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REPRESENTATION THEORY OF WREATH PRODUCTS OF FINITE GROUPS

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Abstract. This is an introduction to the representation theory of wreath products of finite groups. We also discuss in full details a couple of examples.

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1. Introduction

This paper is an introduction to the representation theory of wreath products. Our exposition is inspired to the book by James and Kerber [9]. However, our approach is more analytical and in particular, we interpret the exponentiation and the composition actions in terms of actions on suitable trees. This is done in Section 2, while in Section 3 we use the little group method [5, Theorem 5.2] to determine a complete list of irreducible representations of wreath products. This procedure does not give immediately a concrete description of the matrix coefficients whose determination requires a detailed analysis of the conjugacy classes. We end Section 3 by analyzing the composition action of the wreath product on two permutation representations. In particular, we generalize Theorem 4.2 in [1] in the case the permutation representations are not multiplicity free.

In Section 4 we analyze two particular examples of groups of the form $C_2 \wr G$ where G is the automorphism group of a finite graph X. Then, $C_2 \wr G$ is the automorphism group of the associated lamplighter graph. We give an explicit list of all irreducible representations in the case $G = C_n$, the cyclic group of n elements (which is the automorphism group of the discrete circle), and in the case $G = S_n$ (which is the automorphism group of the complete graph of n vertices).

2. Wreath Products of Finite Groups

Let G be a finite group acting transitively on a set X and let F be another finite group. Denote by F^X the set of all maps $f : X \to F$. Set

$$F \wr G = \{(f, g) : f \in F^X,\ g \in G\} \equiv F^X \times G.$$

The group G acts on F^X in a natural way, by setting $(gf)(x) = f(g^{-1}x)$, for any $g \in G$, $f \in F^X$ and $x \in X$. Moreover, F^X is a group under pointwise multiplication: $(ff')(x) = f(x) \cdot f'(x)$, and $g(ff') = gf \cdot gf'$, $(gf)^{-1} = gf^{-1}$, that is G acts on F^X as a group of automorphisms. Then in $F \wr G$ we can define a multiplication law by setting:

$$(f, g)(f', g') = (f \cdot gf', gg') \tag{2.1}$$

for all $(f, g), (f', g') \in F \wr G$. Clearly, $(f \cdot gf')(x) = f(x)f'(g^{-1}x)$, for all $x \in X$.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 50, Functional Analysis, 2007.

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Lemma 2.1. *The set $F \wr G$ with the multiplication law* (2.1) *is a group. Moreover, the identity element is $(\mathbf{1}_F, 1_G)$, where $\mathbf{1}_F(x) = 1_F$ for all $x \in X$, and the inverse of (f,g) is given by $(g^{-1}f^{-1}, g^{-1})$.*

Proof. It easy to show that $(\mathbf{1}_F, 1_G)$ is the identity. Moreover,

$$(g^{-1}f^{-1}, g^{-1})(f,g) = (g^{-1}f^{-1} \cdot g^{-1}f, 1_G) = (g^{-1}(f^{-1}f), 1_G) = (\mathbf{1}_F, 1_G) = (f,g)(g^{-1}f^{-1}, g^{-1}),$$

and therefore $(g^{-1}f^{-1}, g^{-1})$ is the inverse of (f,g). Finally, if $(f,g), (f',g'), (f'',g'') \in F \wr G$ then

$$[(f,g)(f',g')](f'',g'') = (f \cdot gf' \cdot gg'f'', gg'g'') = (f,g)[(f',g')(f'',g'')]$$

simply because $g(f' \cdot g'f'') = gf' \cdot gg'f''$. □

The group $F \wr G$ is called the *wreath product* of F by (the permutation group) G. The subgroup

$$\{(f, 1_G) : f : X \to F\} \cong F^X$$

is called the *base group*. We will identify it with F^X. It is easy to show that the base group is normal in the wreath product. Moreover, if we identify $\{(\mathbf{1}_F, g) : g \in G\}$ with G then the wreath product may be written as a semidirect product: $F \wr G = F^X \rtimes G$. The diagonal subgroup of the base group F^X is: $\operatorname{diag} F^X = \{f \in F^X : f \text{ is constant on } X\} \cong F$. Clearly, $\operatorname{diag} F^X \cdot G$, as a subgroup of $F \wr G$, is isomorphic to the direct product $F \times G$.

Suppose that F acts transitively on a finite set Y. Now we define a natural action of $F \wr G$ on the product space $X \times Y$.

Lemma 2.2. *For $(f,g) \in F \wr G$ and $(x,y) \in X \times Y$, set*

$$(f,g)(x,y) = (gx, f(gx)y) \equiv \big(gx, [(g^{-1}f)(x)]y\big) \tag{2.2}$$

Then (2.2) *defines a transitive action of $F \wr G$ on $X \times Y$.*

Proof. Clearly, $(\mathbf{1}_F, 1_G)(x,y) = (x,y)$ for any $(x,y) \in X \times Y$. Moreover, if $(f,g), (f',g') \in F \wr G$ and $(x,y) \in X \times Y$ then

$$\begin{aligned}[(f,g)(f',g')](x,y) &= (f \cdot gf', gg')(x,y) = \big(gg'x, \{[(gg')^{-1}f \cdot g'^{-1}f'](x)\}y\big) \\ &= \big(gg'x, [(g^{-1}f)(g'x)]\{[(g'^{-1}f')(x)]y\}\big) = (f,g)(g'x, f'(g'x)y) = (f,g)[(f',g')(x,y)]\end{aligned}$$

and therefore (2.2) is an action. It is obvious that it is transitive. □

The action defined in (2.2) is called the *composition* of the actions of G on X and F on Y. The composition action restricted to $\operatorname{diag} F^X \cdot G$ coincides with the product action of $G \times F$ on $X \times Y$.

The theory of wreath products becomes more transparent if we think of them as groups acting on finite trees. The tree of $X \times Y$ is the finite, two levels rooted tree obtained by taking the empty set $\emptyset$ as the root, X as the first level and then attaching to each $x \in X$ a copy of Y. More precisely, the vertex set is $V = \{\emptyset\} \coprod X \coprod (X \times Y)$ and the edge set is $E = \{\{\emptyset, x\} : x \in X\} \coprod \{\{x, (x,y)\} : x \in X,\ y \in Y\}$.

Then $F \wr G$ acts on the tree (V,E) as a group of isometries: if $(f,g) \in F \wr G$ then (f,g) fixes $\emptyset$, sends $x \in X$ to gx and sends $(x,y) \in X \times Y$ to $(gx, f(gx)y)$ (composition action). In other words, G permutes the first level and, if we fix $g \in G$ and $x \in X$, the group $\{f(gx) : f \in F^X\} \cong F$ permutes $\{(gx,y) : y \in Y\} \cong Y$.

Now let Y^X be the set of all maps $\varphi : X \to Y$. We can also define a natural action of $F \wr G$ on Y^X.

Lemma 2.3. *For $(f,g) \in F \wr G$, $\varphi \in Y^X$ and $x \in X$, set*

$$[(f,g)\varphi](x) = f(x)\varphi(g^{-1}x). \tag{2.3}$$

Then (2.3) *is a transitive action of $F \wr G$ on Y^X.*

Proof. Clearly, $(\mathbf{1}_F, 1_G)\varphi = \varphi$. Moreover, if $(f,g), (f',g') \in F \wr G$, $\varphi \in Y^X$ and $x \in X$, we have

$$\begin{aligned}\big\{[(f,g)(f',g')\varphi]\big\}(x) &= [(f \cdot gf', gg')\varphi](x) = [f(x)f'(g^{-1}x)]\varphi(g'^{-1}g^{-1}x) \\ &= f(x)[f'(g^{-1}x)\varphi(g'^{-1}g^{-1}x)] = f(x)\big\{[(f',g')\varphi](g^{-1}x)\big\} = \big\{(f,g)[(f',g')\varphi]\big\}(x)\end{aligned}$$

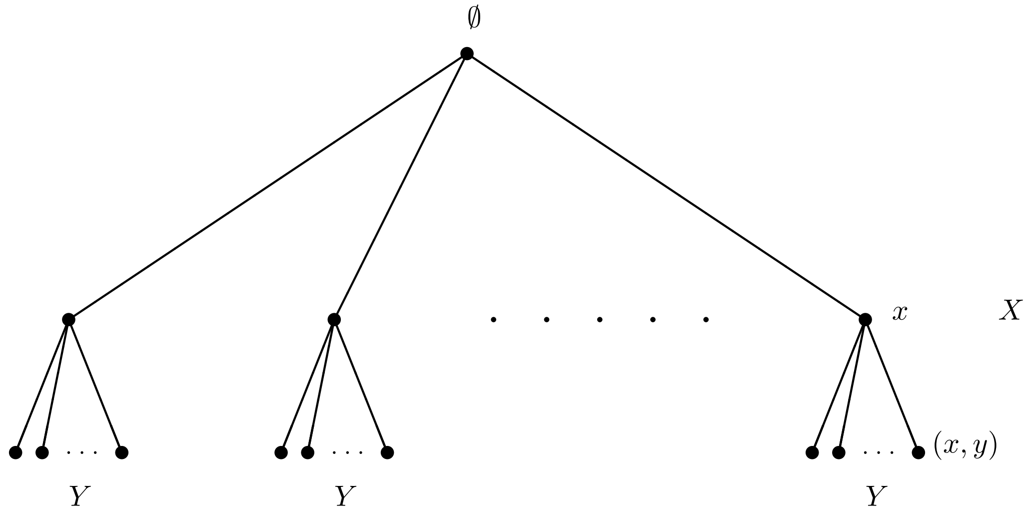

Fig. 1. The tree of $X \times Y$ is obtained by attaching to each $x \in X$ a copy of Y.

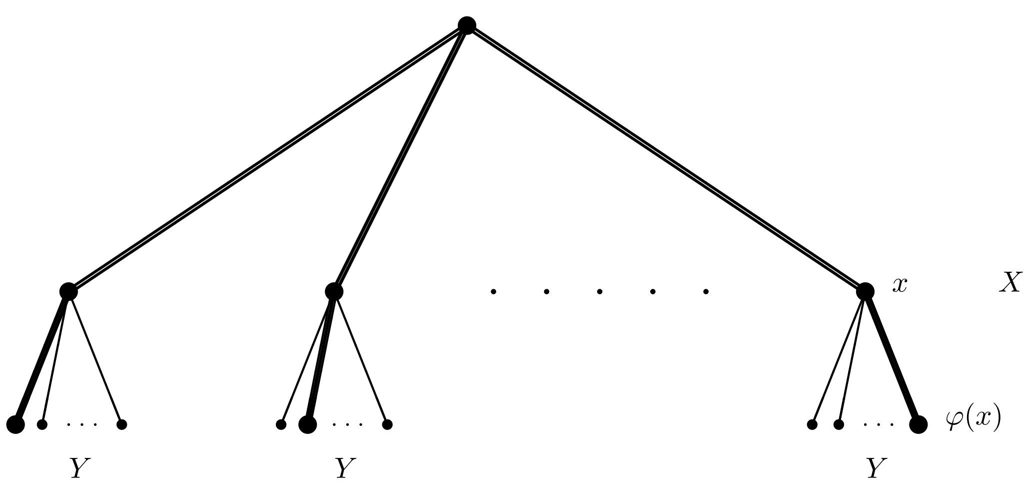

Fig. 2. $\varphi \in Y^X$ may be seen as a subtree of the tree of $X \times Y$.

and therefore (2.3) is an action. It is obvious that it is also transitive (the action of the subgroup F^X is transitive). □

The action defined in (2.3) is called the *exponentiation* of the action of F by the action of G. Its restriction to diag $F^X \cdot G$ is called the *power* of F by G.

Consider again the tree of $X \times Y$. We may identify any $\varphi \in Y^X$ with the subtree $\emptyset \coprod X \coprod \{(x, \varphi(x)) : x \in X)\}$. That is, Y^X may be seen as the family of all subtrees obtained by taking the root $\emptyset$, all the first level X and, for any $x \in X$, exactly one vertex $(x, y) \in X \times Y$. Then the action of $F \wr G$ on this family of subtrees (induced by the composition action) coincides exactly with the exponentiation action.

Now let H be a third group. We can form the wreath products $H \wr F \equiv H^Y \times F$ and then $(H \wr F) \wr G \equiv (H \wr F)^X \times G$. Alternatively, if we see $F \wr G$ as a group acting on $X \times Y$ by the composition action, we can form the wreath product $H \wr (F \wr G) \equiv H^{X \times Y} \times (F \wr G)$. Now we show that both constructions lead to the same result.

Theorem 2.1 (Associativity of the wreath product)**.** *The map*

$$\begin{array}{rccc} \Psi: & H \wr (F \wr G) & \to & (H \wr F) \wr G, \\ & (h, f, g) & \mapsto & (\vartheta, g) \end{array}$$

where $\vartheta : X \to H \wr F$ is defined by setting $\vartheta(x) = (h(x, \cdot), f(x))$ for any $x \in X$, is a group isomorphism.

Proof. Clearly, Ψ is a bijection. Take $(h, f, g), (h', f', g') \in H \wr (F \wr G)$. Their product is

$$(h, f, g)(h', f', g') = (h \cdot (f, g)h', f \cdot gf', gg'), \quad \text{where} \quad [h \cdot (f, g)h'](x, y) = h(x, y)h'(g^{-1}x, f(x)^{-1}y).$$

Then

$$\Psi((h, f, g)(h', f', g')) = (\vartheta'', gg'), \quad \text{where} \quad \vartheta''(x) = (h(x, \cdot)h'(g^{-1}x, f(x)^{-1}\cdot), f(x)f'(g^{-1}x)).$$

On the other hand, if $(\vartheta, g) = \Psi(h, f, g)$ and $(\vartheta', g') = \Psi(h', f', g')$, then $(\vartheta, g)(\vartheta', g') = (\vartheta \cdot g\vartheta', gg')$, and

$$\begin{aligned} (\vartheta \cdot g\vartheta')(x) &= \vartheta(x)\vartheta'(g^{-1}x) = (h(x, \cdot), f(x))(h'(g^{-1}x, \cdot), f'(g^{-1}x)) \\ &= (h(x, \cdot)h'(g^{-1}x, f(x)^{-1}\cdot), f(x)f'(g^{-1}x)), \end{aligned}$$

that is $\vartheta \cdot g\vartheta' = \vartheta''$. □

Then we can write simply $H \wr F \wr G$. More generally, suppose that $G_1, G_2, \ldots, G_m$ are finite groups, with G_i acting on the sets X_i, $i = 1, 2 \ldots, m-1$. Then the *iterated wreath product* $G_m \wr G_{m-1} \wr \ldots G_1$ is the set of all m-tuple $(f_m, f_{m-1}, \ldots, f_2, f_1)$ where $f_1 \in G_1$ and $f_k : X_1 \times \cdots \times X_{k-1} \to G_k$, $k = 2, 3, \ldots, m$, with the multiplication law

$$\begin{aligned} &(f_m, f_{m-1}, \ldots, f_2, f_1)(f'_m, f''_{m-1}, \ldots, f'_2, f'_1) = \\ &\qquad \big(f_m \cdot (f_{m-1}, \ldots, f_2, f_1)f'_m, f_{m-1} \cdot (f_{m-2}, \ldots, f_2, f_1)f'_{m-1}, \ldots, f_2 \cdot f_1 f'_2, f_1 f'_1\big) \end{aligned}$$

where

$$(f_k, f_{k-1}, \ldots, f_2, f_1)(x_1, x_2, \ldots, x_k) = \big(f_1 x_1, f_2(f_1 x_1)x_2, \ldots, f_k(f_1 x_{k-1})x_k\big),$$

for all $(x_1, x_2, \ldots, x_k) \in X_1 \times X_2 \times \cdots \times X_k$, $k = 0, 1, 2 \ldots, m$.

3. Representation Theory of Wreath Products

Let G, X, F be as in the previous sections. We want to describe the irreducible representations of the wreath product $F \wr G$. In virtue of Theorem 9.1.6 and of Corollary 9.1.7 in [3], every irreducible representation of the base group F^X is of the form

$$\bigotimes_{x \in X} \sigma_x$$

where

$$\begin{array}{ccc} X & \to & \widehat{F} \\ x & \mapsto & \sigma_x \end{array}$$

is any map from X to $\widehat{F}$, the dual of F. In other words, if $f_0 \in F^X$ then

$$\left(\bigotimes_{x \in X} \sigma_x\right)(f_0, 1_G) = \bigotimes_{x \in X} \sigma_x(f_0(x))$$

and if $\bigotimes_{x \in X} v_x \in \bigotimes_{x \in X} V_{\sigma_x}$, with V_{σ_x} the space on which acts the representation σ_x, then

$$\left[\left(\bigotimes_{x \in X} \sigma_x\right)(f_0, 1_G)\right]\left(\bigotimes_{x \in X} v_x\right) = \bigotimes_{x \in X} \sigma_x(f_0(x))v_x.$$

Lemma 3.1. *The (f,g) conjugate of $\bigotimes_{x\in X}\sigma_x$ is given by*

$$^{(f,g)}\left(\bigotimes_{x\in X}\sigma_x\right) = \bigotimes_{x\in X} {}^{f(x)}\sigma_{g^{-1}x} \sim \bigotimes_{x\in X}\sigma_{g^{-1}x}.$$

Proof. Since $(f,g)^{-1} = (g^{-1}f^{-1}, g^{-1})$, we have

$$^{(f,g)}\left(\bigotimes_{x\in X}\sigma_x\right)(f_0,1_G) = \left(\bigotimes_{x\in X}\sigma_x\right)[(f,g)^{-1}(f_0,1_G)(f,g)] = \left(\bigotimes_{x\in X}\sigma_x\right)(g^{-1}f^{-1}\cdot g^{-1}(f_0f),1_G)$$
$$= \bigotimes_{x\in X}\sigma_x(f(gx)^{-1}f_0(gx)f(gx)) = \bigotimes_{x\in X}\sigma_{g^{-1}x}(f(x)^{-1}f_0(x)f(x)) = \bigotimes_{x\in X}{}^{f(x)}\sigma_{g^{-1}x}(f_0(x))$$
$$= \left[\bigotimes_{x\in X}{}^{f(x)}\sigma_{g^{-1}x}\right](f_0,1_G)$$

but

$$^{f(x)}\sigma_{g^{-1}x} \sim \sigma_{g^{-1}x}$$

since $f(x)\in F$, and therefore

$$^{(f,g)}\left(\bigotimes_{x\in X}\sigma_x\right) \sim \bigotimes_{x\in X}\sigma_{g^{-1}x}.$$

□

Lemma 3.2. *Let $\sigma = \left(\bigotimes_{x\in X}\sigma_x\right) \in \widehat{F^X}$. Then the inærtia group of σ with respect to $F\wr G$ is given by*

$$I_{F\wr G}(\sigma) = F\wr T_G(\sigma),$$

where $T_G(\sigma) = \{g\in G : \sigma_{gx}\sim\sigma_x \ \forall x\in X\}$.

Proof. From Lemma 3.1 we know that

$$I_{F\wr G}(\sigma) = \{(f,g) : \sigma_{gx}\sim\sigma_x,\ \forall x\in X\}$$

and this is isomorphic to

$$F^X \rtimes T_G(\sigma) = F\wr T_G(\sigma).$$

□

Remark 3.1. Actually we may write

$$T_G(\sigma) = \{g\in G : \sigma_{gx} = \sigma_x\}.$$

Lemma 3.3. *Each $\left(\bigotimes_{x\in X}\sigma_x\right)\in\widehat{F^X}$ has an extension $\widetilde{\sigma}$ to the whole $I_{F\wr G}(\sigma)$: it is given by setting*

$$\widetilde{\sigma}(f,g)\left(\bigotimes_{x\in X}v_x\right) := \bigotimes_{x\in X}\sigma_{g^{-1}x}(f(x))v_{g^{-1}x},$$

for all $(f,g)\in F\wr T_G(\sigma)$ and $\bigotimes_{x\in X}v_x\in\bigotimes_{x\in X}V_{\sigma_x}$.

Proof. From the definition of $\tilde{\sigma}$, we have

$$\widetilde{\sigma}(f,g)\left(\bigotimes_{x\in X}v_x\right) = \bigotimes_{x\in X}\sigma_{g^{-1}x}(f(x))v_{g^{-1}x} = \bigotimes_{x\in X}\sigma_x(f(x))v_{g^{-1}x}$$

where the last equality follows from the definition of $T_G(\sigma)$. Therefore

$$\widetilde{\sigma}((f_1,g_1)\cdot(f_2,g_2))\left(\bigotimes_{x\in X} v_x\right) = \widetilde{\sigma}((f_1\cdot(g_1f_2), g_1g_2)\left(\bigotimes_{x\in X} v_x\right) = \bigotimes_{x\in X}\sigma_x(f_1(x)f_2(g_1^{-1}x))v_{g_2^{-1}g_1^{-1}x}.$$

On the other hand

$$\widetilde{\sigma}(f_1,g_1)\left(\widetilde{\sigma}(f_2,g_2)\left(\bigotimes_{x\in X} v_x\right)\right) = \widetilde{\sigma}(f_1,g_1)\left(\bigotimes_{x\in X}\sigma_x(f_2(x))v_{g_2^{-1}x}\right)$$

$$= \bigotimes_{x\in X}\sigma_x(f_1(x))\sigma_{g_1^{-1}x}(f_2(g_1^{-1}x))v_{g_2^{-1}g_1^{-1}x} = \bigotimes_{x\in X}\sigma_x(f_1(x))\sigma_x(f_2(g_1^{-1}x))v_{g_2^{-1}g_1^{-1}x}$$

$$= \bigotimes_{x\in X}\sigma_x(f_1(x)f_2(g_1^{-1}x))v_{g_2^{-1}g_1^{-1}x}$$

and this shows that $\widetilde{\sigma}$ is a representation ending the proof. □

Let Σ be a system of representatives for the $F\wr G$-conjugacy classes of irreducible representations of $\widehat{F^X}$. For each $\sigma\in\Sigma$, denote by $\widetilde{\sigma}$ its extension to $I_{F\wr G}(\sigma)$ as shown in Lemma 3.3. For each $\psi\in\widehat{T}_G(\sigma)$, denote by $\overline{\psi}$ its inflation (see [5, Equation (3)]) to $I_{F\wr G}(\sigma)$ (using the homomorphism $I_{F\wr G}(\sigma)\to T_G(\sigma)\cong I_{F\wr G}(\sigma)/F^X$). Then, as an immediate consequence of the little group method of Mackey and Wigner (see [5, Theorem 5.2]), we have the following.

Theorem 3.1.

$$\widehat{F\wr G} = \left\{\mathrm{Ind}^G_{I_{F\wr G}(\sigma)}(\widetilde{\sigma}\otimes\overline{\psi}) : \sigma\in\Sigma,\ \psi\in\widehat{T}_G(\sigma)\right\}$$

that is the above is the list of all irreducible representations of $F\wr G$ and for different values of σ, ψ we obtain inequivalent representations.

3.1. The character and the matrix coefficients of the representation $\widetilde{\sigma}$.

If we want to write the character and the matrix coefficients of one of the irreducible representations in Theorem 3.1, the main problem is to compute the character and the matrix coefficients of $\widetilde{\sigma}$. Indeed, the matrix coefficients of $\overline{\psi}$ are easy: they can be obtained by composing those of ψ with the homomorphism $I_{F\wr G}(\sigma)\to T_G(\sigma)\cong I_{F\wr G}(\sigma)/B$. Then, for $\widetilde{\sigma}\otimes\overline{\psi}$ we can use the formulas for the character and the matrix coefficients of a tensor product, and for $\mathrm{Ind}^G_{I_{F\wr G}(\sigma)}(\widetilde{\sigma}\otimes\overline{\psi})$ the formulas for an induced representation. Therefore, this section is entirely devoted to $\widetilde{\sigma}$.

Let G be a finite group and, for $g\in G$, denote by $\mathfrak{C}(g)$ the conjugacy class of G containing g. Suppose that G acts on a finite set X and denote by π this action. That is, for $g\in G$, $\pi(g)$ is the permutation of X associated to g. Denote by $\mathcal{C}(\pi(g))$ the cycles of the permutation $\pi(g)$; then any $c\in\mathcal{C}(\pi(g))$ is of the form $c=(x,\pi(g)^{-1}x,\ldots,\pi(g)^{-\ell(c)+1}x)$, where $\ell(c)$ is the *length* of c (that is the smallest positive integer ℓ such that $\pi(g)^\ell x = x$). Moreover, the *cycle decomposition* of $\pi(g)$ is just

$$\pi(g) = \prod_{c\in\mathcal{C}(\pi(g))} c \equiv \prod_{c\in\mathcal{C}(\pi(g))}(x,\pi(g)^{-1}x,\ldots,\pi(g)^{-\ell(c)+1}x).$$

If $g,h\in G$ then $\mathcal{C}(\pi(hgh^{-1})) = h\mathcal{C}(\pi(g))$ where, if

$$c = (x,\pi(g)^{-1}x,\ldots,\pi(g)^{-\ell(c)+1}x)\in\mathcal{C}(\pi(g)),$$

then

$$hc = (\pi(h)x,\pi(h)\pi(g)^{-1}x,\ldots,\pi(h)\pi(g)^{-\ell(c)+1}x).$$

Now let F be another finite group. Let $\mathfrak{D}$ be the conjugacy classes of F. Form the wreath product $F\wr G = F^X\times G$. In what follows, for the sake of simplicity, we will use the notation gx to denote $\pi(g)x$. Moreover, if H is a group and $a,b\in H$, we will write $a\sim_H b$ to denote that a and b are conjugate in H. For $(f,g)\in F\wr G$ and $c=(x,g^{-1}x,\ldots,g^{-\ell(c)+1}x)\in\mathcal{C}(g)$, we set

$$a_{c,x}(f,g) = f(x)\cdot f(g^{-1}x)\ldots f(g^{-\ell(c)+1}) \equiv [f\cdot(gf)\ldots(g^{-\ell(c)+1}f)](x). \tag{3.1}$$

Suppose that $\Omega_1, \Omega_2, \ldots, \Omega_m$ are the orbits of $T_G(\sigma)$ on X, that is there exist inequivalent representations $\sigma_1, \sigma_2, \ldots, \sigma_m \in \widehat{F}$ such that $\sigma_x = \sigma_i$ for all $x \in \Omega_i$, $i = 1, 2, \ldots, m$. Let $v_1^i, v_2^i \ldots, v_{d_i}^1$ be an orthonormal basis in V_{σ_i} and $d_i = \dim V_{\sigma_i}$, $i = 1, 2, \ldots, m$. Then the character and the matrix coefficients of σ_i are respectively:

$$\chi_i(t) = \sum_{j=1}^{d_i} \langle t v_j^i, v_j^i \rangle_{V_i}, \qquad t \in F$$
$$u_{j,k}^i(t) = \langle \sigma(t) v_k^i, v_j^i \rangle_{V_i}, \qquad t \in F, \quad j,k = 1, 2, \ldots, d_i.$$

Recall that

$$\sum_{k=1}^{d_i} u_{j,k}^i(t) u_{k,h}^i(t) = u_{j,h}^i(t), \qquad \sigma(t) v_k^i = \sum_{j=1}^{d_i} u_{j,k}^i(t) v_j.$$

For $g \in T_G(\sigma)$, denote by $\mathcal{C}_i(g)$ the cycles of the permutation induced by g on Ω_i. Set

$$\mathcal{A} = \big\{\varphi : X \to \mathbb{N} |\ \varphi(x) \in \{1, 2, \ldots, d_i\}\ \forall x \in \Omega_i\big\},$$

and for every $\varphi \in \mathcal{A}$,

$$v_\varphi = \bigotimes_{i=1}^m \bigotimes_{x \in \Omega_i} v_{\varphi(x)}^i.$$

Then $\{v_\varphi : \varphi \in \mathcal{A}\}$ is an orthonormal basis for $\bigotimes_{x \in X} V_{\sigma_x}$. We will use the notation $a_{c,x}(f, g)$ in (3.1).

Theorem 3.2. *The matrix coefficients and the character of the extension* $\widetilde{\sigma}$ *of* σ *are given respectively by*:

$$u_{\psi,\varphi}^{\widetilde{\sigma}}(f, g) = \prod_{i=1}^m \prod_{x \in \Omega_i} u_{\psi(x), \varphi(g^{-1}x)}^i(f(x)), \qquad \varphi, \psi \in \mathcal{A}, \quad (f, g) \in I_{F \wr G}(\sigma),$$
$$\chi_{\widetilde{\sigma}}(f, g) = \prod_{i=1}^m \prod_{c \in \mathcal{C}_i(g)} \chi_{\sigma_i}(a_c(f, g)), \qquad (f, g) \in I_{F \wr G}(\sigma).$$

Proof. From Lemma 3.3, we obtain $((f, g) \in I_{F \wr G}(\sigma))$

$$\widetilde{\sigma}(f, g) v_\varphi = \bigotimes_{i=1}^m \bigotimes_{x \in \Omega_i} \sigma_x(f(x)) v_{\varphi(g^{-1}x)}^i = \bigotimes_{i=1}^m \bigotimes_{x \in \Omega_i} \left(\sum_{j=1}^{d_i} u_{j, \varphi(g^{-1}x)}^i (f(x)) v_j^i \right)$$
$$= \sum_{\psi \in \mathcal{A}} \left(\prod_{i=1}^m \prod_{x \in \Omega_i} u_{\psi(x), \varphi(g^{-1}x)}^i (f(x)) \right) v_\psi,$$

and this proves the formula for the matrix coefficients. Moreover, the character of $\widetilde{\sigma}$ is given by

$$\chi_{\widetilde{\sigma}}(f,g) = \sum_{\varphi\in\mathcal{A}} \langle \widetilde{\sigma}(f,g)v_\varphi, v_\varphi\rangle = \sum_{\varphi\in\mathcal{A}} \left(\prod_{i=1}^{m} \prod_{x\in\Omega_i} u^i_{\varphi(x),\varphi(g^{-1}x)}(f(x)) \right)$$

$$= \sum_{\varphi\in\mathcal{A}} \left(\prod_{i=1}^{m} \prod_{\substack{c\equiv(x,g^{-1}x,\dots,\\ g^{-\ell(c)+1}x)\in\mathcal{C}_i(g)}} u^i_{\varphi(x),\varphi(g^{-1}x)}(f(x)) u^i_{\varphi(g^{-1}x),\varphi(g^{-2}x)}(f(g^{-1}x)) \dots u^i_{\varphi(g^{-\ell(c)+1}x),\varphi(x)}(f(g^{-\ell(c)+1}x)) \right)$$

$$= \prod_{i=1}^{m} \prod_{\substack{c\equiv(x,g^{-1}x,\dots,\\ g^{-\ell(c)+1}x)\in\mathcal{C}_i(g)}} \sum_{\varphi(x)=1}^{d_i} \sum_{\varphi(g^{-1}x)=1}^{d_i} \cdots \sum_{\varphi(g^{-\ell(c)+1}x)=1}^{d_i} u^i_{\varphi(x),\varphi(g^{-1}x)}(f(x)) \times$$

$$\times u^i_{\varphi(g^{-1}x),\varphi(g^{-2}x)}(f(g^{-1}x)) \dots u^i_{\varphi(g^{-\ell(c)+1}x),\varphi(x)}(f(g^{-\ell(c)+1}x))$$

$$= \prod_{i=1}^{m} \prod_{\substack{c\equiv(x,g^{-1}x,\dots,\\ g^{-\ell(c)+1}x)\in\mathcal{C}_i(g)}} \sum_{\varphi(x)=1}^{d_i} u^i_{\varphi(x),\varphi(x)}(a_{c,x}(f,g)) = \prod_{i=1}^{m} \prod_{c\in\mathcal{C}_i(g)} \chi_{\sigma_i}(a_c(f,g)).$$

□

In the following corollary, we examine a particular case of Theorem (3.2).

Corollary 3.1. *Suppose that $\sigma_x = \sigma$ for all $x \in X$ (so that $T_G(\sigma) = G$ and $\widetilde{\sigma} \in \widehat{F \wr G}$). Then*

$$\chi_{\widetilde{\sigma}}(f,g) = \prod_{c\in\mathcal{C}(g)} \chi_\sigma(a_c(f,g)), \qquad (f,g) \in F \wr G.$$

In particular,

$$\chi_{\widetilde{\sigma}}(\mathbf{1}_F, 1_G) = (\dim\sigma)^{|X|}, \quad \chi_{\widetilde{\sigma}}(\mathbf{1}_F, g) = (\dim\sigma)^{|\mathcal{C}(g)|}, \quad \chi_{\widetilde{\sigma}}(f, 1_G) = \prod_{x\in X} \chi_\sigma(f(x)).$$

Finally, if f is constant, $f(x) = t$ for all $x \in X$, and $a_k(g) = |\{c \in \mathcal{C}(g) : \ell(c) = k\}|$, then

$$\chi_{\widetilde{\sigma}}(f,g) = \prod_{k=1}^{|X|} \chi_\sigma(t^k)^{a_k(g)}.$$

3.2. The composition of two permutation representations. In the notation of (2.2), suppose that $L(X) = \bigoplus_{i=0}^{n} a_i V_i$ and $L(Y) = \bigoplus_{j=0}^{m} b_j W_j$ are the decompositions of $L(X)$ and $L(Y)$ respectively into irreducible G- and F-representations. That is, $V_0, V_1, \dots, V_n$ (respectively, $W_0, W_1, \dots, W_m$) are pairwise inequivalent irreducible representations and $a_0, a_1, \dots, a_n$ (respectively, $b_0, b_1, \dots, b_m$) are their multiplicities in $L(X)$ (respectively, $L(Y)$); we also suppose that V_0 (respectively, W_0) is the trivial representation (so that $a_0 = b_0 = 1$). We fix $x_0 \in X$ and define K as the stabilizer of x_0 in G, that is $K = \{g \in G : gx_0 = x_0\}$.

Theorem 3.3. *Consider $X \times Y$ as a permutation module with respect to the composition action (2.2). Then the following*

$$L(X \times Y) = \left[\bigoplus_{i=0}^{n} a_i(V_i \otimes W_0)\right] \bigoplus \left[\bigoplus_{j=1}^{m} b_j(L(X) \otimes W_j)\right] \tag{3.2}$$

is the decomposition of $L(X \times Y)$ into irreducible $F \wr G$-representation.

Proof. First of all, we prove that every subspace in the right hand side of (3.2) is $F \wr G$-invariant. Suppose that $\mathcal{G} \in L(X)$, $\mathcal{F} \in L(Y)$, $(x, y) \in X \times Y$ and $(f, g) \in F \wr G$. Then

$$[(f,g)(\mathcal{G}\otimes\mathcal{F})](x,y) = (\mathcal{G}\otimes\mathcal{F})[(f,g)^{-1}(x,y)] = (\mathcal{G}\otimes\mathcal{F})(g^{-1}x, f(x)^{-1}y) = (g\mathcal{G})(x)\cdot(f(x)\mathcal{F})(y). \quad (3.3)$$

In general, $(f(x)\mathcal{F})(y)$ depends on x, and therefore (3.3) is *not* a tensor product. But there are two special cases. If $v \otimes \mathbf{1}_Y \in V_i \otimes W_0$ then $(f, g)(v \otimes \mathbf{1}_Y) = (gv) \otimes \mathbf{1}_Y$, and therefore each space $V_i \otimes W_0$ is $F \wr G$-invariant. On the other hand, if $(x', y') \in X \times Y$ and $\delta_x \otimes w \in L(X) \otimes W_j$, then

$$[(f,g)(\delta_x \otimes w)](x', y') = \delta_{gx}(x') \cdot [f(x')w](y'),$$

which implies that

$$(f,g)(\delta_x \otimes w) = \delta_{gx} \otimes [f(x)w], \quad (3.4)$$

and each space $L(X) \otimes W_j$ is also invariant.

Now, by mean of Theorem 3.1, we want to prove that each subspace in the right hand side of (3.2) is $F \wr G$-irreducible. A representation of the form $V_i \otimes W_0$ is clearly irreducible, because V_i is G-irreducible and W_0 is trivial. In the setting of Theorem 3.1, take $\sigma =$ the trivial F^X-representation, that is the tensor product $\bigotimes_{x\in X} (W_0)_x$ of $|X|$-times the trivial representation of F. Then σ has the whole $F \wr G$ as its inærtia group, and then tensoring its extension to $F \wr G$ (which is the trivial representation of $F \wr G$) with V_i we obtain exactly $V_i \otimes W_0$.

On the other hand, if we take σ_j= the representation of F^X on the tensor product $\bigotimes_{x\in X} W_{\epsilon(x)}$, where $\epsilon(x_0) = j$ and $\epsilon(x) = 0$ for $x \neq x_0$, then the inærtia group of σ_j is $F \wr K$. Denote by ι the trivial representation of $\dfrac{F \wr K}{F^X} \cong K$ and by $\widetilde{\sigma}_j$, $\bar{\iota}$ respectively the extension of σ_j and the inflation of ι (both to $F \wr K$). We want to show that the representation $\mathrm{Ind}_{F\wr K}^{F\wr G}(\bar{\iota} \otimes \widetilde{\sigma}_j)$ is isomorphic to the representation of $F \wr G$ on $L(X) \otimes W_j$ (clearly, $\bar{\iota} \otimes \widetilde{\sigma}_j \cong \widetilde{\sigma}_j$). For each $x \in X$, choose $t_x \in G$ such that: $t_x x_0 = x$. Then $\{t_x : x \in X\}$ is a system of representatives for the right cosets of K in G, and $\{(\mathbf{1}_X, t_x) : x \in X\}$ is a system of representatives for the right cosets of $F \wr K$ in $F \wr G$. Applying (3.4), we can write:

$$L(X) \otimes W_j = \bigoplus_{x\in X} (\mathbf{1}_X, t_x)\big(L(\{x_0\}) \otimes W_j\big) \quad (3.5)$$

and another application of (3.4) yields

$$(f,k)(\delta_{x_0} \otimes w) = \delta_{x_0} \otimes [f(x_0)w].$$

The last identity shows that the representation of $F \wr K$ on $L(\{x_0\}) \otimes W_j$ is isomorphic to $\bar{\iota} \otimes \tilde{\sigma}_j$ (see Lemma (3.3)). Then (3.5) ensure us that the representation

$$\mathrm{Ind}_{F\wr K}^{F\wr G}(\bar{\iota} \otimes \tilde{\sigma}_j)$$

is isomorphic to the representation of $F \wr G$ on $L(X) \otimes W_j$. It follows that $L(X) \otimes W_j$ is $F \wr G$-irreducible. □

We recall that in any isotypic decomposition like $L(X) = \bigoplus_{i=0}^{m} a_i V_i$, the sum of the squares of the multiplcities of the irreducible representations is equal to the number of orbits of K on X, that is

$$\sum_{i=0}^{m} (a_i)^2 = \#\text{ orbits of } K \text{ on } X.$$

This is called *Wielandt's Lemma* in [1–4]. See also [15, 16].

Exercise 3.1. Show that for any orthogonal decomposition of $L(X)$ into G-invariant subspaces

$$L(X) = \bigoplus_{l=0}^{m} c_l U_l, \quad (3.6)$$

where every block $c_l U_l$ is the orthogonal sum of c_l invariant G-isomorphic subspaces, we have

$$\sum_{l=0}^{h} (c_l)^2 \leq \# \text{ orbits of } K \text{ on } X,$$

with equality if and only if (3.6) is the isotypic decomposition. [Hint: You can obtain the isotypic decomposition in two steps: first decompose each U_l into irreducible representations and then group together equivalent copies. At both steps, the sum of the squares of the multiplicities increases, and remains stationary if and only if the U_l are irreducible (first step) and pairwise inequivalent (second step).]

Exercise 3.2. Fix $y_0 \in Y$ and suppose that H is the stabilizer (in F) of y_0.

(1) Prove that $J = \{(f, g) \in F \wr G : g \in K,\ f(x_0) = 1_F\}$ is the stabilizer of (x_0, y_0).

(2) Suppose that $X = \coprod_{u=0}^{r} \Xi_s$ and $Y = \coprod_{v=0}^{t} \Lambda_v$ are the decompositions of X into K-orbits and of Y into H-orbits, with $\Xi_0 = \{x_0\}$ and $\Lambda_0 = \{y_0\}$. Prove that

$$X \times Y = \left[\coprod_{v=0}^{t} (\Xi_0 \times \Lambda_v)\right] \coprod \left[\coprod_{u=1}^{s} (\Xi_u \times Y)\right]$$

is the decomposition of $X \times Y$ into J-orbits.

Exercise 3.3. Use Exercise 3.1 and Exercise 3.2 to prove that all the representations in the right hand side of (3.2) are irreducible.

4. Representation Theory of Groups of the form $C_2 \wr G$

Let G be a finite group and X a finite homogeneous G-space.

For $\omega, \theta \in C_2^X$, set $\omega \cdot \theta = \sum_{x \in X} \omega(x)\theta(x)$ and define $\chi_\theta \in L(C_2^X)$ by setting $\chi_\theta(\omega) = (-1)^{\omega \cdot \theta}$. Then the dual group of C_2^X is just $\widehat{C_2^Z} = \{\chi_\theta : \theta \in C_2^X\}$ and G acts on it by setting: $g\chi_\theta(\omega) = \chi_\theta(g^{-1}\omega)$, that is $g\chi_\theta = \chi_{g\theta}$. The action of G on $\widehat{C_2^X}$ is equivalent to the action on C_2^X and both are the same thing as the action on the subsets of X. In particular, the stabilizer $G_\theta = \{g \in G : g\chi_\theta = \chi_\theta\}$ coincides with the stabilizer of $Z_\theta = \{x \in X : \theta(x) = 0\}$. The extension of the character χ_θ is simply the character $\tilde{\chi}_\theta$ of $C_2 \wr G_\theta$, given by: $\tilde{\chi}_\theta(\omega, g) = \chi_\theta(\omega)$, for all $\omega \in C_2^Z$, $g \in G_\theta$. Similarly, if $\eta \in \widehat{G_\theta}$ (that is η is an irreducible representation of G_θ) then its *inflation* $\eta^\#$ to $C_2 \wr G_\theta$ is given by: $\eta^\#(\omega, g) = \eta(g)$, for all $\omega \in C_2^Z$, $g \in G_\theta$. Both $\tilde{\chi}_\theta$ and $\eta^\#$ are irreducible $C_2 \wr G_\theta$-representations, and so is their tensor product $\tilde{\chi}_\theta \otimes \eta^\#$; clearly $\tilde{\chi}_\theta \otimes \eta^\#(\omega, g) = \chi_\theta(\omega)\eta(g)$. Now we can apply theorem 3.1.

Theorem 4.1. *Let Θ be a systems of representatives for the orbits of G on C_2^Z (any orbit has exactly one element in Θ). Then*

$$\widehat{C_2 \wr G} = \left\{ \mathrm{Ind}_{C_2 \wr G_\theta}^{C_2 \wr G} \tilde{\chi}_\theta \otimes \eta^\# : \theta \in \Theta \text{ and } \eta \in \widehat{G_\theta} \right\},$$

that is the right hand side is a complete list of irreducible inequivalent representations of $C_2 \wr G$.

4.1. Representation theory of the finite lamplighter group $C_2 \wr C_n$.

Any irreducible representation of C_n is a one-dimensional character of the form: $e_k(h) = \exp\left(2\pi i \frac{hk}{n}\right)$, $h, k \in C_n$.

Think of $\theta \in C_2^n$ as a function $\theta : \mathbb{Z} \to C_2$ satisfying $\theta(k+n) = \theta(k)$ for any $k \in \mathbb{Z}$. Then the *period* of θ is the smallest positive integer $t = t(\theta)$ such that $\theta(k+t) = \theta(k)$ for any $k \in \mathbb{Z}$; clearly t divides n and if $n = mt$ then the stabilizer of θ is the subgroup $C_m = \langle t \rangle$ (recall also that for any divisor m of n, the subgroup of C_n isomorphic to C_m is unique [11]). The characters of the subgroup $\langle t \rangle$ are given by: $e_0|_{\langle t \rangle}, e_1|_{\langle t \rangle}, \ldots, e_{m-1}|_{\langle t \rangle}$, where $e_0, e_1, \ldots, e_{m-1}$ are as above. Indeed, for $0 \leq r, l \leq m-1$ we have:

$$e_r(lt) = \exp\left(2\pi i \frac{rlt}{n}\right) = \exp\left(2\pi i \frac{rl}{m}\right).$$

We set $e_r|_{\langle t\rangle}(k) = e_r(k)$ when $k \in \langle t\rangle$, $e_r|_{\langle t\rangle(k)} = 0$ otherwise. In what follows, we also set $m(\theta) = \frac{n}{t(\theta)}$, but we will write simply t and m when it is clear the θ we are talking about.

Now take $\theta \in C_2^n$ and $0 \le r \le m-1$. If we compute the inflation of $e_r|_{\langle t\rangle}$ and the extension of χ_θ, we obtain the character $\tilde{\chi}_\theta \otimes (e_r|_{\langle t\rangle})^\#$ of $C_2^n \wr \langle t\rangle$ given by:

$$\tilde{\chi}_\theta \otimes (e_r|_{\langle t\rangle})^\#(\omega, lt) = \chi_\theta(\omega)e_r(lt),$$

for $\omega \in C_2^n$ and $l = 0, 1, \dots, m-1$. Let Θ be a set of representatives for the orbits of C_n on C_2^n (such orbits may be enumerated by mean of the so called Polya–Redfield theory; see [12] for an elementary account and [10] for a more comprehensive treatment). Then we can apply Theorem 4.1.

Theorem 4.2. *The set*

$$\left\{\mathrm{Ind}_{C_2\wr\langle t(\theta)\rangle}^{C_2\wr C_n}\left[\tilde{\chi}_\theta \otimes (e_r|_{\langle t(\theta)\rangle})^\#\right] : \theta \in \Theta,\ r = 0, 1, \dots, m(\theta)-1\right\}$$

is a complete list of irreducible inequivalent representations of $C_2^n \wr C_n$.

Now we want to give the matrix expression for $\mathrm{Ind}_{C_2^n\wr\langle t\rangle}^{C_2^n\wr C_n}\tilde{\chi}_\theta \otimes (e_r|_{\langle t\rangle})^\#$. If $(\omega, k) \in C_2^n \wr C_n$ and $0 \le s, j \le t-1$ then $[U_{\theta,r}(\omega, k)]_{s,j}$ will denote the (s, j)-entry of this matrix evaluated at (ω, k). Note that $\{(0, s) : s = 0, 1, \dots, t-1\}$ is a set of representatives for the right cosets of $C_2 \wr \langle t\rangle$ in $C_2 \wr C_n$. Moreover, $((-s)\omega, k+j-s) \in C_2 \wr \langle t\rangle$ if and only if $k+j-s \in \langle t\rangle$, that is, if and only if t divides $k+j-s$, and therefore

$$\begin{aligned}[\tilde{\chi}_\theta \otimes (e_r|_{\langle t\rangle})^\#]((\mathbf{0}_{C_n}, s)^{-1}(\omega, k)(\mathbf{0}_{C_n}, j)) &= \chi_\theta((-s)\omega)e_r|_{\langle t\rangle}(k+j-s) \\ &= \left\{\tilde{\chi}_{s\theta} \otimes [(s-j)(e_r|_{\langle t\rangle})]^\#\right\}(\omega, k)\end{aligned} \tag{4.1}$$

Then we may apply the formula for the matrix of an induced representation ([4, Equation (10)]), getting:

$$[U_{\theta,r}(\omega, k)]_{s,j} = \begin{cases} 0 & \text{if } k+j-s \notin \langle t\rangle \\ \chi_\theta((-s)\omega)e_r(k+j-s) & \text{if } k+j-s \in \langle t\rangle. \end{cases} \tag{4.2}$$

4.2. Representation theory of the hyperoctahedral group $C_2 \wr S_n$. Now $G = S_n$ and $X = \{1, 2, \dots, n\}$. For any $0 \le k \le n$, choose $\theta^{(k)} \in C_2^X$ such that $|\{j \in Z : \theta^{(k)}(j) = 0\}| = k$. Then $\{\theta^{(0)}, \theta^{(1)}, \dots, \theta^{(n)}\}$ is a set of representatives for the orbits of S_n on C_2^X. Moreover, the stabilizer of $\theta^{(k)}$ is isomorphic to $S_k \times S_{n-k}$. We recall that the irreducible representations of the simmetric group S_t are canonically parametrized by the partitions of t; [9, 13]. For $\lambda \vdash t$ (this means that λ is a partition of t), we will denote by ρ_λ the irreducible representation of S_t canonically associated to λ and by S^λ the corresponding representation space. The irreducible representations of the group $S_k \times S_{n-k}$ are all of the form $\rho_\lambda \otimes \rho_\mu$, for $\lambda \vdash k$ and $\mu \vdash n-k$. If we set $\rho_{[\lambda;\mu]} = \mathrm{Ind}_{C_2\wr(S_k\times S_{n-k})}^{C_2\wr S_n}[\tilde{\chi}_{\theta^{(k)}} \otimes (\rho_\lambda\rho_\mu)^\#]$, applying theorem 4.1 we have the next result.

Theorem 4.3.

$$\left\{\rho_{[\lambda;\mu]} : \lambda \vdash k,\ \mu \vdash n-k \text{ and } 0 \le k \le n\right\}$$

is a complete list of inequivalent, irreducible $C_2 \wr S_n$*-representations.*

See also [7, 9].

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