How To Create Cash Flows That Give A Priori IRRs?

Charlie Charoenwong
Nanyang Technological University

Xinxin Li
Ithaca College

Chee K. Ng
Ithaca College

How to Create Cash Flows That Give a Priori IRRs?

Charlie Charoenwong
Nanyang Technological University
acharlie@ntu.edu.sg

Xinxin Li
Ithaca College
xli4@ithaca.edu

Chee Ng
Ithaca College
cng2@ithaca.edu

Keywords: *Capital Budgeting, Internal Rate of Return*

Abstract
We illustrate algebraically the way to numerically derive the cash flows of an investment project yielding a set of desired IRRs. This is significant given the dearth of practice questions with the appropriate cash flow numbers, and their corresponding sign, in widely adopted Finance textbooks that will allow students to find multiple IRRs as learning exercises. The note serves to fill that void by allowing the readers to change the IRR inputs in an Excel spreadsheet and have the corresponding cash flows output ready for an assignment. Instructors can use the Excel codes to create different cash flows that yield multiple IRRs with ease so that students in the same class or those in different semesters need not be given the same set of cash flows where plagiarism could compromise the learning process.

INTRODUCTION

In the teaching of the capital budgeting topic in corporate finance, the internal rate of return (IRR) technique is presented as the most commonly used, though economically flawed, auxiliary to the net present value (NPV) technique in ranking the available projects. One of the major challenges of the IRR technique is the existence of multiple IRRs which complicate, if not totally negate, the ranking's usefulness. Most widely adopted corporate finance textbooks, from Berk and Demarzo, Brigham and Ehrhardt, Copeland et al., Emery et al., Keown et al., Ross et al., to Welsh, all present the simplest 2-IRR case to illustrate the invalidation of the IRR as a comparable technique to the most-tenable NPV technique. We attempt to generalize beyond the 2-IRR case by providing an algorithm in Microsoft Excel® to generate the cash flows that will result in more IRRs as pedagogical tool to benefit instructors and students alike in the learning experience.

In the teaching of the internal rate of return (IRR) for an investment project with unconventional cash flows, i.e., two or more changes of signs in the cash flow series, the Descartes' rule of signs is usually footnoted in most corporate finance textbooks. The maximum number of the IRRs is equal to or fewer than the number of the sign changes. Other possibilities are an even number fewer than that maximum number. Hence, a cash flow series with an odd number of sign changes is bound to have at least one IRR and may have up to that odd number of IRRs. Another cash flow series with an even number of sign changes can have zero IRR or may have up to that even number

of IRRs. In other words, a series of cash flows with 4 sign changes can have 0, 2, or 4 IRRs while a serious of cash flows with 5 sign changes can have 1, 3, or 5 IRRs.

In Section 2, we demonstrate the algebraic derivation of the cash flows associated with 4 desired IRRs *a priori*. In Section 3, we translate mathematical analysis in Section 2 to coding in an Excel spreadsheet that will produce the exact same results and generate a new set of cash flows the moment we change any value, some, or all of the 4 *a priori* IRRs. Section 4 shows the 4 IRRs graphically. Section 5 concludes.

THE DERIVATION OF 4-IRR CASH FLOWS
Let's start with a numerical example. From the basic understanding of functions, if we want to set the left-hand-side of Equation (0) to equal zero, then x can take any one of the four possible values of .07, .25, .30 or .40 as the root to make the left-hand-side equal to zero. That is:

$$(x - .07)(x - .25)(x - .30)(x - .40) = 0 \qquad (0)$$

Expanding the left-hand-side function, we get a polynomial of the fourth degree as:
$$[(x - .07)(x - .25)][(x - .30)(x - .40)] = 0$$

$$[x^2 - .32x + .0175][x^2 - .7x + .12] = 0$$

$$x^4 - .7x^3 + .12x^2 - .32x^3 + .224x^2 - .0384x + .0175x^2 - .01225x + .0021 = 0$$

$$x^4 - 1.02x^3 + .3615x^2 - .05065x + .0021 = 0 \qquad (1)$$

Since we are creating a situation with four roots or IRRs, we need to have a minimum 4-year project that would allow us up to four sign changes. Without losing generality, let's start with all future cash flows as positive, and designate only the initial cash outflow as negative. Later, the signs and sizes of the future cash flows will be dictated by the outcomes when we solve for them algebraically. The general 4-year project's cash flows are shown in Equation (2) below. We set the left-hand-side of Equation (2) which is the net present value, NPV, to equal zero so that x is the internal rate of return, IRR.

$$\frac{C_4}{(1 + x)^4} + \frac{C_3}{(1 + x)^3} + \frac{C_2}{(1 + x)^2} + \frac{C_1}{(1 + x)^1} - C_0 = 0 \qquad (2)$$

Next, we rearrange Equation (2) above into the same format as in the fourth-degree polynomial in Equation (1). That is, we multiply throughout by $(1+x)^4$ in Equation (2), we get:

$$C_4 + C_3(1 + x) + C_2(1 + x)^2 + C_1(1 + x)^2 - C_0(1 + x)^4 = 0$$

Employing the Pascal-triangle numbers for the coefficient numbers in expanding the four $(1 + x)^n$ terms, we get:

C_4					1					

C_3				1		1			
C_2			1		2		1		
C_1		1		3		3		1	
C_0	1		4		6		4		1

$$C_4 + C_3(1+x) + C_2(1+2x+x^2) + C_1(1+3x+3x^2+x^3) - C_0(1+4x+6x^2+4x^3+x^4) = 0$$

Collecting terms and rearranging them in fourth-degree polynomial as in Equation (1), we get:

$$-C_0 x^4 + (C_1 - 4C_0)x^3 + (C_2 + 3C_1 - 6C_0)x^2 + (C_3 + 2C_2 + 3C_1 - 4C_0)x^1$$
$$+ (C_4 + C_3 + C_2 + C_1 - C_0)x^0 = 0 \qquad (3)$$

Now, we *compare and equate* the coefficients of the corresponding terms in Equations (1) and (3), we get:

$$x^4 : -C_0 = 1 \text{ or } C_0 = -1$$
$$x^3 : C_1 - 4C_0 = -1.02$$
$$C_1 - 4(-1) = -1.02$$
$$C_1 = -5.02$$
$$x^2 : C_2 + 3C_1 - 6C_0 = .3615$$
$$C_2 + 3(-5.02) - 6(-1) = .3615$$
$$C_2 = 9.4215$$
$$x^1 : C_3 + 2C_2 + 3C_1 - 4C_0 = -.05065$$
$$C_3 + 2(9.4215) + 3(-5.02) - 4(-1) = -.05065$$
$$C_3 = -7.83365$$
$$x^0 : C_4 + C_3 + C_2 + C_1 - C_0 = .0021$$
$$C_4 - 7.83365 + 9.4215 - 5.02 - (-1) = .0021$$
$$C_4 = 2.43425$$

THE EXCEL CODING
Next, we proceed to code in Excel so that we can solve the above 4-root problem in Excel without having to do the manual calculations painstakingly. The sequential steps are:

Table 1: Inputs in Excel to generate cash flows that have the desired four IRRs.

A	B	C	D	E	F	G	H	I	J	K	L	M
1												
2					Power	Coefficient						
3	IRR				4	1.00000		-1	0	0	0	0
4	0.07	=B5	=C5	=D5	3	-1.02000		-4	1	0	0	0
5	0.25	=B6	=C6		2	0.36150		-6	3	1	0	0

6	0.30	=B7			1	-0.05065		-4	3	2	1	0
7	0.40				0	0.00210		-1	1	1	1	1
8											
9					Solution	CF					
10				C_0	-1.00000	-1.00000					
11				C_1	-5.02000	5.02000					
12				C_2	9.42150	-9.42150					
13				C_3	-7.83365	7.83365					
14				C_4	2.43425	-2.43425					

i.   We manually enter the IRR numbers in decimals in cells B4 through B7.
ii. We use the = plus cell reference syntax to create the top-left triangle of entries in cells C4 through E6. For example, we enter =B5 in cell C4, =B6 in cell C5, etc.
iii. We create the header column in Column F for Powers and Cash flows.
iv. We use algebraic functions to find the coefficients for the 4th power polynomial in cells G3 through G7. Specifically, we type in:
  a. 1 in cell G3
  b. =SUM(B4:B7) in cell G4
  c. =SUM(B4:E4*B5:E7) in cell G5, and press Shift, Control and Enter simultaneously.
  d. =−(SUM(B4:E4*B5:E5*B6:E6)+SUM(B4:E4*B6:E6*B7:E7)+B4*C4*E4) in cell G6
  e. =PRODUCT(B4:B7) in cell G7
v.   We manually enter the Pascal triangle numbers in the lower-half triangle in the rectangular array of cells I3 through M7 for matrix multiplication later.
vi. We apply matrix multiplication in Excel on the coefficients in cells G3 through G7 and the Pascal-triangle matrix in cells I3 through M7 to generate the cash flows, and have them output in cells D10 through D14. Specifically, we highlight cells G10 through F14, and type in =MMULT(MINVERSE(I3:M7),G3:G7) in the formula bar, followed by pressing Control, Shift, Enter simultaneously[1] for the output to appear in cells G10 through G14.
vii. We then flip the signs of cash flow 1 through cash flow 4, keeping the sign for the initial flow intact.

THE GRAPH

Finally, we proceed to open another sheet in Excel, and use the generated cash flows[2] to find all the NPV's for the entire domain plus a couple of percent points beyond the highest IRR.

[1] For users of Office 365, simply pressing the Enter key.
[2] Note that multiplying the values of CF_ts by any constant, such as 1 million or 1 billion, to scale up will not change the values of the IRRs.

	Year, t	Cash Flow, CF$_t$		Discount Rate	NPV, $
	A	B	C	D	E
1	0	-1.00000		0.00	-0.002100
2	1	5.02000		0.01	-0.001565
3	2	-9.42150		0.02	-0.001130
4	3	7.83365		0.03	-0.000781
5	4	-2.43425		0.04	-0.000504
6				0.05	-0.000288
7				0.06	-0.000123
8				0.07	0.000000
9				0.08	0.000088
10				0.09	0.000148
11				0.10	0.000184
12				0.11	0.000203
13				0.12	0.000208
14				0.13	0.000203
15				0.14	0.000190
16				0.15	0.000172
17				0.16	0.000150
18				0.17	0.000128
19				0.18	0.000105
20				0.19	0.000083
21				0.20	0.000063
22				0.21	0.000045
23				0.22	0.000029
24				0.23	0.000017
25				0.24	0.000007
26				0.25	0.000000
27				0.26	-0.000004
28				0.27	-0.000006
29				0.28	-0.000006
30				0.29	-0.000003
31				0.30	0.000000
32				0.31	0.000004
33				0.32	0.000009
34				0.33	0.000014
35				0.34	0.000018
36				0.35	0.000021
37				0.36	0.000022
38				0.37	0.000021
39				0.38	0.000018
40				0.39	0.000011
41				0.40	0.000000

| 42 | | | | 0.41 | -0.000015 |
| 43 | | | | 0.42 | -0.000035 |

We can see from the zero NPV numbers in column E that happen at discount rates of 7%, 25%, 30% and 40%. These rates are the IRRs from which we began our exercise with. There are four changes of signs in the series of cash flows, we obtained four IRRs, and they follow consistently from the Descartes rule.

A plot of the NPV profile follows. We see clearly that the IRRs are at 7%, 25%, 30% and 40% as we started with.
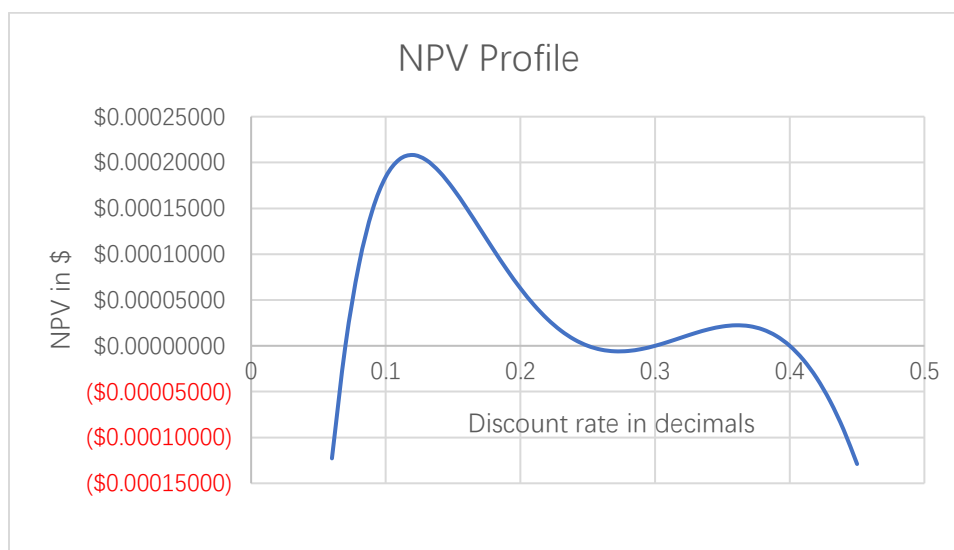
Graph 1: NPV profile of a 4-year project whose cash flows were generated algebraically from 7%, 25%, 30% and 40% IRRs.

To generate different cash flows for different classes or semesters of students, one merely needs to change the entries in cells B4 through B7, and the corresponding cash flows will be output in cells H10 though H14. The authors have a similar Excel coding that will generate the cash flows for five different IRRs which can be requested from us.

To push the learning envelope a step further, we can ask the students using Solver, an add-in program in Excel, to find the cost of capital that gives the maximum NPV for the project. With Solver, we can further ask them to find the cost of capital between 25% and 30% that is associated with the minimum NPV. The respective answers are 11.92% and 27.31% for this particular sets of cash flows. The latter answer on 27.31% shows that the NPV profile is indeed asymmetric between 25% and 30% domain for the minimum NPV value not to be at 27.50%. The 19 basis-point difference proves the asymmetry our naked eyes cannot perceive.

CONCLUSION
In order to generate the cash flows with 4 different values of IRR, one simply needs to manually enter the four different values in cells B4 through B7 while leaving everything else intact in the Excel spreadsheet. The cash flows necessary to create those 4 IRRs will be output in cells H10 through H14. A bonus learning experience for students with this exercise could well be the Solver,

an add-in program in Excel, to find the cost of capital that gives the maximum or minimum value of the NPV within a certain domain of the discount rates.

REFERENCES

Berk, Jonathan, and Peter DeMarzo, 2020, Fifth edition, *Corporate finance*, Pearson.

Brigham, Eugene F., and Michael C. Ehrhardt, 2019, 16th edition, *Financial management: theory and practice*, Cengage Learning.

Copeland, Thomas E., J. Fred Weston, and Kuldeep Shastri, Fourth edition, *Financial theory and corporate policy*, Pearson.

Emery, Douglas R., John D. Finnerty, and John D. Stowe, 2011, Fourth edition, *Corporate financial management*, Wohl Publishing.

Keown, Arthur J., John D. Martin, and J. William Petty, 2020, Tenth edition, *Foundations of finance*, Pearson.

Ross, Stephen, Randolph Westerfield, Jeffrey Jaffe, and Bradford Jordan, 2021, 13th edition, *Corporate finance*, McGraw Hill.

Welch, Ivo, 2009, *Corporate finance: an introduction*, Person.