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Trees, wreath products and finite Gelfand pairs

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Abstract

We present a new construction of finite Gelfand pairs by looking at the action of the full automorphism group of a finite spherically homogeneous rooted tree of type $\mathbf{r}$ on the variety $\mathcal{V}(\mathbf{r}, \mathbf{s})$ of all spherically homogeneous subtrees of type $\mathbf{s}$.

This generalizes well-known examples as the finite ultrametric space, the Hamming scheme and the Johnson scheme.

We also present further generalizations of these classical examples. The first two are based on Harary's notions of composition and exponentiation of group actions. Finally, the generalized Johnson scheme provides the inductive step for the harmonic analysis of our main construction.

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Keywords: Finite Gelfand pairs; Wreath products; Rooted trees; Finite ultrametric space; Hamming scheme; Johnson scheme

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1. Introduction

Let T be a finite rooted tree of depth m and let $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ be an m-tuple of integers $\geqslant 2$. We say that T is of type $\mathbf{r}$ when each vertex at distance k from the root has exactly r_{k+1} sons, for $k = 0, 1, \ldots, m-1$. If $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ is another m-tuple of integers with $1 \leqslant s_k \leqslant r_k$, then we can consider the variety $\mathcal{V}(\mathbf{r}, \mathbf{s})$ of all subtrees of T of type $\mathbf{s}$. The group $\mathrm{Aut}(T)$ of all automorphisms of the tree acts transitively on $\mathcal{V}(\mathbf{r}, \mathbf{s})$, i.e. $\mathcal{V}(\mathbf{r}, \mathbf{s}) = \mathrm{Aut}(T)/K(\mathbf{r}, \mathbf{s})$, where $K(\mathbf{r}, \mathbf{s})$ is the stabilizer of a fixed $T' \in \mathcal{V}(\mathbf{r}, \mathbf{s})$. In this paper we show that the decomposition into irreducibles of the permutation representation of $\mathrm{Aut}(T)$ on $\mathcal{V}(\mathbf{r}, \mathbf{s})$ is multiplicity-free. In other words $(\mathrm{Aut}(T), K(\mathbf{r}, \mathbf{s}))$ is a finite Gelfand pair.

This is a *new example* that includes several other examples of finite Gelfand pairs previously studied (which indeed are particular cases of this construction):

- For $m = 1$ we find the pair $(S_r, S_{r-s} \times S_s)$, the so called Johnson scheme, considered, among the others, by Delsarte [12,13], Dunkl [17–19], Stanton [40–42] and by Diaconis and Shahshahani [15].
- For $m = 2$, $s_1 = r_1$ and $s_2 = 1$ one obtains the so called Hamming scheme, namely the pair $(S_{r_2} \wr S_{r_1}, S_{r_2-1} \wr S_{r_1})$, again considered by Delsarte, Dunkl, Stanton, Letac [31] and many others (in particular, the case $r_2 = 2$ yields the hypercube as homogeneous space, and the literature on it and the associated diffusion problem, the Ehrenfest diffusion model, is vast; see, for instance, the paper by Diaconis, Graham and Morrison [16]).
- For $m = 2$, $1 \leqslant s_1 < r_1$ and $s_2 = 1$ one obtains the so called nonbinary Johnson scheme, considered by Dunkl [17] and Tarnanen, Aaltonen, Goethals [43].
- For $m > 1$ and $\mathbf{s} = (1, 1, \ldots, 1)$ the homogeneous space coincides with the set of all leaves of the tree and one gets the ultrametric space, which was considered by Letac [32], Stanton [41], Figà-Talamanca [22] and by Bekka, de la Harpe and Grigorchuk [8].

Further, we examine several other constructions involving wreath products and semidirect products that give rise to finite Gelfand pairs.

For the general theory of finite Gelfand pairs we refer to the book by Diaconis [14] which has been undoubtedly the most influential, especially in view of the applications, the pioneering monograph [31] by Letac, the book by Klimyk and Vilenkin [30], and our recent survey [11]; see also the book by Terras [44]. The papers by Dunkl [19] and Stanton [40] are very nice surveys on several other examples involving Weyl groups or Chevalley groups over finite fields. The paper by Saxl [35] classifies all finite Gelfand pairs in the symmetric group and also presents some further results for linear groups over finite fields.

For the (equivalent) point of view of the theory of association schemes, started by Delsarte in his epochal thesis [12] and for the theory of the Bose–Mesner algebras we refer to the beautiful book by Bannai and Ito [5]. A more recent account, with a friendly approach and a view towards statistical applications, is [2].

The paper is organized as follows.

In Section 2 we give some preliminaries on finite Gelfand pairs and on wreath products. In particular, we recall the notions of composition and exponentiation for group actions (terminology due to F. Harary [26]).

In Section 3 we present our main construction we alluded to above, namely the pair $(\mathrm{Aut}(T), K(\mathbf{r}, \mathbf{s}))$.

In Section 4 we present our generalization of the ultrametric Gelfand pair [22,32,41]. This construction was already considered by Bailey, Praeger, Rowley and Speed [3] (even in a more general setting, see also [2, Chapters 3 and 9]) and by Hanaki and Hirotsuka [25] (this latter in the language of association schemes). More precisely, given two Gelfand pairs (G, K) and (F, H) with homogeneous spaces $X = G/K$ and $Y = F/H$, then there is a natural action, namely the composition of $F \wr G$ on $X \times Y$. This gives rise to a Gelfand pair. In other words, given two permutation representations G on X and F on Y, we show that the action of $F \wr G$ on $X \times Y$ is multiplicity-free if and only if the previous actions are both multiplicity-free. We give explicit formulas for the spherical functions and we show how the ultrametric space is obtained by iterating this construction starting from the pair (S_q, S_{q-1}).

In Section 5 we present our generalization of the Hamming scheme. Particular cases have been studied recently (mostly from the point of view of the Theory of Special Functions) by Mizukawa [33], Akazawa and Mizukawa [1]. See also Mizukawa and Tanaka [34]. More precisely, given a Gelfand pair (F, H) with $Y = F/H$ and a finite group G acting on a set X, then the natural action, the exponentiation, of $F \wr G$ on Y^X, gives rise to a Gelfand pair, namely $(F \wr G, H \wr G)$. We actually consider a more general construction involving semidirect products.

In Section 6 we present our generalization of the Johnson scheme. More precisely, given a Gelfand pair (F, H) we show that $(F \wr S_n, F \wr S_{n-h} \times H \wr S_h)$ is a Gelfand pair. The corresponding homogeneous space may be identified with $\Theta_h = \bigsqcup_{A \in S_n/(S_{n-h} \times S_h)} Y^A$, i.e. with the set of all functions $\theta : A \to Y$ where A ranges among all the h-subsets of $\{1, 2, \ldots, n\}$. This generalizes a construction given by Dunkl [17] and by Tarnanen, Aaltonen and Goethals [43] who considered the case $F = S_m$ and $H = S_{m-1}$. We also show that our construction is the keypoint for an inductive analysis of $(\mathrm{Aut}(T_{\mathbf{r}}), K(\mathbf{r}, \mathbf{s}))$.

Note that for $h = 1$ the corresponding construction yields a particular case of the generalized ultrametric space (Section 4), while, for $h = n$, we have a particular case of the generalized Hamming scheme (Section 5). In this setting, a more general construction finds an obstruction from a classical result of Beaumont and Peterson [7] (who attribute it to Chevalley): in general, the only nontrivial subgroup of S_n acting transitively on the h-subsets of $\{1, 2, \ldots, n\}$ for all $h = 1, 2 \ldots, n$ is the alternating subgroup A_n.

2. Preliminaries

2.1. *Definition and characterizations of Gelfand pairs*

Let G be a finite group and $K \leqslant G$ a subgroup of G. Denote by $X = G/K = \{gK\colon g \in G\}$ the corresponding *homogeneous space* and by $x_0 \in X$ the point stabilized by K.

Denote by $L(G) = \{f : G \to \mathbb{C}\}$ the convolution algebra of all complex-valued functions on G. We then say that $f \in L(G)$ is *bi-K-invariant* if $f(kgk') = f(g)$ for all $g \in G$ and $k, k' \in K$. The subalgebra of bi-K-invariant functions on G can be identified with $L(K \backslash G/K)$, the algebra of all complex-valued functions on the double-K-cosets of G.

The *permutation representation* λ of G on $L(X) = \{f : X \to \mathbb{C}\}$ is then defined by $[\lambda(g)f](x) = f(g^{-1}x)$ for $g \in G$, $x \in X$ and $f \in L(X)$. We also denote by $\langle f_1, f_2\rangle = \sum_{x\in X} f_1(x)\overline{f_2(x)}$ the *scalar product* of two functions $f_1, f_2 \in L(X)$.

The pair (G, K) is a *Gelfand pair* if the algebra $L(K\backslash G/K)$ of bi-K-invariant functions is commutative or, equivalently, the decomposition $L(X) = \bigoplus_{i=0}^{n} V_i$ into irreducible G-modules is multiplicity-free.

If this is the case, for all $i = 0, 1, \dots, n$, there exists a (unique up to normalization) bi-K-invariant function $\phi_i \in V_i$ whose G-translates span the whole V_i. The ϕ_i's are called *spherical functions* and they constitute a basis for the subspace of bi-K-invariant functions; this way, the number $n + 1$ of irreducible components V_i's equals the number of K-orbits on X. We also observe that there is a bijection between the K-orbits on X and the G-orbits on $X \times X$ with respect to the diagonal action $g(x, x') = (gx, gx')$, $g \in G$, $x, x' \in X$. Indeed, denoting by $\sqcup$ a *disjoint union*, we have that if $X = \bigsqcup_{i=0}^{n} \Lambda_i$ is the partition of X into its K-orbits, with $\Lambda_0 = \{x_0\}$, and more generally for each $1 \leqslant j \leqslant n$, $x_j \in \Lambda_j$ is chosen so that $\Lambda_j = K \cdot \{x_j\}$, then the G-orbits of $X \times X$ are given by the sets $\widetilde{\Lambda}_j := G \cdot \{(x_0, x_j)\}$ (in particular, $\widetilde{\Lambda}_0 = \{(x, x)\colon x \in X\}$): indeed, it is easy to verify that the $\widetilde{\Lambda}_j$'s partition $X \times X$. We indicate by

$$\Lambda_j \to \widetilde{\Lambda}_j \tag{2.1}$$

this correspondence.

On $L(X)$ we now define, for $0 \leqslant i \leqslant n$, the *Markov operators*

$$[M_i f](x) = \sum_{y\in g\Lambda_i} f(y) = \sum_{y\colon (x,y)\in\widetilde{\Lambda}_i} f(y) \tag{2.2}$$

where $f \in L(X)$, $x = gx_0$ and the second equality follows from the previous argument. Observe that M_i is nothing but the convolution operator with kernel the characteristic function of Λ_i: in particular, $M_0 = I_X$ is the identity operator.

For $0 \leqslant i, j \leqslant n$ set

$$\Xi_{i,j}(x, y) = \left\{z \in X\colon\ (x, z) \in \widetilde{\Lambda}_i \text{ and } (z, y) \in \widetilde{\Lambda}_j\right\} \tag{2.3}$$

and observe that $\Xi_{i,j}(gx, gy) = g\Xi_{i,j}(x, y)$ for all $g \in G$, so that $|\Xi_{i,j}(x, y)| =: \xi_{i,j}(s)$ depends only on the G-orbit $\widetilde{\Lambda}_s$ of (x, y). It then follows that $M_i M_j = \sum_{s=0}^{n} \xi_{i,j}(s) M_s$. The algebra generated by the operators M_i's is called the *Bose–Mesner algebra* [2,4,12] and it is clear that (G, K) is a Gelfand pair if and only if this algebra of operators is commutative (*orbit criterion*).

In the following we shall use a criterion (Corollary 2.2) for Gelfand pairs which can be deduced from the following lemma (Proposition 29.2 in [46]).

Lemma 2.1. *Let G be a finite group, $K \leqslant G$ a subgroup and denote by $X = G/K$ the corresponding homogeneous space. Let $L(X) = \bigoplus_{i=0}^{n} m_i V_i$ be a decomposition into irreducible G-subrepresentations where m_i denotes the multiplicity of V_i. Then*

$$\sum_{i=0}^{n} m_i^2 = \text{number of } K\text{-orbits on } X \ (= \text{number of } G\text{-orbits on } X \times X). \tag{2.4}$$

Corollary 2.2. *Let G be a finite group, $K \leqslant G$ a subgroup and denote by $X = G/K$ the corresponding homogeneous space. Suppose we have a decomposition $L(X) = \bigoplus_{t=0}^{h} Z_t$ into G-subrepresentations with $h + 1 =$ the number of K-orbits on X. Then the Z_t's are irreducible and (G, K) is a Gelfand pair.*

Proof. Refine if necessary the decomposition with the Z_t's into irreducibles as in the statement of the previous lemma. Then $h + 1 \leqslant \sum_{i=0}^{n} m_i \leqslant \sum_{i=0}^{n} m_i^2$ and the lemma force $h = n$ and $m_i = 1$ for all i's concluding the proof. □

2.2. Symmetric Gelfand pairs

Let G and $K \leqslant G$ be finite groups and denote by $X = G/K$ the corresponding homogeneous space.

Suppose that for any $g \in G$ one has $g^{-1} \in KgK$. Then (G, K) is a Gelfand pair. This can be shown directly by checking that any two bi-K-invariant functions commute (see, e.g. [11]). The pair (G, K) is then called a *symmetric* Gelfand pair [14,32].

In [11,32] it is shown that symmetry is equivalent to the condition that (x, y) and (y, x) belong to the same G-orbit on $X \times X$ for all $x, y \in X$.

Suppose that G acts on a metric space (X, d) isometrically (i.e. $d(gx, gy) = d(x, y)$ for all $x, y \in X$ and $g \in G$) and that the action is 2-*point homogeneous* (or *distance transitive*), that is, for all $x_1, x_2, y_1, y_2 \in X$ such that $d(x_1, y_1) = d(x_2, y_2)$ there exists $g \in G$ such that $gx_1 = x_2$ and $gy_1 = y_2$. Fix $x_0 \in X$ and denote by $K = \{g \in G\colon\ gx_0 = x_0\}$ the stabilizer of this point. Then (G, K) is a symmetric Gelfand pair: indeed $d(x, y) = d(y, x)$ and the previous argument applies.

In [11] we presented a short proof of the following characterization of symmetric Gelfand pairs due to Garsia [9,23]:

Lemma 2.3 *((Garsia's criterion)). A Gelfand pair (G, K) is symmetric if and only if the spherical functions are real-valued.*

The simplest Gelfand pair, namely $(C_n, \{e\})$, where C_n denotes the cyclic group of order n and e is the unit element, is nonsymmetric for $n \geqslant 3$; note that the spherical functions are the characters $\phi_j(x) = \exp(2\pi i jx/n)$.

From the point of view of the Bose–Mesner algebras, recalling that the Markov operator M_i in (2.2) can be viewed as the convolution operator with kernel the characteristic function of the set $\Lambda_i \subseteq X$ and observing that M_i is selfadjoint (i.e. $\langle M_i f_1, f_2\rangle = \langle f_1, M_i f_2\rangle$, for all $f_1, f_2 \in L(X)$) if and only if $\pi^{-1}(\Lambda_i) \subseteq G$ is symmetric (i.e. $g \in \pi^{-1}(\Lambda_i)$ implies

$g^{-1} \in \pi^{-1}(\Lambda_i)$), where $\pi : G \to X = G/K$ is the canonical projection, one easily deduces the following criterion.

Lemma 2.4. *A Gelfand pair* (G, K) *is symmetric if and only if the Markov operators are selfadjoint, equivalently, if they have real spectrum.*

Another example of a nonsymmetric Gelfand pair is provided by (A_4, K) where A_4 is the alternating group on $\{1, 2, 3, 4\}$ and $K = \{e, (1, 2)(3, 4)\}$. Indeed, letting A_4 act on the set X of all 2-subsets of $\{1, 2, 3, 4\}$ we have that this action is transitive and K is the stabilizer of the point $\{1, 2\}$. By simple calculations one shows that there are exactly four K-orbits on X, namely

$$\begin{aligned}
\Lambda_0 &= \{1, 2\},\\
\Lambda_1 &= \big\{\{2, 3\}, \{1, 4\}\big\},\\
\Lambda_2 &= \big\{\{1, 3\}, \{2, 4\}\big\},\\
\Lambda_3 &= \{3, 4\}.
\end{aligned}$$

By Corollary 2.2 one deduces that (G, K) is a Gelfand pair. However it is not symmetric as $(\{1, 2\}, \{1, 3\})$ and $(\{1, 3\}, \{1, 2\})$ do not belong to the same G-orbit in $X \times X$.

In terms of (normalized) Markov operators, we have, using group algebra notation,

$$\begin{aligned}
M_0 &= \frac{1}{2}\big[e + (12)(34)\big],\\
M_1 &= \frac{1}{4}\big[(124) + (234) + (132) + (143)\big],\\
M_2 &= \frac{1}{4}\big[(123) + (134) + (142) + (243)\big],\\
M_3 &= \frac{1}{2}\big[(13)(24) + (14)(23)\big].
\end{aligned}$$

Then M_0 acts as identity, $M_1^2 = M_2$, $M_2^2 = M_1$, $M_3^2 = M_0$, $M_1M_3 = M_1$, $M_2M_3 = M_2$ and $M_1M_2 = \frac{1}{2}(M_0 + M_3)$.

Using the techniques in [2, Section 2.4], we have that the spectra of the Markov operators are as in the following *character table*:

M_0	1	1	1	1
M_1	1	0	ω	ω^2
M_2	1	0	ω^2	ω
M_3	1	-1	1	1

where $\omega = \exp(2\pi i/3)$.

Applying Lemma 2.4 we again deduce that (A_4, K) is nonsymmetric.

2.3. Composition and exponentiation of group actions

We recall that given a group G acting on a set X and another group F, the *wreath* product $F \wr G$ of F by G is the group whose set of elements is $F^X \times G = \{(f, g)\colon f : X \to F, g \in G\}$ and multiplication $(f, g)(f', g') = (f(gf'), gg')$ where $[gf'](x) = f'(g^{-1}x)$ and $f(gf')$ is the pointwise product: $[f(gf')](x) = f(x)f'(g^{-1}x)$, for all $f, f' \in F^X$, $g \in G$ and $x \in X$. We remind that the unit element of $F \wr G$ is $(\mathbf{1}, e_G)$, where $\mathbf{1}(x) = e_F$ for all $x \in X$ and e_F and e_G are the unit elements in F and G, respectively; moreover the inverse of an element $(f, g) \in F \wr G$ is

$$(f, g)^{-1} = \left(f', g^{-1}\right) \quad \text{where } f'(x) = f(gx)^{-1}. \tag{2.5}$$

Suppose now that F acts on a set Y.

We can define an action of $F \wr G$ on $X \times Y$ by setting

$$(f, g)(x, y) = \bigl(gx, f(gx)y\bigr) \equiv \bigl(gx, \bigl[\bigl(g^{-1}f\bigr)(x)\bigr]y\bigr). \tag{2.6}$$

It is easy to check that (2.6) defines indeed an action; Harary [26] calls it the *composition action*.

We can also define an action of $F \wr G$ on $Y^X = \{\eta : X \to Y\}$ by setting

$$\bigl[(f, g)\eta\bigr](x) = f(x)\eta\bigl(g^{-1}x\bigr). \tag{2.7}$$

It is easy to check that (2.7) indeed defines an action; Harary [26] calls it the *exponentiation action*.

3. The main construction: Gelfand pairs associated with subtrees

3.1. The homogeneous space $\mathcal{V}(\mathbf{r}, \mathbf{s})$

Let $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ be an m-tuple of positive integers (that, as one naturally expects, could be assumed to be $\geqslant 2$). Set $X_0 = \{\emptyset\}$ and $X_k = \{1, 2, \ldots, r_k\}$ so that $|X_k| = r_k$ for all $1 \leqslant k \leqslant m$.

The associated $\mathbf{r}$-*tree* is the graph $T_\mathbf{r} = (V, E)$ where the set of vertices is

$$V = X_0 \sqcup X_1 \sqcup (X_1 \times X_2) \sqcup \cdots \sqcup (X_1 \times X_2 \times \cdots \times X_m)$$

and two vertices $v = (x_1, x_2, \ldots, x_k)$ and $w = (y_1, y_2, \ldots, y_h)$ are adjacent, namely $\{v, w\} \in E$, if $|h - k| = 1$ and $x_i = y_i$ for all $1 \leqslant i \leqslant \min\{h, k\}$; if $h = k + 1$ we say that w is a *son/successor* of v and that v is the *father/predecessor* of w. Clearly every vertex at level i has exactly r_{i+1} successors. The set $V_i = V_i(T_\mathbf{r}) = X_1 \times X_2 \times \cdots \times X_i$ is called the ith *level* of the tree $T_\mathbf{r}$. m is called the *depth* of $T_\mathbf{r}$ and V_m is called the set of *leaves* of $T_\mathbf{r}$.

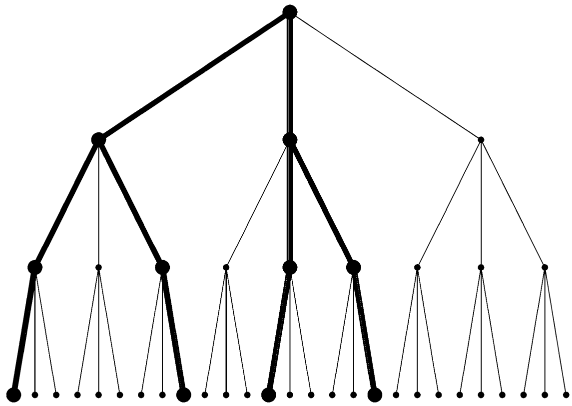

Fig. 1. A tree of type (3, 3, 3) with a subtree of type (2, 2, 1).

Suppose now that $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ is another m-tuple such that $1 \leqslant s_i \leqslant r_i$ for all i's. Denote by $T_\mathbf{s}$ the corresponding $\mathbf{s}$-tree. See, for instance, Fig. 1. Note that there are exactly

$$\binom{r_1}{s_1} \cdot \prod_{i=2}^{m} \binom{r_i}{s_i}^{s_1 s_2 \cdots s_{i-1}}$$

distinct embeddings of $T_\mathbf{s}$ as a subtree of $T_\mathbf{r}$. Indeed, the number of vertices at level i is $s_1 s_2 \cdots s_{i-1}$ and any such subtree is uniquely determined by the m-tuple $(f_0, f_1, \ldots, f_{m-1})$ where f_i is the map that associated with each vertex v at the ith level in $T_\mathbf{s}$ the set of all successors of v (note that for each i there are exactly $\binom{r_{i+1}}{s_{i+1}}^{s_1 s_2 \cdots s_i}$ such maps f_i's). We denote by $\mathcal{V}(\mathbf{r}, \mathbf{s})$ the set of all $\mathbf{s}$-subtrees of $T_\mathbf{r}$.

Denote by S_k the symmetric group on k elements and by $\mathrm{Aut}(T_\mathbf{r})$ the group of all automorphisms of $T_\mathbf{r}$; it is well known that $\mathrm{Aut}(T_\mathbf{r}) = S_{r_m} \wr S_{r_{m-1}} \wr \cdots \wr S_{r_2} \wr S_{r_1}$, see, for instance, [6,24]. Observe that if $g \in \mathrm{Aut}(T_\mathbf{r})$ then g stabilizes the levels V_i's. Moreover, g is uniquely determined by a *labelling* [24], that we continue to denote by g, namely a map $g : V \ni v \mapsto g(v) \in \bigcup_{i=0}^{m-1} S_{r_{i+1}}$ ($g(v) \in S_{r_{i+1}}$ if $v \in V_i$) such that

$$g(x_1, x_2, \ldots, x_k) = \big(g(\emptyset)x_1, g(x_1)x_2, \ldots, g(x_1, x_2, \ldots, x_{k-1})x_k\big).$$

We observe that the group $\mathrm{Aut}(T_\mathbf{r})$ acts on $\mathcal{V}(\mathbf{r}, \mathbf{s})$.

Fix an $\mathbf{s}$-subtree $T_\mathbf{s}^*$ and denote by $K(\mathbf{r}, \mathbf{s}) = \{g \in \mathrm{Aut}(T_\mathbf{r})\colon\ gT_\mathbf{s}^* = T_\mathbf{s}^*\}$ its stabilizer. This way one has the identification $\mathcal{V}(\mathbf{r}, \mathbf{s}) = \mathrm{Aut}(T_\mathbf{r})/K(\mathbf{r}, \mathbf{s})$.

We end this section with an explicit description of the structure of the group $K(\mathbf{r}, \mathbf{s})$. Suppose first that the tree $T_\mathbf{r}$ has depth 1 so that $\mathbf{r} = r_1$ and $\mathbf{s} = s_1$; clearly $\mathrm{Aut}(T_\mathbf{r}) = S_{r_1}$ and $K(\mathbf{r}, \mathbf{s}) = S_{s_1} \times S_{r_1 - s_1}$.

In general, let $\mathbf{r}' = (r_2, r_3, \ldots, r_m)$ and $\mathbf{s}' = (s_2, s_3, \ldots, s_m)$, then, one has the recursive expression:

$$\mathrm{Stab}_{\mathrm{Aut}(T_\mathbf{r})}(T_\mathbf{s}) = \mathrm{Aut}(T_{\mathbf{r}'}) \wr S_{r_1 - s_1} \times K(\mathbf{r}', \mathbf{s}') \wr S_{s_1}. \tag{3.1}$$

In particular, if the tree $T_{\mathbf{r}}$ has depth 2, so that $\mathbf{r} = (r_1, r_2)$, $\mathbf{s} = (s_1, s_2)$ and $\mathrm{Aut}(T_{\mathbf{r}}) = S_{r_2} \wr S_{r_1}$, one has

$$K(\mathbf{r}, \mathbf{s}) = S_{r_2} \wr S_{r_1 - s_1} \times (S_{s_2} \times S_{r_2 - s_2}) \wr S_{s_1}.$$

3.2. The Gelfand pair $(\mathrm{Aut}(T_{\mathbf{r}}), K(\mathbf{r}, \mathbf{s}))$

Given two rooted trees we say that they are *rooted*-isomorphic if there exists a graph isomorphism exchanging the respective roots; note that, more generally, the level of the single elements remains unchanged under such an isomorphism.

Lemma 3.1. *Let* T_1, T_2, T_1' *and* T_2' *be* $\mathbf{s}$*-subtrees inside* $T_{\mathbf{r}}$. *Then* (T_1, T_2) *and* (T_1', T_2') *belong to the same* $\mathrm{Aut}(T_{\mathbf{r}})$*-orbit on* $\mathcal{V}(\mathbf{r}, \mathbf{s}) \times \mathcal{V}(\mathbf{r}, \mathbf{s})$ *if and only if* $T_1 \cap T_2$ *is rooted-isomorphic to* $T_1' \cap T_2'$.

Remark 3.2. Note that if T_1 and T_2 are two $\mathbf{s}$-subtrees inside $T_{\mathbf{r}}$ then their intersection $T_1 \cap T_2$ need not to be an $\mathbf{u}$-subtree of $T_{\mathbf{r}}$ (for some $\mathbf{u} = (u_1, u_2, \ldots, u_k)$, with $u_i \leqslant s_i$).

Proof. Observe first that if $gT_j = T_j'$, $j = 1, 2$, for some $g \in \mathrm{Aut}(T_{\mathbf{r}})$, then $g(T_1 \cap T_2) = T_1' \cap T_2'$ so that the "only if" part follows trivially.

We prove the other implication by induction on the depth m of the tree $T_{\mathbf{r}}$. For $m = 1$, one has $\mathbf{r} = r$ and $\mathbf{s} = s$, $\mathcal{V}(\mathbf{r}, \mathbf{s})$ is simply the set of all s-subsets of an r-set and $\mathrm{Aut}(T_{\mathbf{r}}) = S_r$; this case is easy and well known.

Suppose that $T_1 \cap T_2$ is rooted-isomorphic to $T_1' \cap T_2'$ and denote by $\alpha : V_1(T_1 \cap T_2) \to V_1(T_1' \cap T_2')$ a bijection such that if $x \in V_1(T_1 \cap T_2)$ then the $T_1 \cap T_2$-subtree T_x rooted at x is (rooted-)isomorphic to the $T_1' \cap T_2'$-subtree $T'_{\alpha(x)}$ rooted at $\alpha(x)$. Extend α to a $\sigma \in S_{r_1}$ such that $\sigma(V_1(T_1)) = V_1(T_1')$ and $\sigma(V_1(T_2)) = V_1(T_2')$.

Modulo this permutation σ we now suppose that $T_1 \cap T_2$ and $T_1' \cap T_2'$ coincide at the first level.

By induction, for all $x \in V_1(T_1 \cap T_2) \equiv V_1(T_1' \cap T_2')$ we have an x-rooted isomorphism g_x between the $T_1 \cap T_2$-subtree rooted at x and the corresponding $T_1' \cap T_2'$-subtree with the same root x. It is then clear that the automorphism g with label $g(\emptyset) = \sigma$, $g(x, x_2, \ldots, x_n) = g_x(x_2, \ldots, x_n)$ if $x \in T_1 \cap T_2$ and the identity otherwise, is the desired rooted automorphism. □

Corollary 3.3. $\mathrm{Aut}(T_{\mathbf{r}})$ *acts transitively on* $\mathcal{V}(\mathbf{r}, \mathbf{s})$.

Proof. Apply the lemma to $T_1 = T_2$ and $T_1' = T_2'$. □

Corollary 3.4. $(\mathrm{Aut}(T_{\mathbf{r}}), K(\mathbf{r}, \mathbf{s}))$ *is a symmetric Gelfand pair.*

Proof. Apply previous lemma to T_1, T_2, T_1', T_2' with $T_1' = T_2$ and $T_2' = T_1$ in combination with the arguments from previous section. □

4. Composition of Gelfand pairs: The generalized ultrametric space

This section is devoted to a particular case (namely $\mathbf{s} = (1, 1, \dots, 1)$) of the general construction of previous section, because now a more general theory can be obtained.

4.1. Composition of Gelfand pairs

Let G and F be two finite groups with subgroups $K \leqslant G$ and $H \leqslant F$. Denote by $X = G/K$ and $Y = F/H$ the corresponding homogeneous spaces. Let $x_0 \in X$ and $y_0 \in Y$ be the points stabilized by K and H, respectively. Consider the composition action of $F \wr G$ on $X \times Y$ (2.6) (see, for instance, Fig. 2) and denote by J the stabilizer of the point (x_0, y_0). Also let $X = \bigsqcup_{i=0}^{n} \Xi_i$ and $Y = \bigsqcup_{j=0}^{m} \Lambda_j$ be the decompositions of X and Y into their K- (respectively H-) orbits (with $\Xi_0 = \{x_0\}$ and $\Lambda_0 = \{y_0\}$).

Then we have

Lemma 4.1.

(1) $J = \{(f, k) \in F \wr G\colon\ k \in K,\ f(x_0) \in H\}$.
(2) *The decomposition of $X \times Y$ into its J-orbits is given by*

$$X \times Y = \left[\bigsqcup_{j=0}^{m} (\Xi_0 \times \Lambda_j)\right] \sqcup \left[\bigsqcup_{i=1}^{n} (\Xi_i \times Y)\right]. \tag{4.1}$$

Proof. (1) The characterization of J follows immediately from the definition of the action (2.6).

(2) We determine the J-orbits on $X \times Y$. If $y \in \Lambda_j$, then $J(x_0, y) = \{(f, k)(x_0, y)\colon\ k \in K$ and $f(x_0) \in H\} = \{(x_0, f(x_0)y)\colon\ f(x_0) \in H\} = \Xi_0 \times \Lambda_j$. Analogously, if $x \in \Xi_i$, $i \geqslant 1$, and $y \in Y$, then $J(x, y) = \{(f, k)(x, y)\colon\ k \in K$ and $f(x_0) \in H\} = \{(kx, f_1 y)\colon\ k \in K, f_1 \in F\} = \Xi_i \times Y$. $\square$

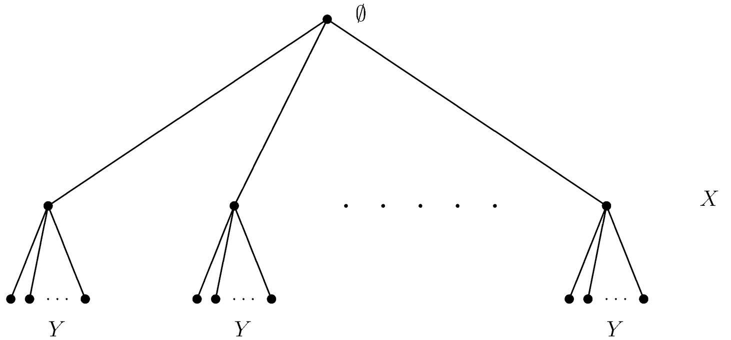

Fig. 2. Composition: $F \wr G$ acts on $X \times Y$ as automorphisms of the tree $\{\emptyset\} \sqcup X \sqcup \{X \times Y\}$.

Suppose that (G, K) and (F, H) are Gelfand pairs and let $L(X) = \bigoplus_{i=0}^n V_i$ and $L(Y) = \bigoplus_{j=0}^m W_j$ be the decomposition into G- (respectively F-) irreducible subrepresentations, where V_0 and W_0 are the one-dimensional subspaces of constant functions. Also denote by $\{\phi_i\}_{i=0}^n$ and $\{\phi'_j\}_{j=0}^m$ the spherical functions of (G, K) and (F, H), respectively with $\phi_0 = \mathbf{1}_X$ and $\phi'_0 = \mathbf{1}_Y$. We then have

Theorem 4.2.

(1) *$(F \wr G, J)$ is a Gelfand pair if (and only if) (G, K) and (F, H) are Gelfand pairs.*
(2) *The decomposition of $L(X \times Y)$ into $(F \wr G)$-irreducibles is given by*

$$L(X \times Y) = \left[\bigoplus_{i=0}^{n}(V_i \otimes W_0)\right] \oplus \left[\bigoplus_{j=1}^{m}\big(L(X) \otimes W_j\big)\right]. \tag{4.2}$$

The spherical functions of $(F \wr G, J)$ are

$$\big\{\phi_i \otimes \phi'_0,\ \delta_{x_0} \otimes \phi'_j\colon\ i = 0, 1, \dots, n,\ j = 1, 2, \dots, m\big\} \tag{4.3}$$

where δ_{x_0} is the Dirac function at $x_0 \in X$.

Proof. (1) We use the orbit criterion (cf. Section 2.1) that yields both implications in an elementary fashion; alternatively, the "if" part may be deduced from the arguments in the proof of (2). Denote, as in (2.1), by $\widetilde{\Xi}_i$, $i \in \mathcal{I} = \{0, 1, \dots, n\}$, and $\widetilde{\Lambda}_j$, $j \in \mathcal{J} = \{0, 1, \dots, m\}$, the G-orbits on $X \times X$ and $Y \times Y$, respectively; set $\mathcal{I}^* = \mathcal{I} \setminus \{0\}$.

In a similar way denote by $\widetilde{\Xi_i \times Y}$ and $\widetilde{\Xi_0 \times \Lambda_j}$ ($i \in \mathcal{I}^*$ and $j \in \mathcal{J}$) the $F \wr G$-orbits on $(X \times Y) \times (X \times Y)$.

The corresponding Markov operators are

$$[M_i\mathcal{F}](x_1) = \sum_{x_2\colon (x_1,x_2)\in\widetilde{\Xi}_i} \mathcal{F}(x_2),$$

$$[N_j\mathcal{G}](y_1) = \sum_{y_2\colon (y_1,y_2)\in\widetilde{\Lambda}_j} \mathcal{G}(y_2),$$

$$[\mathcal{M}_i\mathcal{H}](x_1, y_1) = \sum_{(x_2,y_2)\colon (x_1,x_2,y_1,y_2)\in\widetilde{\Xi_i\times Y}} \mathcal{H}(x_2, y_2),$$

$$[\mathcal{N}_j\mathcal{H}](x_1, y_1) = \sum_{(x_2,y_2)\colon (x_1,x_2,y_1,y_2)\in\widetilde{\Xi_0\times\Lambda_j}} \mathcal{H}(x_2, y_2)$$

where $\mathcal{F} \in L(X), \mathcal{G} \in L(Y)$ and $\mathcal{H} \in L(X \times Y)$.

We need to show that the $\mathcal{M}_i$'s together with the $\mathcal{N}_j$'s generate a commutative algebra if and only if the M_i's and, separately, the N_j's do.

If $\mathcal{G} \in L(X)$ and $\mathcal{F} \in L(Y)$, their tensor product is given by $[\mathcal{G} \otimes \mathcal{F}](x, y) = \mathcal{G}(x)\mathcal{F}(y)$ for all x, y. By linearity we may assume that $\mathcal{H} = \mathcal{F} \otimes \mathcal{G}$.

For $i \in \mathcal{I}^*$ we have

$$\begin{aligned}
[\mathcal{M}_i\mathcal{H}](x_1, y_1) &= \big[\mathcal{M}_i(\mathcal{F} \otimes \mathcal{G})\big](x_1, y_1) \\
&= \sum_{(x_2,y_2):\ (x_1,x_2,y_1,y_2)\in\widetilde{\Xi_i \times Y}} \mathcal{F}(x_2)\mathcal{G}(y_2) \\
&=_* \sum_{\substack{x_2:\ (x_1,x_2)\in\widetilde{\Xi}_i \\ y_2\in Y}} \mathcal{F}(x_2)\mathcal{G}(y_2) \\
&= \big([M_i\mathcal{F}](x_1)\big)\Big(\sum_{j\in\mathcal{J}} [N_j\mathcal{G}](y_1)\Big) \\
&= \Big[\Big(M_i \otimes \Big(\sum_{j\in\mathcal{J}} N_j\Big)\Big)\mathcal{H}\Big](x_1, y_1),
\end{aligned}$$

where $=_*$ comes from the following fact: if we identify $(X \times Y) \times (X \times Y)$ with $(X \times X) \times (Y \times Y)$, then $\widetilde{\Xi_i \times Y} = \widetilde{\Xi}_i \times \widetilde{Y}$ and $\widetilde{\Xi_0 \times \Lambda_j} = \widetilde{\Xi}_0 \times \widetilde{\Lambda}_j$, where $\widetilde{Y} = Y \times Y$.

One then deduces

$$\mathcal{M}_i = M_i \otimes \Big(\sum_{j\in\mathcal{J}} N_j\Big) \tag{4.4}$$

and, similarly,

$$\mathcal{N}_j = M_0 \otimes N_j = I_{L(X)} \otimes N_j. \tag{4.5}$$

From (4.4) and (4.5) point (1) follows immediately.

(2) We now determine the decomposition into $F \wr G$-irreducibles of $L(X \times Y)$. We first observe that

$$L(X \times Y) = L(X) \otimes L(Y) = \Big(\bigoplus_{i=0}^{n} V_i\Big) \otimes \Big(\bigoplus_{j=0}^{m} W_j\Big)$$

so that (4.2) is a decomposition of $L(X \times Y)$.

If $\mathcal{G} \in L(X)$ and $\mathcal{F} \in L(Y)$, then $[\mathcal{G} \otimes \mathcal{F}](x, y) = \mathcal{G}(x)\mathcal{F}(y)$ for all x, y and therefore, if $(f, g) \in F \wr G$, using (2.5) and (2.6)

$$\begin{aligned}
\big[(f, g)(\mathcal{G} \otimes \mathcal{F})\big](x, y) &= (\mathcal{G} \otimes \mathcal{F})\big[(f, g)^{-1}(x, y)\big] \\
&= (\mathcal{G} \otimes \mathcal{F})\big(g^{-1}x, f(x)^{-1}y\big) \\
&= (g\mathcal{G})(x)\big[f(x)\mathcal{F}\big](y).
\end{aligned} \tag{4.6}$$

We are now in position to show that the subspaces $V_i \otimes W_0$ and $L(X) \otimes W_j$ are $F \wr G$-invariant. Let $v \otimes \mathbf{1} \in V_i \otimes W_0$, $\delta_x \otimes w \in L(X) \times W_j$, where δ_x is the Dirac delta at $x \in X$, and $(f, g) \in F \wr G$. We have $(f, g)[v \otimes \mathbf{1}] = gv \otimes \mathbf{1} \in V_i \otimes W_0$ and $(f, g)[\delta_x \otimes w] = \delta_{gx} \otimes f(x)w \in L(X) \otimes W_j$.

From (4.1) we have that the number of $F \wr G$-orbits, namely $n+m+1$ equals the number of $F \wr G$-invariant subspaces in (4.2) and Corollary 2.2 yields the first part of the statement.

We now determine the spherical functions. As $\phi_i \otimes \phi'_0 \in V_i \otimes W_0$ and $\delta_{x_0} \otimes \phi'_j \in L(X) \otimes W_j$ we are only left to the simple verification that these are J-invariant: for $(f, k) \in J$ we have $(f, k)[\phi_i \otimes \phi'_0] = (f, k)[\phi_i \otimes \mathbf{1}_Y] = (k\phi_i \otimes \mathbf{1}_Y) = \phi_i \otimes \mathbf{1}_Y$ since ϕ_i is a spherical function for (G, K) and therefore it is K-invariant. Analogously, recalling that x_0 is the point stabilized by K, one checks that $(f, k)[\delta_{x_0} \otimes \phi'_j] = \delta_{x_0} \otimes \phi'_j$. The proof is now complete. □

From (4.3) and using Garsia's criterion (Lemma 2.3) one easily proves that:

Proposition 4.3. *$(F \wr G, J)$ is symmetric if and only if (G, K) and (F, H) are symmetric.*

4.2. An application: The finite ultrametric space

In this subsection we apply the results of Theorem 4.2 to the case of the ultrametric space. These results were obtained by Letac [32]. See also [8,22,41].

According with the notation of Section 3 denote by $T_{\mathbf{r}}$, $\mathbf{r} = (q, q, \dots, q)$, a finite q-ary tree of depth m and by $T^*_{\mathbf{s}}$, $\mathbf{s} = (1, 1, \dots, 1)$, the $\mathbf{s}$-subtree given by the ray from the root to the leftmost leaf. Notice that the set $\mathcal{V}(\mathbf{r}, \mathbf{s})$ of all $\mathbf{s}$-subtrees can be identified with the set $X = \{0, 1, \dots, q-1\}^m = C_q^m$ of all leaves. We know that the group $\mathrm{Aut}(T_{\mathbf{r}}) = S_q \wr \cdots \wr S_q$ acts transitively on X and if $K = K(\mathbf{r}, \mathbf{s})$ denotes the stabilizer of $T^*_{\mathbf{s}} = z_0$ the pair $(\mathrm{Aut}(T_{\mathbf{r}}), K)$ is Gelfand. In this case one can give a direct proof of this fact by observing that the space X can be endowed with a distance d, by setting, for $x, y \in X$, $d(x, y) = m - h$ where h is the depth of the nearest common ancestor and checking that the action of $\mathrm{Aut}(T_{\mathbf{r}})$ is 2-point homogeneous (see Section 2.2) with respect to this distance [22].

For a tree of depth one $\mathrm{Aut}(T_{\mathbf{r}})$ coincides with S_q and the stabilizer K with the subgroup S_{q-1}. The space $L(S_q/S_{q-1}) = L(C_q)$ splits into two irreducible representations, namely V_0, the subspace of constant functions, and V_1, the subspace of functions of mean zero. Moreover the spherical function associated with V_0 is the constant function $\mathbf{1}$ while the spherical function associated with V_1 is given by

$$\phi(x) = \begin{cases} 1, & \text{if } x = 0, \\ -\frac{1}{q-1}, & \text{if } x \neq 0. \end{cases} \tag{4.7}$$

Setting $\mathbf{r}' = (q, q, \dots, q)$ ($m-1$ times) we have that $\mathrm{Aut}(T_{\mathbf{r}}) = \mathrm{Aut}(T_{\mathbf{r}'}) \wr S_q$. Applying recursively Theorem 4.2 we obtain the following decomposition:

$$L(X) = L\left(C_q^m\right) = \bigoplus_{j=0}^{m} W_j$$

where

$$W_j = \begin{cases} V_0^{\otimes^m}, & \text{if } j = 0, \\ V_1 \otimes V_0^{\otimes^{m-1}}, & \text{if } j = 1, \\ L(C_q^{j-1}) \otimes V_1 \otimes V_0^{\otimes^{m-j}}, & \text{if } j \geqslant 2. \end{cases}$$

From Theorem 4.2 we deduce that the spherical function $\phi_j \in W_j$ is

$$\phi_j(x_1, x_2, \dots x_m) = \delta_0(x_1) \cdots \delta_0(x_{j-1})\phi(x_j).$$

Since the distance of an element $x = (x_1, x_2, \dots, x_m)$ in X from the point z_0 is $m - k$, where k is the largest index such that $x_1 = \cdots = x_k = 0$, taking into account (4.7) we obtain

$$\phi_j(x) = \begin{cases} 1, & \text{if } d(x, z_0) < m - j + 1, \\ -\frac{1}{q-1}, & \text{if } d(x, z_0) = m - j + 1, \\ 0, & \text{if } d(x, z_0) > m - j + 1. \end{cases} \tag{4.8}$$

4.3. Another application: The Kaloujnine group

Let $K(q, m) = C_q \wr C_q \wr \cdots \wr C_q$, the m-iterated wreath product of the cyclic group C_q, be the Kaloujnine group [10]. $K(q, m)$ can be viewed as a subgroup of $\mathrm{Aut}(T_{\mathbf{q}})$ where $\mathbf{q} = (q, q, \dots, q)$, m times; thus it acts on the set of leaves of $T_{\mathbf{q}}$. Denote by $J(q, m)$ the subgroup of $K(q, m)$ which stabilizes the leftmost leaf.

As C_q is abelian, we have that $(C_q, \{e\})$ is a Gelfand pair. We identify the corresponding homogeneous space X with $\{0, 1, 2, \dots, q - 1\}$. Clearly

$$L(X) = \bigoplus_{j=0}^{q-1} V_j,$$

where V_j is the one-dimensional subspace spanned by the character $\phi_j(x) = \exp(2\pi i j x/q)$.

With the notation preceding Lemma 4.1, setting $G = C_q$, $K = \{e\}$, $F = K(q, m - 1)$ and $H = J(q, m - 1)$ we clearly have

$$K(q, m) = F \wr G \quad \text{and}$$

$$J(q, m) = J(q, m - 1) \times \underbrace{K(q, m - 1) \times \cdots \times K(q, m - 1)}_{m-1},$$

so that, combining an induction argument with Theorem 4.2, one has that $(K(q, m), J(q, m))$ is a (nonsymmetric) Gelfand pair.

For $j = 0, 1, 2, \dots, q - 1$ and $s \geqslant 1$ set

$$W_j^{0,s} = V_j \otimes V_0^{\otimes^{s-1}}$$

and, for $j = 1, 2, \dots, q-1$ and $t = 0, 1, \dots, m-1$,

$$W_j^{t,m} = L(X)^{\otimes^t} \otimes W_j^{0,m-t}.$$

This way one has the decomposition into irreducible $K(q, m)$-representations

$$L\big(K(q,m)/J(q,m)\big) \equiv L\big(X^m\big) \equiv L(X)^{\otimes^m} = W_0^{0,m} \oplus \left[\bigoplus_{t=0}^{m-1} \bigoplus_{j=1}^{q-1} W_j^{t,m}\right].$$

The spherical function in $W_j^{t,m}$ is then given by

$$\phi_j^{t,m}(x_1, x_2, \dots, x_m) = \begin{cases} 1, & \text{if } j = 0,\ t = 0, \\ \phi_j(x_1) \equiv \exp(2\pi i j x_1/q), & \text{if } j \neq 0,\ t = 0, \\ \delta_0(x_1) \cdots \delta_0(x_t) \exp(2\pi i j x_{t+1}/q), & \text{otherwise.} \end{cases}$$

5. Exponentiation of Gelfand pairs: The generalized Hamming scheme

This section is devoted to another particular case (namely $s_1 = r_1$) of the construction from Section 3 because, again, more general theories can be developed.

5.1. Gelfand pairs associated with semidirect products

The general construction presented below is inspired by the classical Frobenius theory of representations of semidirect products with abelian groups [37,38].

Let $G = NH = N \ltimes H$ be a finite group, semidirect product of N and H. Suppose that $K \leqslant N$ is an H-invariant subgroup of N and that (N, K) is a Gelfand pair. Denote by $X = N/K$ the homogeneous space associated with (N, K), by $L(X) = \bigoplus_{i=0}^n V_i$ the (multiplicity-free) decomposition of $L(X)$ into N-invariant irreducibles and by $\phi_i \in V_i$ the corresponding spherical functions.

Observe that the map

$$p : N/K \ni nK \mapsto nKH \in G/KH$$

is a bijection; as a consequence, the element nK (viewed as an element in $X \equiv N/K$) can be identified with the element nKH (viewed as an element in G/KH); also observe that $nKH = nhKH$ for any $h \in H$.

The action of G on $X \equiv G/KH$ is given by the rule

$$nh(n_0KH) = nhn_0h^{-1}KH \tag{5.1}$$

and the induced action on $L(X)$ is thus given by $[gf](x) = f(g^{-1}x)$, namely

$$[nhf](n_0KH) = f\big(h^{-1}n^{-1}n_0KH\big) = f\big(h^{-1}n^{-1}n_0hKH\big) = f\big(\big(n^{-1}n_0\big)^h KH\big)$$

for $f \in L(X)$, $g = nh$, $x = n_0 KH \in X$ and $n, n_0 \in N$, $h \in H$. In particular, $[hf](n_0 KH) = f(n_0^h KH)$. Moreover, if V_i is any irreducible N-invariant subspace in $L(X)$, $h \in H$ and $f \in V_i$, then $n[hf] = h[(h^{-1}nh)f]$ for all $n \in N$ which shows that the subspace hV_i is still N-invariant. Moreover it is also irreducible: indeed if ρ_i is the representation of N on V_i, then $\rho_i^h(n) := \rho_i(h^{-1}nh)$ defines a representation of N on V_i which is equivalent to that on hV_i. As a consequence of this, H permutes the V_i's. Denote by Γ_j, $j = 0, 1, \ldots, r$, the H-orbits on $\{V_0, V_1, \ldots, V_n\}$. Then the $L(X)$ subspaces

$$W_j = \bigoplus_{i:\ V_i \in \Gamma_j} V_i \tag{5.2}$$

are clearly G-invariant, G-irreducible and pairwise nonequivalent (the restrictions to $N \leqslant G$ of the representations W_j and $W_{j'}$ decompose into inequivalent subrepresentations for $j \neq j'$).

Thus

Theorem 5.1. *$L(X) = \bigoplus_{j=0}^r W_j$ is multiplicity-free and (G, KH) is a Gelfand pair. The corresponding spherical functions are given by*

$$\Phi_j = \frac{1}{|\Gamma_j|} \sum_{i:\ V_i \in \Gamma_j} \phi_i = \frac{1}{|H|} \sum_{h \in H} h\phi_i. \tag{5.3}$$

Note that $h\phi_i$ is the spherical function for hV_i and that the dimension of W_j is given by

$$\dim(W_j) = |\Gamma_j| \dim(V)$$

where $V \in \Gamma_j$.

Also, if $X = \bigsqcup_{i=0}^n \Xi_i$ is the partition of X into its K-orbits, observe that, as before, for each $h \in H$ the subset $h\Xi_i$ is still K-invariant; in other words H permutes the orbits Ξ_i's. Denote by Λ_j, $j = 0, 1, \ldots, r$, the corresponding KH-orbits, i.e. each Λ_j is the union of the Ξ_i's belonging to a single orbit of H on $\{\Xi_0, \Xi_1, \ldots, \Xi_n\}$.

Analogously let

$$X \times X = \bigsqcup_{i=0}^n \widetilde{\Xi}_i$$

be the partition into N-orbits of $X \times X$ (see (2.1)); observe that, given $h \in H$, $h\widetilde{\Xi}_i$ is still N-invariant and thus H permutes the orbits $\widetilde{\Xi}_i$'s. Denote by $\widetilde{\Lambda}_j$ the corresponding KH-orbits.

We thus have that $X = \bigsqcup_{j=0}^r \Lambda_j$ and $X \times X = \bigsqcup_{j=0}^r \widetilde{\Lambda}_j$ are the partitions of X and $X \times X$ into its KH-orbits and G-orbits, respectively. Note that the correspondence (cf. Section 2.1) between the K-orbits on X and the N-orbits on $X \times X$ parallels the correspondence between the KH-orbits on X and the G-orbits on $X \times X$.

From this it follows immediately that if $M_i : L(X) \to L(X)$ denote the Markov operators for (N, K), then

$$\overline{M_j} = \sum_{i:\ \widetilde{\Xi}_i \subseteq \widetilde{\Lambda}_j} M_i$$

are the Markov operators corresponding to (G, KH).

This gives another proof of the fact that (G, KH) is a Gelfand pair. Note that, by Garsia's criterion (Lemma 2.3), if (N, K) is symmetric, so is also (G, KH).

Remark 5.2. Let G be finite group and set $\widetilde{G} := \{(g, g):\ g \in G\}$. It is well known [14] that $(G \times G, \widetilde{G})$ is a Gelfand pair (if G is *ambivalent* [28], i.e. g^{-1} is conjugate to g, for every $g \in G$, then it is even symmetric). Set $N = \{e\} \times G$, $H = \widetilde{G}$ and $K = \{e\} \times \{e\}$. Then $G \times G$, which is the semidirect product of N by H, and $KH = \widetilde{G}$ constitute a Gelfand pair; however $(N, K) \cong (G, \{e\})$ is a Gelfand pair if and only if G is abelian. This shows that in Theorem 5.1 one does not have the inverse implication.

As the symmetry is concerned we again cannot invert the implication. Indeed, if $K = \{e, (1, 2)(3, 4)\} \cong \mathbb{Z}_2$ in Section 2.2 we showed that (A_4, K), is a nonsymmetric Gelfand pair; however $S_4 = A_4 \ltimes H$, where $H = \{e, (1, 2)\} \cong \mathbb{Z}_2$ and $(S_4, \mathbb{Z}_2 \times \mathbb{Z}_2)$ is well known to be symmetric (in general one has $(S_n, S_{n-h} \times S_h)$ is symmetric for all $1 \leqslant h \leqslant n$, see [11]).

5.2. An application: Gelfand pairs associated with semidirect products with abelian groups

Let $G = AH = A \ltimes H$ be a (finite) group, semidirect product of an *abelian* group A and H. Then Theorem 5.1 implies that (G, H) is a Gelfand pair.

In this case (5.1) corresponds to the action π of G on $A \cong G/H$ given by $\pi(ah)a_0 = aha_0h^{-1}$, $a, a_0 \in A$, $h \in H$. Note that H is the stabilizer of the unit element $e \in A$.

The corresponding representation of G on $L(A) = \{f : A \to \mathbb{C}\}$, the space of complex-valued functions on A, is given by $[(ah)f](a_0) = f(h^{-1}a^{-1}a_0h)$. In particular, if $\chi \in \hat{A}$ is a character of A, then $[(ah)\chi](a_0) = \chi(h^{-1}a^{-1}a_0h) = \chi(h^{-1}a^{-1}h)\chi(h^{-1}a_0h)$.

Denote by $\Gamma_0 = \{1_A\}, \Gamma_1, \ldots, \Gamma_k$ the H-orbits on $\hat{A}$ and by V_i the subspace of $L(A)$ generated by the characters in Γ_i, $i = 0, 1, \ldots, k$; it is then clear that $V_0, V_1, \ldots, V_k$ are G-invariant subspaces of $L(A)$.

From (5.3) the spherical function ϕ_i in V_i is given by

$$\phi_i(a_0) = \frac{1}{|\Gamma_i|} \sum_{\chi \in \Gamma_i} \chi(a_0) = \frac{1}{|H|} \sum_{h \in H} \chi_i\left(h^{-1}a_0^{-1}h\right)$$

where χ_i denotes a fixed character in Γ_i.

Suppose now that $G = A \wr H = A^X \ltimes H$, where A is abelian and H acts transitively on a set X. Then the above considerations yield that (G, H) is a Gelfand pair. The corresponding spherical functions are now given by

$$\phi_\chi(f_0) = \frac{1}{|H|}\sum_{h\in H}\chi\left(h^{-1}f_0h\right)$$

where $f_0 : X \to A$, $h_0 \in H$ and χ is a character of A^X. But $h^{-1}f_0h = [h^{-1}f_0]$ where $[h^{-1}f_0](x) = f_0(hx)$ and $\chi = \prod_{x\in X}\chi_x$, where χ_x is a character of A, for all $x \in X$. Thus, $\chi(h^{-1}f_0h) = \prod_{x\in X}\chi_x[f_0(hx)]$ and

$$\phi_\chi(f_0) = \frac{1}{|H|}\sum_{h\in H}\prod_{x\in X}\chi_x\left[f_0(hx)\right].$$

Example 5.3. As an application let $A = \mathbb{Z}_2$ and $H = S_n$. We identify $\mathbb{Z}_2$ with the multiplicative group $\{-1, +1\}$ and we denote by $\mathbf{1}$ and by $-\mathbf{1}$ the identity and the sign character, respectively. For $x = (x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_2)^n$ and $\chi = (\chi_1, \chi_2, \ldots, \chi_n)$ a character of $(\mathbb{Z}_2)^n$ denote by $w(x)$ the number of components x_i that equal -1 and by $W(\chi)$ the number of components χ_j that equal $-\mathbf{1}$, respectively.

The orbits of S_n on $(\mathbb{Z}_2)^n$ are given by $A_r = \{x\colon w(x) = r\}$, $r = 0, 1, \ldots, n$, and those on $(\mathbb{Z}_2^*)^n$, the dual of $(\mathbb{Z}_2)^n$, by $\mathcal{A}_r = \{\chi\colon W(\chi) = r\}$, $r = 0, 1, \ldots, n$.

Let $f_r \in A_r$ be the (unique) element in $(\mathbb{Z}_2)^n$ for which the first $n-r$ components equal 1 and the remaining r equal -1. Then for the spherical function ϕ_s corresponding to $\mathcal{A}_s$ we have the following expression:

$$\phi_s(f_r) = \frac{1}{\binom{n}{s}}\sum_{\chi\in\mathcal{A}_s}\chi(f_r) = \frac{1}{\binom{n}{s}}\sum_{j=\max\{0,s+r-n\}}^{\min\{s,r\}}(-1)^j\binom{r}{j}\binom{n-r}{s-j} = K_s\left(r; \frac{1}{2}; n\right)$$

where K_s denotes the Krawtchouk polynomial [17].

More generally one might replace $\mathbb{Z}_2$ in the previous example by a cyclic group of higher order (see Durbin [21], Vere-Jones [45], Dunkl and Ramirez [20] and the most recent paper by Mizukawa [33]). In this more general setting the corresponding spherical functions are given by the monomial symmetric functions (Theorem 3.5 in [33]) and have $(n+1, m+1)$-hypergeometrical expressions (Theorem 4.6 in [33]).

5.3. Cartesian product of Gelfand pairs

Let X be a finite set and (F, H) a finite Gelfand pair. Set $N = F^X$ and $K = H^X$ so that the corresponding homogeneous space is $N/K = Y^X$, where $Y = F/H$.

If $\phi_0, \phi_1, \ldots, \phi_n \in L(Y)$ are the spherical functions for (F, H), then

$$\phi_{\mathbf{i}} = \bigotimes_{x\in X}\phi_{\mathbf{i}(x)}, \quad \mathbf{i} \in \{0, 1, \ldots, n\}^X, \tag{5.4}$$

are the spherical function for (N, K). This way, if V_i is the F-invariant subspace containing the spherical function ϕ_i, then the corresponding N-invariant subspaces are

$$V_{\mathbf{i}} = \bigotimes_{x \in X} V_{\mathbf{i}(x)}$$

and

$$L(Y^X) = \bigoplus_{\mathbf{i} \in \{0,1,\dots,n\}^X} V_{\mathbf{i}}$$

is the decomposition into N-irreducibles.

Analogously, if $Y = \bigsqcup_{i=0}^{n} \Xi_i$ is the decomposition into H-orbits, then, setting $\Xi_{\mathbf{i}} = \{f \in Y^X\colon\ f(x) \in \Xi_{\mathbf{i}(x)}, \forall x \in X\}$ one has that

$$Y^X = \bigsqcup_{\mathbf{i} \in \{0,1,\dots,n\}^X} \Xi_{\mathbf{i}}$$

is the decomposition into its K-orbits.

5.4. *Exponentiation of Gelfand pairs: The generalized Hamming scheme*

Let G be a group of permutations acting transitively on a finite set X and (F, H) be a finite Gelfand pair. With the notation from previous section, as G leaves K invariant (by just permuting the H's), we have that $(F \wr G, H \wr G) \equiv (F^X \ltimes G, H^X \ltimes G)$ is a Gelfand pair. This follows directly from Sections 5.1 and 5.3 combined together.

We only explicitly describe the spherical functions, the irreducible decomposition: again this follows directly from Sections 5.1 and 5.3.

The corresponding spherical functions are given by (5.3) and (5.4); thus denoting by Γ_j, $j = 0, 1, \dots, r$, the G-orbits on the set of $V_{\mathbf{i}}$'s we have

$$\Phi_j = \frac{1}{|G|} \sum_{g \in G} g\phi_{\mathbf{i}} \equiv \frac{1}{|G|} \sum_{g \in G} \phi_{g^{-1}\mathbf{i}}$$

where $[g^{-1}\mathbf{i}](x) = \mathbf{i}(gx)$, for $g \in G$ and $x \in X$.

The corresponding invariant subspaces are

$$W_j = \bigoplus_{\mathbf{i}:\ V_{\mathbf{i}} \in \Gamma_j} V_{\mathbf{i}}.$$

We now particularize the construction from previous section with $G = \mathrm{Sym}(X)$ the symmetric group on the set X. This specific choice will lead to the explicit determination of all the orbits involved in the preceding arguments.

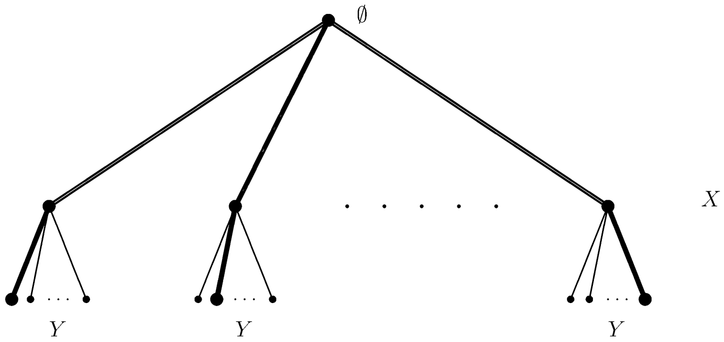

Fig. 3. Y^X coincides with the space of all subtrees of type $(|X|, 1)$ of the tree of $X \times Y$.

We first determine the $\mathrm{Sym}(X)$-orbits on the set $\{0, 1, \dots, n\}^X$. Given $\mathbf{i} \in \{0, 1, \dots, n\}^X$ and $0 \leqslant j \leqslant n$ set $a_j(\mathbf{i}) = |\{x \in X\colon\ \mathbf{i}(x) = j\}|$ and define the *type* of $\mathbf{i}$ as

$$a(\mathbf{i}) = \big(a_0(\mathbf{i}), a_1(\mathbf{i}), \dots, a_n(\mathbf{i})\big).$$

It is easy to see that $\mathbf{i}_1$ and $\mathbf{i}_2$ in $\{0, 1, \dots, n\}^X$ belong to the same $(H \wr \mathrm{Sym}(X))$-orbit if and only if they have the same type: $a(\mathbf{i}_1) = a(\mathbf{i}_2)$.

Then we immediately have that two spherical functions $\phi_{\mathbf{i}_1}$ and $\phi_{\mathbf{i}_2}$ (equivalently two F^X-irreducible spaces $V_{\mathbf{i}_1}$ and $V_{\mathbf{i}_2}$) are in the same G-orbit if and only if $a(\mathbf{i}_1) = a(\mathbf{i}_2)$.

Let now $Y = \bigsqcup_{i=0}^{n} \Lambda_i$ be the decomposition of X into its G-orbits.

Analogously, for $\theta \in Y^X$ and $0 \leqslant j \leqslant n$ set $\tau_j(\theta) = |\{x \in X\colon\ \theta(x) \in \Lambda_j\}|$ and define the *type* of θ as

$$\tau(\theta) = \big(\tau_0(\theta), \tau_1(\theta), \dots, \tau_n(\theta)\big).$$

See, for instance, Fig. 3.

It is easy to see that θ_1 and θ_2 in Y^X belong to the same $(H \wr \mathrm{Sym}(X))$-orbit if and only if they have the same type: $\tau(\theta_1) = \tau(\theta_2)$.

Then, again we immediately have that two H^X-orbits $\Lambda_{\mathbf{i}_1}$ and $\Lambda_{\mathbf{i}_2}$ (equivalently two F^X-orbits $\widetilde{\Lambda}_{\mathbf{i}_1}$ and $\widetilde{\Lambda}_{\mathbf{i}_2}$ in $Y^X \times Y^X \equiv (Y \times Y)^X$) are in the same G-orbit if and only if $\tau(\theta_1) = \tau(\theta_2)$.

Combining these last observations with the preceding sections we get the following description for the generalized Hamming scheme; compare with [2, pp. 297–298].

Theorem 5.4. *Let X be a finite set and (F, H) a finite Gelfand pair. Then*:

(1) *For $\mathbf{a} \in \mathbb{N}^{\{0,1,\dots,n\}}$ such that $\sum_{j=0}^{n} a_j = |X|$ set $W_{\mathbf{a}} = \bigoplus_{\mathbf{i}\colon a(\mathbf{i})=\mathbf{a}} V_{\mathbf{i}}$. Then the $W_{\mathbf{a}}$'s are distinct irreducible representations of $F \wr \mathrm{Sym}(X)$ and*

$$\dim(W_{\mathbf{a}}) = \binom{|X|}{\mathbf{a}} \dim(V_1)^{a_1} \dim(V_2)^{a_2} \cdots \dim(V_n)^{a_n}.$$

(2) $L(X^Y) = \bigoplus_{\mathbf{a}\in\mathbb{N}^{\{0,1,\dots,n\}}:\ \sum_{j=0}^{n} a_j = |X|} W_{\mathbf{a}}$ *is the decomposition of* $L(Y^X)$ *into its irreducible components*; *in particular* $(F \wr \mathrm{Sym}(X), H \wr \mathrm{Sym}(X))$ *is a Gelfand pair.*

(3) *The spherical function in* $W_{\mathbf{a}}$ *is given by*

$$\Phi_{\mathbf{a}} = \frac{1}{\binom{|X|}{\mathbf{a}}} \sum_{\mathbf{i}:\ a(\mathbf{i})=\mathbf{a}} \phi_{\mathbf{i}}.$$

6. The generalized Johnson scheme

6.1. Induced representations and induced operators

We start by recalling the definition of induced representation [37]. Let G be a finite group, $K \leqslant G$ a subgroup, (ρ, V) a representation of G and W a K-invariant subspace of V. Suppose that S is a system of representatives for the set of left cosets G/K, that is $G = \bigsqcup_{s\in S} sK$. V is said to be induced by W if one has the following direct sum decomposition: $V = \bigoplus_{s\in S} \rho(s)W$. The standard notation is $V = \mathrm{Ind}_K^G W$. Note also that $\dim V = |G/K| \cdot \dim W$.

Suppose now that (ρ_1, V_1) and (ρ_2, V_2) are two representations of G, $V_i = \mathrm{Ind}_K^G(W_i)$, $i = 1, 2$, and that $\tau : W_1 \to W_2$ is a K-intertwining operator. We define the operator $\mathrm{Ind}_K^G \tau : V_1 \to V_2$ by setting, if $v = \sum_{s\in S} \rho_1(s) w_s$ is an element of V_1 (thus $w_s \in W_1$ for every $s \in S$)

$$\big(\mathrm{Ind}_K^G \tau\big)(v) = \sum_{s\in S} \rho_2(s)\tau w_s. \tag{6.1}$$

Lemma 6.1.

(1) $\mathrm{Ind}_K^G \tau$ *intertwines* V_1 *and* V_2;

(2) $\ker \mathrm{Ind}_K^G \tau = \mathrm{Ind}_K^G \ker \tau$;

(3) $\mathrm{ran}\, \mathrm{Ind}_K^G \tau = \mathrm{Ind}_K^G \mathrm{ran}\, \tau$, *where* $\mathrm{ran}\, T$ *denotes the range of the operator* T.

Proof. Suppose that $v = \sum_{s\in S} \rho_1(s) w_s \in V_1$. If $g \in G$ then for every $s \in S$ there exist $t_s \in S$ and $k_s \in K$ such that $gs = t_s k_s$. Therefore $\rho_1(g)v = \sum_{s\in S} \rho_1(gs) w_s = \sum_{s\in S} \rho_1(t_s)[\rho_1(k_s) w_s]$ and thus

$$\begin{aligned}
\big[\big(\mathrm{Ind}_K^G \tau\big)\rho_1(g)\big](v) &= \sum_{s\in S} \rho_2(t_s)\tau\big[\rho_1(k_s) w_s\big] \\
&= \sum_{s\in S} \rho_2(t_s)\rho_2(k_s)\tau w_s \\
&= \rho_2(g) \sum_{s\in S} \rho_2(s)\tau w_s \\
&= \big[\rho_2(g)\, \mathrm{Ind}_K^G(\tau)\big](v).
\end{aligned}$$

The points (2) and (3) are obvious. □

6.2. The Johnson scheme

In this subsection, we recall some basic facts on the Johnson scheme, i.e. the Gelfand pair $(S_n, S_{n-h} \times S_h)$; see [17–19]. In what follows, n is a fixed positive integer and, for $0 \leqslant h \leqslant n$, Ω_h denotes the homogeneous space $S_n/(S_{n-h} \times S_h)$, i.e. the space of all h-subsets of $\{1, 2, \ldots, n\}$. The permutation module $L(\Omega_h)$ is denoted by $M^{n-h,h}$. We define the intertwining operator (or Radon transform [36]) $d : M^{n-h,h} \to M^{n-h+1,h-1}$ by setting $(d\gamma)(B) = \sum_{A \in \Omega_h:\ B \subseteq A} \gamma(A)$ for every $B \in \Omega_{h-1}$ and $\gamma \in M^{n-h,h}$. The adjoint of d is the operator d^* defined by setting $(d^*\beta)(A) = \sum_{B \in \Omega_{h-1}:\ B \subseteq A} \beta(B)$. The following theorem is well known (see, for instance, [11]) and gives the basic properties of the Johnson scheme in terms of the operators d and d^*.

Theorem 6.2.

(1) *For* $0 \leqslant k \leqslant n/2$, $M^{n-k,k} \cap \ker d$ *is an irreducible representation of* S_n *and its dimension is equal to* $\binom{n}{k} - \binom{n}{k-1}$;

(2) *If* $0 \leqslant k \leqslant \min\{n-h, h\}$ *then* $(d^*)^{h-k}$ *maps* $M^{n-k,k} \cap \ker d$ *one to one into* $M^{n-h,h}$;

(3) $M^{n-h,h} = \bigoplus_{k=0}^{\min\{n-h,h\}} (d^*)^{h-k}(M^{n-k,k} \cap \ker d)$ *is the decomposition of* $M^{n-h,h}$ *into* S_n*-irreducible representations.*

Using a standard notation in the representation theory of the symmetric group, the irreducible representation $M^{n-k,k} \cap \ker d$ will be denoted by $S^{n-k,k}$. We will also use the following notation: if $0 \leqslant u \leqslant v \leqslant n$ and $A \in \Omega_v$ then $\Omega_u(A)$ will denote the space of all u-subsets of A and $M^{v-u,u}(A)$ the space $L(\Omega_u(A))$ seen as a module over the symmetric group S_v of all permutations of A (but note that when we write Ω_h we indicate $\Omega_h(\{1, 2, \ldots, n\})$).

We recall that if we set, for $A, B \in \Omega_h$, $\delta(A, B) = h - |A \cap B|$, then δ is a metric on Ω_h and the group S_n acts two point homogeneously with respect to δ.

Thus, fixing a point $\overline{A}$ in Ω_h and denoting by $S_{n-h} \times S_h$ its stabilizer, then the spherical functions may be seen as radial functions, i.e. functions of the variable $\delta(A, B)$. Setting, for $0 \leqslant u \leqslant \min\{n-h, h\}$, $\sigma_u = \{A \in \Omega_h\colon\ \delta(A, \overline{A}) = u\}$, then the spherical function $\psi(n, h, k)$ of $(S_n, S_{n-h} \times S_h)$ belonging to the subspace isomorphic to $S^{n-k,k}$ is given by

$$\psi(n, h, k) = \sum_{u=0}^{\min\{n-h,h\}} \psi(n, h, k; u)\chi_{\sigma_u} \tag{6.2}$$

where χ_{σ_u} denotes the characteristic function of the set σ_u and the coefficient $\psi(n, h, k; u)$ can be expressed in terms of the Hahn polynomials: $\psi(n, h, k; u) = Q_k(u; -(n-h) - 1, -h - 1, h)$ where $Q_k(x; \alpha, \beta, N) = \sum_{i=0}^{k} \frac{(-k)_i(k+\alpha+\beta+1)_i(-x)_i}{(-N)_i(\alpha+1)_i i!}$.

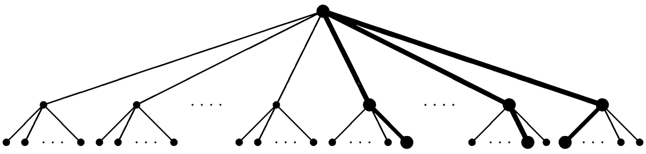

Fig. 4. An element $\theta \in \Theta_h$ coincides with a subtree of type $(h, 1)$ in the tree $\{1, 2, \ldots, n\} \times Y$.

6.3. *The homogeneous space* Θ_h

Let (F, H) be a finite Gelfand pair, $Y = F/H$ and $y_0 \in Y$ the point stabilized by H. Suppose that $Y = \bigsqcup_{i=0}^m \Lambda_i$ is the decomposition of Y into its H-orbits (with $\Lambda_0 = \{y_0\}$), $L(Y) = \bigoplus_{i=0}^m W_i$ is the decomposition of $L(Y)$ into irreducible representations of F (with $W_0 =$ the trivial representation) and ϕ_i is the spherical function in W_i, $i = 0, 1, \ldots, m$. Let S_n be the symmetric group on $\{1, 2, \ldots, n\}$ and for $0 \leqslant h \leqslant n$, let Ω_h be the S_n-homogeneous space ($\equiv S_n/S_{n-h} \times S_h$) consisting of all h-subsets of $\{1, 2, \ldots, n\}$. We consider the wreath product $F \wr S_n$ of F and S_n (with respect to the action of S_n on $\{1, 2, \ldots, n\}$) and we construct a natural homogeneous space of $F \wr S_n$ using the actions of F on Y and of S_n on Ω_h.

Let Θ_h be the set of all functions $\theta : A \to Y$ whose domain is an element of Ω_h ($A \in \Omega_h$) and whose range is Y. See, for instance, Fig. 4. In other words

$$\Theta_h = \bigsqcup_{A \in \Omega_h} Y^A. \tag{6.3}$$

If $\theta \in \Theta_h$ and $\theta : A \to Y$ then we will write $\operatorname{dom}\theta = A$ (the domain of definition of θ). The group $F \wr S_n$ acts on Θ_h in a natural way: if $(f, \pi) \in F \wr S_n$ and $\theta \in \Theta_h$ then $(f, \pi)\theta$ is the function, with domain $\pi \operatorname{dom}\theta$, defined by setting

$$\big[(f, \pi)\theta\big](j) = f(j)\theta\big(\pi^{-1} j\big) \tag{6.4}$$

for every $j \in \pi \operatorname{dom}\theta$. It is clear that this action is transitive.

If $\overline{A}$ is the element in Ω_h stabilized by $S_{n-h} \times S_h$ and we define $\theta_0 \in Y^{\overline{A}} \subseteq \Theta_h$ by setting $\theta_0(j) = y_0$ for every $j \in \overline{A}$, then it easy to check that the stabilizer of θ_0 is equal to $(H \wr S_h) \times (F \wr S_{n-h})$; therefore we can write $\Theta_h = (F \wr S_n)/[(H \wr S_h) \times (F \wr S_{n-h})]$.

We recall [39] that a *weak* $(m + 1)$*-composition* of h is an *ordered* sequence $\mathbf{a} = (a_0, a_1, \ldots a_m)$ of $m + 1$ *nonnegative* integers such that $a_0 + a_1 + \cdots + a_m = h$. In what follows, the set of all weak $(m + 1)$-compositions of h will be denoted by $C(h, m + 1)$ (we also recall that $|C(h, m + 1)| = \binom{m+h}{m}$). For $\mathbf{a} = (a_0, a_1, \ldots, a_m) \in C(h, m + 1)$ we set $\ell(\mathbf{a}) = a_1 + a_2 + \cdots + a_m \equiv h - a_0$.

If $\mathbf{a} \in C(h, m + 1)$ and $A \in \Omega_h$ then a *composition* (or *ordered partition*) of A of type $\mathbf{a}$ is an *ordered* sequence $\mathbf{A} = (A_0, A_1, \ldots, A_m)$ of subsets of A such that $A = \bigsqcup_{i=0}^m A_i$ and $|A_i| = a_i$, $i = 0, 1, \ldots, m$. The set of all compositions of A of type $\mathbf{a}$ will be denoted by $\Omega_{\mathbf{a}}(A)$.

Definition 6.3. For $\theta \in \Theta_h$ we define the *type* of θ as the sequence of nonnegative integers $\mathrm{type}(\theta) = (t, b_0, b_1, \dots, b_m)$ where $t = |\mathrm{dom}\,\theta \cap \overline{A}|$ and $b_i = |\{j \in \mathrm{dom}\,\theta \cap \overline{A}\colon \theta(j) \in \Lambda_i\}|$, $i = 0, 1, \dots, m$.

Lemma 6.4. *The orbits of* $(H \wr S_h) \times (F \wr S_{n-h})$ *on* Θ_h *are parametrized by the set*

$$\big\{(t, \mathbf{b})\colon \max\{0, 2h-n\} \leqslant t \leqslant h,\ \mathbf{b} \in C(t, m+1)\big\} \equiv \bigsqcup_{t=\max\{0,2h-n\}}^{h} C(t, m+1).$$

Proof. Two points $\theta_1, \theta_2 \in \Theta_h$ belong to the same orbit of $(H \wr S_h) \times (F \wr S_{n-h})$ if and only if $\mathrm{type}(\theta_1) = \mathrm{type}(\theta_2)$. Moreover, if $\mathrm{type}(\theta) = (t, b_0, b_1, \dots, b_m)$ then $\sum_{i=0}^m b_i = t$ and $t = |\mathrm{dom}\,\theta \cap \overline{A}|$ is subject (only) to the conditions $\max\{0, 2h-n\} \leqslant t \leqslant h$. □

Clearly, in the case $2h \leqslant n$, $\bigsqcup_{t=0}^{h} C(t, m+1)$ is the same thing as $C(h, m+2)$ and it is bijective to the set $\{(i_1, i_2, \dots, i_h)\colon 0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_h \leqslant m+1\}$.

We end this subsection introducing two intertwining operators (or *Radon transforms* [36]) between the permutation representations on Θ_h and Θ_{h-1}. We will use the following notation: if $\theta \in \Theta_h$ and $\xi \in \Theta_k$, $k < h$, we will write $\xi \subseteq \theta$ when $\mathrm{dom}\,\xi \subseteq \mathrm{dom}\,\theta$ and $\theta|_{\mathrm{dom}\,\xi} = \xi$.

Definition 6.5. We define the intertwining operator $D\colon L(\Theta_h) \to L(\Theta_{h-1})$ by setting

$$(D\mathcal{F})(\xi) = \sum_{\theta \in \Omega_h\colon \xi \subseteq \theta} \mathcal{F}(\theta) \quad \text{for every } \mathcal{F} \in L(\Theta_h),\ \xi \in \Theta_{h-1}.$$

The adjoint $D^*\colon L(\Theta_{h-1}) \to L(\Theta_h)$ of D is the operator defined by

$$(D^*\mathcal{G})(\theta) = \sum_{\xi \in \Omega_{h-1}\colon \xi \subseteq \theta} \mathcal{G}(\xi) \quad \text{for every } \mathcal{G} \in L(\Theta_{h-1}),\ \theta \in \Theta_h.$$

6.4. *On two kinds of tensor product*

Now we introduce two kinds of tensor product. For the first one, suppose that $A \in \Omega_h$. Then there is a natural isomorphism between $L(Y^A)$ and $L(Y)^{\otimes^h}$: if we are given, for every $j \in A$, a function $\mathcal{F}^j \in L(Y)$, then the tensor product $\bigotimes_{j \in A} \mathcal{F}^j$ of the functions $\mathcal{F}^j$ (*over* A) coincides with the function in $L(Y^A)$ defined by setting

$$\Big(\bigotimes_{j \in A} \mathcal{F}^j\Big)(\theta) = \prod_{j \in A} \mathcal{F}^j\big(\theta(j)\big) \quad \text{for every } \theta \in Y^A. \tag{6.5}$$

For the second kind, suppose that $\mathbf{a} \in C(h, m+1)$, $B \in \Omega_{\ell(\mathbf{a})}$, $(A_1, A_2, \dots, A_m) \in \Omega_{(a_1, a_2, \dots, a_m)}(B)$, $\mathcal{F}^j \in W_i$ for every $j \in A_i$, $i = 1, 2, \dots, m$, and that $\gamma \in M^{n-h, a_0}(\complement B)$.

Then we can define the tensor product $\gamma \otimes (\bigotimes_{j\in B} \mathcal{F}^j)$ by setting, for every $\theta \in \Theta_h$ satisfying the condition $\operatorname{dom}\theta \supseteq B$,

$$\left[\gamma \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right](\theta) = \gamma(\operatorname{dom}\theta \setminus B) \cdot \prod_{j\in B} \mathcal{F}^j\big(\theta(j)\big).$$

Clearly, a tensor product of the second kind may be expressed by means of tensor products of the first kind:

$$\gamma \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right) = \sum_{A_0\in\Omega_{a_0}(\complement B)} \gamma(A_0)\left[\left(\bigotimes_{j\in A_0} \phi_0\right) \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right] \tag{6.6}$$

where $(\bigotimes_{j\in A_0} \phi_0)$ is the constant function $\equiv 1$ on Y^{A_0}.

Now we describe the action of the group $F \wr S_n$ on such tensor products.

Lemma 6.6. *An element $(f,\pi) \in F \wr S_n$ acts on the above introduced tensor products obeying the following rules*:

(1) $(f,\pi)\left(\bigotimes_{j\in A} \mathcal{F}^j\right) = \bigotimes_{t\in\pi A} f(t)\mathcal{F}^{\pi^{-1}t}$;

(2) $(f,\pi)\left[\gamma \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right] = (\pi\gamma) \otimes \left(\bigotimes_{t\in\pi B} f(t)\mathcal{F}^{\pi^{-1}t}\right)$.

Proof. If $\theta \in Y^{\pi A}$ then

$$\begin{aligned}
\left[(f,\pi)\left(\bigotimes_{j\in A} \mathcal{F}^j\right)\right](\theta) &= \left(\bigotimes_{j\in A} \mathcal{F}^j\right)\big[(f,\pi)^{-1}\theta\big] = \prod_{j\in A} \mathcal{F}^j\big\{\big[(f,\pi)^{-1}\theta\big](j)\big\} \\
&= \prod_{j\in A} \mathcal{F}^j\big[f(\pi j)^{-1}\theta(\pi j)\big] = \prod_{t\in\pi A} \big[f(t)\mathcal{F}^{\pi^{-1}t}\big]\big(\theta(t)\big) \\
&= \left[\bigotimes_{t\in\pi A} f(t)\mathcal{F}^{\pi^{-1}t}\right](\theta).
\end{aligned}$$

Then (2) may be proved by mean of the decomposition (6.6). □

We end this subsection proving a formula that relates the action of the operators D and D^* on a tensor product of the second kind with the action of the operators d and d^*.

Lemma 6.7.

(1) $D[\gamma \otimes (\bigotimes_{j\in B} \mathcal{F}^j)] = |Y|[(d\gamma) \otimes (\bigotimes_{j\in B} \mathcal{F}^j)]$;
(2) $D^*[\gamma \otimes (\bigotimes_{j\in B} \mathcal{F}^j)] = (d^*\gamma) \otimes (\bigotimes_{j\in B} \mathcal{F}^j)$.

Proof. Since $[\gamma \otimes (\bigotimes_{j\in B} \mathcal{F}^j)](\theta)$ is defined for those θ such that $\operatorname{dom}\theta \supseteq B$, then $\{D[\gamma \otimes (\bigotimes_{j\in B} \mathcal{F}^j)]\}(\xi)$ is defined for those $\xi \in \Theta_{h-1}$ satisfying the condition

$$|B \setminus \operatorname{dom}\xi| \leqslant 1,$$

i.e. for those ξ for which there exists $\theta \in \Theta_h$ such that $\operatorname{dom}\theta \supseteq B$ and $\xi \subseteq \theta$. But if $|B \setminus \operatorname{dom}\xi| = 0$, i.e. $\operatorname{dom}\xi \supseteq B$, then

$$\begin{aligned}
\left\{D\left[\gamma \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right]\right\}(\xi) &= \sum_{\theta\in\Theta_h:\ \theta\supseteq\xi,\ \operatorname{dom}\theta\supseteq B} \left[\gamma \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right](\theta) \\
&= \sum_{\theta\in\Theta_h:\ \theta\supseteq\xi} \gamma(\operatorname{dom}\theta \setminus B) \cdot \prod_{j\in B} \mathcal{F}^j\big(\theta(j)\big) \\
&= \sum_{v\in\complement\operatorname{dom}\xi} \sum_{y\in Y} \gamma\big[\big(\operatorname{dom}\xi \sqcup \{v\}\big) \setminus B\big] \cdot \prod_{j\in B} \mathcal{F}^j\big(\theta(j)\big) \\
&= |Y|(d\gamma)(\operatorname{dom}\xi \setminus B) \cdot \bigsqcup_{j\in B} \mathcal{F}^j\big(\xi(j)\big) \\
&= |Y|\left[(d\gamma) \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right](\xi),
\end{aligned}$$

while if $|B \setminus \operatorname{dom}\xi| = 1$ and u is the unique element in $B \setminus \operatorname{dom}\xi$ then

$$\left\{D\left[\gamma \otimes \left(\bigotimes_{j\in B} \mathcal{F}^j\right)\right]\right\}(\xi) = \gamma\big[\big(\operatorname{dom}\xi \sqcup \{u\}\big) \setminus B\big] \cdot \left(\sum_{y\in Y} \mathcal{F}^u(y)\right) \cdot \prod_{j\in B\setminus\{u\}} \mathcal{F}^j\big(\xi(j)\big) = 0$$

since $\mathcal{F}^u \notin W_0$. In particular, if $a_0 = 0$ then $D[\gamma \otimes (\bigotimes_{j\in B} \mathcal{F}^j)] = 0$. The proof of (2) is similar. □

6.5. The decomposition of $L(\Theta_h)$ into irreducible representations

We recall that $L(Y) = \bigoplus_{i=0}^m W_i$ denotes the decomposition of $L(Y)$ into F-irreducible representations.

Definition 6.8. If $\mathbf{a} \in C(h, m+1)$ and $\mathbf{A} = (A_0, A_1, \ldots, A_m) \in \Omega_{\mathbf{a}}(A)$ then

(1) $W_{\mathbf{a}}(\mathbf{A})$ will denote the subspace of $L(Y^A)$ spanned by all tensor products $\bigotimes_{j\in A}\mathcal{F}^j$ such that $\mathcal{F}^j \in W_i$ for every $j \in A_i$, $i = 0, 1, \ldots, m$;
(2) we define

$$W_{h,\mathbf{a}} = \bigoplus_{A\in\Omega_h} \bigoplus_{\mathbf{A}\in\Omega_{\mathbf{a}}(A)} W_{\mathbf{a}}(\mathbf{A}).$$

Clearly, $W_{h,\mathbf{a}}$ coincides with the subspace of $L(\Theta_h)$ spanned by all the tensor products $\gamma \otimes (\bigotimes_{j\in B}\mathcal{F}^j)$ where $B \in \Omega_{\ell(\mathbf{a})}$, $\gamma \in M^{n-h,a_0}(\complement B)$ and there exists $(A_1, A_2, \ldots, A_m) \in \Omega_{(a_1,a_2,\ldots,a_m)}(B)$ such that $\mathcal{F}^j \in W_i$ for every $j \in A_i$, $i = 1, 2, \ldots, m$. Moreover, from Lemma 6.6 it follows that each $W_{h,\mathbf{a}}$ is an $F \wr S_n$-invariant subspace of $L(\Theta_h)$.

Lemma 6.9.

$$W_{h,\mathbf{a}} = \mathrm{Ind}^{F\wr S_n}_{F\wr S_{n-h}\times F\wr S_{a_0}\times F\wr S_{a_1}\times\cdots\times F\wr S_{a_m}}\left(I_{F\wr S_{n-h}} \otimes W_0^{\otimes a_0} \otimes W_1^{\otimes a_1} \otimes \cdots \otimes W_m^{\otimes a_m}\right)$$

where $I_{F\wr S_{n-h}}$ is the identity representation of $F \wr S_{n-h}$.

Proof. We first observe the following simple facts on wreath products (for the notation see Section 2.3).

Claim 6.10. *Let F and G be finite groups. Suppose that G acts on a finite set X and that $H \leqslant G$ is a subgroup. Then*

$$(F \wr G)/(F \wr H) \cong G/H. \tag{6.7}$$

Proof. For $g_1, g_2 \in G$ write $g_1 \sim_H g_2$ if there exists $h \in H$ such that $g_1 = hg_2$, equivalently if g_1 and g_2 belong to the same H-lateral: $Hg_1 = Hg_2$. Analogously, for $f_1, f_2 \in F^X$ and $g_1, g_2 \in G$, write $(f_1, g_2) \sim_{F\wr H} (f_2, g_2)$ if there exists $(f, h) \in F \wr H$ such that $(f_1, g_2) = (f, h)(f_2, g_2)$. Denoting as usual by $\mathbf{1} \in F^X$ the constant function $\mathbf{1}(x) = e_F$, where e_F is the unit element in F, one easily shows that $(f, g) \sim_{F\wr H} (\mathbf{1}, g)$ for all $f \in F^X$ and $g \in G$ and then that, for all $g_1, g_2 \in G$, $(\mathbf{1}, g_1) \sim_{F\wr H} (\mathbf{1}, g_2)$ if and only if $g_1 \sim_H g_2$. □

Claim 6.11. *Let F, G_1 and G_2 be finite groups. Suppose that G_i acts on a finite set X_i for $i = 1, 2$. Then $G_1 \times G_2$ acts on $X = X_1 \sqcup X_2$ and*

$$F \wr (G_1 \times G_2) \cong F \wr G_1 \times F \wr G_2. \tag{6.8}$$

Proof. One easily checks that the map

$$F \wr (G_1 \times G_2) \equiv F^X \times (G_1 \times G_2) \to \left(F^{X_1} \times G_1\right) \times \left(F^{X_2} \times G_2\right) \equiv (F \wr G_1) \times (F \wr G_2),$$

$$\left(f, (g_1, g_2)\right) \mapsto \left((f|_{X_1}, g_1), (f|_{X_2}, g_2)\right)$$

is an isomorphism. □

Applying (6.7) and (6.8) we have

$$\begin{aligned} &F \wr S_n/(F \wr S_{n-h} \times F \wr S_{a_0} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}) \\ &\quad \equiv S_n/(S_{n-h} \times S_{a_0} \times S_{a_1} \times \cdots \times S_{a_m}) \\ &\quad \equiv \bigsqcup_{A \in \Omega_h} \bigsqcup_{\mathbf{A} \in \Omega_{\mathbf{a}}(A)} \mathbf{A}. \end{aligned}$$

Moreover, if $\overline{\mathbf{A}} = (A_0, A_1, \ldots, A_m) \in \Omega_{\mathbf{a}}(\overline{A})$ is stabilized by $S_{n-h} \times S_{a_0} \times S_{a_1} \times \cdots \times S_{a_m}$ (i.e. S_{a_i} is the symmetric group on A_i, $i = 0, 1, \ldots, m$) then $W_{\mathbf{a}}(\overline{\mathbf{A}})$, as a representation of $F \wr S_{n-h} \times F \wr S_{a_0} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}$, is clearly equivalent to $I_{F \wr S_{n-h}} \otimes W_0^{\otimes a_0} \otimes W_1^{\otimes a_1} \otimes \cdots \otimes W_m^{\otimes a_m}$. Then the lemma follows from the definition of $W_{h,\mathbf{a}}$. □

The following corollary is a consequence of (6.1) and Lemmas 6.7, 6.9.

Corollary 6.12.

(1) $D = |Y| \cdot \mathrm{Ind}_{F \wr S_{n-\ell(\mathbf{a})} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}}^{F \wr S_n} d \otimes I \otimes \cdots \otimes I$;
(2) $D^* = \mathrm{Ind}_{F \wr S_{n-\ell(\mathbf{a})} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}}^{F \wr S_n} d^* \otimes I \otimes \cdots \otimes I$.

Definition 6.13. For $0 \leqslant k \leqslant (n - \ell(\mathbf{a}))/2$ we set

$$W_{h,\mathbf{a},k} = \mathrm{Ind}_{F \wr S_{n-\ell(\mathbf{a})} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}}^{F \wr S_n} S^{n-\ell(\mathbf{a})-k,k} \otimes W_1^{\otimes a_1} \otimes \cdots \otimes W_m^{\otimes a_m}.$$

Clearly,

$$\begin{aligned} \dim W_{h,\mathbf{a},k} = &\binom{n}{n-\ell(\mathbf{a}), a_1, \ldots, a_m} \left[\binom{n-\ell(\mathbf{a})-k}{k} - \binom{n-\ell(\mathbf{a})-k}{k-1} \right] \\ &\times (\dim W_1)^{a_1} (\dim W_2)^{a_2} \cdots (\dim W_m)^{a_m}. \end{aligned}$$

Lemma 6.14.

$$W_{h,\mathbf{a}} = \bigoplus_{k=0}^{\min\{n-h, h-\ell(\mathbf{a})\}} W_{h,\mathbf{a},k}.$$

Proof. By transitivity of induction, we can write:

$$\begin{aligned} &\mathrm{Ind}_{F \wr S_{n-h} \times F \wr S_{a_0} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}}^{F \wr S_n} \\ &\quad = \mathrm{Ind}_{F \wr S_{n-h+a_0} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}}^{F \wr S_n} \mathrm{Ind}_{F \wr S_{n-h} \times F \wr S_{a_0} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}}^{F \wr S_{n-h+a_0} \times F \wr S_{a_1} \times \cdots \times F \wr S_{a_m}} \end{aligned}$$

and since

$$\begin{aligned}&\mathrm{Ind}_{F\wr S_{n-h}\times F\wr S_{a_0}\times F\wr S_{a_1}\times\cdots\times F\wr S_{a_m}}^{F\wr S_{n-h+a_0}\times F\wr S_{a_1}\times\cdots\times F\wr S_{a_m}}\big(I_{F\wr S_{n-h}}\otimes W_0^{\otimes a_0}\otimes W_1^{\otimes a_1}\otimes\cdots\otimes W_m^{\otimes a_m}\big)\\ &\quad= M^{n-h,a_0}\otimes W_1^{\otimes a_1}\otimes\cdots\otimes W_m^{\otimes a_m}\end{aligned}$$

($I_{F\wr S_{n-h}}\otimes W_0^{\otimes a_0}$ is the trivial representation), the lemma follows from the decomposition $M^{n-h,a_0}=\bigoplus_{k=0}^{\min\{n-h,a_0\}} S^{n-h+a_0-k,k}$ (Theorem 6.2). □

The following corollary is a consequence of Lemma 6.1, Theorem 6.2, Corollary 6.12 and Lemma 6.14. It shows how to construct the representations $W_{h,\mathbf{a},k}$ using the operators D and D^*. We set $\mathbf{a}^{(-k)}=(a_0-k,a_1,\dots,a_m)$.

Corollary 6.15.

(1) $W_{k+\ell(\mathbf{a}),\mathbf{a},k}=\ker D\cap W_{k+\ell(\mathbf{a}),\mathbf{a}}$.
(2) *If* $0\leqslant k\leqslant\min\{n-h,h-\ell(\mathbf{a})\}$ *then* $(D^*)^{h-k-\ell(\mathbf{a})}$ *is an isomorphism of* $W_{k+\ell(\mathbf{a}),\mathbf{a}^{(-k)},k}$ *onto* $W_{h,\mathbf{a},k}$.

Theorem 6.16.

(1) $\{W_{h,\mathbf{a},k}\colon \mathbf{a}\in C(h,m+1),\ 0\leqslant k\leqslant\min\{n-h,h-\ell(\mathbf{a})\}\}$ *is a set of pairwise inequivalent irreducible representations of* $F\wr S_n$.
(2) $(F\wr S_n,(H\wr S_h)\times(F\wr S_{n-h}))$ *is a Gelfand pair.*
(3) *The decomposition of* $L(\Theta_h)$ *into irreducible representations is given by*

$$L(\Theta_h)=\bigoplus_{\mathbf{a}\in C(h,m+1)}\ \bigoplus_{k=0}^{\min\{n-h,h-\ell(\mathbf{a})\}} W_{h,\mathbf{a},k}. \tag{6.9}$$

Proof. From (6.3) we obtain immediately the following decomposition of $L(\Theta_h)$:

$$L(\Theta_h)=\bigoplus_{A\in\Omega_h} L\big(Y^A\big). \tag{6.10}$$

Moreover, from the decomposition $L(Y)=\bigoplus_{i=0}^m W_i$ of $L(Y)$ into irreducible representations and from the definition of $W_{\mathbf{a}}(\mathbf{A})$ it follows that

$$\begin{aligned}L\big(Y^A\big)\cong L(Y)^{\otimes h}&=\bigoplus_{l_1=0}^m\bigoplus_{l_2=0}^m\cdots\bigoplus_{l_h=0}^m W_{l_1}\otimes W_{l_2}\otimes\cdots\otimes W_{l_h}\\ &=\bigoplus_{\mathbf{a}\in C(h,m+1)}\ \bigoplus_{\mathbf{A}\in\Omega_{\mathbf{a}}(A)} W_{\mathbf{a}}(\mathbf{A}).\end{aligned} \tag{6.11}$$

From (6.10), (6.11) and the definition of $W_{h,\mathbf{a}}$ it follows that $L(\Theta_h) = \bigoplus_{\mathbf{a}\in C(h,m+1)} W_{h,\mathbf{a}}$ and therefore Lemma 6.14 ensures that (6.9) is an orthogonal decomposition of $L(\Theta_h)$ into invariant subspaces.

But the map

$$T(t, b_0, b_1, \ldots, b_m) = \begin{cases} (t+n-2h, b_0+h-t, b_1, \ldots, b_m), & \text{if } n-h < h-\ell(\mathbf{b}), \\ (b_0, b_0+h-t, b_1, \ldots, b_m), & \text{if } n-h \geqslant h-\ell(\mathbf{b}), \end{cases}$$

is a bijection between the set in Lemma 6.4 and the set $\{(k, a_0, a_1, \ldots, a_m)\colon 0 \leqslant k \leqslant \min\{n-h, h-\ell(\mathbf{a})\}, \mathbf{a} \in C(h, m+1)\}$ that parametrizes the representations in (6.9). Indeed, its inverse is given by

$$\begin{aligned} &T^{-1}(k, a_0, a_1, \ldots, a_m) \\ &\quad = \begin{cases} (k-n+2h, a_0+k-n+h, a_1, \ldots, a_m), & \text{if } n-h < h-\ell(\mathbf{a}), \\ (k+h-a_0, k, a_1, \ldots, a_m), & \text{if } n-h \geqslant h-\ell(\mathbf{a}). \end{cases} \end{aligned}$$

Therefore we can end the proof by invoking Corollary 2.2. □

Remark 6.17. The point (1) may be also obtained from the general representation theory of wreath products applied to $F \wr S_n$. Using the terminology (but not the notation) in [27,28], $V = W_1^{\otimes a_1} \otimes \cdots \otimes W_m^{\otimes a_m}$ is an irreducible representation of the base group $F^{\times^n}$, the inertia group of V is $F \wr (S_{n-h+a_0} \times S_{a_1} \otimes \cdots \otimes S_{a_m})$, $S^{n-h+a_0-k,k}$ is an irreducible representation of $S_{n-h+a_0} \times S_{a_1} \times \cdots \times S_{a_m}$ (trivial on $S_{a_1} \times \cdots \times S_{a_m}$) and $W_{h,\mathbf{a},k}$ is obtained inducing up $S^{n-h+a_0-k,k} \otimes V$ from the inertia group to $F \wr S_n$. An interesting paper on permutation representations of wreath products that might be translated into the framework of group actions on subtrees is [29].

6.6. The spherical functions

For $\mathbf{a} = (a_0, a_1, \ldots, a_m) \in C(h, m+1)$ we set $\tilde{\mathbf{a}} = (a_1, a_2, \ldots, a_m)$ which clearly is an element of $C(\ell(\mathbf{a}), m)$. Moreover, for $0 \leqslant u \leqslant \min\{n-h, h-\ell(\mathbf{a})\}$ we define the function

$$\begin{aligned} &\Phi(h, \mathbf{a}, u) \\ &\quad = \sum_{(A_1, A_2, \ldots, A_m) \in \Omega_{\tilde{\mathbf{a}}}(\overline{A})} \\ &\qquad \times \sum_{A_0 \in \Omega_{a_0}(\complement(A_1 \cup \cdots \cup A_m))\colon |A_0 \setminus \overline{A}| = u} \left[\left(\bigotimes_{j\in A_0} \phi_0 \right) \otimes \left(\bigotimes_{j\in A_1} \phi_1 \right) \otimes \cdots \otimes \left(\bigotimes_{j\in A_m} \phi_m \right) \right] \end{aligned} \qquad (6.12)$$

where $(\bigotimes_{j\in A_i}\phi_i)$ indicates the tensor product of a_i times the function ϕ_i over A_i. From Lemma 6.6 it follows that each $\Phi(h,\mathbf{a},u)$ is $(H\wr S_h)\times(F\wr S_{n-h})$-invariant. It is also easy to show that the set

$$\left\{\Phi(h,\mathbf{a},u)\colon 0\leqslant u\leqslant \min\left\{n-h, h-\ell(\mathbf{a})\right\}\right\} \tag{6.13}$$

constitutes an orthogonal basis for the $(H\wr S_h)\times(F\wr S_{n-h})$-invariant functions in the module $W_{h,\mathbf{a}}$. Indeed, $\Phi(h,\mathbf{a},u)$ belongs to $\bigoplus_{B_1\in\Omega_{h-u}(\overline{A}),\ B_2\in\Omega_u(\complement\overline{A})} L(Y^{B_1\sqcup B_2})$ and these spaces are orthogonal for different values of u. Now we want to express the spherical functions as linear combinations of the $\Phi(h,\mathbf{a},u)$'s. We will use the notation in (6.2).

Theorem 6.18. *The spherical function $\Psi(n,h,\mathbf{a},k)$ in $W_{h,\mathbf{a},k}$ is given by*

$$\Psi(n,h,\mathbf{a},k)=\frac{1}{\binom{h}{a_0,a_1,\dots,a_m}}\sum_{u=0}^{\min\{n-h,h-\ell(\mathbf{a})\}}\psi\big(n-\ell(\mathbf{a}),h-\ell(\mathbf{a}),k;u\big)\Phi(h,\mathbf{a},u).$$

Proof. The function $\Psi(n,h,\mathbf{a},k)$ defined above is $(H\wr S_h)\times(F\wr S_{n-h})$-invariant because it is a linear combination of invariant functions and its value on θ_0, the point stabilized by $(H\wr S_h)\times(F\wr S_{n-h})$, is equal to 1. Moreover, from (6.12) it follows that

$$\begin{aligned}
&\sum_{u=0}^{\min\{n-h,a_0\}}\psi(n-h+a_0,a_0,k;u)\Phi(h,\mathbf{a},u)\\
&=\sum_{(A_1,A_2,\dots,A_m)\in\Omega_{\tilde{\mathbf{a}}}(\overline{A})}\ \sum_{u=0}^{\min\{n-h,a_0\}}\psi(n-h+a_0,a_0,k;u)\\
&\quad\times\sum_{A_0\in\Omega_{a_0}(\complement(A_1\cup\cdots\cup A_m))\colon |A_0\setminus\overline{A}|=u}\left[\left(\bigotimes_{j\in A_0}\phi_0\right)\otimes\left(\bigotimes_{j\in A_1}\phi_1\right)\otimes\cdots\otimes\left(\bigotimes_{j\in A_m}\phi_m\right)\right]\\
&=\sum_{(A_1,A_2,\dots,A_m)\in\Omega_{\tilde{\mathbf{a}}}(\overline{A})}\left[\psi(n-h+a_0,a_0,k)\otimes\left(\bigotimes_{j\in A_1}\phi_1\right)\otimes\cdots\otimes\left(\bigotimes_{j\in A_m}\phi_m\right)\right]
\end{aligned}$$

since

$$\sum_{u=0}^{\min\{n-h,a_0\}}\psi(n-h+a_0,a_0,k;u)\sum_{A_0\in\Omega_{a_0}(\complement(A_1\cup\cdots\cup A_m))\colon |A_0\setminus\overline{A}|=u}\left(\bigotimes_{j\in A_0}\phi_0\right)$$

coincides with the spherical function of the Gelfand pair $(S_{n-h+a_0},S_{n-h}\times S_{a_0})$ (where S_{n-h+a_0} is the symmetric group on $\complement(A_1\cup\cdots\cup A_m)$ and $S_{n-h}\times S_{a_0}$ is the stabilizer of $\overline{A}\setminus(A_1\cup A_2\cup\cdots\cup A_m)$) belonging to the irreducible representation $S^{n-h+a_0-k,k}$.

Therefore $\psi(n - h + a_0, a_0, k) \otimes (\bigotimes_{j\in A_1}\phi_1) \otimes \cdots \otimes (\bigotimes_{j\in A_m}\phi_m)$ belongs to $S^{n-h+a_0-k,k} \otimes W_1^{\otimes a_1} \otimes \cdots \otimes W_m^{\otimes a_m}$ and the theorem follows from Definition 6.13. □

6.7. More explicit formulas for the spherical functions

In what follows, $\phi_i(j)$ will denote the value of the spherical function ϕ_i on the orbit Λ_j. The value of $\Phi(h, \mathbf{a}, u)$ on a θ with type$(\theta) = (t, \mathbf{b})$ clearly equals 0 if $t = |\mathrm{dom}\,\theta \cap \overline{A}| \neq h - u$ while, if $t = h - u$, it equals

$$\Phi(h, \mathbf{a}, u; \mathbf{b}) = \sum_{\boldsymbol{\alpha}} \prod_{j=0}^{m} \binom{b_j}{\alpha_{0j}, \alpha_{1j}, \ldots, \alpha_{mj}} \prod_{i=0}^{m} [\phi_i(j)]^{\alpha_{ij}} \tag{6.14}$$

where the sum is over all nonnegative integer-valued matrices $\boldsymbol{\alpha} = (\alpha_{ij})_{i=0,1,\ldots,m,\ j=0,1,\ldots,m}$ such that $\sum_{i=0}^{m} \alpha_{ij} = b_j$, $j = 0, 1, \ldots, m$, $\sum_{j=0}^{m} \alpha_{ij} = a_i$, $i = 1, 2, \ldots, m$, and $\sum_{j=0}^{m} \alpha_{0j} = a_0 - t$. Indeed, if $A_0 \cup A_1 \cup \cdots \cup A_m = \mathrm{dom}\,\theta$ and $B_j = \{r \in \mathrm{dom}\,\theta \cap \overline{A}\colon \theta(r) \in \Lambda_j\}$ then

$$\left[\left(\bigotimes_{w\in A_0}\phi_0\right) \otimes \left(\bigotimes_{w\in A_1}\phi_1\right) \otimes \cdots \otimes \left(\bigotimes_{w\in A_m}\phi_m\right)\right](\theta) = \prod_{i=0}^{m}\prod_{j=0}^{m} [\phi_i(j)]^{\alpha_{ij}}$$

where

$$\alpha_{ij} = |A_i \cap B_j| \tag{6.15}$$

and, for a fixed intersection matrix (α_{ij}) we have $\prod_{j=0}^{m} \binom{b_j}{\alpha_{0j}, \alpha_{1j}, \ldots, \alpha_{mj}}$ ways to chose the subsets $A_i \cap B_j$ inside B_j, and

$$A_0 = \left[\mathrm{dom}\,\theta \setminus \overline{A}\,\right] \cap \left[\bigcup_{j=0}^{m} (A_0 \cap B_j)\right].$$

Therefore, the value of $\Psi(n, h, \mathbf{a}, k)$ on a θ of type$(\theta) = (t, \mathbf{b})$ is given by

$$\Psi(n, h, \mathbf{a}, k; t, \mathbf{b}) = \frac{1}{\binom{h}{a_0, a_1, \ldots, a_m}} \psi\big(n - \ell(\mathbf{a}), h - \ell(\mathbf{a}), k; h - t\big) \Phi(h, \mathbf{a}, h - t; \mathbf{b}).$$

6.8. The end of the story

To end the section and the paper we indicate the relations between the construction of this section, namely the generalized Johnson scheme, and the Gelfand pairs associated with subtrees from Section 3.2.

The classical Johnson scheme $(S_n, S_h \times S_{n-h})$ clearly corresponds to the Gelfand pair $(\mathrm{Aut}(T_{\mathbf{r}}, T_{\mathbf{s}}), K(\mathbf{r}, \mathbf{s}))$ where $\mathbf{r} = n$ and $\mathbf{s} = h$.

More generally, given the Gelfand pair (F, H) where $F = \mathrm{Aut}(T_{\mathbf{r}'})$ and $H = K(\mathbf{r}', \mathbf{s}')$, $\mathbf{r}' = (r_2, r_3, \ldots, r_m)$ and $\mathbf{s}' = (s_2, s_3, \ldots, s_m)$, the homogeneous space Θ_h in Section 6.3 is nothing but $\mathcal{V}(\mathbf{r},\mathbf{s})$, where now $\mathbf{r}=(n,r_2,r_3,\dots,r_m)$ and $\mathbf{s}=(h,s_2,s_3,\dots,s_m)$. Indeed, the subgroup $(H\wr S_h)\times(F\wr S_{n-h})$ coincides with $K(\mathbf{r},\mathbf{s})$ since its expression coincides with that given in (3.1). The point that it stabilizes, namely $\theta_0\in\Theta_h$ (which corresponds to an h-subset $\overline{A}\subset\{1,2,\dots,n\}$) is given by $\theta_0(j)=y_0$ for all $j\in\overline{A}$ where y_0 is the $\mathbf{s}'$-subtree stabilized by $H\equiv K(\mathbf{r}',\mathbf{s}')$.

Example 6.19. We end this subsection by giving an example in which formula (6.14) becomes more simple. Suppose that (F,H) is the Gelfand pair of the ultrametric space (see Section 4.2) and (to simplify notation) that $2h\leqslant n$. The homogeneous space Θ_h now coincides with the space of all h-subsets $\{z_1,z_2,\dots,z_h\}$ of the ultrametric space such that $d(z_i,z_j)=m$ (maximum distance) for $i\neq j$. It coincides also with the homogeneous space $\mathcal{V}(\mathbf{r},\mathbf{s})$, with $\mathbf{r}=(n,q,\dots,q)$ (m times), and $\mathbf{s}=(h,1,\dots,1)$.

First observe that from (4.8) it follows that in this case in (6.14)

$$\prod_{i=0}^{m}[\phi_i(j)]^{\alpha_{ij}}=\begin{cases}\left(-\frac{1}{q-1}\right)^{\alpha_{m,1}+\alpha_{m-1,2}+\cdots+\alpha_{1,m}}, & \text{if } \alpha_{i,j}=0 \text{ for } i+j>m+1,\\ 0, & \text{otherwise}\end{cases}$$

that is in (6.15) we must have $A_i\subseteq B_0\cup B_1\cup\cdots\cup B_{m-i+1}$, $i=1,2,\dots,m$, and the value of $\prod_{i=0}^{m}[\phi_i(j)]^{\alpha_{ij}}$ is determined by the cardinalities $\gamma_j=|A_{m-j+1}\cap B_j|$, $j=1,2,\dots,m$. Therefore we have:

$$\Phi(h,\mathbf{a},u;\mathbf{b})=\sum_{\boldsymbol{\gamma}}\prod_{j=1}^{m}\binom{b_j}{\gamma_j}\binom{\sum_{w=0}^{j-1}b_w-\sum_{v=1}^{j-1}a_{m-v+1}}{a_{m-j+1}-\gamma_j}\left(-\frac{1}{q-1}\right)^{\gamma_1+\cdots+\gamma_m}$$

where the sum is over all the $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\dots,\gamma_m)$ such that

$$\max\left\{0,\sum_{v=1}^{j}a_{m-v+1}-\sum_{w=0}^{j-1}b_w\right\}\leqslant\gamma_j\leqslant\min\{b_j,a_{m-j+1}\}$$

(in particular, we have $\Phi(h,\mathbf{a},t;\mathbf{b})=0$ when the conditions $\sum_{v=1}^{j}a_{m-v+1}\leqslant\sum_{w=0}^{j}b_w$, $j=1,2,\dots,m-1$, are not satisfied). Indeed, to compute $\Phi(h,\mathbf{a},u;k)$ we have to choose, in all possible ways,

- the subset $A_{m-j+1}\cap B_j$ in B_j, for $j=1,2,\dots,m$,
- the subset $A_{m-j+1}\setminus B_j$ in $(\bigcup_{w=0}^{j-1}B_w\setminus\bigcup_{v=1}^{j-1}A_{m-v+1})$, for $j=1,2,\dots,m$,

and then necessarily $A_0=[\bigcup_{w=0}^{m}B_w\setminus\bigcup_{v=1}^{m}A_{m-v+1}]\cup[\operatorname{dom}\theta\setminus\overline{A}\,]$.

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