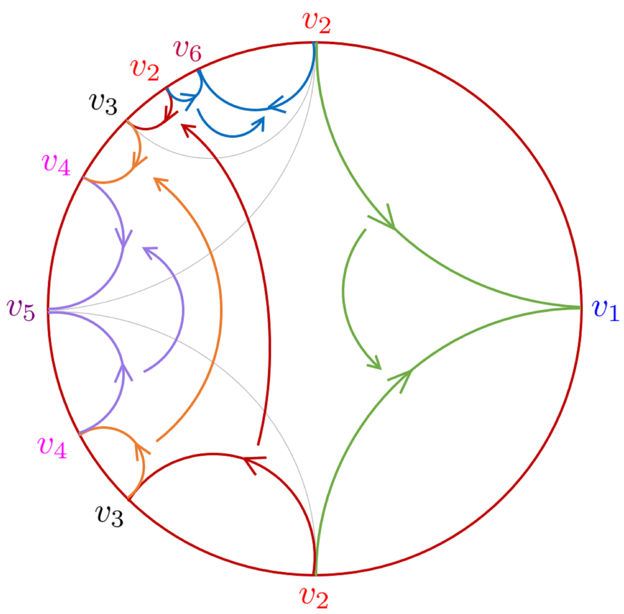


Figure 6.5: The hyperbolic polygon F for $K = G_\infty \cap G_\infty^{M_3}$ in the Poincaré disc model

Also, we have

$$M_3 A_1 M_3^{-1} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} = T^3 \in G_\infty,$$
$$M_3 A_2 M_3^{-1} = \begin{pmatrix} 5 & -24 \\ 4 & -19 \end{pmatrix} = VVT^{-1}T^{-1} \in G_\infty,$$
$$M_3 A_3 M_3^{-1} = \begin{pmatrix} -7 & 30 \\ -4 & 17 \end{pmatrix} = VT^{-1}VT^{-1}T^{-1} \in G_\infty,$$
$$M_3 A_4 M_3^{-1} = \begin{pmatrix} -5 & 18 \\ -2 & 7 \end{pmatrix} = TV^{-1}T^{-1} \in G_\infty$$

and

$$M_3 A_5 M_3^{-1} = \begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix} = V^{-1}V^{-1}V^{-1} \in G_\infty.$$

Therefore, the generators of K satisfy the condition

$$M_3 A_j M_3^{-1} \in G_\infty$$

for $j = 1, \ldots, 5$ and $M_3 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

6.3.2 Proof of Theorem 6.2

It is a simple task to see that the Fuchsian group $K = G_\infty \cap G_\infty^{M_3}$ has six inequivalent cusps. Thus, the quotient Riemann surface $Z = K\backslash\mathbb{H}$ is a six-times punctured sphere. In this case, φ, $\psi : Z \to X$ extend to rational functions of degree 4. For the canonical projection $\rho : \mathbb{H} \to Z$, we may normalize the punctures so that

$$\begin{aligned} &\rho(\infty) = 0, \\ &\rho(0) = \rho(2/5) = \rho(2) = 1, \\ &\rho(1/3) = \infty, \\ &\rho(1/2) = \rho(3/2) = b, \\ &\rho(1) = c, \\ &\rho(2/3) = \rho(4/3) = d. \end{aligned}$$

Thus, we have $\varphi(0) = \pi_\infty(\infty) = 0$. Similarly, we obtain

$$\varphi(b) = 0, \quad \varphi(1) = \varphi(d) = 1, \quad \varphi(\infty) = \varphi(c) = \infty.$$

We have to compute multiplicities of these values. For instance, we have

$$\varphi^{-1}(\infty) = \{\infty, c\}.$$

The part of the basic fundamental domain $\tilde{F}_\infty$ corresponding to ∞ under π_∞ is the cusp neighbourhood $D = \tilde{F}_\infty \cap \{\tau \in \mathbb{H} : |\tau - 1| < \varepsilon\}$ for a small enough $\varepsilon > 0$. Since V and V^{-1} fix 1 while W sends 1 to 1/3, the multiplicity of φ at $\rho(1) = c$ is 3 and that is 1 at $\rho(1/3) = \infty$. We write

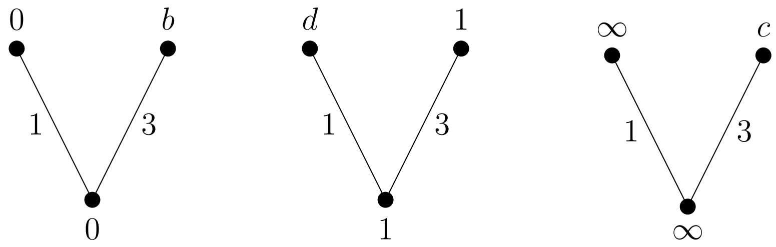
$$(\varphi)_\infty = 3 \cdot c + 1 \cdot \infty$$

as a divisor for short[1]. In the same way, we have

$$(\varphi)_0 = 3 \cdot b + 1 \cdot 0 \quad \text{and} \quad (\varphi)_1 = 1 \cdot d + 3 \cdot 1.$$

We can express the above observations by the following ramification data, where the ramification indices are expressed by the numbers attached to the lines.

[1] When the equation $\varphi(z) = w$ has solutions z_j with multiplicities m_j for $j = 1, 2, \ldots, N$, we write $(\varphi)_w = m_1 \cdot z_1 + m_2 \cdot z_2 + \cdots + m_N \cdot z_N$ as a divisor on Z.



Consequently, φ may be expressed by

$$\varphi(z) = e\frac{z(z-b)^3}{(z-c)^3} \quad \text{and} \quad \varphi(z) - 1 = e'\frac{(z-d)(z-1)^3}{(z-c)^3}$$

for constants e and e'. Since b, c, d are different from 1, we have the unique solution

$$b = -2, \quad c = -\frac{1}{2}, \quad d = -1 \quad \text{and} \quad e = e' = \frac{1}{8}.$$

Hence,

$$\alpha = \varphi(z) = \frac{z(z+2)^3}{(2z+1)^3}.$$

Next we determine the form of ω given in Lemma 4.5. Since SM_3 swaps 0 and ∞ (respectively $\frac{1}{3}$ and -1), ω swaps $\rho(0) = 1$ and $\rho(\infty) = 0$ (respectively, $\rho(\frac{1}{3}) = \infty$ and $\rho(-1) = \rho(1) = -\frac{1}{2}$). Therefore, the involution $\omega : Z \to Z$ is given by

$$\omega(z) = \frac{1-z}{1+2z}.$$

By Lemma 4.5, we obtain

$$\beta = \psi(z) = 1 - \varphi\big(\omega(z)\big) = \frac{z^3(z+2)}{2z+1}.$$

We compute

$$\alpha\beta = \frac{z^4(z+2)^4}{(2z+1)^4} \quad \text{and} \quad (1-\alpha)(1-\beta) = \frac{(1-z^2)^4}{(2z+1)^4}.$$

Note that φ and ψ both map the interval $[0,1]$ onto itself homeomorphically. Hence, for α, $\beta \in [0,1]$, we obtain the relation

$$(\alpha\beta)^{1/4} + \big\{(1-\alpha)(1-\beta)\big\}^{1/4} = 1.$$

Chapter 7

Modular Equations in the Theory of Signature 3

In Ramanujan's theory of signature $\frac{1}{t} = 3$, the generalized modular equation of degree p is given by

$$\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \beta)} = p\,\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)}, \tag{7.1}$$

where $p > 1$ is an integer. In this chapter, we derive geometrically the modular equations corresponding to the cases $p = 2$ and $p = 3$. Recall that $q = \frac{1}{1-2t}$. Therefore, the case of signature $\frac{1}{t} = 3$ corresponds to the case $q = 3$.

7.1 The Subgroup G_3

For $t = \frac{1}{3}$,

$$f_{\frac{1}{3}}(z) = i\,\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-z)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)}$$

maps the upper half-plane $\mathbb{H}$ conformally onto the curvilinear triangle $\Delta_{\frac{1}{3}}$ whose interior angles are 0, 0 and $\frac{\pi}{3}$ at the vertices $f_{\frac{1}{3}}(0) = \infty$, $f_{\frac{1}{3}}(1) = 0$ and $f_{\frac{1}{3}}(\infty) = e^{i\frac{\pi}{6}}$, respectively. Thus, we have

$$\Delta_{\frac{1}{3}} = \left\{ \tau \in \mathbb{H} : 0 < \operatorname{Re} \tau < \frac{\sqrt{3}}{2},\ \left|\tau - \frac{1}{\sqrt{3}}\right| > \frac{1}{\sqrt{3}} \right\}.$$

Let us denote the reflection of $\Delta_{\frac{1}{3}}$ across the geodesic joining $f_{\frac{1}{3}}(\infty) = e^{i\frac{\pi}{6}}$ to ∞, i.e., the line $\operatorname{Re} z = \frac{\sqrt{3}}{2}$ by $\Delta'_{\frac{1}{3}}$. Let $\overline{\Delta}_{\frac{1}{3}}$ denote the closure of $\Delta_{\frac{1}{3}}$. Suppose

$$\tilde{F}_3 = \overline{\Delta}_{\frac{1}{3}} \cup \overline{\Delta}'_{\frac{1}{3}}.$$

Then, we may choose $\tilde{F}_3$ as the fundamental domain for G_3 whose generators are given by

$$T := T_6 = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V := V_3 = \begin{pmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

Recall that V is an elliptic element of order 3 with fixed point at

$$\tau_0 := e^{i\frac{\pi}{6}} = \frac{\sqrt{3}+i}{2}.$$

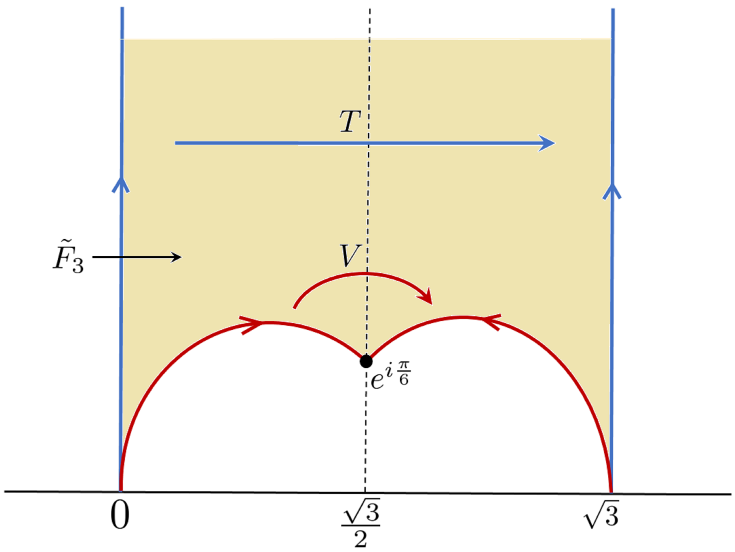


Figure 7.1: Fundamental domain for $G_3 = \langle T, V \rangle$

Recall also that the canonical projection $\pi_3 : \mathbb{H} \to X = G_3 \backslash \mathbb{H} = \widehat{\mathbb{C}} \setminus \{0, 1\}$ satisfies

$$\pi_3(0) = 1, \quad \pi_3(\infty) = 0 \quad \text{and} \quad \pi_3(\tau_0) = \infty \tag{7.2}$$

by (4.2). The following result is useful.

Lemma 7.1. $A \in \mathrm{PSL}(2, \mathbb{R})$ *belongs to* G_3 *precisely when* A *is represented by a* *matrix of the form*

$$\begin{pmatrix} a & b\sqrt{3} \\ c\sqrt{3} & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \ ad - 3bc = 1. \tag{7.3}$$

Proof. We denote by G the group of Möbius transformations represented by the matrices in (7.3). Then it is well-known that G is a proper subgroup of H_6 (see [37]). On the other hand, it is obvious that G_3 is contained in G. Since $|H_6 : G_3| = 2$, we have $G = G_3$ as required. $\square$

7.2 Case $p = 2$

In this section, we prove the following theorem (see [19, (i) of Theorem 7.1]) by applying the geometric approach described in Chapter 4.

Theorem 7.2. *In the theory of signature $\frac{1}{t} = 3$, if $p = 2$, then the moduli α and β are related parametrically as*

$$\alpha := \varphi(z) = \frac{z(z+3)^2}{2(z+1)^3} \quad \textit{and} \quad \beta := \psi(z) = \frac{z^2(z+3)}{4} \tag{7.4}$$

for $z \in \widehat{\mathbb{C}} \setminus \{0, 1, -2, -3\}$. The modular equation is given by

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = 1. \tag{7.5}$$

We remark that the expressions of α and β in (7.4) are found in Ramanujan's notebook (see [19, Theorem 6.1]). In this case, we obtain the polynomial $P(x, y)$ in Theorem 4.1 as

$$P(x, y) = (2x-1)^3y^3 - 3x(4x^2 - 13x + 10)y^2 + 3x(2x^2 - 10x + 9)y - x^3,$$

which is equivalent to (7.5).

7.2.1 The Subgroup $G_3 \cap G_3^{M_2}$

For the case $p = 2$ in the theory of signature 3, we have to consider the Hecke subgroup $K = G_3 \cap G_3^{M_2}$, which is a subgroup of G_3 and $|G_3 : K| = 3$ (see Table 5.1). Let $F_K = (F, \mathcal{A}_K)$ be the admissible fundamental domain for K, where F is a hyperbolic polygon and $\mathcal{A}_K$ is the set of side-pairing transformations of F. In this case, we need to take three copies of $\tilde{F}_3$ to construct the hyperbolic polygon F. Let

$$F = \tilde{F}_3 \cup V(\tilde{F}_3) \cup V^2(\tilde{F}_3).$$

Then F is the hyperbolic polygon with six vertices at

$$0,\ \frac{1}{\sqrt{3}},\ \frac{\sqrt{3}}{2},\ \frac{2}{\sqrt{3}},\ \sqrt{3},\ \infty$$

in the counterclockwise order. If the geodesic joining 0 to ∞ is identified with the geodesic joining $\sqrt{3}$ to ∞ by A_1, the geodesic joining $\sqrt{3}$ to $\frac{2}{\sqrt{3}}$ is identified with the geodesic joining 0 to $\frac{1}{\sqrt{3}}$ by A_2, and the geodesic joining $\frac{2}{\sqrt{3}}$ to $\frac{\sqrt{3}}{2}$ is identified with the geodesic joining $\frac{1}{\sqrt{3}}$ to $\frac{\sqrt{3}}{2}$ by A_3, then we have

$$\begin{aligned}
A_1 &= T,\\
A_2 &= V^2TV^2 = \begin{pmatrix} 1 & -\sqrt{3} \\ 2\sqrt{3} & -5 \end{pmatrix},\\
A_3 &= (SV)^{-1}TSV = \begin{pmatrix} -5 & 3\sqrt{3} \\ -4\sqrt{3} & 7 \end{pmatrix}
\end{aligned}$$

as the side pairing transformations of F, i.e., $\mathcal{A}_K = \{A_1, A_2, A_3\}$ (see Figure 7.2). Figure 7.3 illustrates the hyperbolic polygon F in the Poincaré disc model, where

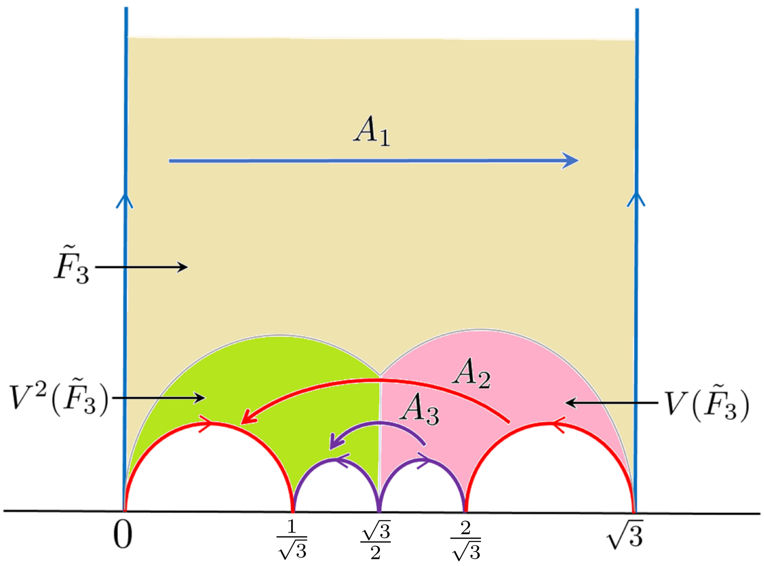


Figure 7.2: Fundamental domain for $K = G_3 \cap G_3^{M_2}$

$v_1,\ v_2,\ v_3$, and v_4 represent the inequivalent cusps.

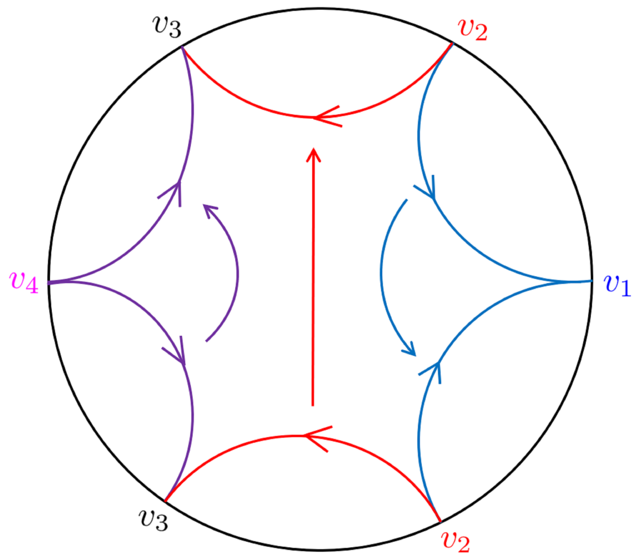


Figure 7.3: The hyperbolic polygon F for $K = G_3 \cap G_3^{M_2}$ in the Poincaré disc model

Also, we verify that

$$M_2 A_1 M_2^{-1} = \begin{pmatrix} 1 & 2\sqrt{3} \\ 0 & 1 \end{pmatrix} = T^2 \in G_3,$$

$$M_2 A_2 M_2^{-1} = \begin{pmatrix} 1 & -2\sqrt{3} \\ \sqrt{3} & -5 \end{pmatrix} = -V^{-1}T^{-1} \in G_3,$$

and

$$M_2 A_3 M_2^{-1} = \begin{pmatrix} -5 & 6\sqrt{3} \\ -2\sqrt{3} & 7 \end{pmatrix} = VT^{-1}VT^{-1} \in G_3.$$

Therefore, A_1, A_2, and A_3 generate the torsion-free group $K = G_3 \cap G_3^{M_2}$.

7.2.2 Proof of Theorem 7.2

In this case, we also see that the Hecke subgroup $K = G_3 \cap G_3^{M_2}$ has four inequivalent cusps. Thus, the quotient Riemann surface $Z = K\backslash\mathbb{H}$ is a four-times punctured sphere. We normalize the map $\rho : \mathbb{H} \to Z$ so that

$$\rho(\infty) = 0,$$
$$\rho(0) = \rho\big(\sqrt{3}\big) = 1,$$

$$\begin{aligned}
&\rho(\tau_0) = -1,\\
&\rho\Big(\frac{1}{\sqrt{3}}\Big) = \rho\Big(\frac{2}{\sqrt{3}}\Big) = b,\\
&\rho\Big(\frac{\sqrt{3}}{2}\Big) = c
\end{aligned}$$

for some constants b and c. Since

$$\pi_3(\tau) = \varphi\big(\rho(\tau)\big)$$

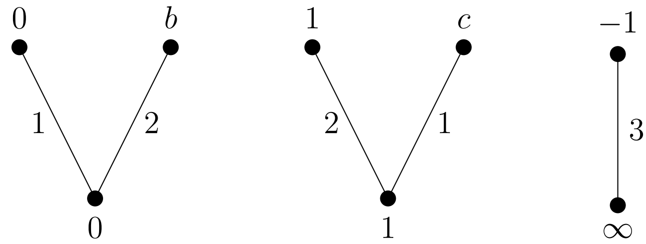
has a branch point of order 3 at $\tau = \tau_0$, the map $\varphi(z)$ has a branch point of order 3 at $z = -1$, that is, $\varphi^{-1}(\infty) = \{-1\}$. Since

$$V(\infty) = \frac{2}{\sqrt{3}} \quad \text{and} \quad V^2(\infty) = \frac{1}{\sqrt{3}},$$

we have

$$(\varphi)_0 = 1 \cdot 0 + 2 \cdot b.$$

We get the following ramification data based on the above observations (the ramification indices are expressed by the numbers attached to the lines).



Therefore, the rational map φ of degree 3 should have the form

$$\varphi(z) = \frac{mz(z-b)^2}{(z+1)^3}$$

for a constant $m \neq 0$. Observe that $V(\tilde{F}_3)$ and $V^2(\tilde{F}_3)$ share the cusps at $\sqrt{3}$ and 0, respectively, with $\tilde{F}_3$. Hence, we see that $(\varphi)_1 = 2 \cdot 1 + 1 \cdot c$ and

$$\varphi(z) - 1 = \frac{m'(z-c)(z-1)^2}{(z+1)^3}$$

for a constant $m' \neq 0$. Then we have

$$b = -3, \quad c = -2, \quad m = \frac{1}{2} \quad \text{and} \quad m' = -\frac{1}{2}.$$

Thus,

$$\varphi(z) = \frac{z(z+3)^2}{2(z+1)^3}.$$

Next we will determine ω in Lemma 4.5. Since SM_2 swaps 0 and ∞ (respectively, $1/\sqrt{3}$ and $-\sqrt{3}/2 \equiv \sqrt{3}/2 \bmod K$), ω swaps 1 and 0 (respectively, $b = -3$ and $c = -2$). Therefore, the involution ω is given by

$$\omega(z) = \frac{1-z}{1+z}.$$

Now Lemma 4.5 yields

$$\psi(z) = 1 - \varphi\big(\omega(z)\big) = \frac{1}{4}z^2(z+3).$$

We summarize the results as

$$\alpha = \varphi(z) = \frac{z(z+3)^2}{2(z+1)^3} \quad \text{and} \quad \beta = \psi(z) = \frac{z^2(z+3)}{4}.$$

Since

$$\alpha\beta = \frac{z^3(z+3)^3}{8(z+1)^3} \quad \text{and} \quad (1-\alpha)(1-\beta) = \frac{(1-z)^3(z+2)^3}{8(z+1)^3},$$

it is now easy to obtain the modular equation (7.5) in Theorem 7.2.

7.3 Case $p = 3$

In this section, we consider the case when the modulus β has degree 3 over the modulus α in the theory of signature 3. By applying the geometric approach developed in Chapter 4, we prove the following theorem (see [19, Lemma 7.4]).

Theorem 7.3. *If β has degree 3 over α in the theory of signature $\frac{1}{t} = 3$, then the moduli α and β are related parametrically as follows:*

$$\alpha := \varphi(z) = 1 - z^3 \quad \textit{and} \quad \beta := \psi(z) = \frac{(1-z)^3}{(1+2z)^3}$$

for $z \in \widehat{\mathbb{C}} \setminus \left\{0, 1, \frac{1}{2}(-1+i\sqrt{3}), \frac{1}{2}(-1-i\sqrt{3})\right\}$. The modular equation is given by

$$(1-\alpha)^{1/3} = \frac{1-\beta^{1/3}}{1+2\beta^{1/3}}. \tag{7.6}$$

Note that, for $p = 3$ in the theory of signature 3, the polynomial $P(x, y)$ in

Theorem 4.1 can be computed as

$$P(x,y) = (8x-9)^3y^3 + 3\left(64x^3 + 504x^2 - 1053x + 486\right)y^2 \\ + 3\left(8x^3 - 171x^2 + 405x - 243\right)y + x^3.$$

7.3.1 Construction of Fundamental Domain for $G_3 \cap G_3^{M_3}$

For the case $p = 3$ in the theory of signature 3, the Hecke subgroup $G_3 \cap G_3^{M_3}$ has index 3 in G_3, i.e., $|G_3 : (G_3 \cap G_3^{M_3})| = 3$ (see Table 5.1). The hyperbolic polygon F is the same as in the case for $p = 2$. We need to find the set

$$\mathcal{A}_K = \{A_1, A_2, A_3\}$$

of side-pairing transformations of F for $K = \langle A_1, A_2, A_3\rangle$. In this case, we choose the side-pairing transformations A_1, A_2, A_3 so that they satisfy the condition

$$M_3 A_j M_3^{-1} \in G_3$$

for $j = 1,\ 2,\ 3$ and $M_3 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Let us identify the geodesic joining 0 to ∞ with the geodesic joining $\sqrt{3}$ to ∞ by A_1, the geodesic joining $\sqrt{3}$ to $\frac{2}{\sqrt{3}}$ with the geodesic joining $\frac{\sqrt{3}}{2}$ to $\frac{2}{\sqrt{3}}$ by A_2, and the geodesic joining $\frac{\sqrt{3}}{2}$ to $\frac{1}{\sqrt{3}}$ with the geodesic joining 0 to $\frac{1}{\sqrt{3}}$ by A_3, then we have

$$\begin{aligned} A_1 &= T, \\ A_2 &= VTV^{-1} = \begin{pmatrix} -5 & 4\sqrt{3} \\ -3\sqrt{3} & 7 \end{pmatrix}, \\ A_3 &= V^{-1}TV = \begin{pmatrix} -2 & \sqrt{3} \\ -3\sqrt{3} & 4 \end{pmatrix} \end{aligned}$$

as the side pairing transformations of F (see Figure 7.4). Figure 7.5 shows the hyperbolic polygon F in the Poincaré disc model, where v_1, v_2, v_3 and v_4 represent the inequivalent cusps. It is easy to see that $K = G_3 \cap G_3^{M_3}$ by Lemma 7.1 and that K is torsion-free. For $M_3 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$M_3 A_1 M_3^{-1} = \begin{pmatrix} 1 & 3\sqrt{3} \\ 0 & 1 \end{pmatrix} = T^3 \in G_3,$$

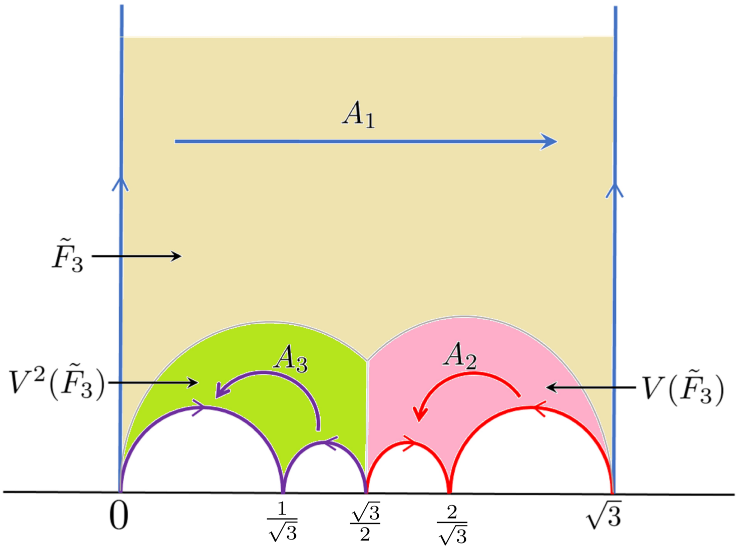


Figure 7.4: Fundamental domain for $K = G_3 \cap G_3^{M_3}$

$$M_3 A_2 M_3^{-1} = \begin{pmatrix} -5 & 12\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix} = TVT^{-1}T^{-1} \in G_3,$$

and

$$M_3 A_3 M_3^{-1} = \begin{pmatrix} -2 & 3\sqrt{3} \\ -\sqrt{3} & 4 \end{pmatrix} = -VT^{-1} \in G_3.$$

Thus, the generators A_1, A_2, and A_3 of the subgroup K also satisfy the condition that

$$M_3 A_j M_3^{-1} \in G_3$$

for $j = 1,\ 2,\ 3.$

7.3.2 Proof of Theorem 7.3

First, we observe that the Hecke subgroup $K = \langle A_1,\ A_2,\ A_3 \rangle$ has four inequivalent cusps. Hence, the quotient Riemann surface $Z = K\backslash\mathbb{H}$ is a four-times punctured sphere. We may normalize $\rho : \mathbb{H} \to Z$ so that

$$\begin{aligned}
&\rho(0) = \rho\Big(\frac{\sqrt{3}}{2}\Big) = \rho(\sqrt{3}) = 0, \\
&\rho(\infty) = 1, \\
&\rho\Big(\frac{1}{\sqrt{3}}\Big) = a,
\end{aligned}$$

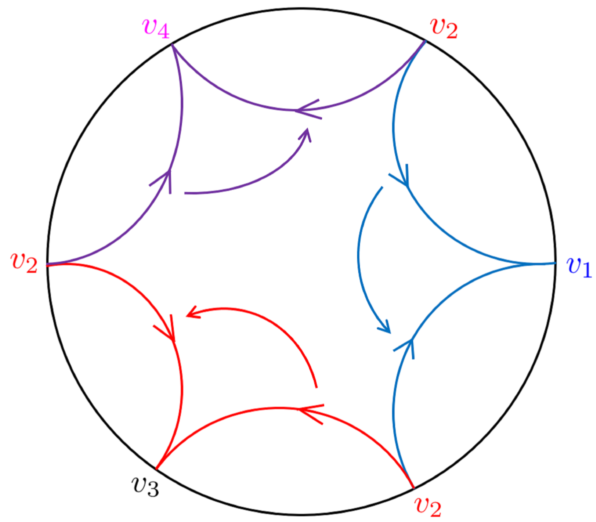


Figure 7.5: The hyperbolic polygon F for $K = G_3 \cap G_3^{M_3}$ in the Poincaré disc model

$$\rho\Big(\frac{2}{\sqrt{3}}\Big) = b,$$
$$\rho(\tau_0) = \infty.$$

By the form of K, the element V normalizes K, that is, $V^{-1}KV = K$. Thus V induces an analytic automorphism $v : Z \to Z$. We recall that V is a rotation about τ_0 of angle $-\frac{2\pi}{3}$. It is clear that $v(0) = 0$ and $v(\infty) = \infty$. Since V satisfies

$$V(\infty) = \frac{2}{\sqrt{3}}, \quad V\Big(\frac{2}{\sqrt{3}}\Big) = \frac{1}{\sqrt{3}} \quad \text{and} \quad V\Big(\frac{1}{\sqrt{3}}\Big) = \infty,$$

the map v should have the form

$$v(z) = e^{-i\frac{2\pi}{3}} z.$$

Hence,

$$b = e^{-i\frac{2\pi}{3}} = \frac{1}{2}(-1 - i\sqrt{3}) \quad \text{and} \quad a = \overline{b} = \frac{1}{2}(-1 + i\sqrt{3}).$$

We will determine the forms of rational maps

$$\varphi,\, \psi : \hat{Z} = \widehat{\mathbb{C}} \to \hat{X} = \widehat{\mathbb{C}}$$

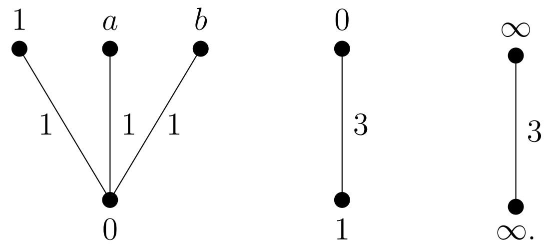
of degree 3. Since φ has a branch point at ∞ of order 3, we have

$$\varphi^{-1}(\infty) = \{\infty\}.$$

In particular, φ is a polynomial of degree 3. We also have

$$\varphi^{-1}(1) = \{0\}.$$

By the above observations, we obtain the following ramification data:



Here, the ramification indices are expressed by the numbers attached to the lines. Therefore, φ should be of the form

$$\varphi(z) = 1 + cz^3$$

for a nonzero constant c and $z \in \hat{Z}$. Since

$$\varphi(1) = \pi_3(\infty) = 0,$$

we obtain $c = -1$. Next we determine the involution ω in Lemma 4.5. Since SM_3 swaps 0 and ∞ (respectively, $1/\sqrt{3}$ and $-1/\sqrt{3} \equiv 2/\sqrt{3} \pmod{K}$), ω swaps 1 and 0 (respectively, a and b). Hence, after some computations, we get the form of ω as

$$\omega(z) = \frac{1-z}{1+2z}.$$

We now apply Lemma 4.5 to obtain

$$\psi(z) = 1 - \varphi\big(\omega(z)\big) = \big(\omega(z)\big)^3 = \frac{(1-z)^3}{(1+2z)^3}.$$

In summary, we have

$$\alpha = \varphi(z) = 1 - z^3$$

and

$$\beta = \psi(z) = \frac{(1-z)^3}{(1+2z)^3}.$$

By eliminating z, we obtain the modular equation (7.6) easily.

7.4 Case $p = 5$

In the theory of signature $\frac{1}{t} = 3$, if β has degree 5 over α, then α and β are related by

$$(\alpha\beta)^{\frac{1}{3}} + \{(1-\alpha)(1-\beta)\}^{\frac{1}{3}} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{\frac{1}{6}} = 1 \tag{7.7}$$

(see [19, Theorems 7.6]).

7.4.1 Construction of Fundamental Domain for $G_3 \cap G_3^{M_5}$

For the case $p = 5$ in the theory of signature 3, the Hecke subgroup $G_3 \cap G_3^{M_5}$ has index 6 in G_3, i.e., $|G_3 : (G_3 \cap G_3^{M_5})| = 6$ (see Table 5.1). Consequently, we suitably choose six copies of $\tilde{F}_3$ to construct the fundamental domain for $G_3 \cap G_3^{M_5}$. Let

$$F = \tilde{F}_3 \cup V(\tilde{F}_3) \cup V^2(\tilde{F}_3) \cup W(\tilde{F}_3) \cup WV(\tilde{F}_3) \cup WV^2(\tilde{F}_3),$$

where

$$W = -V^{-1}T = \begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix}.$$

Then F is a hyperbolic 10-gon with vertices at

$$0,\ \frac{1}{2\sqrt{3}},\ \frac{\sqrt{3}}{5},\ \frac{2}{3\sqrt{3}},\ \frac{\sqrt{3}}{4},\ \frac{1}{\sqrt{3}},\ \frac{\sqrt{3}}{2},\ \frac{2}{\sqrt{3}},\ \sqrt{3},\ \infty$$

in the counterclockwise order. Let $F_K = (F, \mathcal{A}_K)$ be the admissible fundamental domain for K, where $\mathcal{A}_K$ is the set of side-pairing transformations of F. Let us identify the geodesic joining 0 to ∞ with the geodesic joining $\sqrt{3}$ to ∞ by A_1, the geodesic joining $\sqrt{3}$ to $\frac{2}{\sqrt{3}}$ with the geodesic joining $\frac{\sqrt{3}}{4}$ to $\frac{1}{\sqrt{3}}$ by A_2, the geodesic joining $\frac{2}{\sqrt{3}}$ to $\frac{\sqrt{3}}{2}$ with the geodesic joining $\frac{2}{3\sqrt{3}}$ to $\frac{\sqrt{3}}{4}$ by A_3, the geodesic joining $\frac{\sqrt{3}}{2}$ to $\frac{1}{\sqrt{3}}$ with the geodesic joining 0 to $\frac{1}{2\sqrt{3}}$ by A_4, and the geodesic joining $\frac{2}{3\sqrt{3}}$ to $\frac{\sqrt{3}}{5}$ with the geodesic joining $\frac{1}{2\sqrt{3}}$ to $\frac{\sqrt{3}}{5}$ by A_5, then we have

$$\begin{aligned}
A_1 &= T, \\
A_2 &= WTV^{-1} = \begin{pmatrix} -4 & 3\sqrt{3} \\ -5\sqrt{3} & 11 \end{pmatrix}, \\
A_3 &= WVT^{-1}V^{-1} = \begin{pmatrix} 7 & -4\sqrt{3} \\ 10\sqrt{3} & -17 \end{pmatrix},
\end{aligned}$$

$$A_4 = WV^{-1}TV = \begin{pmatrix} 2 & -\sqrt{3} \\ 5\sqrt{3} & -7 \end{pmatrix},$$
$$A_5 = WV^{-1}T^{-1}(WV)^{-1} = \begin{pmatrix} 14 & -3\sqrt{3} \\ 25\sqrt{3} & -16 \end{pmatrix}$$

as the side pairing transformations of F, i.e., $\mathcal{A}_K = \{A_1, A_2, A_3, A_4, A_5\}$ (see Figure 7.6).

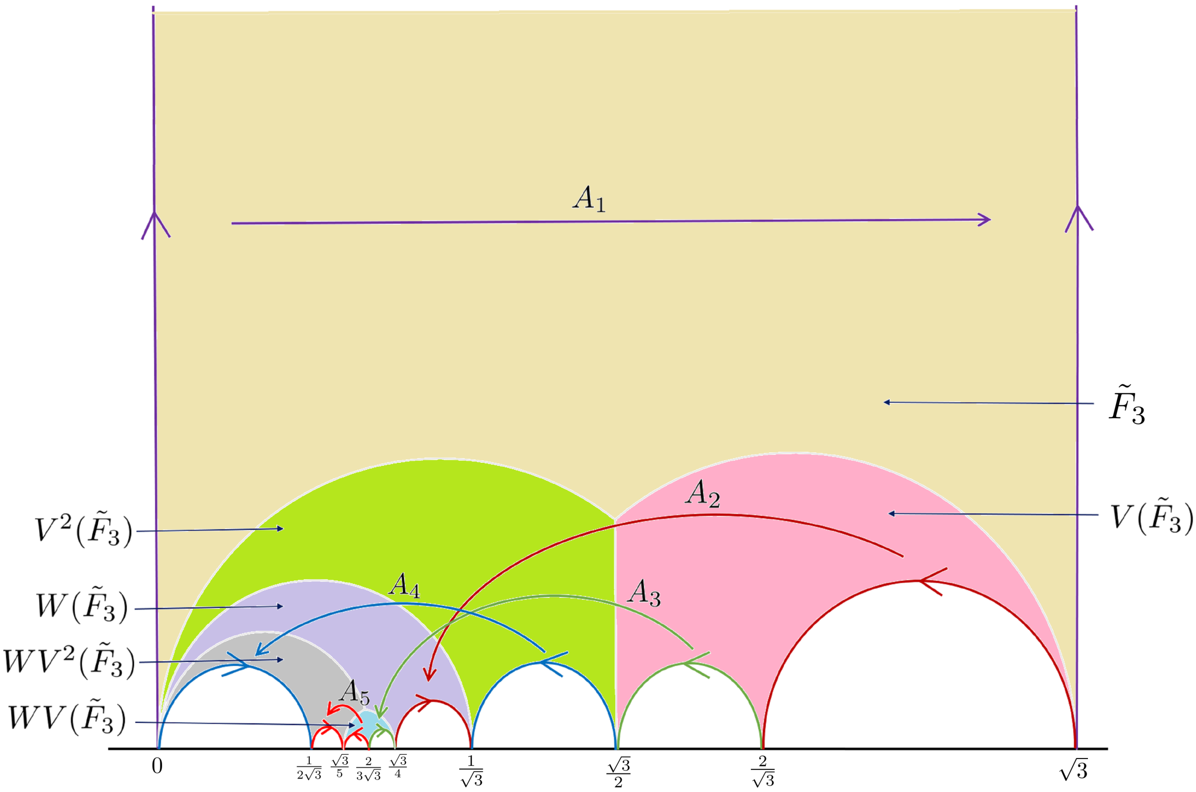


Figure 7.6: Fundamental domain for $K = G_3 \cap G_3^{M_5}$

For $M_5 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$M_5 A_1 M_5^{-1} = \begin{pmatrix} 1 & 5\sqrt{3} \\ 0 & 1 \end{pmatrix} = T^5 \in G_3,$$
$$M_5 A_2 M_5^{-1} = \begin{pmatrix} -4 & 15\sqrt{3} \\ -\sqrt{3} & 11 \end{pmatrix} = TV^{-1}T^{-1}T^{-1}T^{-1} \in G_3,$$

$$M_5A_3M_5^{-1} = \begin{pmatrix} 7 & -20\sqrt{3} \\ 2\sqrt{3} & 17 \end{pmatrix} = TV^{-1}TV^{-1}T^{-1}T^{-1} \in G_3,$$
$$M_5A_4M_5^{-1} = \begin{pmatrix} 2 & -5\sqrt{3} \\ \sqrt{3} & -7 \end{pmatrix} = VT^{-1}T^{-1} \in G_3,$$

and

$$M_5A_5M_5^{-1} = \begin{pmatrix} 14 & -15\sqrt{3} \\ 5\sqrt{3} & -16 \end{pmatrix} = (VT^{-1})^5 \in G_3.$$

Thus, if K is the group generated by the side pairing transformations A_1, A_2, A_3, A_4 and A_5, then the generators of K satisfy the condition that $M_5A_jM_5^{-1} \in G_3$ for $j = 1, \ldots, 5$. Therefore, we deduce that $K = G_3 \cap G_3^{M_5}$, which is torsion-free.

7.4.2 The Quotient Riemann Surface $(G_3 \cap G_3^{M_5})\backslash\mathbb{H}$

It is not difficult to see that the Hecke subgroup

$$K = (G_3 \cap G_3^{M_5}) = \langle A_1,\, A_2,\, A_3,\, A_4,\, A_5\rangle$$

has four inequivalent cusps. Let us denote the inequivalent cusps of K by v_1, v_2, v_3, and v_4. Figure 7.7 shows the hyperbolic polygon F in the Poincaré disc model.

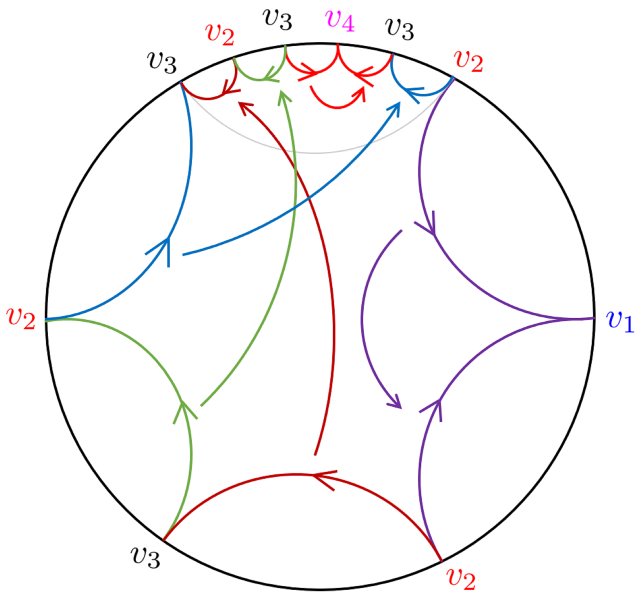


Figure 7.7: The hyperbolic polygon F for $K = G_3 \cap G_3^{M_5}$ in the Poincaré disc model

Let N_v, N_e and N_f denote the numbers of vertices, edges and faces, respectively, of Z, then we have

$$N_v = 4, \quad N_e = 5 \quad \text{and} \quad N_f = 1.$$

The quotient Riemann surface $Z = K\backslash\mathbb{H}$ has four punctures. Let $\hat{Z}$ be the compactification of Z. The Euler characteristic $\chi(\hat{Z})$ of the quotient surface $\hat{Z}$ is given by

$$\chi(\hat{Z}) = N_v - N_e + N_f = 0.$$

Since the quotient Riemann surface $\hat{Z}$ is compact, connected and orientable, the Euler characteristic $\chi(\hat{Z})$ and the genus g of $\hat{Z}$ are related by (see [30, p. 66])

$$\chi(\hat{Z}) = 2 - 2g.$$

Thus, we obtain $g = 1$, that is, $\hat{Z}$ is a genus one Riemann surface. When Z is not a planar surface, it is technically difficult to find an explicit form of the polynomial $P(x, y)$ in Theorem 4.1. We hope to give further applications when Z is a non-planar surface in the future work.

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