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On the presentations of the trivial group

Fabio Scarabotti

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1 Introduction

Presentations of the trivial group are studied in low-dimensional topology and combinatorial group theory. A major open problem, the Andrews–Curtis conjecture, asserts that if $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$ is a balanced presentation of the trivial group (balanced means that the number of relators is equal to the number of generators) then the set of words $R = \{r_1, \ldots, r_n\}$ may be reduced to the set $A = \{a_1, \ldots, a_n\}$ by the following transformations:

(i) $r_i \to r_i^{-1}$, r_j unchanged for $j \neq i$,

(ii) $(r_i, r_j) \to (r_i r_j, r_j)$, r_k unchanged for $k \neq i$,

(iii) $r_i \to w^{-1} r_i w$, r_j unchanged for $j \neq i$, $w \in A$.

We refer to [1], [2] and [3].

In [4] C. P. Rourke proved that in order to check whether a (not necessarily balanced) presentation defines the trivial group, it is sufficent to consider only those consequences of R that can be obtained by the operations of cyclic permutation and of cyclically reduced product (see below for a formal statement). Rourke proved this theorem using methods from algebraic topology; in the present paper we give a very elementary and short proof of Rourke's theorem, based only on simple algebraic manipulations. Our proof also produces an algorithm which allows us to compute n such that a generator a belongs to R_n if a is expressed as a product of conjugates of the relators and their inverses.

There are two main differences between an Andrews–Curtis trivialization and the Rourke process. In the first method the set of words obtained has constant size equal to n at each step, while in the second method the set of word obtained grows very fast. Moreover, in the first method every conjugation is admitted, while in the second method only cyclic permutations are used. In any case, Rourke's method is much stronger and has an elementary proof, while the Andrews–Curtis conjecture is still a very hard and outstanding open problem.

Notation. If v is a reduced word, its length will be denoted by $|v|$. If $v = v_1 v_2 \dots v_k$ with $|v| = |v_1| + |v_2| + \dots + |v_k|$, we will write

$$v = v_1 v_2 \dots v_k \quad \text{w.c.}$$

where w.c. means without cancellations; we will also say that the product $v = v_1 v_2 \dots v_k$ is reduced.

2 Statement and proof of Rourke's theorem

In this section, after a formal statement of Rourke's theorem, we will present our algebraic proof. Let R be a set of cyclically reduced words in an alphabet a, a^{-1}, b, b^{-1}, c, $c^{-1}, \dots$. Denote by R^+ the set of elements of R together with all their cyclic permutations and all the cyclic permutations of their inverses, and denote by $\tilde{R}$ the set of elements of R together with all cyclically reduced products of pairs of elements of R. Then define the set R_n as follows:

$$R_1 = R^+ \quad \text{and} \quad R_n = (\tilde{R}_{n-1})^+ \quad \text{for } n = 2, 3, \dots.$$

Now let $\langle a, b, c, \dots \mid R \rangle$ be a presentation of a group G (where R is again a set of cyclically reduced words). In [4] Rourke proved the following result:

Theorem. *If G is the trivial group then every generator $a, b, c, \dots$ belongs to the union* $\bigcup_{n=1}^{\infty} R_n$

Proof of the theorem. Let $\langle a, b, c, \dots \mid R \rangle$ be a presentation of the trivial group. Then we can express the generator a as a product of conjugates

$$a = t_1^{-1} r_1 t_1 \cdot t_2^{-1} r_2 t_2 \cdot t_3^{-1} r_3 t_3 \dots t_m^{-1} r_m t_m,$$

where $r_1, r_2, r_3, \dots, r_m$ are elements of R or their inverses. But if r is cyclically reduced, the conjugate $t^{-1} r t$, if not reduced, may be written in a reduced form $\bar{t}^{-1} \bar{r} \bar{t}$ where $\bar{r}$ is a cyclic permutation of r. Thus a may be written as a product of conjugates of elements of R_1 where every conjugate is expressed in a reduced form. Then Rourke's theorem follows from the following lemma.

Lemma. *Let R be a set of cyclically reduced words in an alphabet a, a^{-1}, b, b^{-1}, c, $c^{-1}, \dots$. Suppose that a can be expressed in the form*

$$a = t_1^{-1} r_1 t_1 \cdot t_2^{-1} r_2 t_2 \cdot t_3^{-1} r_3 t_3 \dots t_m^{-1} r_m t_m \tag{1}$$

where $r_1, r_2, r_3, \dots, r_m$ belong to R_n and $m > 1$. Then we can obtain from (1) an expression of the form

$$a = s_1^{-1} u_1 s_1 \cdot s_2^{-1} u_2 s_2 \cdot s_3^{-1} u_3 s_3 \dots s_h^{-1} u_h s_h$$

where $u_1, u_2, u_3, \dots, u_h$ belong to R_{n+1} and $h < m$.

Proof. We may suppose that in (1) every block $t_i^{-1}r_it_i$ is reduced; a consequence is that if there is cancellation between $(t_{i-1}t_i^{-1})$ and r_i (thus t_{i-1} deletes all t_i^{-1}) there cannot be cancellation between r_{i-1} and $(t_{i-1}t_i^{-1})$ and conversely. Since the length of

$$t_1^{-1} \cdot r_1 \cdot (t_1t_2^{-1}) \cdot r_2 \cdot (t_2t_3^{-1}) \cdot r_3 \dots (t_{m-1}t_m^{-1}) \cdot r_m \cdot t_m$$

as a reduced word is one and $m > 1$, there is a word in this product that is deleted by its neighbours. We have to examine two possible cases:

(i) there is a word $(t_{i-1}t_i^{-1})$ that is trivial or is deleted by r_i or by r_{i-1};

(ii) there is a word r_i that is deleted, possibly part by $(t_{i-1}t_i^{-1})$ and part by $(t_it_{i+1}^{-1})$.

Case (i). Suppose that $(t_{i-1}t_i^{-1})$ is deleted by r_i. Since there is no cancellation between t_i^{-1} and r_i, there must exist words s_{i-1} and u_i such that $t_{i-1} = s_{i-1}t_i$ w.c. and $r_i = s_{i-1}^{-1}u_i$ w.c.. Therefore we have

$$t_{i-1}^{-1}r_{i-1}t_{i-1} \cdot t_i^{-1}r_it_i = t_i^{-1}(s_{i-1}^{-1}r_{i-1}u_i)t_i \tag{2}$$

But $t_i^{-1}(s_{i-1}^{-1}r_{i-1}u_i)t_i$ is a conjugate of an element of R_{n+1} and in this case we can prove the lemma using transformation (2) in (1). We can proceed similarly if $(t_{i-1}t_i^{-1})$ is trivial or is deleted by r_{i-1}.

Case (ii). Now suppose that r_i is deleted in the expression $(t_{i-1}t_i^{-1}) \cdot r_i \cdot (t_it_{i+1}^{-1})$. Since the word $t_i^{-1}r_it_i$ is reduced, there exist v_i, w_i, s_{i-1}, and s_{i+1} such that $r_i = v_iw_i$ w.c., $t_{i-1} = s_{i-1}v_i^{-1}t_i$ w.c. and $t_{i+1} = s_{i+1}w_it_i$ w.c. (clearly it is possible that $w_i = 1$ or $v_i = 1$). We have to examine three possible subcases.

First subcase. If we have $|s_{i-1}w_it_i| < |t_{i-1}|$ (for example, if $|w_i| < |v_i|$) then we will use in (1) the identity

$$t_{i-1}^{-1}r_{i-1}t_{i-1} \cdot t_i^{-1}r_it_i = t_i^{-1}r_it_i \cdot (s_{i-1}w_it_i)^{-1}r_{i-1}(s_{i-1}w_it_i), \tag{3}$$

obtaining in place of (1) a similar expression where m is the same but $\sum_{i=1}^m |t_i|$ is smaller.

Second subcase. If we have $|s_{i-1}w_it_i| \geqslant |t_{i-1}|$ and $|s_{i+1}v_i^{-1}t_i| < |t_{i+1}|$ we will use in (1) the identity

$$t_i^{-1}r_it_i \cdot t_{i+1}^{-1}r_{i+1}t_{i+1} = (s_{i+1}v_i^{-1}t_i)^{-1}r_{i+1}(s_{i+1}v_i^{-1}t_i) \cdot t_i^{-1}r_it_i, \tag{4}$$

obtaining again in place of (1) a similar expression where m is the same but $\sum_{i=1}^m |t_i|$ is smaller.

Third subcase. If $|s_{i-1}w_it_i| = |t_{i-1}|$ and $|s_{i+1}v_i^{-1}t_i| = |t_{i+1}|$ (therefore $|v_i| = |w_i|$) we will use (3) in (1), obtaining an expression where both m and $\sum_{i=1}^m |t_i|$ are the same but the cardinality of the set $\{(t_h, t_k) : |t_h| > |t_k| \text{ and } h < k\}$ is smaller.

Therefore the third subcase may occur consecutively in an expression like (1) at most $\frac{1}{2}m(m-1)$ times. Similarly, in a sequence of application of transformations (3) and (4) to (1), the first and second subcases may occur at most $\sum_{i=1}^m |t_i|$ times. In any case, starting from (1), after a finite sequence of transformations of type (3) or (4) (if they are necessary), we must find an expression like (1) with the same m and where case (i) occurs. This proves the lemma.

3 An application

As was mentioned above, the given proof produces an algorithm for finding a suitable number n. For example, if we consider the potential counter-example of Akbulut and Kirby [1] to the Andrews–Curtis conjecture

$$\{x, y \mid r, s\}, \quad \text{where } r = yx^{-1}y^{-1}x^{-1}yx \quad \text{and} \quad s = y^{-4}x^5$$

we have

$$y = y^5x^{-1}s^{-1}xy^{-5} \cdot y^5x^{-1}y^{-1}sy\,xy^{-5} \cdot y^4ry^{-4} \cdot y^3ry^{-3} \cdot y^2ry^{-2} \cdot yr\,y^{-1} \cdot r \cdot x^5sx^{-5}$$

and

$$x = yx\,r^{-1}x^{-1}y^{-1} \cdot yx\,yx^{-1}y^{-1}.$$

Using transformations like (2) as many times as possible, it is easy to see that if $R = \{r, s\}$, then $y \in R_4$ and $x \in R_5$. Similar results, but with many tedious calculations, may be obtained for the other potential counter-examples indicated in [2].

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References

[1] S. Akbulut and R. Kirby. A potential smooth counterexample in dimension 4 to the Poincaré conjecture, the Schoenflies conjecture and the Andrews–Curtis conjecture. *Topology* **24** (1985), 375–390.

[2] C. Hog-Angeloni and W. Metzler. The Andrews–Curtis conjecture and its generalizations. In *Two-dimensional homotopy and combinatorial group theory*, London Math. Soc. Lecture Notes Ser. no. 197 (Cambridge University Press, 1993), pp. 365–380.

[3] R. I. Grigorchuk and P. F. Kurchanov. Some questions of group theory related to geometry. In *Algebra VII*, Encyclopedia of Mathematical Sciences, vol. 58 (Springer-Verlag, 1993), pp. 167–240.

[4] C. P. Rourke. Presentations and the trivial group. In *Topology of low-dimensional manifolds*, Lecture Notes in Math. vol. 722 (Springer-Verlag, 1979), pp. 134–143.

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F. Scarabotti, Università di Roma "La Sapienza", Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Via Antonio Scarpa, 16, 00161 Roma, Italy
E-mail: scarabotti@dmmm.uniroma1.it