NONLINEAR ENERGY FORMS AND LIPSCHITZ SPACES ON THE INFINITE KOCH CURVE

Raffaela Capitanelli* and Maria Rosaria Lancia†

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Università degli Studi di Roma "La Sapienza"
Via A. Scarpa 16, 00161 Roma, Italy

الخلاصـة

نعتبر في هذا المقال الأشكال المُحدّبة اللاخطية للطاقة المتواجدة على مُنحنى "كوخ" (Koch) اللانهائي ونُبرهن أنّ المجـالات المُـتـناظرة تـتطابـق وفضـاءات "لبسيش" $Lip_{\delta,d_f}(p,\infty,K^{<\infty>})$ أين $\delta = \frac{\log 4}{\log 3}$.

ABSTRACT

We consider the nonlinear convex energy forms $\mathcal{E}^{(p)}_{K^{<\infty>}}$ on the infinite Koch curve $K^{<\infty>}$ and we prove that the corresponding domains $\mathcal{F}^{(p)}_{K^{<\infty>}}$ coincide with the spaces $Lip_{\delta,d_f}(p,\infty,K^{<\infty>})$, where $\delta = \frac{\log 4}{\log 3}$.

*To whom correspondence should be addressed.
e-mail: raffaela.capitanelli@uniroma1.it

†e-mail: lancia@dmmm.uniroma1.it

NONLINEAR ENERGY FORMS AND LIPSCHITZ SPACES ON THE INFINITE KOCH CURVE

1. INTRODUCTION

Nonlinear energy forms

$$\mathcal{E}^{(p)},\ 1 < p < \infty$$

on a fractal set, the unit Koch curve, were constructed for the first time in [1], where the properties of their domains $\mathcal{F}^{(p)}$ were studied and it was shown that they can be considered to be the analogues of the usual Sobolev spaces $W^{1,p}$.

In [2], the spaces $W^{1,p} \equiv \mathcal{F}^{(p)}$ were put in relation to the spaces $Lip_{\alpha,d_f}(p,\infty,K)$ and the following characterization was given:

$$W^{1,p}(K) = Lip_{\delta,d_f}(p,\infty,K),\ \text{for every } 1 < p < \infty, \tag{1.1}$$

where d_f is the fractal (Hausdorff) dimension of K and $\delta = d_f = \frac{\log 4}{\log 3}$. In the quadratic case $p = 2$, analogous results have been proved for various fractals K such as the Sierpinski gasket, the Koch curve, and more general fractals first in [3] and then in [4–6]. More precisely, the domain $\mathcal{F}^{(2)}$ of the energy form $\mathcal{E}^{(2)}$ on K — which is the fractal analogue of the space $H^1 = W^{1,2}$ — has been put in relation with the space $Lip_{\alpha,d_f}(2,\infty,K)$ and the following identification is proved:

$$W^{1,2}(K) = Lip_{\delta,d_f}(2,\infty,K),$$

where δ is a parameter depending on the structural constants of the fractal (see [4]).

The purpose of this paper is to extend the nonlinear results in [2] to the case of the infinite Koch curve $K^{<\infty>}$ (see Theorem 5.5 in Section 5). Following the lines in [7], we define on $K^{<\infty>}$ the nonlinear energy forms

$$\mathcal{E}^{(p)}_{K^{<\infty>}},\ \ 1 < p < \infty$$

with domains $\mathcal{F}^{(p)}_{K^{<\infty>}}$ — which can still be considered to be the analogue of the Sobolev space $W^{1,p}$ on the real line — and we prove the following characterization:

$$W^{1,p}(K^{<\infty>}) = Lip_{\delta,d_f}(p,\infty,K^{<\infty>}),\ \text{for every } 1 < p < \infty, \tag{1.2}$$

where $\delta = \frac{\log 4}{\log 3}$, as in the previous case, and d_f is the Hausdorff dimension of $K^{<\infty>}$.

This characterization allows us to regard the functions of finite energy on the fractal $K^{<\infty>}$ as *traces* of functions belonging to suitable Sobolev spaces on the plane (see [8] and [3]). These trace results are of great importance in some applications: for instance, in the second order transmission problems of the type studied in [9–11] where the fractal set is a layer separating two domains.

Our results in the present paper can be used in 3-dimensional second order transmission problems with *infinite* fractal layers (namely, the fractal surface $S = K^{<\infty>} \times [0,h], h > 0$), of the kind described in [11]. They can also be useful in dealing with nonlinear fractal transmission problems involving the fractal *p-Laplacian* in the transmission condition (in this regard, see [12]).

Nonlinear energy forms (and the related p-Laplacians) naturally arise in many application problems such as in the study of non-Newtonian fluids or flows in porous media, in nonlinear elasticity, in glaceology, in petroleum extraction, as well as in some reaction diffusion problems (for an exhaustive discussion, see [13] and the references listed therein).

The plan of the paper is the following. In Section 2, we recall the definitions of the Koch curve K, of the invariant measure μ and, for $1 < p < \infty$, of the nonlinear energy forms $\mathcal{E}^{(p)}$ with domains $\mathcal{F}^{(p)}$.

In Section 3, the definition of the Lipschitz spaces is given according to [3]; moreover, from [2], we recall, the identification of the domain $\mathcal{F}^{(p)}$ of the associated nonlinear energy form $\mathcal{E}^{(p)}$ with the Lipschitz space $Lip_{\delta,d_f}(p,\infty,K)$, where $\delta = d_f = \frac{\log 4}{\log 3}$.

In Section 4, according to [7], we introduce the infinite Koch curve $K^{<\infty>}$, the invariant measure μ and, for $1 < p < \infty$, the nonlinear energy forms $\mathcal{E}^{(p)}_{K^{<\infty>}}$ with domains $\mathcal{F}^{(p)}_{K^{<\infty>}}$.

In the last section, we prove that, for $1 < p < \infty$, the domains $\mathcal{F}^{(p)}_{K^{<\infty>}}$ of $\mathcal{E}^{(p)}_{K^{<\infty>}}$ coincide with the spaces $Lip_{\delta,d_f}(p,\infty,K^{<\infty>})$, where $\delta = \frac{\log 4}{\log 3}$.

2. THE NONLINEAR ENERGY FORM ON THE KOCH CURVE

In this section, we recall the nonlinear energy forms on the Koch curve, whose construction has been developed by one of the authors in [1].

We start by recalling the construction of the unit Koch curve K. Let $\Psi = \{\psi_i, i = 1,...,4\}$ denote the set of the $N = 4$ contractive similitudes $\psi_i : \mathbb{C} \to \mathbb{C}$, with contraction factor $l^{-1} = \frac{1}{3}$ given by $\psi_1 = \frac{z}{3}$, $\psi_2 = \frac{z}{3}e^{i\frac{\pi}{3}} + \frac{1}{3}$, $\psi_3 = \frac{z}{3}e^{-i\frac{\pi}{3}} + \frac{1}{2} + i\frac{\sqrt{3}}{6}$, $\psi_4 = \frac{z+2}{3}$.

Let $A \subset \mathbb{R}^2$, we define, for arbitrary n-tuples of indices $i_1,...,i_n \in \{1,...,4\}$, $\psi_{i_1...i_n} := \psi_{i_1} \circ ... \circ \psi_{i_n}$, $A_{i_1...i_n} := \psi_{i_1...i_m}(A)$, $A^{(0)} = A$, and

$$A^{(n)} = \bigcup_{i_1,...,i_n=1}^{4} A_{i_1...i_n}.$$

Let us denote by z_0 and z_1 the points $(0,0)$ and $(1,0)$. Let $V = \{z_0, z_1\}$ and let $V_\star = \cup_{n\geq 0} V^{(n)}$; the set $K = \bar{V}_\star$, that is, the closure in $\mathbb{R}^2$ of $V_\star$, is the so-called unit Koch curve.

On the Koch curve K, there exists an invariant measure μ (see [14]) which is given, after normalization, by the restriction to K of the d_f-dimensional Hausdorff measure of $\mathbb{R}^2$ normalized, that is,

$$\mu = (H^{d_f}(K))^{-1} H^{d_f}(\cdot)\lfloor K \ , \tag{2.1}$$

where $d_f = \frac{\log 4}{\log 3}$.

For $f : V_\star \to \mathbb{R}$, we define for $1 < p < \infty$:

$$\mathcal{E}^{(p)}_n[f] = \frac{1}{p} 4^{(p-1)n} \sum_{i_1,...,i_n=1}^{4} \sum_{\xi,\eta\in V} |f(\psi_{i_1...i_n}(\xi)) - f(\psi_{i_1...i_n}(\eta))|^p. \tag{2.2}$$

It is shown in [1] that the sequence $\mathcal{E}_n^{(p)}(f,f)$ is non-decreasing, and by defining for $f : V_\star \to \mathbb{R}$:

$$\mathcal{E}^{(p)}[f] = \lim_{n\to\infty} \mathcal{E}_n^{(p)}[f], \tag{2.3}$$

the set

$$\mathcal{F}_\star^{(p)} = \{f : V_\star \to \mathbb{R} : \mathcal{E}^{(p)}[f] < \infty\} \tag{2.4}$$

does not degenerate to a space containing only constant functions. As proved in [1], each $f \in \mathcal{F}_\star^{(p)}$ can be uniquely extended in $C(K)$. We denote this extension on K still by f and we define the space

$$\mathcal{F}^{(p)} = \{f \in C(K) : \mathcal{E}^{(p)}[f] < \infty\}, \tag{2.5}$$

where $\mathcal{E}^{(p)}[f] := \mathcal{E}^{(p)}[f|_{V^\star}]$. Hence $\mathcal{F}^{(p)} \subset C(K) \subset L^p(K,\mu)$. Moreover, $(\mathcal{E}^{(p)}, \mathcal{F}^{(p)})$ is a non-negative energy functional in $L^p(K,\mu)$ and the following result holds (see [1]).

Theorem 2.1

(i) $\mathcal{F}^{(p)}$ *is complete in the norm* $\|f\|_{\mathcal{F}^{(p)}} := \|f\|_{L^p(K,\mu)} + (\mathcal{E}^{(p)}[f])^{1/p}$.

(ii) $\mathcal{F}^{(p)}$ *is dense in* $L^p(K,\mu)$.

(iii) $\mathcal{F}^{(q)} \subset \mathcal{F}^{(p)}$, *for* $1 < p \le q < \infty$.

3. THE LIPSCHITZ SPACES $Lip_{\alpha,d_f}(p,q,K)$

In this section we recall the definition of the Lipschitz spaces introduced by Jonsson in [3].

Let $B_e(x,r)$ denote the closed Euclidean ball with center $x \in \mathbb{R}^D$ and radius r. According to [8], we first recall the definition of d_f-set.

Definition 3.1. *A closed non-empty subset* $F \subset \mathbb{R}^D$ *is a* d_f*-set* $(0 < d_f \le D)$ *if there exists a Borel measure* μ *in* $\mathbb{R}^D$ *with* $supp\mu = F$*, such that for some positive constants* $c_1 = c_1(F)$ *and* $c_2 = c_2(F)$*:*

$$c_1 r^{d_f} \le \mu(B_e(x,r)) \le c_2 r^{d_f} \quad \text{for } x \in F,\ 0 < r \le 1. \tag{3.1}$$

Such a μ *is called a* d_f*-measure on* F.

If F is a d_f-set, then the restriction to F of the d_f-dimensional Hausdorff measure of $\mathbb{R}^D$ is a d_f-measure on F and thus the Hausdorff dimension of F is d_f (for details and proofs see [8]).

We recall that the measure μ on the Koch curve has the property that there exist two positive constants c_1, c_2 such that

$$c_1 r^{d_f} \le \mu(B_e(x,r)) \le c_2 r^{d_f},\ \forall x \in K,$$

and so the Koch curve is a d_f-set with $d_f = \frac{\log 4}{\log 3} > 1$ and the space $Lip_{\alpha,d_f}(p,q,K)$ is well-defined.

Let $F \subset \mathbb{R}^D$ be a d_f-set, $0 < d_f \le D$, let μ be the d_f-measure on F.

In order to define Lipschitz spaces which we shall consider here it will be convenient to introduce the following notation. Given an appropriately defined function θ over a set F, we write for convenience

$$I(F,r,\theta,p) \equiv \int\int_{x,\,y\in F,\,|x-y|<c_0 3^{-h+r}} |\theta(x)-\theta(y)|^p d\mu(x)d\mu(y)$$

for some numbers $c_0,\ h,\ p$, and r. Here and elsewhere $c_0>0,\ 1<p<\infty,\ r=0,\ n$. We first define Lipschitz spaces $Lip_{\alpha,d_f}(p,q,F)$ in the following.

Definition 3.2. *Let $c_0>0$, $\alpha>0$, $1<p<\infty$, $1\le q\le\infty$, $Lip_{\alpha,d_f}(p,q,F)$ is the space of those functions f such that $f\in L^p(F,\mu)$,*

$$\|\{a_h\}\|_{l_q} = \left(\sum_{h=0}^{\infty} a_h^q\right)^{1/q} < \infty \ \text{ for } 1\le q<\infty \tag{3.2}$$

$$\|\{a_h\}\|_{l_\infty} = \sup_{h\ge 0}|a_h| < \infty \ \text{ for } q=\infty \tag{3.3}$$

where, for each $h\in\mathbb{N}$,

$$a_h = \left(3^{h\alpha p+hd_f} I(F,0,f,p)\right)^{1/p}. \tag{3.4}$$

The norm in $Lip_{\alpha,d_f}(p,q,F)$ is defined as:

$$\||f\||_{Lip_{\alpha,d_f}(p,q,F)} := \|f\|_{L^p(F,\mu)} + \|\{a_h\}\|_{l_q}. \tag{3.5}$$

In Jonsson's notations these spaces are denoted by $Lip(\alpha,p,q,F)$; we modified this notation in $Lip_{\alpha,d_f}(p,q,F)$ to put in evidence also the dependence on the fractal dimension. Moreover, the constant 3 in (3.4) replaces the constant 2 in Jonsson's definition: this clearly gives equivalent spaces with equivalent norms.

Considering the Koch curve K we proved the following results in [2] and the statement of Theorem 3.3 may be given.

Theorem 3.3. *Let $1<p<\infty$. Let K denote the Koch curve, $\mathcal{F}^{(p)}$ the domain of the associated nonlinear energy form $\mathcal{E}^{(p)}$, then*

$$\mathcal{F}^{(p)} = Lip_{\delta,d_f}(p,\infty,K), \tag{3.6}$$

where $\delta=\frac{\log 4}{\log 3}$, with equivalent norms.

We note that the smoothness index δ does not depend on p.

Corollary 3.4. *Let K be the Koch curve and $\delta=\frac{\ln 4}{\ln 3}$. Then the space $Lip_{\delta,d_f}(p,\infty,K)$ does not consist only of constant functions.*

Remark 3.5. In order to prove that the Lipschitz space $Lip_{\delta,d_f}(p,\infty,K)$ is not trivial it obviously suffices to construct an explicit example of a non-constant function belonging to this space. For the case considered in the previous corollary, such a function has been constructed in [15]. A more difficult task is to give significant characterizations of all functions belonging to these Lipschitz spaces. Therefore, it is remarkable that such characterizations are available — in terms of spaces of "finite energy" — both in the linear and in the nonlinear case considered in the previous mentioned papers. In this regard, this characterization also shows that there exist a whole family of non-trivial functions, namely the harmonic (or p-harmonic) functions, which can be also explicitly constructed by a suitable harmonization procedure.

4. THE NONLINEAR ENERGY FORM ON THE INFINITE KOCH CURVE

In this section, we recall the definition of the infinite Koch curve $K^{<\infty>}$ (for the general case of expanded nested fractals, see [7]).

We set $K^{<0>} = K$, $K^{<n>} = 3^n K$ for $n \geq 1$ and $K^{<\infty>} = \bigcup_{n=0}^{\infty} K^{<n>}$. We have that

$$K^{<n>} = \bigcup_{i_1 \dots i_n = 1}^{4} K^{<n>}_{i_1 \dots i_n},$$

where

$$K^{<n>}_{i_1 \dots i_n} = \phi_{i_1 \dots i_n} K$$

with

$$\phi_{i_1 \dots i_n} = 3^n \psi_{i_1 \dots i_n}.$$

We note that the infinite Koch curve is confined to the sector of amplitude $\pi/6$. We define the mapping σ_n by $(\sigma_n f)(x) = f(3^n x)$ for $x \in K$ which maps a function f on $K^{<n>}$ to a function $\sigma_n f$ on the unit Koch curve.

We extend the Hausdorff measure μ of K to $K^{<\infty>}$ by defining its value on a Borel set B to be $\mu(\phi^{-1}_{i_1 \dots i_n}(B))$ if $B \subset K^{<n>}_{i_1 \dots i_n}$.

We also define nonlinear energy forms $\mathcal{E}^{(p)}_{K^{<n>}}$ with domains $\mathcal{F}^{(p)}_{K^{<n>}}$ on $L^p(K^{<n>}, \mu)$ by

$$\mathcal{F}^{(p)}_{K^{<n>}} = \sigma_n^{-1} \mathcal{F}^{(p)} \tag{4.1}$$

and

$$\mathcal{E}^{(p)}_{K^{<n>}}[f] = \sum_{i_1 \dots i_n = 1}^{4} \mathcal{E}^{(p)}[f(\phi_{i_1 \dots i_n} \cdot)] \tag{4.2}$$

for $f \in \mathcal{F}^{(p)}_{K^{<n>}}$. We set $\mathcal{F}^{(p)}_{K^{<0>}} = \mathcal{F}^{(p)}$ and $\mathcal{E}^{(p)}_{K^{<0>}}[f] = \mathcal{E}^{(p)}[f]$; moreover, we set

$$\|f\|_{\mathcal{F}^{(p)}_{K^{<n>}}} := \|f\|_{L^p(K^{<n>}, \mu)} + (\mathcal{E}^{(p)}_{K^{<n>}}[f])^{1/p}.$$

From definitions (4.1) and (4.2), we have that if $n \leq m$ and $f \in \mathcal{F}^{(p)}_{K^{<m>}}$, then $\mathcal{E}^{(p)}_{K^{<n>}}[f\lfloor_{K^{<n>}}] \leq \mathcal{E}^{(p)}_{K^{<m>}}[f\lfloor_{K^{<m>}}]$.

We recall the following scaling properties from [7].

Proposition 4.1.

(*a*) *For a function f defined on $K^{<n>}$,*

$$\int_{K^{<n>}} f d\mu = 4^n \int_K \sigma_n f d\mu.$$

(*b*) *For a function $f \in \mathcal{F}^{(p)}_{K^{<n>}}$,*

$$(4^{p-1})^n \mathcal{E}^{(p)}_{K^{<n>}}[f] = \mathcal{E}^{(p)}[\sigma_n f].$$

We define the space $\mathcal{F}^{(p)}_{K^{<\infty>}}$ of functions f defined on the infinite Koch curve $K^{<\infty>}$ by

$$\mathcal{F}^{(p)}_{K^{<\infty>}} = \{f : f\lfloor_{K^{<n>}} \in \mathcal{F}^{(p)}_{K^{<n>}} \text{for each } n \text{ and } \lim_{n\to\infty} \mathcal{E}^{(p)}_{K^{<n>}}[f\lfloor_{K^{<n>}}] < \infty\} \bigcap L^p(K^{<\infty>}, \mu).$$

We let

$$\mathcal{E}^{(p)}_{K^{<\infty>}}[f] = \lim_{n\to\infty} \mathcal{E}^{(p)}_{K^{<n>}}[f\lfloor_{K^{<n>}}]$$

for $f \in \mathcal{F}^{(p)}_{K^{<\infty>}}$ and

$$\|f\|_{\mathcal{F}^{(p)}_{K^{<\infty>}}} := \|f\|_{L^p(K^{<\infty>},\mu)} + (\mathcal{E}^{(p)}_{K^{<\infty>}}[f])^{1/p}.$$

Denote by $C(K^{<\infty>})$ the space of continuous functions on $K^{<\infty>}$ and by $C_0(K^{<\infty>})$ the space of continuous functions with compact support on $K^{<\infty>}$. We have $\mathcal{F}^{(p)}_{K^{<\infty>}} \subset C(K^{<\infty>})$.

We recall the following property (see [7]): if $f \in C_0(K^{<\infty>})$, supp$f \subset K^{<n>}$ and $f \in \mathcal{F}^{(p)}_{K^{<n>}}$, then $f \in \mathcal{F}^{(p)}_{K^{<\infty>}}$ and $\mathcal{E}^{(p)}_{K^{<\infty>}}[f] = \mathcal{E}^{(p)}_{K^{<n>}}[f]$. By the previous property, the spaces $\mathcal{F}^{(p)}_{K^{<\infty>}}$ are not trivial: it is sufficient to consider functions $f \in \mathcal{F}^{(p)}$, $f \in C_0(K^{<\infty>})$ with supp$f \subset K$. Moreover, by proceeding as in [7], it turns out that the forms $\mathcal{E}^{(p)}_{K^{<\infty>}}$ with domains $\mathcal{F}^{(p)}_{K^{<\infty>}}$ are regular on $L^p(K^{<\infty>}, \mu)$.

5. THE LIPSCHITZ SPACES $Lip_{\alpha,d_f}(p, \infty, K^{<\infty>})$

In this section, we give the definition of Lipschitz spaces on $K^{<\infty>}$ and we prove that for $1 < p < \infty$ the domains $\mathcal{F}^{(p)}_{K^{<\infty>}}$ coincide with the spaces $Lip_{\delta,d_f}(p, \infty, K^{<\infty>})$ where $\delta = \frac{\log 4}{\log 3}$ (see for $p = 2$ also [16]). In order to define the spaces $Lip_{\alpha,d_f}(p, \infty, K^{<\infty>})$ we now first define the Lipschitz spaces $Lip_{\alpha,d_f}(p, \infty, K^{<n>})$ on $K^{<n>}$.

Definition 5.1. *Let $c_0 > 0$, $\alpha > 0$, $1 < p < \infty$, $n \in \mathbb{N}$. We denote by $Lip_{\alpha,d_f}(p, \infty, K^{<n>})$ the space of those functions f such that $f \in L^p(K^{<n>}, \mu)$ and*

$$\sup_{h\geq -n} \left(3^{h(\alpha p+d_f)} I(K^{<n>}, 0, f\lfloor_{K^{<n>}}, p)\right)^{\frac{1}{p}} < \infty.$$

We also let

$$|||f|||_{Lip_{\alpha,d_f}(p,\infty,K^{<n>})} = \|f\|_{L^p(K^{<n>},\mu)} + \sup_{h\geq -n}\left(3^{h(\alpha p+d_f)}I(K^{<n>},0,f\lfloor_{K^{<n>}},p)\right)^{\frac{1}{p}}$$

for $f \in Lip_{\alpha,d_f}(p,\infty,K^{<n>})$.

Definition 5.2. *Let* $c_0 > 0$, $\alpha > 0$, $1 < p < \infty$. *We denote by* $Lip_{\alpha,d_f}(p,\infty,K^{<\infty>})$ *the space of those functions* f *such that* $f \in L^p(K^{<\infty>},\mu)$, $f\lfloor_{K^{<n>}} \in Lip_{\alpha,d_f}(p,\infty,K^{<n>})$ *for each* n *and*

$$\lim_{n\to\infty} |||f|||_{Lip_{\alpha,d_f}(p,\infty,K^{<n>})} < \infty.$$

We also let

$$|||f|||_{Lip_{\alpha,d_f}(p,\infty,K^{<\infty>})} = \|f\|_{L^p(K^{<\infty>},\mu)} + \sup_{h\in Z}\left(3^{h(\alpha p+d_f)}I(K^{<\infty>},0,f,p)\right)^{\frac{1}{p}}$$

for $f \in Lip_{\alpha,d_f}(p,\infty,K^{<\infty>})$.

In order to prove Theorem 5.5, we now prove some scaling properties of the terms that appear in the definitions of the Lipschitz spaces on $K^{<n>}$.

Proposition 5.3. *Let* f *be a function defined on* $K^{<n>}$, *with fixed* n. *Let* $\delta = \frac{\log 4}{\log 3}$. *Then*

$$3^{h\delta p+hd_f}I(K,0,\sigma_n f,p) = (4^{p-1})^n 3^{(h-n)(\delta p+d_f)}I(K^{<n>},n,f,p).$$

Proof. From (a) of Proposition 4.1

$$\begin{aligned}3^{h\delta p+hd_f}I(K,0,\sigma_n f,p) &= 3^{h(\delta p+d_f)}4^{-2n}I(K^{<n>},n,f,p)\\ &= 3^{(h-n)(\delta p+d_f)}3^{n(\delta p+d_f)}4^{-2n}I(K^{<n>},n,f,p).\end{aligned}$$ □

Proposition 5.4. *Let* $\delta = \frac{\log 4}{\log 3}$, $1 < p < \infty$, *and* $c_0 > 0$. *Then there exist two positive constants* c_1 *and* c_2 *such that*

$$c_1\|f\|_{\mathcal{F}^{(p)}_{K^{<n>}}} \leq |||f|||_{Lip_{\delta,d_f}(p,\infty,K^{<n>})} \leq c_2\|f\|_{\mathcal{F}^{(p)}_{K^{<n>}}}. \tag{5.1}$$

Proof. By Theorem 3.3, there exist two positive constants c_1 and c_2 such that

$$c_1\left(\mathcal{E}^{(p)}[\sigma_n f]\right)^{\frac{1}{p}} \leq \sup_{h\geq 0}\left(3^{h\delta p+hd_f}I(K,0,\sigma_n f,p)\right)^{\frac{1}{p}} \leq c_2\left(\mathcal{E}^{(p)}[\sigma_n f]\right)^{\frac{1}{p}}.$$

From (b) of Proposition 4.1 and Proposition 5.3, we obtain that there exist two positive constants c_1 and c_2 such that

$$c_1\left(\mathcal{E}^{(p)}_{K^{<n>}}[f]\right)^{\frac{1}{p}} \leq \sup_{h\geq -n}\left(3^{h\delta p+hd_f}I(K^{<n>},0,f,p)\right)^{\frac{1}{p}} \leq c_2\left(\mathcal{E}^{(p)}_{K^{<n>}}[f]\right)^{\frac{1}{p}}.$$ □

Theorem 5.5. *Let $1 < p < \infty$. Let $K^{<\infty>}$ denote the infinite Koch curve, $\mathcal{F}^{(p)}_{K^{<\infty>}}$ the domain of the associated nonlinear energy form $\mathcal{E}^{(p)}_{K^{<\infty>}}$, then*

$$\mathcal{F}^{(p)}_{K^{<\infty>}} = Lip_{\delta,d_f}(p,\infty,K^{<\infty>}),$$

where $\delta = \frac{\log 4}{\log 3}$, with equivalent norms.

Proof. Let $f \in \mathcal{F}^{(p)}_{K^{<\infty>}}$. By definition, $f\lfloor_{K^{<n>}} \in \mathcal{F}^{(p)}_{K^{<n>}}$ for each n, $f \in L^p(K^{<\infty>},\mu)$ and $\lim_{n\to\infty} \mathcal{E}^{(p)}_{K^{<n>}}[f\lfloor_{K^{<n>}}] = \sup_n \mathcal{E}^{(p)}_{K^{<n>}}[f\lfloor_{K^{<n>}}] < \infty$. Passing to the limit in (5.1) we have that $Lip_{\delta,d_f}(p,\infty,K^{<\infty>}) \subset \mathcal{F}^{(p)}_{K^{<\infty>}}$. We proceed in a similar way for the other inclusion. □

Corollary 5.6. *Let $K^{<\infty>}$ be the infinite Koch curve and $\delta = \frac{\log 4}{\log 3}$. Then the space $Lip_{\delta,d_f}(p,\infty,K^{<\infty>})$ does not consist only of constant functions.*

5.1. The "Bilateral" Infinite Koch Curve $\hat{K}$.

We now define the "bilateral" infinite Koch curve $\hat{K}$. For every fixed n, let $K^{\leq n>}_{-}$ be the curve obtained by symmetrization with respect to the origin of the axes the curve $K^{<n>}$. Let $K^{\leq\infty>}_{-} = \bigcup_{n=0}^{\infty} K^{\leq n>}_{-}$. Clearly, $K^{\leq\infty>}_{-}$ is symmetric with respect to the origin of the axes of the curve $K^{<\infty>}$. Set now $K^{\leq\infty>}_{+} = K^{<\infty>}$, we define

$$\hat{K} = K^{\leq\infty>}_{+} \bigcup K^{\leq\infty>}_{-}.$$

As in section 4, we define the nonlinear energy forms $\mathcal{E}^{(p)}_{K^{\leq\infty>}_{-}}$ with domains $\mathcal{F}^{(p)}_{K^{\leq\infty>}_{-}}$. Similarly to Theorem 5.5, a characterization in terms of Lipschitz spaces on $K^{\leq\infty>}_{-}$ holds, namely, $\mathcal{F}^{(p)}_{K^{\leq\infty>}_{-}} = Lip_{\delta,d_f}(p,\infty,K^{\leq\infty>}_{-})$.

By analogy with the Euclidean case, on the domains

$$\mathcal{F}^{(p)}_{\hat{K}} = \left\{ f : \hat{K} \to \mathbb{R}, f\lfloor_{K^{\leq\infty>}_{-}} \in \mathcal{F}^{(p)}_{K^{\leq\infty>}_{-}}, f\lfloor_{K^{\leq\infty>}_{+}} \in \mathcal{F}^{(p)}_{K^{\leq\infty>}_{+}},\ f \text{ continuous in } (0,0) \right\}$$

we can define the energy forms

$$\mathcal{E}^{(p)}_{\hat{K}} = \mathcal{E}^{(p)}_{K^{\leq\infty>}_{-}} + \mathcal{E}^{(p)}_{K^{\leq\infty>}_{+}}.$$

Finally, we define by

$$Lip_{\alpha,d_f}(p,\infty,\hat{K})$$

(with $\alpha > 0$, $1 < p < \infty$) the space of those functions f such that $f : \hat{K} \to \mathbb{R}$,

$$f\lfloor_{K^{\leq\infty>}_{-}} \in Lip_{\alpha,d_f}(p,\infty,K^{\leq\infty>}_{-}),$$

$$f\lfloor_{K^{\leq\infty>}_{+}} \in Lip_{\alpha,d_f}(p,\infty,K^{\leq\infty>}_{+}),$$

$$f \ \text{ continuous in } (0,0).$$

With this characterization, Theorem 5.5 can be extended to the case of $\hat{K}$ as follows.

Corollary 5.7. *Let $1 < p < \infty$. Let $\hat{K}$ denote the bilateral infinite Koch curve, $\mathcal{F}^{(p)}_{\hat{K}}$ the domain of the associated nonlinear energy form $\mathcal{E}^{(p)}_{\hat{K}}$, then*

$$\mathcal{F}^{(p)}_{\hat{K}} = Lip_{\delta,d_f}(p,\infty,\hat{K})$$

where $\delta = \frac{\log 4}{\log 3}$, with equivalent norms.

REFERENCES

[1] R. Capitanelli, "Nonlinear Energy Forms on Certain Fractal Curves", *J. Nonlinear Convex Anal.*, **3**(**1**) (2002), pp. 67–80.

[2] R. Capitanelli and M.R. Lancia, "Nonlinear Energy Forms and Lipschitz Spaces on the Koch Curve", *J. Convex Anal.*, **9**(**1**) (2002), pp. 245–257.

[3] A. Jonsson, "Brownian Motion on Fractals and Function Spaces", *Math. Z.*, **6** (1996), pp. 495–504.

[4] M.R. Lancia and M.A. Vivaldi, "Lipschitz Spaces and Besov Traces on Self-Similar Fractals", *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.*, **117**(**XXIII**) (1999), pp. 101–116.

[5] K.P. Paluba, "Some Function Spaces Related to the Brownian Motion on Simple Nested Fractals", *Stoch. and Stoch. Rep.*, **67** (1999), pp. 267–285.

[6] T. Kumagai, "Brownian Motion Penetrating Fractals: an Application of the Trace Theorem of Besov Spaces", *J. Funct. Anal.*, **170** (2000), pp. 69–92.

[7] M. Fukushima, "Dirichlet Forms, Diffusion Processes and Spectral Dimensions for Nested Fractals", in *Ideas and Methods in Mathematical Analysis, Stochastics, and Applications (Oslo, 1988)*. Cambridge: Cambridge Univ. Press, 1992, pp. 151–161.

[8] A. Jonsson and H. Wallin, "Function Spaces on Subsets of $\mathbb{R}^n$", *Math. Rep.*, **2**(**1**) (1984), pp. 1–221.

[9] M.R. Lancia, "A Transmission Problem with a Fractal Interface", *Z. Anal. Anwendungen*, **21** (2002), pp. 113–133.

[10] M.R. Lancia and M.A. Vivaldi, "Asymptotic Convergence of Transmission Energy Forms", *Adv. Math. Sc. Appl.*, **13**(**1**) (2003), pp. 315–341.

[11] M.R. Lancia, "On Some Second Order Transmission Problems" (preprint MeMoMat 15/2002) to appear in AJSE.

[12] R. Capitanelli, "Lagrangians on Homogeneous Spaces", *PhD Thesis, Università di Roma "La Sapienza"*, 2001.

[13] J.I. Dìaz, "Nonlinear Partial Differential Equations and Free Boundaries", *Elliptic Equations, vol. I. Research Notes in Mathematics, 106.* Boston, MA: Pitman (Advanced Publishing Program), 1985.

[14] J.E. Hutchinson, "Fractals and Selfsimilarity", *Indiana Univ. Math. J.*, **30** (1981), pp. 713–747.

[15] A. Jonsson, "Haar Wavelets of Higher Order on Fractals and Regularity of Functions", *Research reports n.7, Umeå University*, 1998.

[16] B.M. Hambly and T. Kumagai, "Diffusion Process on Fractal Fields: Heat Kernel Estimates and Large Deviations", *Probab. Theory Relat. Fields*, **127**(**3**) (2003), pp. 305–352.

Paper Received 13 November 2002; Revised 18 May 2003; Accepted 26 October 2003.