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Noncommutative Korteweg–de Vries and modified Korteweg–de Vries hierarchies via recursion methods

Sandra Carillo[1,a)] and Cornelia Schiebold[2,b)]
[1]*Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Sapienza University of Rome, Via A. Scarpa 16, 00161 Rome, Italy*
[2]*Department of Natural Sciences, Engineering, and Mathematics, Mid Sweden University, S-851 70 Sundsvall, Sweden*

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Here, noncommutative hierarchies of nonlinear equations are studied. They represent a generalization to the operator level of corresponding hierarchies of scalar equations, which can be obtained from the operator ones via a suitable projection. A key tool is the application of Bäcklund transformations to relate different operator-valued hierarchies. Indeed, in the case when hierarchies in 1+1-dimensions are considered, a "Bäcklund chart" depicts links relating, in particular, the Korteweg–de Vries (KdV) to the modified KdV (mKdV) hierarchy. Notably, analogous links connect the hierarchies of operator equations. The main result is the construction of an operator soliton solution depending on an infinite-dimensional parameter. To start with, the potential KdV hierarchy is considered. Then Bäcklund transformations are exploited to derive solution formulas in the case of KdV and mKdV hierarchies. It is remarked that hierarchies of matrix equations, of any dimension, are also incorporated in the present framework. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3155080]

I. INTRODUCTION

Noncommutative integrable systems have been an area of increasing activities in recent years, both as a topic of independent interest and as a powerful tool for the investigation of the classical integrable systems. The present article is devoted to various noncommutative Korteweg–de Vries (KdV)-type hierarchies, their operator solitons, and the Bäcklund transformations relating them. Here these topics are studied in their own right, but the results are formulated and established in a way that prepares concrete application both to the classical scalar and matrix hierarchies. The results are presented within the framework of operators on Banach spaces. This approach was pioneered in a seminal work by Marchenko[24] and placed in the context of modern Banach space geometry by Aden and Carl.[1] For the further development of the theory the reader is referred to Refs. 8, 9, 36, and 37. Alternative approaches in the same spirit may be found in Refs. 17, 11, 12, 25, 34, and 35. General references are Refs. 4, 20, 23, 38, and 39 to mention but a few.

The method this work is based on involves solution of a noncommutative (operator-valued) system which is then used to extract information on the original integrable system via projection methods. In general terms, the first step is of mostly algebraic character, whereas the second step makes a link to advanced operator theory. In the development of the theory it turns out that Bäcklund transformation methods (see Refs. 2 and 3 for the original source and Refs. 31 and 32 for modern introductions) are still available within this approach. One of the principal motivations of the present work is to exploit this key tool systematically in the noncommutative context.

Here, the focus is on the noncommutative hierarchies of the potential KdV (pKdV) equation, the KdV equation, and the modified KdV (mKdV) equation, according to Refs. 13, 15, 27, 19, and

[a)]Electronic mail: carillo@dmmm.uniroma1.it or sandra.carillo@uniroma1.it.
[b)]Electronic mail: cornelia.schiebold@miun.se.

40. The main goal of the present article is to exhibit operator soliton solutions, depending on an infinite-dimensional parameter, which solve the equations of a whole hierarchy (see Theorems 1, 10, and 12). In the case of the pKdV hierarchy, which still seems to be the most accessible with respect to the relevant computations, the proof relies on an involved inductive argument (see Ref. 1 for the pKdV equation itself). Then, in the case of the KdV and mKdV hierarchies, Bäcklund transformations are applied. This choice not only shortens the proof but also provides additional insight into the connections among the hierarchies and their operator solitons.[10]

Bäcklund and reciprocal transformations represent a powerful tool in the investigation of integrable nonlinear differential equations. Indeed, they play a key role both in establishing structural properties enjoyed by a nonlinear system (Hamiltonian and/or bi-Hamiltonian structure, symmetry properties, etc.), as well as in the generation of solutions. Since their introduction by Bäcklund[2,3] such transformations have been applied to a variety of physically meaningful problems, and lately also the link to differential geometry has been exploited[31] (see the monographies[29,31,32] and references therein).

Application of reciprocal transformations concerning interconnection of nonlinear evolution equations has its origin in the work of Rogers and Wong.[33] Since then links among nonlinear evolution equations, as well as among the hierarchies of the related symmetry generators, have been extensively studied.[5,14] Specifically, nonlinear evolution equations connected to each other via Bäcklund charts admit a symmetry, as well as the bi-Hamiltonian structure and recursion operators, which are again related. The link relating the KdV, mKdV, and Harry Dym[22] (HD) equations is analogous to that one which relates the Caudrey-Dodd-Gibbon-Sawada-Kotera (CDGSK), Kaup-Kupershmidt (KK), and Kawamoto equation,[5,16,18,21,30] to mention only work more related to the present one. Here the attention focuses on links between KdV-type equations, but an extension to further related noncommutative equations is suggested.

The proof that the "operator soliton" indeed is a solution of the KdV hierarchy follows directly from the pKdV result. By contrast, the mKdV hierarchy requires subtle use of the Miura transformation. As a preparation, a factorization of the Miura transformation is obtained locally on operator functions contained in the image of the Miura transformation. As an immediate corollary, solutions of some equation in the noncommutative mKdV hierarchy are mapped to solutions of the corresponding equation in the KdV hierarchy. These results on the Miura transformation, presented in Sec. IV, may be of independent interest. Then, connections among the operator solitons are established. Combination of the previous results now leads to a conceptual proof in the mKdV case.

The material is organized as follows. In Sec. II, the pKdV hierarchy in operator form is introduced, then it is proved to admit soliton solutions in Theorem 1. In Sec. III, the KdV and mKdV hierarchies are introduced, and the main results on their operator solitons are stated [see Theorem 10 (KdV case) and Theorem 12 (mKdV case)]. The proof of Theorem 10 is then given. Section IV explores crucial general properties on the noncommutative Miura transformation. Section V explains how operator solitons are connected to each other. In Sec. VI, all the previous results are combined to establish Theorem 12.

II. THE NONCOMMUTATIVE pKdV HIERARCHY AND ITS SOLITON SOLUTION

This section treats solutions of the noncommutative pKdV hierarchy. Theorem 1 provides the formula for the operator soliton, which depends on two operator-valued parameters. The solution property is proved for the pKdV hierarchy by an inductive argument. In Sec. III, corresponding results for the noncommutative KdV and mKdV hierarchies will be deduced by Bäcklund transformations.

Consider the noncommutative pKdV hierarchy, which, in recursion operator formulation, reads as

$$V_{t_{2n-1}} = \Psi(V)^{n-1} V_x \quad (E_{2n-1})$$

with $n \geq 1$, where the recursion operator is defined by

$$\Psi(V) = D^2 + A_{V_x} + D^{-1}A_{V_x}D + D^{-1}C_{V_x}D^{-1}C_{V_x}, \tag{1}$$

and V is a function whose values are bounded linear operators on some Banach space. D denotes the derivative with respect to x, while C_T and A_T denote the commutator and anticommutator, namely,

$$C_T(S) = [T,S], \quad A_T(S) = \{T,S\}.$$

The choice of the recursion operator is inspired by related recursion operators admitted by the KdV and mKdV equations.[13,15,27,19,40] Actually the relation $\Psi(V)=D^{-1}\Phi(U)D$ with $U=DV$ is valid, where Φ is the recursion operator of the KdV hierarchy[27] (see also Sec. III). In particular, this shows that Ψ indeed is a recursion operator in the sense of Ref. 26.

The lowest members $n=1,2,3$ of the hierarchy (E_{2n-1}) read as

$$(E_1) \;\; V_{t_1} = V_x,$$

$$(E_3) \;\; V_{t_3} = V_{xxx} + 3V_x^2,$$

$$(E_5) \;\; V_{t_5} = V_{xxxxx} + 5\{V_x, V_{xxx}\} + 5V_{xx}^2 + 10V_x^3,$$

where, as usual, $t_1=x$, $t_3=t$. In general (E_{2n-1}) explicitly depends on t_1, t_{2n-1}. The equations (E_{2n-1}), $1 \le n \le N$, are regarded as a system of partial differential equations where the unknown V depends on the variables $t_1, t_3, \ldots, t_{2N-1}$. In literature, it is customary to consider $\{(E_{2n-1})\}_{n\ge 1}$ as a system for formal functions in infinitely many variables $t_1, t_3, \ldots$. Since construction of explicit solutions by analytical methods is intended, it is preferable to work with truncated expressions in finitely many variables.

Let E be a Banach space and A, B bounded linear operators on E. Consider the operator-valued function

$$V_N = (I + L_N)^{-1}(AL_N + L_NA), \tag{2}$$

where

$$L_N = L_N(t_1, \ldots, t_{2N-1}) = \exp\left(\sum_{k=1}^{N} A^{2k-1}t_{2k-1}\right)B \tag{3}$$

and I denotes the identity operator on E. It is known[1] that V_2 provides a solution of the noncommutative pKdV, which can be understood as an operator analog of the 1-soliton. A direct proof of the corresponding fact that V_3 solves the noncommutative potential fifth order KdV (Ref. 9) is already very involved.

Here, relying on recursion methods, it is proved that (2), henceforth termed operator soliton, represents a solution of all the equations in the pKdV hierarchy. Specifically, the following result holds.

Theorem 1: *The operator soliton V_N in (2) solves the system of noncommutative pKdV equations $\{(E_{2n-1})\}_{1\le n\le N}$ for any $N \in \mathbb{N}$.*

In the above statement no explicit assumptions on regularity are included at this stage since the proof relies on purely algebraic manipulations with the operators D, ∂_{t_n}, and D^{-1}. In applications of course all these operations need to be defined. As a rule this will be done by verifying that the partial derivatives of sufficient high order are defined in the classical sense and decay sufficiently fast for $x \to -\infty$ (where D^{-1} is realized by $\int_{-\infty}^{x} \cdots dx$). Actually many solutions that will be constructed in the forthcoming paper[7] will belong to some generalized Schwartz space.

The theorem will be proved by induction on n. The essence of the inductive argument is contained in the following proposition.

Proposition 2: *The operator soliton (2) satisfies*

$$V_{t_{n+2}} = \Psi(V) V_{t_n}$$

for all odd numbers $n \in \mathbb{N}$ *with* $n+2 \leq 2N-1$.

To simplify the notation, the dependence of V_N on N has been suppressed. Let $W_n, n \in \mathbb{N}$ denote the following operator functions:

$$W_n = (I+L)^{-1}A^n. \tag{4}$$

The proof relies on various auxiliary lemmas as follows.

Lemma 3: *The operator functions given by (4) satisfy*

$$(W_n)_{t_m} = -W_{n+m} + W_m W_n$$

for all odd numbers $m \in \mathbb{N}$ *with* $m \leq 2N-1$ *and all numbers* $n \in \mathbb{N}$.

Proof: The identity

$$L(I+L)^{-1} = ((I+L)-I)(I+L)^{-1} = I - (I+L)^{-1},$$

allows us to deduce

$$\begin{aligned}(W_n)_{t_m} &= ((I+L)^{-1}A^n)_{t_m} = ((I+L)^{-1})_{t_m}A^n = -(I+L)^{-1}L_{t_m}(I+L)^{-1}A^n \\ &= -(I+L)^{-1}A^m L(I+L)^{-1}A^n = -(I+L)^{-1}A^m(I-(I+L)^{-1})A^n \\ &= -W_m(A^n - W_n) = -W_{n+m} + W_m W_n,\end{aligned}$$

where the last reformulation is obtained upon use of

$$W_n A^m = ((I+L)^{-1}A^n)A^m = (I+L)^{-1}A^{m+n} = W_{m+n}.$$

Next some results involving derivatives of the operator soliton are proved. To start with, the subsequent lemma holds true. □

Lemma 4: *The operator soliton (2) satisfies*

$$V_{t_n} = W_n V,$$

$$V_{t_n t_m} = -W_{n+m}V + \{W_n, W_m\}V$$

for all odd numbers $n, m \in \mathbb{N}$ *with* $n, m \leq 2N-1$.

Proof: Lemma 3, in particular, implies that

$$((I+L)^{-1})_{t_n} = -W_n(I-(I+L)^{-1}).$$

The latter, together with

$$(I+L)^{-1}(AL+LA)_{t_n} = (I+L)^{-1}A^n(AL+LA) = W_n(AL+LA),$$

gives

$$\begin{aligned}V_{t_n} &= ((I+L)^{-1})_{t_n}(AL+LA) + (I+L)^{-1}(AL+LA)_{t_n} = -W_n(I-(I+L)^{-1})(AL+LA) + W_n(AL+LA) \\ &= W_n(I+L)^{-1}(AL+LA) = W_n V.\end{aligned}$$

Taking another derivative and applying Lemma 3, one obtains

$$V_{t_n t_m} = (W_n V)_{t_m} = (W_n)_{t_m}V + W_n(V)_{t_m} = (-W_{m+n} + W_m W_n)V + W_n W_m V = -W_{m+n}V + \{W_m, W_n\}V.$$

□

The first step to prove Proposition 2 is the comparison of the term $V_{t_{n+2}}$ with $D^2 V_{t_n}$ [the term comprising the highest power of D in $\Psi(V)V_{t_n}$].

Lemma 5: *The operator soliton V given in (2) satisfies*

$$V_{t_{n+2}} - D^2 V_{t_n} = 2\{W_1, W_{n+1}\}V + \{W_2, W_n\}V - 2(W_1 W_1 W_n + W_1 W_n W_1 + W_n W_1 W_1)V$$

for all odd numbers $n \in \mathbb{N}$ with $n+2 \le 2N-1$.

Proof: Substitution of $t_1 = x$ in Lemma 4 gives

$$DV_{t_n} = V_{t_n t_1} = -W_{n+1}V + \{W_n, W_1\}V = -V_{t_{n+1}} + \{W_n, W_1\}V.$$

It follows from Lemmas 3 and 4 that

$$\begin{aligned}
D^2 V_{t_n} &= -V_{t_{n+1}t_1} + (\{(W_n)_{t_1}, W_1\}V + \{W_n, (W_1)_{t_1}\}V + \{W_n, W_1\}V_{t_1}) \\
&= V_{t_{n+2}} - \{W_{n+1}, W_1\}V + (\{-W_{n+1} + W_1 W_n, W_1\}V + \{W_n, -W_2 + W_1 W_1\}V + \{W_n, W_1\}W_1 V) \\
&= V_{t_{n+2}} - 2\{W_{n+1}, W_1\}V - \{W_n, W_2\}V + \{W_1 W_n, W_1\}V + \{W_n, W_1 W_1\}V + \{W_n, W_1\}W_1 V \\
&= V_{t_{n+2}} - 2\{W_{n+1}, W_1\}V - \{W_n, W_2\}V + 2(W_1 W_1 W_n + W_1 W_n W_1 + W_n W_1 W_1)V.
\end{aligned}$$

□

The next step is to collect some auxiliary results on products, commutators, and anticommutators.

Lemma 6: *The operator soliton (2) and the operator functions given in (4) satisfy*

$$VW_m = W_{m+1} + AW_m - 2W_1 W_m$$

for all numbers $m \in \mathbb{N}$.

Proof: The crucial identity to deal with anticommutators is once again

$$L(I+L)^{-1} = I - (I+L)^{-1},$$

which yields

$$\begin{aligned}
VW_m &= (I+L)^{-1}(AL + LA)(I+L)^{-1}A^m = (I+L)^{-1}A(L(I+L)^{-1})A^m + ((I+L)^{-1}L)A(I+L)^{-1}A^m \\
&= (I+L)^{-1}A^{m+1} + A(I+L)^{-1}A^m - 2(I+L)^{-1}A(I+L)^{-1}A^m = W_{m+1} + AW_m - 2W_1 W_m.
\end{aligned}$$

□

Lemma 7: *The operator soliton (2) satisfies*

$$\{V_{t_m}, V_{t_n}\} = \{W_m, W_{n+1}\}V + \{W_{m+1}, W_n\}V - 2(W_m W_1 W_n + W_n W_1 W_m)V,$$

$$[V_{t_m}, V_{t_n}] = [W_m, W_{n+1}]V + [W_{m+1}, W_n]V - 2(W_m W_1 W_n - W_n W_1 W_m)V$$

for all odd numbers $n, m \in \mathbb{N}$ with $n, m \le 2N-1$.

Proof: Since $W_m A = W_{m+1}$, Lemma 6, together with Lemma 4, implies that

$$\begin{aligned}
\{V_{t_m}, V_{t_n}\} &= \{W_m V, W_n V\} = W_m(VW_n)V + W_n(VW_m)V \\
&= W_m(W_{n+1} + AW_n - 2W_1 W_n)V + W_n(W_{m+1} + AW_m - 2W_1 W_m)V \\
&= (W_m W_{n+1} + W_{m+1}W_n - 2W_m W_1 W_n)V + (W_n W_{m+1} + W_{n+1}W_m - 2W_n W_1 W_m)V \\
&= \{W_m, W_{n+1}\}V + \{W_{m+1}, W_n\}V - 2(W_m W_1 W_n + W_n W_1 W_m)V,
\end{aligned}$$

and the first identity is proved. The second one follows from analogous computations (modulo obvious sign changes). □

Lemma 8: *The operator soliton (2) satisfies*

$$[V_{t_n}, V_x] = D([W_1, W_n]V)$$

for all odd numbers $n \in \mathbb{N}$ *with* $n \le 2N-1$.

Proof: Observe that a combination of Lemmas 3 and 4 implies that

$$\begin{aligned} D([W_1,W_n]V) &= [(W_1)_x, W_n]V + [W_1,(W_n)_x]V + [W_1,W_n]V_x \\ &= [-W_2 + W_1W_1, W_n]V + [W_1, -W_{n+1} + W_1W_n]V + [W_1,W_n]W_1V \\ &= -[W_1,W_{n+1}]V - [W_2,W_n]V + 2[(W_1)^2, W_n]V. \end{aligned}$$

Then Lemma 7 shows that the latter coincides with $[V_{t_n}, V_x]$. □

Now all the needed tools have been collected to prove Proposition 2.

Proof of Proposition 2: Write $\Psi = \Psi_1 + \Psi_2$, where

$$\Psi_1(V) = D^2 + A_{V_x},$$

$$\Psi_2(V) = D^{-1}(A_{V_x}D + C_{V_x}D^{-1}C_{V_x}).$$

On application of Lemma 7,

$$\{V_x, V_{t_n}\} = \{W_1, W_{n+1}\}V + \{W_2, W_n\}V - 2\{(W_1)^2, W_n\}V.$$

The latter, combined with Lemma 5, allows us to infer

$$\Psi_1(V)V_{t_n} = D^2V_{t_n} + \{V_x, V_{t_n}\} = V_{t_{n+2}} - \{W_1, W_{n+1}\}V + 2W_1W_nW_1V.$$

Hence, it remains to show that $\Psi_2(V)V_{t_n} = \{W_1, W_{n+1}\}V - 2W_1W_nW_1V$. To this end, note that

$$\Psi_2(V)V_{t_n} = D^{-1}(\{V_x, V_{t_n x}\} + [V_x, D^{-1}[V_x, V_{t_n}]]).$$

Lemma 8, again combined with Lemma 4, implies that

$$\begin{aligned} [V_x, D^{-1}[V_x, V_{t_n}]] &= [W_1V, [W_n, W_1]V] = W_1V(W_nW_1 - W_1W_n)V - (W_nW_1 - W_1W_n)VW_1V \\ &= W_1\{V, W_n\}W_1V - \{W_1VW_1, W_n\}V \end{aligned}$$

and

$$\begin{aligned} \{V_x, V_{t_n x}\} &= \{W_1V, -W_{n+1}V + \{W_1, W_n\}V\} \\ &= -\{W_1V, W_{n+1}V\} + W_1V(W_1W_n + W_nW_1)V + (W_1W_n + W_nW_1)VW_1V \\ &= -\{W_1V, W_{n+1}V\} + W_1\{V, W_n\}W_1V + \{W_1VW_1, W_n\}V. \end{aligned}$$

Lemma 6 allows replacing all occurrences of W_nVW_m via the identity

$$W_nVW_m = W_nW_{m+1} + W_{n+1}W_m - 2W_nW_1W_m.$$

With this, a straightforward computation yields

$$\begin{aligned} \{V_x, V_{t_n x}\} + [V_x, D^{-1}[V_x, V_{t_n}]] &= -\{W_1V, W_{n+1}V\} + 2W_1\{V, W_n\}W_1V \\ &= -(W_1VW_{n+1} + W_{n+1}VW_1)V + 2((W_1VW_n)W_1 + W_1(W_nVW_1))V \\ &= (-\{W_1, W_{n+2}\} - \{W_2, W_{n+1}\} + 2\{(W_1)^2, W_{n+1}\})V + 2(2W_1W_{n+1}W_1 \\ &\quad + W_2W_nW_1 + W_1W_nW_2 - 2W_1\{W_1, W_n\}W_1)V \\ &= -\{W_2, W_{n+1}\}V - \{W_1, W_{n+2}\}V + 2((W_1)^2W_{n+1} + 2W_1W_{n+1}W_1 \\ &\quad + W_{n+1}(W_1)^2)V + 2(W_2W_nW_1 + W_1W_nW_2)V - 4((W_1)^2W_nW_1 \\ &\quad + W_1W_n(W_1)^2)V \end{aligned}$$

$$
\begin{aligned}
&= \{-W_2+(W_1)^2, W_{n+1}\}V + \{W_1, -W_{n+2}+W_1W_{n+1}\}V + \{W_1, W_{n+1}\}W_1V \\
&\quad -2((-W_2+(W_1)^2)W_nW_1V + W_1(-W_{n+1}+W_1W_n)W_1V \\
&\quad + W_1W_n(-W_2+(W_1)^2)V + W_1W_nW_1(W_1V)) \\
&= \{D(W_1), W_{n+1}\}V \\
&\quad + \{W_1, D(W_{n+1})\}V + \{W_1, W_{n+1}\}D(V) - 2(D(W_1)W_nW_1V \\
&\quad + W_1D(W_n)W_1V + W_1W_nD(W_1)V + W_1W_nW_1D(V)) \\
&= D(\{W_1, W_{n+1}\}V - 2W_1W_nW_1V).
\end{aligned}
$$

Hence, the identity

$$\{V_x, V_{t_n x}\} + [V_x, D^{-1}[V_x, V_{t_n}]] = D(\{W_1, W_{n+1}\}V - 2W_1W_nW_1V)$$

is verified and the proof of Proposition 2 is complete. □

Now, on use of Proposition 2, a short proof of Theorem 1 is possible.

Proof of Theorem 1: The statement follows immediately via induction. Indeed (E_1) is trivially satisfied. The induction step from n to $n+1$ follows from Proposition 2,

$$V_{t_{2n+1}} = \Psi(V)V_{t_{2n-1}} = \Psi(V)(\Psi(V)^{n-1}V_x) = \Psi(V)^n V_x,$$

completing the proof of Theorem 1. □

A closing result, which will be applied in the sequel, is the following one.

Corollary 9: *The operator-valued function*

$$\hat{V} = -(I-L)^{-1}(AL+LA) \tag{5}$$

solves the noncommutative pKdV hierarchy.

Proof: Note that the only property the function L in (3) must satisfy to prove Theorem 1 is that it solves the base equations,

$$L_{t_n} = A^n L$$

for odd numbers $n \in \mathbb{N}$. Hence, in (2) L can be replaced by $-L$. □

III. NONCOMMUTATIVE KdV AND mKdV HIERARCHIES AND THEIR OPERATOR SOLITONS

In this section the operator solitons of the noncommutative KdV and mKdV hierarchies are stated. In the KdV case the relation $U=V_x$ between pKdV and KdV leads to a quick proof. In contrast, the proof of the result in the mKdV case requires considerable work. The problem is that the Miura transformation between KdV and mKdV is only in one direction given by an explicit formula. However, the argument is a subtle combination of an inductive verification in the spirit of the proof of Theorem 1 and some partial information which can nevertheless be extracted from the Miura transformation. The latter is based on a factorization result for the recursion operator of the KdV, which is only proved locally at those functions contained in the image of the Miura transformation. These results are contained in Sec. IV. In Sec. V it is proved that the operator soliton of the mKdV hierarchy is mapped to the operator soliton of the KdV hierarchy under the Miura transformation (and some similar further relations). Finally combination of all these results allows completing the proof in Sec. VI.

Consider the noncommutative KdV hierarchy,

$$U_{t_{2n-1}} = \Phi(U)^{n-1}U_x, \quad n \geq 1$$

generated by the recursion operator

$$\Phi(U) = D^2 + 2A_U + A_{U_x}D^{-1} + C_U D^{-1} C_U D^{-1}, \tag{6}$$

see Ref. 27, Eq. (6.23). For example, the case $n=3$ yields the noncommutative KdV equation,

$$U_t = U_{xxx} + 3\{U, U_x\}.$$

Actually (6) is related to the recursion operator (1) of the noncommutative pKdV hierarchy via the simple link $D\Psi(V)=\Phi(U)D$, where $U=DV$. This allows the derivation of the operator soliton simultaneously for all equations of the noncommutative KdV hierarchy. In view of later analytic applications, the result is again formulated for truncated systems of N equations. Recall also the remarks following Theorem 1.

Theorem 10: *The operator soliton*

$$U_N = (I + L_N)^{-1} A (I + L_N)^{-1} (AL_N + L_N A), \tag{7}$$

where $L_N = L_N(t_1, \ldots, t_{2N-1})$ is given by (3), solves the system of noncommutative KdV equations $U_{t_{2n-1}} = \Phi(U)^{n-1} U_x$, $1 \le n \le N$, for any $N \in \mathbb{N}$.

Proof: The statement follows directly from

$$U_{t_{2n-1}} = DV_{t_{2n-1}} = D\Psi(V)^{n-1}V_x = D\Psi(D^{-1}U)^{n-1}D^{-1}U_x = (D\Psi(D^{-1}U)D^{-1})^{n-1}U_x = \Phi(U)^{n-1}U_x.$$

□

Combined with Corollary 9, Theorem 10 immediately yields the following.

Corollary 11: *The operator function*

$$\hat{U}_N = -(I - L_N)^{-1} A (I - L_N)^{-1} (AL_N + L_N A) \tag{8}$$

also solves the system $U_{t_{2n-1}} = \Phi(U)^{n-1} U_x$, $1 \le n \le N$, for any $N \in \mathbb{N}$.

Consider finally the mKdV hierarchy

$$\tilde{U}_{t_{2n-1}} = \tilde{\Psi}(\tilde{U})^{n-1} \tilde{U}_x, n \ge 1,$$

generated by the recursion operator

$$\tilde{\Psi}(\tilde{U}) = (D + C_{\tilde{U}} D^{-1} C_{\tilde{U}})(D + A_{\tilde{U}} D^{-1} A_{\tilde{U}}), \tag{9}$$

see Ref. 19. When $n=3$, the noncommutative mKdV equation is obtained, namely,

$$\tilde{U}_t = \tilde{U}_{xxx} + 3\{\tilde{U}^2, \tilde{U}_x\}.$$

Theorem 12: *The operator soliton*

$$\tilde{U}_N = -i(I - (L_N)^2)^{-1} (AL_N + L_N A), \tag{10}$$

where $L_N = L_N(t_1, \ldots, t_{2N+1})$ is given by (3), solves the noncommutative system of mKdV equations $\tilde{U}_{t_{2n-1}} = \tilde{\Psi}(\tilde{U})^{n-1} \tilde{U}_x$, $1 \le n \le N$, for any $N \in \mathbb{N}$.

IV. A NONCOMMUTATIVE MIURA TRANSFORMATION

The natural analog of the Miura transformation M, in the noncommutative setting, is

$$M(\tilde{U}) = \tilde{U}^2 + i\tilde{U}_x. \tag{11}$$

In Ref. 36, see also Ref. 9, it is verified that M maps solutions of the noncommutative mKdV equation to solutions of the noncommutative KdV equation. The main goal of this section is to extend this fact to solutions of the corresponding hierarchies. The following property of the Miura transformation will be used.

Lemma 13: *The Miura transformation satisfies the identity*

$$(M(\widetilde{U}))_{t_n} = (A_{\widetilde{U}} + iD)\widetilde{U}_{t_n}.$$

To establish that the Miura transformation preserves the solution property, a crucial observation is that the recursion operator $\Phi(U)$ of the KdV hierarchy can be explicitly factorized if the argument U lies in the image of the Miura transformation, specifically as follows.

Proposition 14: *If* U, $\widetilde{U}$ *are operator functions related by the Miura transform, i.e.,* $U = M(\widetilde{U})$*, then*

$$\Phi(U)D = (D - iA_{\widetilde{U}})(D + C_{\widetilde{U}}D^{-1}C_{\widetilde{U}})(D + iA_{\widetilde{U}}).$$

In operator theoretic terminology, the content of Proposition 14 is that Φ and $\widetilde{\Psi}$ are related operators. Recall that two operators S and T are said to be related if there are operators A and B such that $S=AB$ and $T=BA$. See Ref. 28 concerning results on common structural properties of related operators.

Proof: The argument uses several auxiliary identities concerning commutators and anticommutators with arbitrary operator functions S, T, to start with

$$DA_T = A_T D + A_{T_x}, \tag{12}$$

$$D^2 A_T = A_T D^2 + 2A_{T_x} D + A_{T_{xx}}. \tag{13}$$

Next the identities

$$DC_T = C_T D + C_{T_x} \tag{14}$$

and $A_T C_T = C_T A_T = C_{T^2}$ yield

$$DC_T - iA_T C_T = C_T D - iC_{M(T)}, \tag{15}$$

$$C_T D + iC_T A_T = DC_T + iC_{M(T)}. \tag{16}$$

Now, from $A_T A_S = (R_{ST} + L_{TS}) + (L_T R_S + L_S R_T)$ and $C_S C_T = (R_{TS} + L_{ST}) - (L_T R_S + L_S R_T)$, where L_T and R_T denote the left and right multiplication with T, respectively,

$$A_T A_S + C_S C_T = R_{\{S,T\}} + L_{\{S,T\}} = A_{\{T,S\}}. \tag{17}$$

Note finally that

$$[C_{M(T)}, C_T] = C_{[M(T),T]} = C_{[T^2 + iT_x, T]} = iC_{[T_x,T]} = i[C_{T_x}, C_T] = -i[C_T, C_{T_x}]. \tag{18}$$

In the next calculation the identities indicated are used to replace the expressions within the boxes,

$$
\begin{aligned}
&(D - iA_{\tilde{U}})(D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})(D + iA_{\tilde{U}}) \\
&\quad = D^3 - iA_{\tilde{U}}D^2 + i\boxed{D^2A_{\tilde{U}}} + A_{\tilde{U}}\boxed{DA_{\tilde{U}}} + (D - iA_{\tilde{U}})C_{\tilde{U}}D^{-1}C_{\tilde{U}}(D + iA_{\tilde{U}}) \\
&\quad \overset{(12),(13)}{=} D^3 - iA_{\tilde{U}}D^2 + i(A_{\tilde{U}}D^2 + 2A_{\tilde{U}_x}D + A_{\tilde{U}_{xx}}) + A_{\tilde{U}}(A_{\tilde{U}}D + A_{\tilde{U}_x}) \\
&\qquad + (\boxed{DC_{\tilde{U}} - iA_{\tilde{U}}C_{\tilde{U}}})D^{-1}(\boxed{C_{\tilde{U}}D + iC_{\tilde{U}}A_{\tilde{U}}}) \\
&\quad \overset{(15),(16)}{=} D^3 + (2iA_{\tilde{U}_x} + \boxed{A_{\tilde{U}}A_{\tilde{U}}})D + (iA_{\tilde{U}_{xx}} + \boxed{A_{\tilde{U}}A_{\tilde{U}_x}}) + (C_{\tilde{U}}D - iC_U)D^{-1}(DC_{\tilde{U}} + iC_U) \\
&\quad \overset{(17)}{=} D^3 + (2iA_{\tilde{U}_x} + (2A_{\tilde{U}^2} - C_{\tilde{U}}C_{\tilde{U}}))D + (iA_{\tilde{U}_{xx}} + (A_{\{\tilde{U},\tilde{U}_x\}} - C_{\tilde{U}_x}C_{\tilde{U}})) \\
&\qquad + (C_{\tilde{U}}D - iC_U)D^{-1}(DC_{\tilde{U}} + iC_U) \\
&\quad = D^3 + 2A_{\tilde{U}^2 + i\tilde{U}_x}D + A_{\{\tilde{U},\tilde{U}_x\} + i\tilde{U}_{xx}} - C_{\tilde{U}}C_{\tilde{U}}D - C_{\tilde{U}_x}C_{\tilde{U}} + (C_{\tilde{U}}DC_{\tilde{U}} + iC_{\tilde{U}}C_U - iC_UC_{\tilde{U}} + C_UD^{-1}C_U) \\
&\quad = D^3 + 2A_{\tilde{U}^2 + i\tilde{U}_x}D + A_{\{\tilde{U},\tilde{U}_x\} + i\tilde{U}_{xx}} + C_UD^{-1}C_U - i[C_U, C_{\tilde{U}}] - (C_{\tilde{U}}(\boxed{C_{\tilde{U}}D - DC_{\tilde{U}}}) + C_{\tilde{U}_x}C_{\tilde{U}}) \\
&\quad \overset{(14)}{=} D^3 + 2A_{\tilde{U}^2 + i\tilde{U}_x}D + A_{(\tilde{U}^2 + i\tilde{U}_x)_x} + C_UD^{-1}C_U - i[C_U, C_{\tilde{U}}] + [C_{\tilde{U}}, C_{\tilde{U}_x}] \\
&\quad \overset{(18)}{=} D^3 + 2A_UD + A_{U_x} + C_UD^{-1}C_U = \Phi(U)D.
\end{aligned}
$$

□

Changing the sign in the Miura transformation, a similar factorization result can be obtained by analogous arguments as follows.

Corollary 15: *If $\hat{U}$, $\tilde{U}$ are operator functions related by $\hat{U} = \tilde{U}^2 - i\tilde{U}_x$, then*

$$\Phi(\hat{U})D = (D + iA_{\tilde{U}})(D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})(D - iA_{\tilde{U}}).$$

Proposition 14 allows to prove that, for the whole hierarchies, the solution property is preserved under the Miura transform.

Proposition 16: *The Miura transform M maps the solutions $\tilde{U}$ of some equation of the noncommutative mKdV hierarchy to solutions $M(\tilde{U})$ of the corresponding equation of the noncommutative KdV hierarchy.*

The main concern of this paper lies in solutions to whole hierarchies. However, note that the proposition even relates solutions of corresponding member equations in the two hierarchies.

Proof: On use of Proposition 14, observe that

$$
\begin{aligned}
(D - iA_{\tilde{U}})\tilde{\Psi}(\tilde{U}) &= (D - iA_{\tilde{U}})(D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})(D + A_{\tilde{U}}D^{-1}A_{\tilde{U}}) \\
&= ((D - iA_{\tilde{U}})(D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})(D + iA_{\tilde{U}}))D^{-1}(D - iA_{\tilde{U}}) \\
&= (\Phi(M(\tilde{U}))D)D^{-1}(D - iA_{\tilde{U}}) = \Phi(M(\tilde{U}))(D - iA_{\tilde{U}}).
\end{aligned}
$$

Thus an induction argument yields

$$(D - iA_{\tilde{U}})\tilde{\Psi}(\tilde{U})^n = \Phi(M(\tilde{U}))^n(D - iA_{\tilde{U}})$$

or

$$(A_{\tilde{U}} + iD)\tilde{\Psi}(\tilde{U})^n = \Phi(M(\tilde{U}))^n(A_{\tilde{U}} + iD).$$

Therefore,

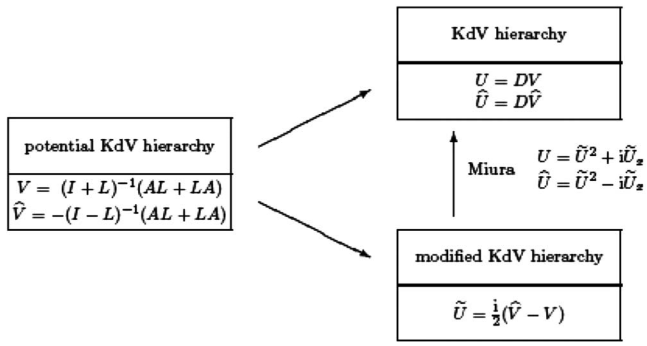

FIG. 1. Noncommutative solitons and their links.

$$\begin{aligned}M(\tilde{U})_{t_{2n+1}} &= (\tilde{U}^2 + i\tilde{U}_x)_{t_{2n+1}} = \{\tilde{U}, \tilde{U}_{t_{2n+1}}\} + iD\tilde{U}_{t_{2n+1}} = (A_{\tilde{U}} + iD)\tilde{U}_{t_{2n+1}} = (A_{\tilde{U}} + iD)(\tilde{\Psi}(\tilde{U})^n\tilde{U}_x)\\ &= ((A_{\tilde{U}} + iD)\tilde{\Psi}(\tilde{U})^n)\tilde{U}_x = (\Phi(M(\tilde{U}))^n(A_{\tilde{U}} + iD))\tilde{U}_x = \Phi(M(\tilde{U}))^n((A_{\tilde{U}} + iD)\tilde{U}_x)\\ &= \Phi(M(\tilde{U}))^n M(\tilde{U})_x,\end{aligned}$$

the latter on application of Lemma 13. □

V. BÄCKLUND LINKS BETWEEN THE OPERATOR SOLITONS

So far it has been examined how solutions of the considered hierarchies are related via Bäcklund transformations. However, it has not been discussed how the operator solitons behave under these transformations. The following diagram (see Fig. 1) resumes what is proved in this section.

The upper arrow in Fig. 1 is evident. The crucial link represented by the lower arrow is the content of the following.

Lemma 17: *The operator functions $\tilde{U}$ given in (10) and V and $\hat{V}$ given in* (2) and (5) *are related via*

$$\tilde{U} = \frac{i}{2}(\hat{V} - V).$$

Proof: The assertion follows immediately from

$$\begin{aligned}(I-L)^{-1} + (I+L)^{-1} &= (I-L)^{-1}(I + (I-L)(I+L)^{-1}) = (I-L)^{-1}((I+L) + (I-L))(I+L)^{-1}\\ &= 2(I-L^2)^{-1}.\end{aligned}$$

□

A consequence of the above relation is the Miura link between the solitons of the noncommutative KdV and mKdV hierarchies (see Ref. 36 in the case of the noncommutative KdV and mKdV equations), which justifies the vertical arrow.

Corollary 18: *The operator functions $\tilde{U}$ given in (10) and U and $\hat{U}$ given in (7) and (8) are related via*

$$\tilde{U}^2 - i\tilde{U}_x = \hat{U},$$

$$\tilde{U}^2 + i\tilde{U}_x = U.$$

Proof: Recall the abbreviation $W_n = (I+L)^{-1}A^n$, $n \in \mathbb{N}_0$, and set in addition $\hat{W}_n = (I-L)^{-1}A^n$. Note that W_n and $\hat{W}_n$ commute.

Then, by Lemmas 4 and 17,

$$\widetilde{U}_x = \frac{i}{2}(\hat{V} - V)_x = \frac{i}{2}(\hat{W}_1\hat{V} - W_1 V) = \frac{i}{2}(\hat{U} - U). \tag{19}$$

On the other hand, since $(I-L^2)^{-1} = \frac{1}{2}(\hat{W}_0 + W_0) = W_0\hat{W}_0$ and

$$W_0(AL + LA)\hat{W}_0 = (I + L)^{-1}(-A(I - L) + (I + L)A)(I - L)^{-1} = -(I + L)^{-1}A + A(I - L)^{-1}$$
$$= -W_0 A + A\hat{W}_0,$$

$$\hat{W}_0(AL + LA)W_0 = \hat{W}_0 A - AW_0,$$

it follows that

$$\begin{aligned}\widetilde{U}^2 &= -W_0\hat{W}_0(AL + LA)\tfrac{1}{2}(\hat{W}_0 + W_0)(AL + LA)\\ &= -\tfrac{1}{2}(\hat{W}_0 W_0(AL + LA)\hat{W}_0 + W_0\hat{W}_0(AL + LA)W_0)(AL + LA)\\ &= -\tfrac{1}{2}(\hat{W}_0(A\hat{W}_0 - W_0 A) + W_0(\hat{W}_0 A - AW_0))(AL + LA)\\ &= \tfrac{1}{2}(W_0 A W_0 - \hat{W}_0 A\hat{W}_0)(AL + LA) = \tfrac{1}{2}(\hat{U} + U).\end{aligned} \tag{20}$$

In summary, (19) and (20) show that $\widetilde{U}^2 - i\widetilde{U}_x = \hat{U}$ and $\widetilde{U}^2 + i\widetilde{U}_x = U$. □

VI. SOLVING THE mKdV HIERARCHY

Here results of Secs. IV and V are combined to prove Theorem 12. The argument is inductive as in the proof of Theorem 1. However, the induction step is very different in spirit. It is based both on the link between the operator solitons and the factorization of the recursion operator generating the noncommutative KdV hierarchy.

Proof of Theorem 12: Since the statement is trivial in the case $n=1$, it remains to provide the induction step $\widetilde{U}_{t_{n+2}} = \widetilde{\Psi}(\widetilde{U})\widetilde{U}_{t_n}$. To this end, the link between the operator solitons given in Lemma 17 is crucial. Namely, since

$$\widetilde{\Psi}(\widetilde{U})\widetilde{U}_{t_n} = \frac{i}{2}(\widetilde{\Psi}(\widetilde{U})\hat{V}_{t_n} - \widetilde{\Psi}(\widetilde{U})V_{t_n}),$$

and, by Proposition 2

$$\widetilde{U}_{t_{n+2}} = \frac{i}{2}(\hat{V}_{t_{n+2}} - V_{t_{n+2}}) = \frac{i}{2}(\Psi(\hat{V})\hat{V}_{t_n} - \Psi(V)V_{t_n}),$$

it is sufficient to prove

$$(\widetilde{\Psi}(\widetilde{U}) - \Psi(\hat{V}))\hat{V}_{t_n} = (\widetilde{\Psi}(\widetilde{U}) - \Psi(V))V_{t_n}. \tag{21}$$

The recursion operators, in turn, satisfy

$$\widetilde{\Psi}(\widetilde{U}) = (D + C_{\widetilde{U}}D^{-1}C_{\widetilde{U}})(D + A_{\widetilde{U}}D^{-1}A_{\widetilde{U}}) = (D + C_{\widetilde{U}}D^{-1}C_{\widetilde{U}})(D - iA_{\widetilde{U}})D^{-1}(D + iA_{\widetilde{U}})$$

and, by Proposition 14

$$\Psi(V) = D^{-1}\Phi(U)D = D^{-1}(D - iA_{\widetilde{U}})(D + C_{\widetilde{U}}D^{-1}C_{\widetilde{U}})(D + iA_{\widetilde{U}}).$$

Hence,

$$\tilde{\Psi}(\tilde{U}) - \Psi(V) = \Upsilon(\tilde{U})(D + iA_{\tilde{U}}),$$

where

$$\Upsilon(\tilde{U}) = (D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})(D - iA_{\tilde{U}})D^{-1} - D^{-1}(D - iA_{\tilde{U}})(D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})$$
$$= i(D^{-1}A_{\tilde{U}}(D + C_{\tilde{U}}D^{-1}C_{\tilde{U}}) - (D + C_{\tilde{U}}D^{-1}C_{\tilde{U}})A_{\tilde{U}}D^{-1}).$$

Analogously, the factorization $D - A_{\tilde{U}}D^{-1}A_{\tilde{U}} = (D + iA_{\tilde{U}})D^{-1}(D - iA_{\tilde{U}})$ of the recursion operator $\tilde{\Psi}(\tilde{U})$ together with Corollary 15 for $\Psi(\hat{V}) = D^{-1}\Phi(\hat{U})D$ leads to

$$\tilde{\Psi}(\tilde{U}) - \Psi(\hat{V}) = -\Upsilon(\tilde{U})(D - iA_{\tilde{U}}).$$

In summary,

$$(\tilde{\Psi}(\tilde{U}) - \Psi(V))V_{t_n} - (\tilde{\Psi}(\tilde{U}) - \Psi(\hat{V}))\hat{V}_{t_n} = \Upsilon(\tilde{U})((D + iA_{\tilde{U}})V_{t_n} + (D - iA_{\tilde{U}})\hat{V}_{t_n})$$
$$= \Upsilon(\tilde{U})(D(V + \hat{V})_{t_n} + i\{\tilde{U},(V - \hat{V})_{t_n}\})$$
$$= 2\Upsilon(\tilde{U})((\tilde{U}^2)_{t_n} - \{\tilde{U},\tilde{U}_{t_n}\}) = 0,$$

where the operator links $V - \hat{V} = 2i\tilde{U}$ and $D(\hat{V} + V) = \hat{U} + U = 2\tilde{U}^2$ have been used (see Lemma 17 and Corollary 18). Hence (21) has been verified, and the proof of Theorem 12 is complete. □

VII. CONCLUSIONS AND PERSPECTIVES

The results presented in this work show that Bäcklund transformations are a powerful tool for investigation also in the noncommutative setting. Indeed, the general idea behind this investigation is to show that the noncommutative pKdV, KdV, and mKdV hierarchies are connected via Bäcklund transformations. They link corresponding members in the hierarchies and, in addition, their recursion operators are also related to each other; finally, solutions too are mapped into solutions.

Hence, comparison between the Bäcklund chart here obtained and the wider one comprised in Ref. 6 suggests that further noncommutative nonlinear hierarchies whose base member is a third order evolution equation can be introduced and related to the hierarchies here considered. This aspect is currently under investigation together with possible generalizations to noncommutative hierarchies whose base member is a fifth order equation.

As mentioned in Sec. I, the operator theoretic strategy followed in this article suggests an application to the solution theory of matrix-valued integrable systems via projection methods. The reader may refer to Ref. 9 for a survey on known applications to the solution theory of classical integrable systems. The forthcoming article[7] will deal with applications of the above results about hierarchies, with a certain emphasis on matrix integrable systems.

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