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THE DISCRETE SINE TRANSFORM AND THE SPECTRUM OF THE FINITE q-ARY TREE*

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Abstract. Recently, He, Liu, and Strang [*Stud. Appl. Math.*, 110 (2003), pp. 123–138] have computed the spectrum of the adjacency matrix of a class of finite trees. In this paper, we propose a different method and apply it to the slightly different class of finite q-ary trees.

Key words. tree, spectrum, discrete sine transform, Radon transform

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1. Introduction.

In [6], He, Liu, and Strang computed the spectrum of the finite trees that can be obtained by taking a ball of finite radius in an infinite homogeneous tree. These trees are rooted, all the leaves (end points) have the same distance from the root, and all the internal vertices have the same degree. Their method is based on a factorization of the characteristic polynomial obtained through a recursion on the diameter of the tree.

In the present paper, we deal with a slightly different kind of tree: the q-ary tree of height n. This means that we have a root which has q sons, q^2 grandsons, etc., for n generations; in this case the root has degree q, while all other internal vertices have degree $q+1$. For these trees we propose a method that is based on a preliminary decomposition of the space of all complex valued functions defined on the vertex set of the tree.

On each level of the tree, we use the decomposition into irreducible representations of the group of automorphisms of the tree $\mathrm{Aut}(T)$ [5], [7]. But note that our proof is very elementary: no knowledge of representation theory is required, only some elementary linear algebra. We obtain a decomposition by means of suitable Radon transforms that intertwine the representations on the various levels of the tree. They are strictly connected with the adjacency operator and the geometry of the tree. To get the spectrum, we apply the discrete sine transform to the action of the adjacency operator on such a decomposition.

Our method has a close resemblance to the proof of a theorem of Stanley [9, Theorem 4.14].

2. The tree and its adjacency operator.

A *tree* T is a connected graph without circuits. We say that T is *rooted* if it has a distinguished vertex x_0, called the root. We say that T is *q-ary* of *height* n if it satisfies the following three conditions: the root has degree q; a vertex is a leaf (i.e., it has degree 1) if and only if its distance from the root is equal to n; all the remaining vertices have degree $q+1$. Figure 1 is the ternary tree of height 3. In what follows, T will be a q-ary tree of height n. We will identify T with the set of all its vertices, and we will write $x \sim y$ to denote that $x, y \in T$ are *adjacent*, i.e., they are connected by an edge. We will denote by Ω_k the set

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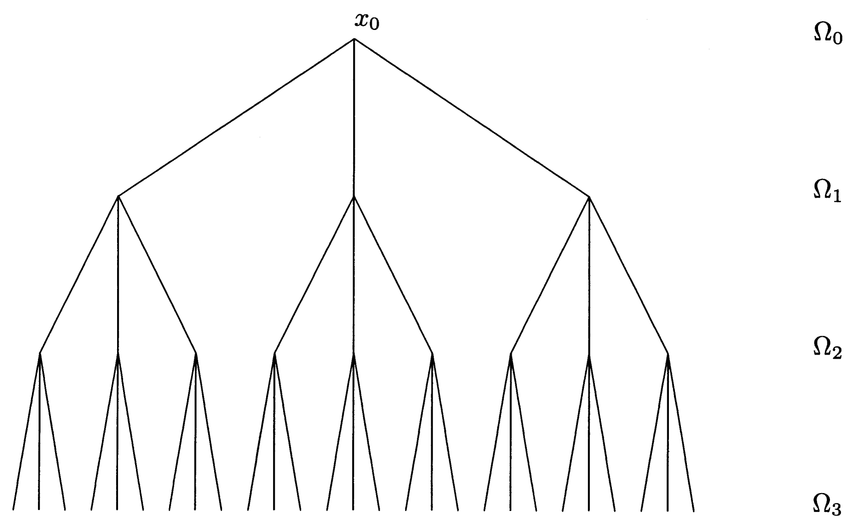

FIG. 1.

of vertices whose distance from the root is equal to k, $k = 0, 1, \ldots, n$ (the *k-level* of the tree). When $x \sim y$ and x belongs to a higher level than y, e.g., $x \in \Omega_k$ and $y \in \Omega_{k+1}$, we will say that x is the *father* of y and that y is a *son* of x, and we will write $x \succ y$. The space $\{f : T \to \mathbb{C}\}$ of all complex valued functions defined on T will be denoted by $L(T)$; it will be endowed with the scalar product $\langle f_1, f_2 \rangle = \sum_{x \in T} f_1(x)\overline{f_2(x)}$. The *adjacency operator* A of T is defined by setting $(Af)(x) = \sum_{y \in T : x \sim y} f(y)$ for all $x \in T$ and $f \in L(T)$. By definition [2], the spectrum of the tree coincides with the spectrum of its adjacency operator A.

3. The discrete sine transform and the spectrum of the path. Let B_n be the $n \times n$ tridiagonal matrix

$$B_n = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}.$$

Set $\alpha = \frac{\pi}{n+1}$. Then the $n \times n$ matrix

$$S_n = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\alpha & \sin 2\alpha & \ldots & \sin(n-1)\alpha & \sin n\alpha \\ \sin 2\alpha & \sin 4\alpha & \ldots & \sin 2(n-1)\alpha & \sin 2n\alpha \\ \vdots & \vdots & & \vdots & \vdots \\ \sin(n-1)\alpha & \sin 2(n-1)\alpha & \ldots & \sin(n-1)^2\alpha & \sin n(n-1)\alpha \\ \sin n\alpha & \sin 2n\alpha & \ldots & \sin(n-1)n\alpha & \sin n^2\alpha \end{pmatrix}$$

is symmetric and orthogonal and diagonalizes B_n:

$$S_n B_n S_n = \begin{pmatrix} 2\cos\alpha & & & \\ & 2\cos 2\alpha & & \\ & & \ddots & \\ & & & 2\cos n\alpha \end{pmatrix}. \tag{1}$$

This is the *discrete sine transform* (DST) [10]. Moreover, (1) is the computation of the spectrum of the tree T in the case $q = 1$ (the path): B_n is the matrix representing the adjacency operator of the path if we take the standard basis $\{\delta_x : x \in T\}$ for $L(T)$, where $\delta_x(y) = 1$ if $x = y$, $\delta_x(y) = 0$ if $x \neq y$.

Remark. The characteristic polynomial $\det(\lambda I - B_n)$ of B_n, also called the characteristic polynomial of the path, may be expressed by the *Chebyshev polynomials of the second kind* [2, p. 11]: $\det(\lambda I - B_n) = U_n(\lambda/2)$. The computation of the spectrum of the tree in [6] is based in a factorization of the characteristic polynomial of the tree in terms of (rescaled) Chebyshev polynomials of the second kind: in the notations of [6], $p_n(\lambda) = (k-1)^{n/2} U_n(\frac{\lambda}{2\sqrt{k-1}})$; see also [3, section 1.4].

4. The Radon transforms *R* and *R.** First note that $T = \sqcup_{k=0}^{n} \Omega_k$ (where $\sqcup$ denotes a disjoint union) leads to the orthogonal decomposition $L(T) = \oplus_{k=0}^{n} L(\Omega_k)$. Then we define the linear operator $R : \oplus_{k=1}^{n} L(\Omega_k) \to \oplus_{k=0}^{n-1} L(\Omega_k)$ by setting

$$(Rf)(x) = \sum_{y \in T : y \prec x} f(y)$$

for every $f \in \oplus_{k=1}^{n} L(\Omega_k)$ and $x \in \sqcup_{k=0}^{n-1} \Omega_k$. In other words, the value of Rf on x is the sum of the values of f on the sons of x. The adjoint of R is the linear operator $R^* : \oplus_{k=0}^{n-1} L(\Omega_k) \to \oplus_{k=1}^{n} L(\Omega_k)$ given by

$$(R^* f)(x) = f(y), \qquad \text{where } y \text{ is the father of } x,$$

for every $f \in \oplus_{k=0}^{n-1} L(\Omega_k)$ and $x \in \sqcup_{k=1}^{n} \Omega_k$.

Clearly R is surjective and R^* is injective. Moreover, R maps $L(\Omega_k)$ onto $L(\Omega_{k-1})$ and R^* maps $L(\Omega_{k-1})$ into $L(\Omega_k)$, $k = 1, 2, \ldots, n$. In particular, $(R^*)^{k-h}(L(\Omega_h))$ is a homomorphic image of $L(\Omega_h)$ in $L(\Omega_k)$: it consists of all functions in $L(\Omega_k)$ that are constant on the leaves of each q-ary subtree of T of height $k - h$ rooted on a vertex in Ω_h.

We also define $W_k = L(\Omega_k) \cap \ker R$, $k = 1, 2, \ldots, n$ and $W_0 = L(\Omega_0) \equiv \mathbb{C}$. Note that $\dim W_0 = 1$ and that $\dim W_k = q^k - q^{k-1}$.

The following identity is easy but important:

$$RR^* f = qf. \tag{2}$$

Indeed, $(RR^*f)(x) = \sum_{y \in T : y \prec x} (R^* f)(y) = qf(x)$.

LEMMA 4.1. *For $k = 1, 2, \ldots, n$ we have an orthogonal decomposition of $L(\Omega_k)$:*

$$L(\Omega_k) = (R^*)^k (W_0) \oplus (R^*)^{k-1}(W_1) \oplus \cdots \oplus (R^*)(W_{k-1}) \oplus W_k.$$

Proof. First note that a consequence of (2) is that

$$\langle R^* f_1, R^* f_2 \rangle = q \langle f_1, f_2 \rangle, \tag{3}$$

and this is also easy to prove directly.

Using (3), we can iterate the decomposition $L(\Omega_k) = R^*(L(\Omega_{k-1})) \oplus [\ker R \cap L(\Omega_k)] \equiv R^*(L(\Omega_{k-1})) \oplus W_k$:

$$\begin{aligned} L(\Omega_k) =& R^*(L(\Omega_{k-1})) \oplus W_k \\ =& (R^*)^2(L(\Omega_{k-2})) \oplus R^*(W_{k-1}) \oplus W_k \\ & \cdots \\ =& (R^*)^k(W_0) \oplus (R^*)^{k-1}(W_1) \oplus \cdots \oplus (R^*)(W_{k-1}) \oplus W_k. \qquad \square \end{aligned}$$

In other words, $(R^*)^k(W_0)$ is the space of constant functions on Ω_k and $(R^*)^{k-h}(W_h)$ is the space of all functions in $L(\Omega_k)$ that are constant on the leaves of each q-ary subtree of T of height $k-h$ rooted on a vertex in Ω_h and whose sum on the leaves of every q-ary subtree of height $k-h+1$ rooted on a vertex in Ω_{h-1} is equal to zero.

Another fundamental identity relates the adjacency operator A to the Radon transforms R and R^*: if $f \in L(T)$ and $f = f_0 + f_1 + \cdots + f_n$ with $f_h \in L(\Omega_h)$, then

$$Af = Rf_1 + \sum_{h=1}^{n-1}(R^* f_{h-1} + Rf_{h+1}) + R^* f_{n-1}, \tag{4}$$

where $Rf_1 \in L(\Omega_0)$, $R^* f_{h-1} + Rf_{h+1} \in L(\Omega_h)$, and $R^* f_{n-1} \in L(\Omega_n)$. For instance, if $x \in \Omega_h$ with $1 \le h \le n-1$, then

$$\begin{aligned} (Af)(x) &= \sum_{y\sim x} f(y) = \sum_{z\in\Omega_{h+1}:z\sim x} f(z) + \sum_{y\in\Omega_{h-1}:y\sim x} f(y) \\ &= (Rf)(x) + (R^*f)(y) \equiv (Rf_{h-1})(x) + (R^* f_{h+1})(x). \end{aligned}$$

Remarks. (1) We call R and R^* Radon transforms because they are (natural) operators intertwining $L(\Omega_k)$ and $L(\Omega_{k+1})$ as permutation representations of $\mathrm{Aut}(T)$, the group of automorphisms of T; see [8]. The decomposition in Lemma 4.1 is well known and coincides with the decomposition of $L(\Omega_k)$ into irreducible representations of $\mathrm{Aut}(T)$; see [5], [7], and also [1, pp. 152–156], which has a more algebraic form. But in our case we are not on a homogeneous space: $\mathrm{Aut}(T)$ does not act transitively on T. Therefore we may not apply the finite Fourier transform (for which we refer to [4]) to get the spectrum of T. Nevertheless, A is $\mathrm{Aut}(T)$-invariant, and therefore the eigenspaces of A must be direct sums of irreducible representations of $\mathrm{Aut}(T)$, as we will show in the next section.

(2) The operators R^* and R can also be seen as instances of "up" and "down" operators as in [9] (but note that Stanley would draw the tree with the root at the bottom and the leaves at the top; therefore in his terminology R goes down and R^* goes up). However, our tree is not a differential poset of Stanley: it is easy to see that in our case

$$(RR^* - R^*R)f = \begin{cases} qf & \text{if} \quad f \in W_k, \\ 0 & \text{if} \quad f \in L(\Omega_k), f \perp W_k, \end{cases}$$

while the definition of differential poset requires that the commutator $RR^* - R^*R$ is always a multiple of the identity. Nevertheless, our computation of the spectrum of the tree in the following section has a close resemblance to the proof of Theorem 4.14 in [9].

5. The spectrum of the tree.

LEMMA 5.1. *For $k = 0, 1, \ldots, n$ and $l = 1, 2, \ldots, n-k+1$ set*

$$W_{k,l} = \left\{ \sum_{h=0}^{n-k} \frac{1}{q^{h/2}} \sin \frac{(h+1)l\pi}{n-k+2} \cdot f \quad : \quad f \in W_k \right\}.$$

Then each $W_{k,l}$ is an eigenspace of A. The corresponding eigenvalue is equal to $2\sqrt{q} \cos \frac{\pi l}{n-k+2}$ and $\oplus_{h=0}^{n-k} (R^*)^h W_k = \oplus_{l=1}^{n-k+1} W_{k,l}$.

Proof. If $f \in W_k$ and $a_0, a_1, \ldots, a_{n-k} \in \mathbb{C}$, then from (2) and (4) it follows that

$$\begin{aligned} &A(a_0 f + a_1 R^* f + \cdots + a_{n-k} (R^*)^{n-k} f) \\ &= a_0 R f + a_1 R R^* f + \sum_{h=k+1}^{n-1} \left[a_{h-k-1} R^* (R^*)^{h-k-1} f + a_{h-k+1} R (R^*)^{h-k+1} f \right] \\ &\quad + a_{n-k-1} R^* (R^*)^{n-k-1} f \\ &= a_1 q f + \sum_{h=k+1}^{n-1} \left[a_{h-k-1} + q a_{h-k+1} \right] (R^*)^{h-k} f + a_{n-k-1} (R^*)^{n-k} f. \end{aligned}$$

Therefore $F = a_0 f + a_1 R^* f + \cdots + a_{n-k} (R^*)^{n-k} f$ is an eigenvector of A; i.e., $AF = \lambda F$ if and only if the coefficients $a_0, a_1, \ldots, a_{n-k}$ solve the eigenvalue problem

$$\text{(5)} \qquad \begin{cases} a_{h-1} + q a_{h+1} = \lambda a_h & \text{for} \quad h = 1, 2, \ldots, n-k-1, \\ q a_1 = \lambda a_0; \quad a_{n-k-1} = \lambda a_{n-k}. \end{cases}$$

With the substitutions $b_h = q^{h/2} a_h$, $h = 0, 1, \ldots, n-k$, and $\mu = \frac{\lambda}{\sqrt{q}}$ (5) becomes

$$\begin{cases} b_{h-1} + b_{h+1} = \mu b_h & \text{for} \quad h = 1, 2, \ldots, n-k-1, \\ b_1 = \mu b_0; \quad b_{n-k-1} = \mu b_{n-k}, \end{cases}$$

which is the eigenvalue problem solved by the DST. Therefore from section 3 one recovers the eigenvalues and the eigenspaces in the statement. Finally, $\oplus_{h=0}^{n-k} (R^*)^h W_k \equiv \{a_0 f + a_1 R^* f + \cdots + a_{n-k} (R^*)^{n-k} f : f \in W_k, \ a_0, a_1, \ldots, a_{n-k} \in \mathbb{C}\}$ is clearly equal to $\oplus_{l=1}^{n-k+1} W_{k,l}$, because the rows of the matrix of the DST form an orthogonal basis. ∎

Now we can state and prove the main theorem on the spectral analysis of A. We will write $(a, b) = 1$ to indicate that the integers a and b are relatively prime.

THEOREM 5.2.

1. *The spectrum of A coincides with the set $\{2\sqrt{q} \cos \frac{\pi l}{n-k+2} : k = 0, 1, \ldots, n; \ l = 1, 2, \ldots, n-k+1; (l, n-k+2) = 1\}$.*
2. *Suppose that $0 \le k \le n$, $1 \le l \le n-k+1$, and $(l, n-k+2) = 1$. If $k = (n-k+2)s + r$, with $0 \le r \le n-k+1$, then the eigenspace corresponding to $2\sqrt{q} \cos \frac{\pi l}{n-k+2}$ is*

$$\oplus_{t=0}^{s} W_{k-t(n-k+2), l(t+1)}.$$

3. *The multiplicity of $2\sqrt{q} \cos \frac{\pi l}{n-k+2}$ is equal to*

$$\begin{aligned} (q^r - q^{r-1}) \frac{q^{(n-k+2)(s+1)} - 1}{q^{n-k+2} - 1} \quad &if \quad 1 \le r \le n-k+1, \\ 1 + (q^{n-k+2} - q^{n-k+1}) \frac{q^{(n-k+2)s} - 1}{q^{n-k+2} - 1} \quad &if \quad r = 0. \end{aligned}$$

Proof. From the decomposition $L(T) = \oplus_{k=0}^{n} L(\Omega_k)$ and Lemmas 4.1 and 5.1 we have

$$L(T) = \oplus_{k=0}^{n} \oplus_{h=0}^{n-k} (R^*)^h W_k = \oplus_{k=0}^{n} \oplus_{l=1}^{n-k+1} W_{k,l},$$

and therefore Lemma 5.1 yields part 1. To prove part 2, observe first that $k = s(n-k+2)+r$ is equivalent to $n+2 = (s+1)(n-k+2)+r$. Therefore $(t+1)(n-k+2)$ is equal to $n - k_1 + 2$ with $0 \leq k_1 \leq n$ if and only if $k_1 = k - t(n-k+2)$ with $0 \leq t \leq s$. Exactly for those values of t the eigenvalue $2\sqrt{q}\cos\frac{l\pi}{n-k+2}$ appears again in the form $2\sqrt{q}\cos\frac{l(t+1)\pi}{(t+1)(n-k+2)}$, and the corresponding eigenspace is $W_{k-t(n-k+2),l(t+1)}$. Moreover, $\sum_{t=0}^{s} \dim W_{k-t(n-k+2),l(t+1)}$ is equal to

$$\sum_{t=0}^{s}(q^{k-t(n-k+2)} - q^{k-t(n-k+2)-1})$$

$$= (q^r - q^{r-1})\sum_{w=0}^{s} q^{(n-k+2)w} = (q^r - q^{r-1})\frac{q^{(n-k+2)(s+1)}-1}{q^{n-k+2}-1} \qquad \text{for} \quad r \geq 1,$$

$$1 + \sum_{t=0}^{s-1}(q^{k-t(n-k+2)} - q^{k-t(n-k+2)-1}) = 1 + (q^{n-k+2} - q^{n-k+1})\sum_{w=0}^{s-1} q^{(n-k+2)w}$$

$$= 1 + (q^{n-k+2} - q^{n-k+1})\frac{q^{(n-k+2)s}-1}{q^{n-k+2}-1} \qquad \text{for} \quad r = 0. \qquad \square$$

Our method might be applied to other classes of rooted trees. For instance, consider a tree where each vertex at level k has q_k sons, $k = 0, 1, \ldots, n-1$. In this case (5) is replaced by the more general eigenvalue problem

$$\text{(6)} \qquad \begin{cases} a_{h-1} + q_{k+h}a_{h+1} = \lambda a_h & \text{for} \quad h = 1, 2, \ldots, n-k-1, \\ q_k a_1 = \lambda a_0; \quad a_{n-k-1} = \lambda a_{n-k}. \end{cases}$$

In general, this problem does not have an explicit elementary solution. Nevertheless, in particular cases some of the eigenvalues are computable. For $q_0 = q+1$ and $q_k = q$, $k = 1, 2, \ldots, n$ we obtain the trees in [6], and in this case almost all eigenvalues are computable; those missing correspond to the subspace $\oplus_{h=0}^{n}(R^*)^h(W_0)$: now for $k \geq 1$ (6) reduces to (5).

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