MATRIX GENERALISED KUMMER RELATION

José A. Díaz-García [1]
Department of Statistics and Computation, Universidad Autonoma Agraria Antonio Narro, 25315 Buenavista, Saltillo, Coahuila, Mexico.
e-mail: *jadiaz@uaaan.mx; jadiazg@ugr.es*
and
Francisco J. Caro-Lopera
Department of Basic Sciences, Universidad de Medellín, Carrera 87 No.30-65, of. 5-103, Medellín, Colombia.
e-mail: *fjcaro@udem.edu.co*

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Summary: An extension of the well known matrix Kummer relation of Herz (1955) is proposed in this paper by assuming a general model which admits a Taylor expansion. Under certain conditions of the involved parameters, the addressed generalisation can be used for deriving efficiently computable matrix variate densities based on non Gaussian models, avoiding some open problems of the literature in the context of shape theory and other related fields.

1. Introduction

The work of Herz (1955), which generalised the classical special functions of hypergeometric type to matrix variables, opened a perspective of important applications in many fields of knowledge. Then, Constantine (1963) placed the work of Herz in the context of the well known zonal polynomial theory and the applications in multivariate analysis and other areas were possible. For example, the information theory became one of the novel fields where these relations were applied successfully, see the works of Ratnarajah and Villancourt (2005a), Ratnarajah and Villancourt (2005b) and the references there in. In fact, some studies of Goodall and Mardia (1993), Caro-Lopera, Díaz-García and González-Farías (2009b) and Caro-Lopera and Díaz-García (2012) considered new relations in order to perform exact inference, which avoids the classical approximations. Some advances, including new relations are studied in the context of shape theory based on affine transformations, see Caro-Lopera, Díaz-García and González-Farías (2009a); explicitly, the general relations transform the elliptical configuration densities, which are infinite series of zonal polynomials, see Caro-Lopera et al. (2009b), into polynomials of lower degree easily computable.

Before the matrix variate elliptical models appeared, the matrix Kummer and Euler relations were the only efficient transformation for certain matrix variate Gaussian-based distributions which

[1] Corresponding author.

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avoided infinite series (hypergeometric) of zonal polynomials and allowed that the existing tables for such polynomials up to 12-th degree, could be used for deriving some exact densities, see Muirhead (1982). However, when the referred transformations did not fit for a polynomial density, the problem of computing hypergeometric series implemented by Koev and Edelman (2006), could give the solution; however as they highlight on page 845:

> *"... Several problems remain open, among them automatic detection of convergence....Another open problem is to determine the best way to truncate the series".*

Moreover, when we consider non gaussian models, the corresponding generalised hypergeometric functions increase the problem and their computation is so out of the scope of the existing numerical algorithms. In fact, recently, Caro-Lopera, Díaz-García and González-Farías (2014), proved that and extension of the Koev and Edelman's algorithm suitable for non Gaussian models fails for getting the exact optimum solution in shape theory, which is easily reached by certain polynomial density based on a Kummer type relation. Thus, an analytical solution to this problem under certain conditions and a general setting, involving a class of distributions with generator function accepting a Taylor expansion, is the motivation of the present work.

Now, the classical Kummer relation can be generalised if we replace its exponential model by a function which admits a Taylor expansion in zonal polynomials; then a sort of new and useful expressions can be derived easily.

However some applications demand the extension of the parameter definition domain, and this implies new integral representations of the series, new integral transforms, and new induction constructions. So, the domain extension problem places the generalised Kummer relations in an interesting mathematical task with promissory applications.

This work develops the above discussion by generalising the classical Kummer relation in section 2, then a number of known results are derived as corollaries, and a source for new relations is established by using some partitional formulae. Finally, a domain extension study, which in particular is useful for applications in statistical shape theory, is proposed at the end of the paper in section 3.

2. Generalized Kummer relation

In this section, we study an extension of the generalised Kummer relation. First, let us consider the following definition from James (1964) and Muirhead (1982).

Fix complex numbers $a_1,\dots,a_p$ and $b_1,\dots,b_q$, and for all $1 \le i \le q$ and $1 \le j \le m$ do not allow $-b_i + (j-1)/2$ to be a nonnegative integer. Then the *hypergeometric function with one matrix argument* ${}_pF_q$ is defined to be the real-analytic function of a complex symmetric $m \times m$ matrix $\mathbf{X}$ given by the series

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;\mathbf{X}) = \sum_{t=0}^{\infty}\sum_{\tau} \frac{(a_1)_\tau \cdots (a_p)_\tau}{(b_1)_\tau \cdots (b_q)_\tau}\, \frac{C_\tau(\mathbf{X})}{t!}.$$

where $\sum_\tau$ denotes summation over al partitions $\tau = (t_1,\dots,t_m), t_1 \ge \cdots \ge t_m \ge 0$, $C_\tau(\mathbf{X})$ is the zonal polynomial of $\mathbf{X}$ corresponding to τ and the generalised hypergeometric coefficient $(b)_\tau$ is given by

$$(b)_\tau = \prod_{i=1}^{m} \left(b - \frac{1}{2}(i-1) \right)_{t_i},$$

where

$$(b)_t = b(b+1)\cdots(b+t-1), \quad (b)_0 = 1.$$

In particular for $p = q = 1$, we have

$${}_1F_1(a,;c;\mathbf{X}) = \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a)_\tau}{(c)_\tau} \frac{C_\tau(\mathbf{X})}{t!},$$

moreover, it is known that ${}_1F_1$ has the integral representation, see Muirhead (1982, Theorem 7.4.2, p. 264)

$${}_1F_1(a;c;\mathbf{X}) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_{0<\mathbf{Y}<\mathbf{I}_m} \operatorname{etr}(\mathbf{XY}) |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2} (d\mathbf{Y}). \quad (1)$$

valid for all symmetric matrix $\mathbf{X}$, $\operatorname{Re}(a) > (m-1)/2$, $\operatorname{Re}(c) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$; where $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$, and $0 < \mathbf{Y}$, denotes that $\mathbf{Y}$ is a $m \times m$ positive definite matrix and $\mathbf{Y} < \mathbf{I}_m$ denotes that $\mathbf{I}_m - \mathbf{Y}$ also is a $m \times m$ positive definite matrix.

Then, Herz (1955, equation (2.8), p. 488) states that

$${}_1F_1(a;c;\mathbf{X}) = \operatorname{etr}(\mathbf{X})\, {}_1F_1(c-a;c;-\mathbf{X}), \quad (2)$$

which is termed generalised Kummer relation, see also Muirhead (1982, equation (6), p. 265).

The first step to obtain a generalisation of Euler's relation is to consider a generalisation of the hypergeometric function with one matrix argument ${}_1F_1$, which is termed hypergeometric generalised function with one matrix argument and is denoted as ${}_1P_1$. From Caro-Lopera et al. (2009b) we have the following definition.

Definition 1 Let $\mathbf{X}$ be an $m \times m$ positive definite matrix. The hypergeometric generalised function ${}_1P_1$ of matrix argument is defined by

$${}_1P_1(f(t,\operatorname{tr}(\mathbf{X})) : a;c;\mathbf{X}) = \sum_{t=0}^{\infty} \frac{f(t,\operatorname{tr}(\mathbf{X}))}{t!} \sum_{\tau} \frac{(a)_\tau}{(c)_\tau} C_\tau(\mathbf{X}), \quad (3)$$

where the function $f(t,\operatorname{tr}(\mathbf{X}))$ is independent of τ.

Here the parameter c can not be zero or an integer or a half-integer $\leq (m-1)/2$. If the parameter a is a negative integer, say, $a = -l$ for $l > 0$, then the function (3) is a polynomial of degree ml, because for $t \geq ml+1$, $(a)_\tau = (-l)_\tau = 0$, see Muirhead (1982, p. 258). In particular note that, ${}_1P_1(1 : a;c;\mathbf{X}) = {}_1F_1(a;c;\mathbf{X})$.

Then, using this notation we see that the Kummer relation (2) is a particular case of a general type of expressions with the following form

$${}_1P_1\left(f^{(t)}(0) : a;c;\mathbf{X}\right) = {}_1P_1\left(f^{(t)}(\operatorname{tr}(\mathbf{X})) : c-a;c;-\mathbf{X}\right),$$

where $f^{(t)}(y)$ denotes the t-th derivative of the function $f(y)$.

In the next theorem is stated a generalisation of (1) by replacing the function $\operatorname{etr}(\mathbf{XY})$ with a function $f(\operatorname{tr}(\mathbf{XY}))$ which has a convergent power series expansion in zonal polynomials.

Theorem 1 Let $\mathbf{X} < \mathbf{I}$, $\text{Re}(a) > (m-1)/2$, $\text{Re}(c) > (m-1)/2$ and $\text{Re}(c-a) > (m-1)/2$. If the function $f(y)$ admits a Taylor expansion in zonal polynomials, then the following integral representation holds

$$
\begin{aligned}
{}_1P_1\left(f^{(t)}(0):a;c;\mathbf{X}\right) &= \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \\
&\quad \times \int_{0<\mathbf{Y}<\mathbf{I}_m} f(\text{tr}(\mathbf{XY}))|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}). \qquad (4)
\end{aligned}
$$

Proof. First, we use an expansion power series in zonal polynomials

$$
\begin{aligned}
f(\text{tr}(\mathbf{XY})) &= \sum_{t=0}^{\infty}\frac{f^{(t)}(0)}{t!}[\text{tr}(\mathbf{XY})]^t \\
&= \sum_{t=0}^{\infty}\frac{f^{(t)}(0)}{t!}\sum_{\tau}C_\tau(\mathbf{XY}).
\end{aligned}
$$

Then integrating term by term using Muirhead (1982, theorem 7.2.10, p. 254), we have that

$$
\begin{aligned}
&\int_{0<\mathbf{Y}<\mathbf{I}_m} f(\text{tr}(\mathbf{XY}))|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}) \\
&= \sum_{t=0}^{\infty}\frac{f^{(t)}(0)}{t!}\sum_{\tau}\int_{0<\mathbf{Y}<\mathbf{I}_m}|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}C_\tau(\mathbf{XY})(d\mathbf{Y}) \\
&= \sum_{t=0}^{\infty}\frac{f^{(t)}(0)}{t!}\sum_{\tau}\frac{(a)_\tau}{(c)_\tau}\frac{\Gamma_m(a)\Gamma_m(c-a)}{\Gamma_m(c)}C_\tau(\mathbf{X}) \\
&= \frac{\Gamma_m(a)\Gamma_m(c-a)}{\Gamma_m(c)}\sum_{t=0}^{\infty}\frac{f^{(t)}(0)}{t!}\sum_{\tau}\frac{(a)_\tau}{(c)_\tau}C_\tau(\mathbf{X}) \\
&= \frac{\Gamma_m(a)\Gamma_m(c-a)}{\Gamma_m(c)}\,{}_1P_1\left(f^{(t)}(0):a;c;\mathbf{X}\right),
\end{aligned}
$$

and the required result follows. □

Now we propose an expression for the Kummer relation based on the generalised hypergeometric function (3), which shall be termed generalised Kummer relation.

Theorem 2 Let $\mathbf{X}$ be an $m \times m$ positive definite matrix, $\text{Re}(a) > (m-1)/2$, $\text{Re}(c) > (m-1)/2$ and $\text{Re}(c-a) > (m-1)/2$. If the function $f(y)$ admits a Taylor expansion in zonal polynomials, then the generalised Kummer relation is given by

$$
{}_1P_1\left(f^{(t)}(0):a;c;\mathbf{X}\right) = {}_1P_1\left(f^{(t)}(\text{tr}(\mathbf{X})):c-a;c;-\mathbf{X}\right). \qquad (5)
$$

Proof. Consider $\mathbf{W} = \mathbf{I} - \mathbf{Y}$ in (4), then we obtain

$$\begin{aligned}
{}_1P_1\left(f(t,\mathrm{tr}(\mathbf{X})) : a;c;\mathbf{X}\right) &= \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \\
&\quad\times \int_{\mathbf{0}<\mathbf{W}<\mathbf{I}_m} f(\mathrm{tr}[\mathbf{X}(\mathbf{I}-\mathbf{W})])|\mathbf{W}|^{c-a-(m+1)/2}|\mathbf{I}-\mathbf{W}|^{a-(m+1)/2}(d\mathbf{W}) \\
&= \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \\
&\quad\times \int_{\mathbf{0}<\mathbf{W}<\mathbf{I}_m} \sum_{t=0}^{\infty} \frac{f^{(t)}(\mathrm{tr}\mathbf{X})}{t!} \sum_{\tau} C_\tau(-\mathbf{X}\mathbf{W}) \frac{|\mathbf{W}|^{c-a-(m+1)/2}}{|\mathbf{I}-\mathbf{W}|^{-(a-(m+1))/2}}(d\mathbf{W}) \\
&= \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \left[\frac{\Gamma_m(a)\Gamma_m(c-a)}{\Gamma_m(c)}\, {}_1P_1\left(f^{(t)}(\mathrm{tr}\mathbf{X}) : c-a;c;-\mathbf{X}\right)\right],
\end{aligned}$$

and the proof is completed. □

Theorem 2 gives a number of new relations, including the classical Kummer, see equation (2.8), p. 488 in Herz (1955) and a so termed Kummer-Pearson VII relation. We end this section by deriving them as corollaries and proposing other particular expressions.

First, we start with the classical Kummer relation:

Corollary 1 In Theorem 2 assume that $f(y) = \exp(y)$, then

$$ {}_1P_1(1 : a;c;\mathbf{X}) = {}_1P_1(\mathrm{etr}(\mathbf{X}) : c-a;c;-\mathbf{X}) = \mathrm{etr}(\mathbf{X})\, {}_1P_1(1 : c-a;c;-\mathbf{X}) \tag{6}$$

Proof. The result follows by taking $f(y) = \exp(y)$ in (5) , which implies that $f^{(t)}(0) = 1$ and $f^{(t)}(\mathrm{tr}(\mathbf{X})) = \mathrm{etr}(\mathbf{X})$. This is the known Kummer relation derived by Herz (1955) and placed in the zonal polynomial theory by Constantine (1963) (widely used by Muirhead (1982)). □

Corollary 2 Assume that $f(y) = (1-y)^{-b}$, $b \in \mathbb{R}$ in Theorem 2, then

$$ {}_1P_1\left((b)_t : a;c;\mathbf{X}\right) = (1-\mathrm{tr}\mathbf{X})^{-b}\, {}_1P_1\left((b)_t(1-\mathrm{tr}\mathbf{X})^{-t} : c-a;c;-\mathbf{X}\right).$$

Proof. In this case the result follows by taking the Pearson VII model $f(y) = (1-y)^{-b}$, where $f^{(t)}(0) = (b)_t$ and $f^{(t)}(\mathrm{tr}(\mathbf{X})) = (b)_t(1-\mathrm{tr}(\mathbf{X}))^{-b-t}$. □

The above expression is referred as the Kummer-Pearson VII relation because it is related with a Pearson VII distribution.

Now, the following result of Caro-Lopera et al. (2009b) can be used in the derivation of a sort of new Kummer relations.

Lemma 1 Let $f(y) = y^{T-1}\exp(-Ry^s)$, $R, s, T \in \mathbb{R}$; if $\sum_{\kappa \in P_r}$ denotes the summation over all the partitions $\kappa = (k^{\nu_k}, (k-1)^{\nu_{k-1}}, \cdots, 3^{\nu_3}, 2^{\nu_2}, 1^{\nu_1})$ of r, then

$$\begin{aligned} f^{(k)}(y) &= y^{T-1}\exp(-Ry^s)\left\{\sum_{\kappa\in P_k}\frac{k!(-R)^{\sum_{i=1}^k \nu_i}\prod_{j=0}^{k-1}(s-j)^{\sum_{i=j+1}^k \nu_i}}{\prod_{i=1}^k \nu_i!(i!)^{\nu_i}} y^{\sum_{i=1}^k (s-i)\nu_i} \right. \\ &\quad + \sum_{m=1}^k \binom{k}{m}\left[\prod_{i=0}^{m-1}(T-1-i)\right] \\ &\quad \left. \times \sum_{\kappa\in P_{k-m}} \frac{(k-m)!(-R)^{\sum_{i=1}^{k-m}\nu_i}\prod_{j=0}^{k-m-1}(s-j)^{\sum_{i=j+1}^{k-m}\nu_i}}{\prod_{i=1}^{k-m}\nu_i!(i!)^{\nu_i}} y^{\sum_{i=1}^{k-m}(s-i)\nu_i - m}\right\}, \end{aligned} \tag{7}$$

thus the corresponding expressions for $f^{(t)}(0)$ and $f^{(t)}(\mathrm{tr}(\mathbf{X}))$ give the required Kotz relations. Then, considering different values for $R, s, T \in \mathbb{R}$ in Lemma 1 and using Theorem 2 we have:

Corollary 3 In Theorem 2 assume that $f(y) = y^{T-1}\exp(-Ry)$, $R, T \in \mathbb{R}$ where T is an integer such that $1 \le T-1 \le t$. Then

$${}_1P_1\left(\frac{t!(-R)^{t-T+1}}{(t-T+1)!} : a; c; X\right) = {}_1P_1\left(f^{(t)}(\mathrm{tr}(\mathbf{X})) : c-a; c; -X\right),$$

where $f^{(t)}(\mathrm{tr}(\mathbf{X}))$ is given by

$$(-R)^t\,\mathrm{tr}^{T-1}\mathbf{X}\exp(-R\,\mathrm{tr}(\mathbf{X}))\left\{1+\sum_{m=1}^t\binom{t}{m}\left[\prod_{i=0}^{m-1}(T-1-i)\right](-R\,\mathrm{tr}(\mathbf{X}))^{-m}\right\}.$$

In addition,

Corollary 4 Let $f(y) = \exp(-Ry^s)$ in Theorem 2, with $R, s \in \mathbb{R}$. Then

$$\begin{aligned} {}_1P_1&\left(\sum_{\kappa^*}\frac{t!(-R)^{\sum_{i=1}^t \nu_i}\prod_{j=0}^{t-1}(s-j)^{\sum_{i=j+1}^t \nu_i}}{\prod_{i=1}^t \nu_i!(i!)^{\nu_i}} : a; c; \mathbf{X}\right) \\ &= {}_1P_1\left(f^{(t)}(\mathrm{tr}(\mathbf{X})) : c-a; c; -\mathbf{X}\right), \end{aligned}$$

where $\sum_{\kappa^*}$ represents the summation over all the partitions κ^* of t such that $t = s\sum_{i=1}^k \nu_i$ and $f^{(t)}(\mathrm{tr}(\mathbf{X}))$ is given by

$$\exp(-R\,\mathrm{tr}^s\mathbf{X})\sum_{\kappa\in P_t}\frac{t!(-R)^{\sum_{i=1}^t \nu_i}\prod_{j=0}^{t-1}(s-j)^{\sum_{i=j+1}^t \nu_i}}{\prod_{i=1}^t \nu_i!(i!)^{\nu_i}}(\mathrm{tr}(\mathbf{X}))^{\sum_{i=1}^t (s-i)\nu_i},$$

in this case the summation $\sum_{\kappa\in P_t}$ holds for all the partitions of t.

The general Kotz relation involved in (7) certainly generalises (6), the classical Kummer relation, which in this sense is based on a simpler exponential model.

Finally, consider the following generalisation of (7), see Caro-Lopera et al. (2009b).

Lemma 2 Let $f(t) = s(t)r(g(t))$, where $s(\cdot)$, $r(\cdot)$ and $g(\cdot)$ have derivatives of all orders, if $w^{(k)}$ denotes $\frac{d^k w}{dt^k}$ then

$$f^{(k)} = \sum_{m=0}^{k} \binom{k}{m} s^{(m)} [r(g(t))]^{(k-m)},$$

where

$$[r(g(t))]^{(k)} = \sum_{\kappa=(k^{\nu_k},(k-1)^{\nu_{k-1}},\cdots,3^{\nu_3},2^{\nu_2},1^{\nu_1})} \frac{k!}{\prod_{i=1}^{k} \nu_i!(i!)^{\nu_i}} r^{(\sum_{i=1}^{k} \nu_i)} \prod_{i=1}^{k} (g^{(i)})^{\nu_i}.$$

Note that the function f admits a Taylor expansion then the above expressions always exist for all k.

Then a so termed Kummer logistic relation can be obtained by setting

$$h(y) = \exp(-y)\,(1+\exp(-y))^{-2},$$

and computing the t-th derivative by using Lemma 2, i.e. $f^{(t)}(\text{tr}(\mathbf{X}))$ is given by

$$\sum_{m=0}^{t} \binom{t}{m} \sum_{\kappa \in P_{t-m}} \frac{(t-m)!\left(\sum_{i=1}^{t-m} \nu_i + 1\right)!\exp(-(1+\sum_{i=1}^{t-m} \nu_i)\,\text{tr}(\mathbf{X}))}{(-1)^{m+\sum_{i=1}^{t-m}(1+i)\nu_i} \prod_{i=1}^{t-m} \nu_i!(i!)^{\nu_i} \left[1+\text{etr}(-\mathbf{X})\right]^{2+\sum_{i=1}^{t-m} \nu_i}}.$$

Thus

$$_1P_1\left(\sum_{m=0}^{t} \binom{t}{m} \sum_{\kappa \in P_{t-m}} \frac{(t-m)!\left(\sum_{i=1}^{t-m} \nu_i + 1\right)!}{(-1)^{m+\sum_{i=1}^{t-m}(1+i)\nu_i} \prod_{i=1}^{t-m} \nu_i!(i!)^{\nu_i} 2^{2+\sum_{i=1}^{t-m} \nu_i}} : a; c; \mathbf{X}\right)$$

$$= {}_1P_1\left(f^{(t)}(\text{tr}\mathbf{X}) : c-a; c; -\mathbf{X}\right).$$

3. Domain Extensions

Unfortunately, many references which cite and use certain important properties and relations involving hypergeometric functions, do not provide the domain of the corresponding parameters; the classical Kummer relation is an important example, see Muirhead (1982, Theorem 7.4.3, p. 265) and James (1964, eq. (51)), among many others. However, Herz (1955) studies deeply this relations according to the respective domains; for example, he proposes two ways for deriving the confluent hypergeometric function ${}_1F_1$. The first one, via the Cauchy inversion formula

$$_1F_1(a;c;\mathbf{X}) = \frac{\Gamma_m(c)}{(2\pi i)^{m(m+1)/2}} \int_{\text{Re}(\mathbf{Z})=\mathbf{X}_0} \text{etr}(\mathbf{Z})\,{}_1F_0(a;\mathbf{X}\mathbf{Z}^{-1})|\mathbf{Z}|^{-c}(d\mathbf{Z}), \tag{8}$$

whose integral on the right side is absolutely convergent for arbitrary complex matrix $\mathbf{X}$ and a provided we take $\mathbf{X}_0 > \mathbf{0}$, $\mathbf{X}_0 > \text{Re}(\mathbf{M})$ and $\text{Re}(c) > (m-1)/2$. And the second one, via the most important integral representation of ${}_1F_1(a;c;\mathbf{X})$, see Herz (1955, eq. (2.9)); which is given by

$$\frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_{0<\mathbf{Y}<\mathbf{I}_m} \text{etr}(\mathbf{X}\mathbf{Y})|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}) \tag{9}$$

holding for all X, $\text{Re}(a) > (m-1)/2$, $\text{Re}(c) > (m-1)/2$ and $\text{Re}(c-a) > (m-1)/2$.

Some authors derived the classical Kummer relation (2) based on the integral (9), see Muirhead (1982, Theorem 7.4.3); so, any posterior results established with that relation inherits the domain of the parameters according to the absolute convergence of (9). In this case the Kummer relation holds in a weaker domain

$$\text{Re}(a) > (m-1)/2, \;\; \text{Re}(c) > (m-1)/2 \text{ and } \text{Re}(c-a) > (m-1)/2. \tag{10}$$

However, Muirhead (1982, Theorem 10.3.7) uses the same Kummer relation when the restrictions (10) are not satisfied. This can be explained by noting that the Kummer relation is still valid in wider domains, for example: it is valid for

$$\text{arbitrary } a \;\text{ and }\; \text{Re}(c) > (m-1)/2, \tag{11}$$

when the integral representation of the confluent hypergeometric is obtained by applying the Laplace transform to (8), see Herz (1955, eq. (2.8)); or it is valid for

$$\text{Re}(a) > (m-1)/2, \;\text{ and }\; \text{Re}(c) > (m-1)/2,$$

if the integral representation for ${}_1F_0$ is used, see Herz (1955, p. 485) or Muirhead (1982, Corollary 7.3.5).

The Laplace procedure of Herz (1955) (based on the domain (11)) can be applied to the generalised Kummer relation, then we have:

Theorem 3 If $f(y)$ admits a Taylor expansion in zonal polynomials, then the generalised Kummer relation is given by

$$ {}_1P_1\left(f^{(t)}(0) : a; c; \mathbf{X}\right) = {}_1P_1\left(f^{(t)}(\text{tr}(\mathbf{X})) : c-a; c; -\mathbf{X}\right), \tag{12}$$

where $\mathbf{X} > 0$, $\text{Re}(c) > (m-1)/2$ and a is arbitrary (or at least $\text{Re}(a) > (m-1)/2$, if the integral representation of ${}_1F_0$ is used, see Herz (1955, p. 485) or Muirhead (1982, Corollary 7.3.5)).

The result follows by applying the Laplace transform to the left hand side of (12)

$$\int_{X>0} \text{etr}(\mathbf{XZ})|\mathbf{X}|^{c-(m+1)/2} {}_1P_1(f^{(t)}(0) : a; c; \mathbf{X})(d\mathbf{X}),$$

and noting that this integral converges absolutely for $\text{Re}(c) > (m-1)/2$ and arbitrary a (or at least for $\text{Re}(a) > (m-1)/2$), see Herz (1955, p. 487) and Muirhead (1982, Theorem 7.2.7). Finally, by applying the same procedure to the right hand side and noting the same facts for the domain, the required result follows.

4. Conclusions

This work proposes an extension of the classical matrix Kummer relation based on a function $f(y)$ which admits a Taylor expansion in zonal polynomials. From the theoretical point of view, it provides a class of families of identities which contain as particular element, the typical Gaussian identity, besides, any new group of identities can be obtain simply by computing the derivative of the kernel function. However, the main motivation for this work has its origin in the efficient computation of series of zonal polynomials and their applications; in fact, the work lies under the open problems for computing hypergeometric functions of zonal polynomials addressed in Koev and Edelman (2006). Other works, as Caro-Lopera and Díaz-García (2012), Caro-Lopera et al. (2009a), Caro-Lopera et al. (2014), derived Kummer relations based on some non Gaussian models which were applied successfully in the context of shape theory and they claimed for a general setting, which was finally achieved in the present work. The generalised Kummer relation derived in this paper also have potential applications in other fields of knowledge, see for example, Ratnarajah and Villancourt (2005a), Ratnarajah and Villancourt (2005b).

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