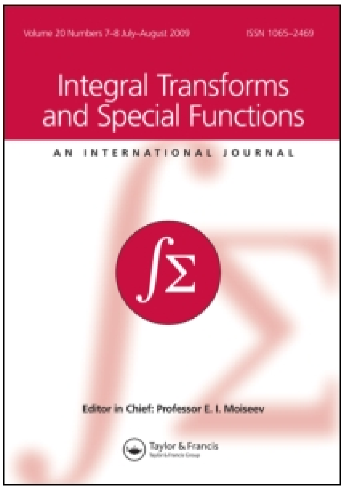

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ASYMPTOTICS FOR THE RELATIVISTIC HERMITE POLYNOMIALS ZEROS

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By using the Liouville - Stekloff method asymptotic estimates for the zeros of the relativistic Hermite polynomials $H_n^{(N)}$ are derived.

KEY WORDS: asymptotics, zeros of the polynomials

MSC (1991): 33C25, 33C45

1. INTRODUCTION

The wave functions of the quantum relativistic harmonic oscillator in configuration space have been shown [1] to be expressable by means of a one-parameter family of polynomials $\{H_n^{(N)}\}_{n=0}^{\infty}$, called Relativistic Hermite Polynomials (RHP):

$$\psi_n(t,x,p;N) = \exp\frac{\imath f}{\hbar}\exp(-\imath n\omega t)2^{-n/2}\alpha^{n+N}H_n^{(N)}(x),$$

where the following notation has been used:

$$N = \frac{mc^2}{\hbar\omega}, \quad \alpha(x,N) = (1+\frac{x^2}{N})^{1/2}, \quad P^0(x,p;n) = \sqrt{p^2+m^2c^2\alpha(x;N)^2},$$

$$f(x,p;N) = \frac{2mc^2}{\omega}\arctan\left(\frac{\sqrt{N}(P^0-p+mc)}{mcx}\right).$$

In the preceding formulas n is the principal quantum number, ω is the frequency of the oscillator and p the momentum.

The RHP reduce to the well-known classical Hermite polynomials in the non-relativistic limit ($N \to \infty$) and the dimensionless real parameter N must be greater than $1/2$ because of the square integrability of the wavefunction.

B. Nagel has shown the existence of a connection between these polynomials and the well known Gegenbauer polynomials in [3].

In this paper we will obtain asymptotic estimates for the zeros of the Relativistic Hermite Polynomials. Our result is Proposition II.

The RHP satisfy a certain differential equation which can be transformed, by a change of variables, into a differential equation to which it is possible to apply the Liouville–Stekloff procedure [2]. This method will give an asymptotic representation of the solutions which allows us to use Tricomi's method [4].

For sake of completeness we will give a brief description of both methods.

2. A RESULT DUE TO F.G. TRICOMI

In paper [4], F.G. Tricomi proved the following result:

Proposition 1. *Suppose the continuous function $f(x)$ admits (uniformly with respect to x), the asymptotic representation:*

$$f(x) = \sum_{k=0}^{m} g_k(x)\, \eta^k + \mathcal{O}(\eta^{m+1}), \qquad (\eta \to 0), \tag{1}$$

where the functions $g_k(x)$ are differentiable m-k+1 times in a neighborhood of a point x_0 which is a simple zero of the function $g_0(x)$ $(g_0(x_0) = 0,\ g_0'(x_0) \neq 0)$. Suppose further that $g_m(x) \in C^1$ in the same neighborhood. Then $\forall \epsilon > 0$ and for $|\mu|$ less than a suitable $\delta > 0$, the equation $f = 0$ is satisfied at least by a value x_0^* such that: $|x_0^* - x_0| < \epsilon$, and the following expansion holds:

$$x_0^* = x_0 + \sum_{k=1}^{m} w_{k-1}\, \eta^k + \mathcal{O}(\eta^{m+1}), \tag{2}$$

where the cofficients $w_0, w_1, w_2, \dots$ are rational functions of the values:

$$G_{k,\ell} := \frac{1}{\ell!}\, g_k^{(\ell)}(x_0),$$

and are determinated by the system:

$$\begin{cases} 0 = G_{10} + G_{01}w_0 \\ 0 = G_{20} + G_{01}w_1 + G_{11}w_0 + G_{02}w_0^2 \\ 0 = G_{30} + G_{01}w_2 + G_{11}w_1 + 2G_{02}w_0w_1 + G_{21}w_0 + G_{12}w_0^2 + G_{03}w_0^3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

(see [4], for the general equation).

Remark. Note that we have written $g_k(x)$ in place of $g_k(x, \eta)$ in order to simplify the notation; nevertheless the method also works in the case when the functions g_k depend on the parameter η. In the following we deal with this more general case.

The first expression for the w_k are given by:

$$w_0 = -\frac{G_{10}}{G_{01}} = -\frac{g_1(x_0)}{g_0'(x_0)}, \tag{3}$$

$$\begin{aligned} w_1 &= -\frac{G_{10}^2 G_{02} - G_{10}G_{01}G_{11} + G_{01}^2 G_{20}}{G_{01}^3} \\ &= -\frac{\frac{1}{2}(g_1(x_0))^2 g_0''(x_0) - g_1(x_0)g_0'(x_0)g_1'(x_0) + (g_0'(x_0))^2 g_2(x_0)}{(g_0'(x_0))^3}. \end{aligned} \tag{4}$$

Note that formula (2) provides an asymptotic estimate for a zero x_0^* of the function f in terms of the zero x_0 of g_0, provided that the representation (1) is known.

3. THE LIOUVILLE – STEKLOFF METHOD

In a book by Erdélyi [2], a procedure can be found, due to Liouville and Stekloff, to approximate the zeros of solutions of a certain differential equation. Such a method can be summarized as follows.

Let's consider the ODE:

$$y'' + [\lambda^2 p(x) + r(x)]y = 0 \tag{5}$$

where $\lambda \rightarrow \infty$.

Here, x is a real variable, $a \le x \le b$, $p(x)$ is positive and twice continuosly differentiable, and $r(x)$ is continuous, for $a \le x \le b$.

New variables, ξ and η, are introduced by the substitutions

$$\xi = \int [p(x)]^{1/2}\, dx,$$

$$\eta = [p(x)]^{1/4} y,$$

which carry the interval $a \le x \le b$ into $\alpha \le \xi \le \beta$ and the differential equation (5) into

$$\frac{d^2\eta}{d\xi^2} + \lambda^2 \eta = \rho(\xi)\eta, \tag{6}$$

where

$$\rho(\xi) = \frac{1}{4}\frac{p''}{p^2} - \frac{5}{16}\frac{p'^2}{p^3} - \frac{r}{p}$$

is a continuous function of ξ, $\alpha \leq \xi \leq \beta$.

It is well known that solutions of (6) satisfy the Volterra integral equation:

$$\eta(\xi) = c_1 \cos \lambda\xi + c_2 \sin \lambda\xi + \frac{1}{\lambda} \int_{\gamma}^{\xi} \sin \lambda(\xi - t)\rho(t)\eta(t; N, n)\, dt \tag{7}$$

where $\alpha \leq \gamma \leq \beta$ and c_1, c_2 are arbitrary. Note that $\eta(\xi)$ and $c_1 \cos \lambda\xi + c_2 \sin \lambda\xi$ have the same value, and the same derivative, at $\xi = \gamma$.

The solutions of (7) can be obtained by successive approximations in the form

$$\eta(\xi, \lambda) = \sum_{\ell=0}^{\infty} \eta_\ell(\xi\lambda) \tag{8}$$

where

$$\eta_0(\xi, \lambda) = c_1 \cos \lambda\xi + c_2 \sin \lambda\xi$$

$$\eta_{\ell+1}(\xi, \lambda) = \frac{1}{\lambda} \int_{\gamma}^{\xi} \sin \lambda(\xi - t)\rho(t)\eta_\ell(t, \lambda)dt, \qquad \ell = 0, 1, \dots .$$

If $|\rho(\xi)| \leq A$, by using the induction method, it can easily be proved that

$$|\eta_\ell(\xi, \lambda)| \leq \frac{|c_1| + |c_2|}{\ell!} \frac{A^\ell |\xi - \gamma|^\ell}{\lambda^\ell}, \qquad \ell = 1, 2, \dots,$$

and in the case of a finite interval (α, β) it follows that (8) is uniformly convergent, and is an asymptotic expansion of $\eta(\xi, \lambda)$ as $\lambda \to \infty$.

4. APPLICATION TO THE PROBLEM

The RHP have been shown [1] to satisfy the following second order differential equation

$$\left(1 + \frac{x^2}{N}\right) y_n'' - \frac{2}{N}(N + n - 1)xy_n' + \frac{n}{N}(2N + n - 1)y_n = 0, \tag{9}$$

where

$$y_n(x; N) \equiv H_n^{(N)}(x).$$

The explicit expression of these polynomials is the following [1]:

$$H_n^{(N)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} A_{n,n-2k}^{(N)} (2x)^{n-2k}, \tag{10}$$

where

$$A_{n,n-2k}^{(N)} = \frac{(-1)^k n! N^k (N-1/2)!(2N+n-1)!}{k!(n-2k)!(N+k-1/2)!(2N)^n(2N-1)!}.$$

Here we will study the distribution of zeros of the RHP: these zeros describe the nodes of the wave functions of the quantum relativistic harmonic oscillator.

In order to study the distribution of zeros of the RHP, we start by inserting the following change of variable [5]:

$$y_n(x,N) = (1+\frac{x^2}{N})^{(N+n-1)/2} u_n(x,N)$$

in the differential equation (9), which gives us the form:

$$u_n''(x;N) + S(x;N,n)u_n(x;N) = 0, \tag{11}$$

where

$$S(x;N,n) = \frac{(1-N)x^2 + n^2 + 2nN + N - 1}{N(1+x^2/N)^2}.$$

The zeros of $y_n (\equiv H_n^{(N)})$ coincide with the ones of u_n. Moreover if $S(x;N,n) < 0$, u_n has, at most, two zeros that are symmetric with respect to the origin. So we are interested in the relevant case $S(x;N,n) > 0$ and hence [5]:

$$\forall x \in \left(-\sqrt{\frac{n^2+2nN+N-1}{N-1}}, \sqrt{\frac{n^2+2nN+N-1}{N-1}} \right)$$

Here, we observe that the equation (11) can be written in the following form:

$$u_n'' + [\lambda_n^2 p(N,x) + r(N,x)]u_n = 0, \tag{12}$$

where

$$p(N,x) = \frac{N}{(N+x^2)^2},$$

$$r(N,x) = \frac{N(N-1)(1-x^2)}{(N+x^2)^2},$$

and

$$\lambda_n^2 = (n^2 + 2nN).$$

This allows us to apply the Liouville–Stekloff method [2]. Therefore we introduce the new variables ξ, η by the following substitution:

$$\xi = \int [p(N,x)]^{1/2} dx = \arctan \frac{x}{\sqrt{N}},$$

$$\eta = [p(N,x)]^{1/4} u,$$

carrying the differential equation (12) into

$$\frac{d^2\eta}{d\xi^2} + \lambda_n^2 \eta = \rho(N,\xi)\eta,$$

where

$$\rho(N,\xi) = \frac{1}{4}\frac{p''}{p^2} - \frac{5}{16}\frac{p'^2}{p^3} - \frac{r}{p} = (N-1)N\tan^2\xi - N.$$

That is:

$$\frac{d^2\eta}{d\xi^2} + (n^2 + 2nN)\eta = (N(N-1)\tan^2\xi - N)\eta. \tag{13}$$

Clearly zeros of u are obtained by taking the tangent of zeros of η. Therefore we will study the zero distribution with respect to the variable η.

The solutions of (13) satisfy the Volterra integral equation:

$$\eta(\xi;N,n) = c_1 \cos\sqrt{n^2+2nN}\xi + c_2 \sin\sqrt{n^2+2nN}\xi$$
$$+\frac{1}{\sqrt{n^2+2nN}}\int_0^\xi \sin\sqrt{n^2+2nN}(\xi - t)(N(N-1)\tan^2 t - N)\eta(t;N,n)\,dt, \tag{14}$$

where

$$c_1 = c_1(N,n) = N^{(2N+2n-3)/4} H_n^{(N)}(0),$$

and

$$c_2 = c_2(N,n) = N^{(2N+2n-3)/4} \frac{n(2N+n-1)}{N\sqrt{n^2+2nN}} H_{n-1}^{(N)}(0).$$

Notice that if n is even, $c_2 = 0$ and if n is odd, $c_1 = 0$. The solutions of (14) can be obtained by successive approximations in the following form [2]:

$$\eta(\xi;N,n) = \sum_{\ell=0}^{\infty} \eta_\ell(\xi;N,n),$$

where, if n is even:

$$\eta_0(\xi;N,n) = c_1(N,n)\cos\lambda_n\xi,$$

$$\eta_1(\xi;N,n) = \frac{c_1(N,n)}{\lambda_n}\int_0^\xi \sin\lambda_n(\xi - t)\rho(N,t)\cos\lambda_n t\,dt,$$

$$\eta_2(\xi;N,n) = \frac{c_1(N,n)}{\lambda_n^2}\int_0^\xi \sin\lambda_n(\xi - t)\rho(N,t)\int_0^t \sin\lambda_n(t-s)\rho(N,s)\cos\lambda_n s\,ds\,dt$$

$$\eta_3(\xi; N, n) = \frac{c_1(N, n)}{\lambda_n^3} \int_0^\xi \sin \lambda_n(\xi - t)\rho(N, t) \int_0^t \sin \lambda_n(t - r)\rho(N, r)$$

$$\times \int_0^r \sin \lambda_n(r - s)\rho(N, s) \cos \lambda_n s \, ds \, dr \, dt$$

...

and, if n is odd:

$$\eta_0(\xi; N, n) = c_2(N, n) \sin \lambda_n \xi,$$

$$\eta_1(\xi; N, n) = \frac{c_2(N, n)}{\lambda_n} \int_0^\xi \sin \lambda_n(\xi - t)\rho(N, t) \sin \lambda_n t \, dt,$$

$$\eta_2(\xi; N, n) = \frac{c_2(N, n)}{\lambda_n^2} \int_0^\xi \sin \lambda_n(\xi - t)\rho(N, t) \int_0^t \sin \lambda_n(t - s)\rho(N, s) \sin \lambda_n s \, ds \, dt,$$

$$\eta_3(\xi; N, n) = \frac{c_2(N, n)}{\lambda_n^3} \int_0^\xi \sin \lambda_n(\xi - t)\rho(N, t) \int_0^t \sin \lambda_n(t - r)\rho(N, r)$$

$$\times \int_0^r \sin \lambda_n(r - s)\rho(N, s) \sin \lambda_n s \, ds \, dr \, dt,$$

.... .

Therefore in both cases we can write the asymptotic representation in the following form:

$$\eta(\xi; N, n) = \sum_{\ell=0}^{m} g_\ell(\xi; N, n) \frac{1}{\lambda_n^\ell} + \mathcal{O}\left(\frac{1}{\lambda_n^{m+1}}\right),$$

where the functions g_ℓ are differentiable $m - \ell + 1$ times in a neighbourhood of every simple zero of the function g_0.

If n is even:

$$g_0(\xi; N, n) = \eta_0(\xi; N, n) = c_1(N, n) \cos \lambda_n \xi = N^{\frac{2N+2n-3}{4}} H_n^{(N)}(0) \cos \sqrt{n^2 + 2nN}\xi,$$

and

$$g_0(\xi_n) = 0, \qquad \text{if} \qquad \xi_n = \pm\frac{1}{\sqrt{n^2+2nN}}\left(\frac{\pi}{2}+k\pi\right).$$

Choosing $k \in \mathbf{N}$ and $k \leq \frac{n-2}{2}$, since $\sqrt{n^2+2nN} \simeq n \ (n \to \infty)$, we obtain n simple symmetric real zeros $\xi_{n,1}, \ldots, \xi_{n,n}$ of g_0 all satisfying $|\xi_{n,i}| < \frac{\pi}{2}$ for $i = 1, \ldots, n$.

If n is odd:

$$\begin{aligned} g_0(\xi; N, n) &= \eta_0(\xi; N, n) = c_2(N, n) \sin \lambda_n \xi \\ &= N^{\frac{2N+2n-3}{4}} \frac{n(2N+n-1)}{N\sqrt{n^2+2nN}} H_{n-1}^{(N)}(0) \sin \sqrt{n^2+2nN}\xi, \end{aligned}$$

and

$$g_0(\xi_n) = 0 \qquad \text{if} \qquad \xi_n = \pm\frac{k\pi}{\sqrt{n^2+2nN}}.$$

And, similarly as above, we choose $k \in \mathbf{N}$ and $k \leq \frac{n-1}{2}$.

5. ASYMPTOTIC ESTIMATES FOR THE ZEROS OF THE RHP

Here we apply Tricomi's method [4] (since the hypotheses are clearly satisfied) to obtain an asymptotic representation of any order of accuracy for all the zeros of η and therefore of $\{H_n^{(N)}(x)\}$ in terms of the zeros of these functions g_0.

That is : $\forall \epsilon > 0$, $\forall h \in \mathbf{N}$ and $\forall h \leq \frac{n-1}{2}$ if n is odd ($h \leq \frac{n-2}{2}$ if n is even), the equation $\eta(\xi; N, n) = 0$ is satisfied by a value $\overline{\xi_n}$ s.t. $|\overline{\xi_n} - \xi_n^h| < \epsilon$ where

$$\xi_n^h = \begin{cases} \pm\dfrac{1}{\sqrt{n^2+2nN}}\left(\dfrac{\pi}{2}+h\pi\right), & \text{if } n \text{ is even,} \\ \pm\dfrac{h\pi}{\sqrt{n^2+2nN}}, & \text{if } n \text{ is odd,} \end{cases}$$

and the following expansion holds:

$$\overline{\xi_n} = \xi_n^h + \sum_{\ell=1}^{m} \omega_{\ell-1}\frac{1}{\lambda^\ell} + \mathcal{O}\left(\frac{1}{\lambda^{m+1}}\right)$$

where

$$\omega_0 = -\frac{g_1(\xi_n^h; N, n)}{g_0'(\xi_n^h; N, n)},$$

$$\begin{aligned} \omega_1 = &-\frac{1/2(g_1(\xi_n^h; N, n))^2 g_0''(\xi_n^h; N, n) - g_1(\xi_n^h; N, n) g_0'(\xi_n^h; N, n) g_1'(\xi_n^h; N, n)}{g_0(\xi_n^h; N, n)^3} \\ &+ \frac{(g_0'(\xi_n^h; N, n))^2 g_2(\xi_n^h; N, n)}{g_0(\xi_n^h; N, n)^3}. \end{aligned}$$

Since the zero distribution of the RHP is symmetric, we will only show how to apply the method for positive $\overline{\xi_n}$.

To compute the explicit expression of ω_0 and ω_1 for $\overline{\xi_n}$ in both cases of n odd or even, let us consider the zero ξ_n^h of $g_0(\xi; N, n)$ s.t. $\forall\epsilon > 0\ |\overline{\xi_n} - \xi_n^h| < \epsilon$.

5.1. *Case n even*

$$g_0(\xi; N, n) = c_1(N, n)\cos\lambda_n\xi,$$

$$g_0'(\xi; N, n) = -\lambda_n c_1(N, n)\sin\lambda_n\xi,$$

$$g_0'(\xi_n^h) = \begin{cases} -\lambda_n c_1(N, n), & \text{if } h \text{ is even,} \\ \lambda_n c_1(N, n), & \text{if } h \text{ is odd,} \end{cases}$$

$$g_1(\xi; N, n) = c_1(N, n)\int_0^{\xi}\sin\lambda_n(\xi - t)\rho(N, t)\cos\lambda_n t\, dt,$$

$$g_1(\xi_n^h) = \begin{cases} c_1(N, n)\displaystyle\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)}\cos^2\lambda_n t\rho(N, t)\, dt, & \text{if } h \text{ is even,} \\ -c_1(N, n)\displaystyle\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)}\cos^2\lambda_n t\rho(N, t)\, dt, & \text{if } h \text{ is odd.} \end{cases}$$

Therefore we obtain:

$$\omega_0^{\text{even}} = \frac{1}{\lambda_n}\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)}\cos^2\lambda_n t\rho(N, t)\, dt. \tag{15}$$

Moreover we have:

$$g_0''(\xi) = -\lambda_n^2 c_1(N, n)\cos\lambda_n\xi,$$

$$g_0''(\xi_n^h) = 0,$$

$$g_1'(\xi; N, n) = c_1(N, n)\lambda_n\int_0^{\xi}\cos\lambda_n(\xi - t)\rho(N, t)\cos\lambda_n t\, dt,$$

$$g_1'(\xi_n^h) = \begin{cases} -c_1(N, n)\lambda_n\displaystyle\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)}\sin\lambda_n t\cos\lambda_n t\rho(N, t)\, dt, & \text{if } h \text{ is even,} \\ c_1(N, n)\lambda_n\displaystyle\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)}\sin\lambda_n t\cos\lambda_n t\rho(N, t)\, dt, & \text{if } h \text{ is odd,} \end{cases}$$

$$g_2(\xi) = c_1(N,n) \int_0^{\xi} \sin \lambda_n(\xi - t)\rho(N,t) \int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \cos \lambda_n s \, ds \, dt$$

$$g_2(\xi_n^h) = \begin{cases} c_1(N,n) \displaystyle\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)} \cos \lambda_n t \rho(N,t) \\ \qquad \times \displaystyle\int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \cos \lambda_n s \, ds \, dt, \quad \text{if } h \text{ is even}, \\ -c_1(N,n) \displaystyle\int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)} \cos \lambda_n t \rho(N,t) \\ \qquad \times \displaystyle\int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \cos \lambda_n s \, ds \, dt, \quad \text{if } h \text{ is odd}. \end{cases}$$

Finally the following expression is obtained:

$$\begin{aligned} \omega_1^{\text{even}} &= -\frac{1}{\lambda_n} \int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)} \cos^2 \lambda_n t \rho(N,t) \, dt \int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)} \sin \lambda_n t \cos \lambda_n t \rho(N,t) dt \\ &\quad + \frac{1}{\lambda_n} \int_0^{\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)} \cos \lambda_n t \rho(N,t) \int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \cos \lambda_n s \, ds \, dt. \end{aligned} \tag{16}$$

5.2. *Case n odd*

$$g_0(\xi; N, n) = c_2(N,n) \sin \lambda_n \xi,$$

$$g_0'(\xi; N, n) = \lambda_n c_2(N,n) \cos \lambda_n \xi,$$

$$g_0'(\xi_n^h) = \begin{cases} \lambda_n c_2(N,n), & \text{if } h \text{ is even}, \\ -\lambda_n c_2(N,n), & \text{if } h \text{ is odd}, \end{cases}$$

$$g_1(\xi; N, n) = c_2(N,n) \int_0^{\xi} \sin \lambda_n(\xi - t)\rho(N,t) \sin \lambda_n t \, dt,$$

$$g_1(\xi_n^h) = \begin{cases} -c_2(N,n) \int_0^{\frac{h\pi}{\lambda_n}} \sin^2 \lambda_n t \rho(N,t)\, dt, & \text{if } h \text{ is even,} \\ c_2(N,n) \int_0^{\frac{h\pi}{\lambda_n}} \sin^2 \lambda_n t \rho(N,t)\, dt, & \text{if } h \text{ is odd.} \end{cases}$$

Therefore we obtain:

$$\omega_0^{\text{odd}} = \frac{1}{\lambda_n} \int_0^{\frac{h\pi}{\lambda_n}} \sin^2 \lambda_n t \rho(N,t)\, dt. \tag{17}$$

Furthermore:

$$g_0''(\xi) = \lambda_n^2 c_2(N,n) \sin \lambda_n \xi,$$

$$g_0''(\xi_n^h) = 0,$$

$$g_1'(\xi; N, n) = c_2(N,n)\lambda_n \int_0^{\xi} \cos \lambda_n(\xi - t)\rho(N,t) \sin \lambda_n t \, dt,$$

$$g_1'(\xi_n^h) = \begin{cases} c_2(N,n)\lambda_n \int_0^{\frac{h\pi}{\lambda_n}} \sin \lambda_n t \cos \lambda_n t \rho(N,t)\, dt & \text{if } h \text{ is even,} \\ -c_2(N,n)\lambda_n \int_0^{\frac{h\pi}{\lambda_n}} \sin \lambda_n t \cos \lambda_n t \rho(N,t)\, dt, & \text{if } h \text{ is odd,} \end{cases}$$

$$g_2(\xi) = c_2(N,n) \int_0^{\xi} \sin \lambda_n(\xi - t)\rho(N,t) \int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \sin \lambda_n s \, ds\, dt$$

$$g_2(\xi_n^h) = \begin{cases} -c_2(N,n) \int_0^{\frac{h\pi}{\lambda_n}} \sin \lambda_n t \rho(N,t) \int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \sin \lambda_n s \, ds\, dt, & \text{if } h \text{ is even,} \\ c_2(N,n) \int_0^{\frac{h\pi}{\lambda_n}} \sin \lambda_n t \rho(N,t) \int_0^{t} \sin \lambda_n(t-s)\rho(N,s) \sin \lambda_n s \, ds\, dt & \text{if } h \text{ is odd,} \end{cases}$$

Finally we obtain:

$$\omega_1^{\text{odd}} = -\frac{1}{\lambda_n}\int_0^{\frac{h\pi}{\lambda_n}} \sin^2\lambda_n t\rho(N,t)\,dt \int_0^{\frac{h\pi}{\lambda_n}} \sin\lambda_n t\cos\lambda_n t\rho(N,t)\,dt$$

$$+\frac{1}{\lambda_n}\int_0^{\frac{h\pi}{\lambda_n}} \sin\lambda_n t\rho(N,t)\int_0^{t}\sin\lambda_n(t-s)\rho(N,s)\sin\lambda_n s\,ds\,dt. \qquad (18)$$

Returning to the original variable x, we can lastly proclaim the following result:

Proposition 2. *For all the zeros of the nth (n even) RHP, we can write the asymptotic estimate*:

$$\overline{x_n} = \sqrt{N}\tan\left[\frac{1}{\lambda_n}(\frac{\pi}{2}+h\pi)+\omega_0^{\text{even}}\frac{1}{\lambda_n}+\omega_1^{\text{even}}\frac{1}{\lambda_n^2}\right]+\mathcal{O}\left(\frac{1}{\lambda_n^3}\right). \qquad (19)$$

For all the zeros of the n-th (n odd) RHP, we can write the asymptotic estimate:

$$\overline{x_n} = \sqrt{N}\tan\left[\frac{h\pi}{\lambda_n}+\omega_0^{\text{odd}}\frac{1}{\lambda_n}+\omega_1^{\text{odd}}\frac{1}{\lambda_n^2}\right]+\mathcal{O}\left(\frac{1}{\lambda_n^3}\right). \qquad (20)$$

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