Uniqueness of Stochastic User Equilibrium

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Abstract

Travel demand assignment to road and/or transit networks is one of the main tools for transportation system analysis. Assignment models are usually specified following a user equilibrium approach, i.e. searching for reciprocally consistent flows and costs. In Stochastic User Equilibrium (SUE) travellers' routing behaviour is modelled by taking into account several sources of uncertainty through discrete choice models, typically derived from random utility theory. This paper presents a comprehensive analysis of SUE uniqueness, proposing conditions which are weaker than those commonly referred to in literature, and includes other existing results.

1. Introduction

Assignment models have been first specified following a (Wardrop or deterministic) *user equilibrium* (UE) approach under perfect information and rational behaviour assumptions searching for reciprocally consistent flows and costs, as introduced by Wardrop (1952). Later, Daganzo and Sheffi (1977) introduced the so-called *stochastic user equilibrium* (SUE), where travellers' route choice behaviour is reproduced through more realistic behavioural models, which take into account several sources of uncertainty by assuming the perceived utility of travel alternatives as jointly distribute random variables.

SUE may effectively be formulated through fixed-point models (Daganzo, 1983; Cantarella 1997), which can be easily extended to deal with several types of assignment (variable demand, multi-mode congestion, multi-classes, …) and allow for weaker uniqueness conditions when compared with optimization models and/or Wardrop user equilibrium.

This paper presents a comprehensive analysis of SUE pattern uniqueness, proposing conditions which are weaker than those in Daganzo (1983) and Cantarella (1997), and include other existing results (Cantarella, 2002; Gentile, 2003; Cantarella and Velonà, 2009). For instance, according to the proposed conditions, the arc cost (vector) function does not need to be monotone increasing. Weak uniqueness conditions may not be stated for Wardrop user equilibrium, as can be shown by very simple examples, thus UE is not addressed.

In this paper user' choice behaviour includes route choice only, assuming constant demand flows. Extension of presented results to assignment with variable demand will be discussed in a future paper. Moreover, (within-day) steady-state conditions are assumed.

Section 2 reviews basic definitions, notations and equations, whilst section 3 describes some fixed-point models for SUE useful to state the SUE uniqueness conditions proposed described in section 4. Then, section 5 describes results of applications to a toy networks which allow for pictorial representation. Finally, major findings are reviewed in section 6 together with some research perspectives.

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2. Definitions, Notations, Equations[1]

This section briefly reviews definitions, notations, and equations underling most approaches for modelling supply-demand interaction, that is travel demand assignment. Users travelling between the same origin-destination pair with common behavioural features are assumed grouped into user classes. User choice behaviour concerns routing alternatives. Let

K_i be the set of available route for user class i, assumed non-empty and finite with $m_i = | K_i |$; each set K_i can be assumed containing at least two routes with no loss of generality (flows resulting from a user class connected by one route only can be included within the vector of arc base flows, see below).

Connections between origins and destinations are modelled through a directed graph, i.e. a collection of *arcs* (each one connecting one node to another node) in the simple case of pre-trip choice behaviour, where each route is represented by a path. In the more general case of mixed pre-trip en-route strategic behaviour (Spiess and Florian, 1989), where each route is represented by a hyperpath (Nguyen and Pallottino, 1988), it is necessary to introduce instead a hypergraph, i.e. a collection of *hyperarcs*, each one connecting one node to several nodes (Gallo et al., 1993). The latter case typically occurs in transit networks to reproduce passengers waiting at stops to board the first arriving carrier of an *attractive* set of lines. Results in this paper apply to both cases, an *arc* may also represent a hyperarc, depending on the context. Let

$\mathbf{B}_i$ be the arc-route matrix for user class i; with entries $b_{ak} \in]0,1]$ if arc a belongs to route k ($b_{ak} = 1$ if route k is represented by a path), $b_{ak} = 0$ otherwise; with no loss of generality in the following it is assumed that in matrix $\mathbf{B}_i$ no row is null, each arc belongs to at least one route, and no column is null, each route contains at least one arc.

Only elementary (or with a limited number of internal loops) routes will be considered in the following to guarantee that upper bounded route flows imply upper bounded arc flows.

2.1 *Modelling Transportation Supply*.

Under steady-state conditions transportation supply is usually modelled through a flow network, where a transportation cost c_a and a flow f_a are associated to each arc a. (Node costs can be introduced by duly modifying the graph). Let

$\mathbf{h}_i \geq \mathbf{0}$ be the vector of route flows for user class i, with entries h_k, $k \in K_i$;
$\mathbf{c}$ be the vector of arc costs, with entries c_a;
$\mathbf{f} \geq \mathbf{0}$ be the vector of arc flows, with entries f_a;
$\mathbf{f_b} \geq \mathbf{0}$ be the vector of arc base flows, with entries $f_{b,a}$, that is the flow that traverses arc a in any case, and does not result from the modelled demand-supply interaction (e.g. trucks, ...);
$\mathbf{w}_i$ be the vector of route costs for user class i, with entries w_k, $k \in K_i$;
$\mathbf{w}_{o,i}$ be the vector of specific route costs for user class i, with entries $w_{o,k}$, $k \in K_i$, that is route cost which cannot be obtained by summing up arc costs (e.g. O-D based tolls, ...).

The *arc-route flows consistency* is expressed by an affine transformation of route flows:

$$\mathbf{f} = \Sigma_i \ \mathbf{B}_i \ \mathbf{h}_i + \mathbf{f_b} \tag{2.1}$$

[1] The contents of this section are intentionally similar to the second section of other papers authored or co-authored by Giulio E. Cantarella. The aim is twofold: helping readers to catch original contributions of different papers about related topics, and proposing standard definitions and notations to the community of transportation system analysis.

Congestion is generally modelled assuming that arc costs depend on arc flows, through the *arc cost function*:

$$\mathbf{c} = \mathbf{c}(\mathbf{f}) \tag{2.2}$$

This vector function is assumed continuous and continuously differentiable w.r.t. arc flows, with Jacobian matrix **Jac**[**c**(**f**)]. (In order to get continuity and c. differentiability a function with a vertical asymptote at capacity can be substituted by a polynomial approximation from a point close to the capacity.)

The *arc-route cost consistency* for each user class i is expressed by an affine transformation of arc costs:

$$\mathbf{w}_\mathrm{i} = \mathbf{B}_\mathrm{i}^\mathrm{T}\, \mathbf{c} + \mathbf{w}_\mathrm{o,i} \quad \forall i \tag{2.3}$$

2.2 *Modelling Travel Demand*.

Route choice behaviour can be modelled assuming that a user of class i

~ examines all routes in the (non-empty) choice set K_i,

~ associates to each route $k \in K_i$ a value of *perceived utility* U_k,

~ chooses a route with the maximum perceived utility.

If the perceived utilities are modelled by a continuous jointly distributed random variable, to represent several sources of uncertainty, the demand model complies with random utility theory (Domencich and McFadden, 1975). These hypotheses imply that the probability that a user of class i chooses route k is the probability that route k has the highest utility. The perceived utility can be expressed as the sum of two terms, the expected value, or *systematic utility* v_k, and the *random residual*. In this paper *probabilistic choice models* are considered, which are obtained when the co-variance matrix of perceived utilities is non-singular. Let

$\mathbf{v}_\mathrm{i}$ be the vector of route systematic utilities for user class i, with entries v_k, $k \in K_i$;

$\mathbf{p}_\mathrm{i} \geq \mathbf{0}$ be the vector of route choice probabilities for user class i, such that $\mathbf{1}^\mathrm{T}\, \mathbf{p}_\mathrm{i} = 1$, with entries p_k, given by the probability p_k that a user of class i chooses route $k \in K_i$;

$d_i \geq 0$ be the demand flow for users belonging to class i; it is assumed that demand flows are measured in common units, using homogenization coefficients for users with different effects on congestion.

The systematic utility values depend on the corresponding route costs through the *route utility function*, generally expressed by an affine transformation of route costs:

$$\mathbf{v}_\mathrm{i} = -\beta_\mathrm{i}\, \mathbf{w}_\mathrm{i} + \mathbf{v}_\mathrm{o,i} \quad \forall i$$

where

$\mathbf{v}_\mathrm{o,i}$ is the vector of the part of route systematic utilities for user class i depending on attributes other than costs; it has entries $v_{o,k}$, $k \in K_i$; in the following this vector is assumed included into vector $\mathbf{w}_\mathrm{o,i}$ thus it is not explicitly reported for the sake of simplicity;

$\beta_\mathrm{i} > 0$ is a positive coefficient.

For above assumptions, with no loss of generality, the *route utility function* is given by:

$$\mathbf{v}_\mathrm{i} = -\beta_\mathrm{i}\, \mathbf{w}_\mathrm{i} \quad \forall i \tag{2.4}$$

According with the random utility theory the route choice probabilities depends on route choice systematic utilities for each user class i, through the *route choice function*:

$$\mathbf{p}_\mathrm{i} = \mathbf{p}_\mathrm{i}(\mathbf{v}_\mathrm{i}) \quad \forall i \tag{2.5}$$

This function is continuous and c. differentiable for usually adopted probabilistic choice models. *Strictly positive choice functions* always give probability greater than zero to all routes considered by the user class i, $\mathbf{p}_i(\mathbf{v}_i) > \mathbf{0}$, whichever are the systematic utilities. An example is the well-known Logit model:

$$p_k = \exp(v_k / \theta_i) / \sum_{j \in Ki} \exp(v_j / \theta_i)$$

where $\theta_i = (\sqrt{6} / \pi)\ \sigma_i > 0$ is the scale or dispersion parameter, and the variances for all routes are assumed equal: $\mathrm{Var}[U_k] = \sigma_i^{\,2}$.

A choice function is called *invariant* if the choice set K_i of all user classes and the parameters of perceived utility distribution, such as θ_i for a Logit model, do not depend on the route systematic utilities. In such models the route choice probabilities depend on differences between systematic utility values only, moreover the resulting route choice functions (for proofs see Cantarella, 1997):

~ are monotone increasing with respect to systematic utility $\mathbf{v}_i$;
~ have symmetric positive semi-definite Jacobian, $\mathbf{Jac}[\mathbf{p}_i(\mathbf{v}_i)]\quad 0$.

Demand conservation for each user class i can be expressed by the following equation regarding route flows:

$$\mathbf{h}_i = d_i\, \mathbf{p}_i \quad \forall i \tag{2.6}$$

Combining equations (2.4-6) leads to the *route flow function* for each user class i:

$$\mathbf{h}_i(\mathbf{w}_i) = d_i\, \mathbf{p}_i(-\beta_i\, \mathbf{w}_i) \quad \forall i \tag{2.7}$$

Properties of the route flow function the can be derived from those of the route choice function, such as the Jacobian $\mathbf{Jac}[\mathbf{h}_i(\mathbf{w}_i)] = -d_i\, \beta_i\, \mathbf{Jac}[\mathbf{p}_i(-\beta_i\, \mathbf{w}_i)]$. This function is continuous and c. differentiable for usually adopted probabilistic choice models; moreover, for invariant choice functions, it

~ is monotone decreasing w.r.t. route costs, $\mathbf{w}_i$;
~ has symmetric negative semi-definite Jacobian, $\mathbf{Jac}[\mathbf{h}_i(\mathbf{w}_i)]\quad 0$.

2.3 *Modelling Travel Demand through Independent Route Variables*.

The set of available routes for each user class i, K_i, has been assumed containing at least two routes, thus flows for the first $(m_i - 1)$ routes, called *independent route* (*iro)* flows, are enough to define the route flow vector, due to the demand conservation equation (2.6). Let

$\tilde{\mathbf{p}}_i \geq \mathbf{0}$ be the vector of the choice probabilities for the independent, say the first $(m_i - 1)$, routes of user class i, with $\mathbf{1}^{\mathrm{T}}\, \tilde{\mathbf{p}}_i \leq 1$;
$\tilde{\mathbf{h}}_i \geq \mathbf{0}$ be the vector of the first $(m_i - 1)$ route flows for user class i;

Demand conservation yields:

$$\tilde{\mathbf{h}}_i = d_i\, \tilde{\mathbf{p}}_i \quad \forall i \tag{2.6'}$$

Assuming that the route choice probabilities depend on systematic utility differences only, as for invariant choice models, the *independent route choice function* may be introduced as:

$$\tilde{\mathbf{p}}_i = \tilde{\mathbf{p}}_i(\tilde{\mathbf{v}}_i) \quad \forall i \tag{2.5'}$$

where $\tilde{\mathbf{v}}_i$ is the vector of the differences between the first $(m_i - 1)$ route systematic utilities and the last one for user class i.

A strictly positive route choice function, say $\tilde{\mathbf{p}}_i(\tilde{\mathbf{v}}_i) > \mathbf{0}$, yields: $\mathbf{1}^{\mathrm{T}}\, \tilde{\mathbf{p}}_i(\tilde{\mathbf{v}}_i) < 1$. An invariant s. positive independent route choice function (Gentile, 2003; Cantarella and Velonà, 2009):

~ is monotone strictly increasing with respect to systematic utilities $\tilde{\mathbf{v}}_i$;
~ has symmetric positive definite Jacobian, $\mathbf{Jac}[\tilde{\mathbf{p}}_i(\tilde{\mathbf{v}}_i)]\quad 0$.

If it also assumed that systematic utilities are linearly decreasing with costs, as in eqn (2.4), it turns out:

$$\tilde{\mathbf{v}}_i = -\beta_i\, \tilde{\mathbf{w}}_i \quad \forall i \qquad (2.4')$$

where $\tilde{\mathbf{w}}_i$ be the vector of the differences between the first $(m_i - 1)$ route costs and the last one, for user class i.

The *independent route flow function* for each user class i may be defined by combining equations (2.4'- 2.6') as:

$$\tilde{\mathbf{h}}_i(\tilde{\mathbf{w}}_i) = d_i\, \tilde{\mathbf{p}}_i(-\beta_i\, \tilde{\mathbf{w}}_i) \quad \forall i \qquad (2.8)$$

This function is continuous and c. differentiable for usually adopted probabilistic choice models; moreover, for invariant strictly positive choice functions, it

- ~ is strictly monotone decreasing (thus invertible) w.r.t. route cost differences, $\tilde{\mathbf{w}}_i$;
- ~ has symmetric negative definite (thus non-singular) Jacobian, $\mathbf{Jac}[\tilde{\mathbf{h}}_i(\tilde{\mathbf{w}}_i)]$ 0.

It is noteworthy that the vector of the iro flows it is linked to the vector of route flows by an affine transformation:

$$\mathbf{h}_i = \mathbf{L}_i\, \tilde{\mathbf{h}}_i + \mathbf{h}_{\mathbf{b}i} \quad \forall i \qquad (2.9)$$

where $\mathbf{L}_i$ and $\mathbf{h}_{\mathbf{b}i}$ are a suitable matrix and vector respectively (details in Gentile, 2003; Cantarella and Velonà, 2010). Far any given vector $\tilde{\mathbf{h}}_i$ vector $\mathbf{h}_i$ can be easily obtained.

In addition, the vector of the iro cost differences is linked to the vector of route costs by a linear transformation:

$$\tilde{\mathbf{w}}_i = \mathbf{L}_i^{\mathsf{T}}\, \mathbf{w}_i \quad \forall i \qquad (2.10)$$

It is noteworthy that there exist infinite many cost vectors, $\mathbf{w}_i$ that yield the same cost difference vector, $\tilde{\mathbf{w}}_i$.

Combing equation (2.8) with the two above equations (2.9, 2.10) eqn (2.7) can be expressed in a different way:

$$\mathbf{h}_i(\mathbf{w}_i) = \mathbf{L}_i\, \tilde{\mathbf{h}}_i(\mathbf{L}_i^{\mathsf{T}}\, \mathbf{w}_i) + \mathbf{h}_{\mathbf{b}i} \quad \forall i$$

It should be stressed that the above formulation w.r.t to independent route variables holds only when for each user class:

- ~ route choice probabilities depend on systematic utility differences only, as for invariant choice models,
- ~ utility function is linear w.r.t. costs.

3. Specification of fixed-point models

Probabilistic route choice functions lead to the stochastic user equilibrium (SUE) which may effectively be formulated through fixed-point models. In sub-section 2.1 transportation supply has been described by three equations, then in sub-section 2.2. travel demand has been described by three further equations. Thus when all the equations (2.1-6) are put together a standard *six equation assignment model* (SEAM) is obtained w.r.t. to six unknown vectors. This model might be analysed and solved as such, but it is easier to reduce it to a fixed-point problem w.r.t. to one unknown vector through successive substitution of one of the six equations into the others. Moreover, according to subsection 2.3, if route choice probabilities depend on systematic utility differences only, as for invariant choice models, and the utility function is linear w.r.t. costs, as in equation (2.4) is linear, models may be formulated w.r.t. independent route variables.

Several equivalent models may be obtained this way, which provide consistent solutions, thus the selection among them mainly is a matter of mathematical convenience. Existence or uniqueness of any vector of flows or costs assure existence and uniqueness of all the others. (Actually, there exist infinite many cost vectors, $\mathbf{w}$, for any single cost difference vector, $\tilde{\mathbf{w}}$, nonetheless given the vector $\tilde{\mathbf{w}}^*$ at the equilibrium, the corresponding route flows, arc flows, arc costs and route cost $\mathbf{w}^*$ can be computed univocally.)

Fixed-point models w.r.t. iro variables are described below, after useful notations.

Let $m = \sum_i m_i$ be the total number of routes. Route flow vectors belong to the *feasible route flow set* given by: $S_h = \{\mathbf{h} = [\mathbf{h}_i]_i : \mathbf{h}_i \geq \mathbf{0},\ \mathbf{1}^T \mathbf{h}_i = d_i \ \ \forall i\} \subseteq R^m_+$, which is compact (closed and bounded) and convex, and also non-empty if each user class is connected by at least one route (connection hypothesis). The interior $S_h°$ of this set is empty.

Let $\tilde{m} = \sum_i (m_i - 1)$, be the total number of iros. Iro flow vectors belong to the *feasible iro flow set* given by: $S_{\tilde{h}} = \{\tilde{\mathbf{h}} = [\tilde{\mathbf{h}}_i]_i : \tilde{\mathbf{h}}_i \geq \mathbf{0},\ \mathbf{1}^T \tilde{\mathbf{h}}_i \leq d_i \ \ \forall i\} \subseteq R^{\tilde{m}}_+$, which is compact and convex, and also non-empty if connection hypothesis holds. In this case the interior $S_{\tilde{h}}°$ is non-empty and, for s. positive choice function, is a bounded open set $\{\tilde{\mathbf{h}} = [\tilde{\mathbf{h}}_i]_i : \tilde{\mathbf{h}}_i > \mathbf{0},\ \mathbf{1}^T \tilde{\mathbf{h}}_i < d_i \ \ \forall i\}$.

The *route cost function* is given by combining equation 2.1-3:

$$\mathbf{w}(\mathbf{h}) = \mathbf{B}^T \mathbf{c}(\mathbf{B}\ \mathbf{h} + \mathbf{f_b}) + \mathbf{w}_o \geq \mathbf{0} \quad \forall\ \mathbf{h} \in S_h \tag{3.1}$$

The *iro cost difference function* is given by combining (3.1) with (2.9) and (2.10):

$$\tilde{\mathbf{w}}(\tilde{\mathbf{h}}) = \mathbf{L}^T \mathbf{B}^T \mathbf{c}(\mathbf{B}\ \mathbf{L}\ \tilde{\mathbf{h}} + \mathbf{B}\ \mathbf{h_b} + \mathbf{f_b}) + \mathbf{L}^T \mathbf{w}_o \ \ \forall\ \tilde{\mathbf{h}} \in S_{\tilde{h}} \tag{3.2}$$

where $\mathbf{B} = [\mathbf{B}_i]_i$, $\mathbf{L} = [\mathbf{L}_i]_i$.

Models w.r.t. to independent route variables can be written down by combining the iro flow function (2.8), with the iro cost difference function (3.2) w.r.t. flows or costs as:

$$\tilde{\mathbf{h}}^* = \tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}^*)) \quad \text{with } \tilde{\mathbf{h}}^* \in S_{\tilde{h}} \tag{3.3}$$

$$\tilde{\mathbf{w}}^* = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}(\tilde{\mathbf{w}}^*)) \tag{3.4}$$

where $\tilde{\mathbf{h}}(\tilde{\mathbf{w}}) = [\tilde{\mathbf{h}}_i(\tilde{\mathbf{w}}_i)]_i$,

Fixed-point models w.r.t. arc variables are described below, after useful notations.

Let n be the total number of arcs. Arc flow vectors belong to the *feasible arc flow set* given by: $S_f = \{\mathbf{f} = \sum_i \mathbf{B}_i\, \mathbf{h}_i + \mathbf{f_b}\!: \mathbf{h}_i \geq \mathbf{0},\ \mathbf{1}^T \mathbf{h}_i = d_i \ \ \forall i\} \subseteq R^n_+$, which is compact and convex, and also non-empty if connection hypothesis holds.

The *arc flow function* (aka network loading map), yielding arc flows for a given arc costs, can be defined by combining equations (2.1, 2.3-6):

$$\mathbf{f}(\mathbf{c}) = \sum_i \mathbf{B}_i\, \mathbf{h}_i(\mathbf{B}_i^T \mathbf{c} + \mathbf{w}_{o,i}) + \mathbf{f_b} \in S_f \quad \forall\ \mathbf{c} \geq \mathbf{0} \tag{3.5}$$

where the arc costs have been assumed non-negative, $\mathbf{c} \geq \mathbf{0}$. This function is continuous and c. differentiable for usually adopted route choice models; it

~ is monotone decreasing with respect to arc costs $\mathbf{c}$

~ has a symmetric negative semi-definite Jacobian for invariant choice models, $\mathbf{Jac}[\mathbf{f}(\mathbf{c})] = - \sum_i \beta_i\, d_i\, \mathbf{B}_i\, \mathbf{Jac}[\mathbf{p}_i(-\beta_i\, (\mathbf{B}_i^T \mathbf{c} + \mathbf{w}_{o,i}))]\, \mathbf{B}_i^T \quad 0$.

The arc flow function can easily be computed when route can explicitly be enumerated and in several cases, also without explicit route enumeration.

Models w.r.t. to arc variables can be easily written down by combining the arc cost function (2.2) with the arc flow function (3.5), remembering that the latter actually is a combination of equations (2.1, 3-6). SUE can be formulated w.r.t. flows or costs:

$$\mathbf{f}^* = \mathbf{f}(\mathbf{c}(\mathbf{f}^*)) \quad \text{with } \mathbf{f}^* \in S_f \tag{3.6}$$

$$\mathbf{c}^* = \mathbf{c}(\mathbf{f}(\mathbf{c}^*)) \quad \text{with } \mathbf{c}^* \in R^n_+ \tag{3.7}$$

4. Existence and uniqueness conditions

Sufficient conditions for existence of solutions can be easily derived by applying model (3.6), through Brouwer theorem (theorem E in appendix), requiring that set S_f is non-empty and both the arc cost function and the arc flow function (i.d. the path choice functions) are continuous.

Applying models (3.6) or (3.7) uniqueness conditions w.r.t arc variables can be derived; some of them are discussed below (details and other conditions are in Cantarella and Velonà, 2010). Proofs are based on *reductio ad absurdum.* If the route choice functions, $\mathbf{p}_i(\mathbf{v}_i)$, and the arc cost function, $\mathbf{c}(\mathbf{f})$, are c. differentiable[2] conditions can be expressed w.r.t. Jacobians.

CONDITION Ar-arc (Daganzo, 1983). According to most literature, uniqueness is guaranteed if the arc flow function is monotone decreasing, as for invariant probabilistic route choice functions, and the arc cost function is strictly monotone increasing. For differentiable functions sufficient conditions are:

$$\mathbf{Jac}[\mathbf{c}(\mathbf{f})] \quad 0 \tag{4.1a}$$

$$\mathbf{Jac}[\mathbf{f}(\mathbf{c})] \quad 0 \quad \text{(as it occurs for invariant choice functions)} \tag{4.1b}$$

CONDITION A-arc (Cantarella, 2002). The above condition Ar-arc turns out to be a particular instance of a more general condition which refer to the composed function obtained combining together the arc cost and the arc flow functions:

$$(\mathbf{c}(\mathbf{f}_1) - \mathbf{c}(\mathbf{f}_2))^{\mathrm{T}} (\mathbf{f}_1 - \mathbf{f}_2) > (\mathbf{f}(\mathbf{c}(\mathbf{f}_1)) - \mathbf{f}(\mathbf{c}(\mathbf{f}_2)))^{\mathrm{T}} (\mathbf{c}(\mathbf{f}_1) - \mathbf{c}(\mathbf{f}_2))$$

$$\forall\ \mathbf{f}_1 \neq \mathbf{f}_2 \in S_f : \mathbf{c}(\mathbf{f}_1) \neq \mathbf{c}(\mathbf{f}_2)$$

Condition A-arc also allow us to support conditions for the convergence of algorithms based on the method of successive averages (Cantarella and Velonà, 2009). If the arc cost function is invertible with $\mathbf{q}(\mathbf{c}) = \mathbf{c}^{-1}(\mathbf{c})$ the above condition become:

$$(\mathbf{c}_1 - \mathbf{c}_2)^{\mathrm{T}} (\mathbf{q}(\mathbf{c}_1) - \mathbf{q}(\mathbf{c}_2)) > (\mathbf{f}(\mathbf{c}_1) - \mathbf{f}(\mathbf{c}_2))^{\mathrm{T}} (\mathbf{c}_1 - \mathbf{c}_2) \quad \forall \mathbf{c}_1 \neq \mathbf{c}_2 \in \mathbf{c}(S_f) \subseteq R^n;_+$$

with $\mathbf{f}_1 = \mathbf{q}(\mathbf{c}_1) \neq \mathbf{q}(\mathbf{c}_2) = \mathbf{f}_2\ \forall \mathbf{c}_1 \neq \mathbf{c}_2 \in \mathbf{c}(S_f)$. For differentiable functions a sufficient condition is[3]:

$$\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{q}(\mathbf{c}))]^{-1} - \mathbf{Jac}[\mathbf{f}(\mathbf{c})] \quad 0 \quad \forall \mathbf{c} \in \mathbf{c}(S_f) \subseteq R^n;_+ \tag{4.2}$$

CONDITION Br-arc. Other conditions, without any relationship with the previous ones, can be derived from Banach theorem (THEOREM Br in appendix), requiring that the composed function $\mathbf{c}(\mathbf{f}(\mathbf{c}))\ \forall \mathbf{c} \in \mathbf{c}(S_f) \subseteq R^n;_+$ is strictly non-expansive. For differentiable functions a sufficient condition is:

$$\| \mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{f}(\mathbf{c}))]\ \mathbf{Jac}[\mathbf{f}(\mathbf{c})] \|_2 < 1 \quad \forall \mathbf{c} \in \mathbf{c}(S_f) \subseteq R^n;_+ \tag{4.3}$$

Similar conditions hold w.r.t. the composed function $\mathbf{f}(\mathbf{c}(\mathbf{f}))\ \forall \mathbf{f} \in S_f$. Condition Br may rarely be applied, since generally involved functions are not strictly non-expansive.

CONDITION B-arc (Cantarella, 2002). Condition Br turns out to be a particular instance of a more general condition requiring that function $\mathbf{c} - \mathbf{c}(\mathbf{f}(\mathbf{c}))$ is s. monotone increasing:

$$(\mathbf{c}_1 - \mathbf{c}_2)^{\mathrm{T}} (\mathbf{c}_1 - \mathbf{c}_2) > (\mathbf{c}(\mathbf{f}(\mathbf{c}_1)) - \mathbf{c}(\mathbf{f}(\mathbf{c}_2)))^{\mathrm{T}} (\mathbf{c}_1 - \mathbf{c}_2) \quad \forall \mathbf{c}_1 \neq \mathbf{c}_2 \in \mathbf{c}(S_f) \subseteq R^n;_+$$

For differentiable functions a sufficient condition is:

$$(\mathbf{I} - \mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{f}(\mathbf{c}))]\ \mathbf{Jac}[\mathbf{f}(\mathbf{c})]) \quad 0 \quad \forall \mathbf{c} \in \mathbf{c}(S_f) \subseteq R^n;_+ \tag{4.4}$$

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[2] Differentiability is assumed to hold over an open set; if this is not the case, an open superset of the relevant set and/or an extrapolation of derivatives to the boundary is considered.

[3] It is noteworthy that this expression appeared in Daganzo (1983) only to prove conditions called Ar in this paper.

CONDITION U-arc. Both conditions A-arc and B-arc are instances of the general uniqueness condition (THEOREM U in appendix) requiring that the composed function $(\mathbf{c} - \mathbf{c}(\mathbf{f}(\mathbf{c})))$ is invertible for $\mathbf{c} \in \mathbf{c}(S_f) \subseteq R^n;_+$.

CONDITION K-arc. For differentiable functions, a condition, somehow related to condition U-arc, can be expressed by applying the Kellogg[4] theorem (THEOREM K in appendix).

At this aim let us assume that existence conditions hold: set $S_f \subseteq R^n;_+$ is non-empty and the arc cost function and the arc flow function are continuous. Since arc cost function $\mathbf{c}(\mathbf{f})$ is continuous over the non-empty, bounded (since compact) set S_f, set $\mathbf{c}(S_f) \subseteq R^n;_+$ is non-empty and bounded, hence there exists a non-empty, compact (say bounded and closed), convex superset $S° \supset \mathbf{c}(S_f)$; moreover if arc cost function $\mathbf{c}(\mathbf{f})$ is strictly positive, $\mathbf{c}(\mathbf{f}) > \mathbf{0}$, superset $S°$ may be chosen as a proper subset of the set of non-negative real vector, $S° \subset R^n;_+$.

Now let us apply the Kellogg theorem (THEOREM K in appendix) to the fixed-point model (3.7) defined over set $S° \subseteq R^n;_+$: $\mathbf{c}^* = \mathbf{c}(\mathbf{f}(\mathbf{c}^*))$ with $\mathbf{c}^* \in S° \subseteq R^n;_+$. At first it is noteworthy that 1) set $S°$ is non-empty, 2) the composed function $\mathbf{c}(\mathbf{f}(\mathbf{c}))$ has no fixed-point $\mathbf{c}^*$ on the boundary of set $S°$, since all fixed-points must belong to set $\mathbf{c}(S_f) \subseteq S°$. Thus if 3) the composed function $\mathbf{c}(\mathbf{f}(\mathbf{c}))$ is c. differentiable on $S°$ with Jacobian $\mathbf{Jac}[\mathbf{c}(\mathbf{f}(\mathbf{c}))] = \mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{f}(\mathbf{c}))]\ \mathbf{Jac}[\mathbf{f}(\mathbf{c})]$, uniqueness is guaranteed if 4) matrix $\mathbf{Jac}[\mathbf{c}(\mathbf{f}(\mathbf{c}))]$ has no eigenvalue equal to one, that is:

$$| \mathbf{I} - \mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{f}(\mathbf{c}))]\ \mathbf{Jac}[\mathbf{f}(\mathbf{c})] | \neq 0 \quad \forall \mathbf{c} \in S° \subseteq R^n;_+ \tag{4.5}$$

Condition K-arc (4.5) includes as special case condition A (4.2).

Under the assumptions of c. differentiable functions, due to the Bolzano (sign-preserving) theorem, condition (4.5) actually yields either of the following two conditions, but not both:

K1-arc $$| \mathbf{I} - \mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{f}(\mathbf{c}))]\ \mathbf{Jac}[\mathbf{f}(\mathbf{c})] | > 0 \quad \forall \mathbf{c} \in S° \subseteq R^n;_+ \tag{4.5.1}$$

K2-arc $$| \mathbf{I} - \mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{f}(\mathbf{c}))]\ \mathbf{Jac}[\mathbf{f}(\mathbf{c})] | < 0 \quad \forall \mathbf{c} \in S° \subseteq R^n;_+ \tag{4.5.2}$$

Condition K1-arc (4.5.1) includes as special case conditions Ar (4.1), Br (4.3), B (4.4).

Applying models (3.1) or (3.2) other uniqueness conditions w.r.t independent route (iro) variables can be derived. Some of them are discussed below (details and other conditions are in Cantarella and Velonà, 2010) under the assumption of invariant strictly positive choice functions, which leads to s. monotone, hence invertible, iro flow function $\tilde{\mathbf{h}}(\tilde{\mathbf{w}})$. Proofs are based on *reductio ad absurdum.* If the iro choice functions, $\tilde{\mathbf{p}}_i(\tilde{\mathbf{v}}_i)$, and the arc cost function, $\mathbf{c}(\mathbf{f})$, are c. differentiable, conditions can be expressed w.r.t. Jacobians matrices.

CONDITION Ar-iro. For invariant strictly positive probabilistic route choice functions, uniqueness is guaranteed if the arc cost function is monotone increasing (but not necessarily s. monotone), thus Ar-arc implies Ar-iro. For differentiable functions a sufficient condition is:

$$\mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})] \quad 0 \tag{4.6a}$$

$$\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}})] \quad 0 \quad \text{(as it occurs for invariant s. positive choice functions)} \tag{4.6b}$$

CONDITION A-iro. The above condition Ar-iro turns out to be a particular instance of a more general conditions which refer to the composed function combining together the inverse of the iro flow function and the iro cost difference function:

$$(\tilde{\mathbf{h}}_1 - \tilde{\mathbf{h}}_2)^{\mathsf{T}} (\tilde{\mathbf{h}}^{-1}(\tilde{\mathbf{h}}_1) - \tilde{\mathbf{h}}^{-1}(\tilde{\mathbf{h}}_2)) < (\tilde{\mathbf{w}}(\tilde{\mathbf{h}}_1) - \tilde{\mathbf{w}}(\tilde{\mathbf{h}}_2))^{\mathsf{T}} (\tilde{\mathbf{h}}_1 - \tilde{\mathbf{h}}_2)$$

---

[4] Gentile (2003) pointed out that Kellogg theorem can be used to assess uniqueness conditions of fixed-points.

$$\forall\ \tilde{\mathbf{h}}_1 \neq \tilde{\mathbf{h}}_2 \in S_{\tilde{h}} \qquad (\text{thus } \tilde{\mathbf{w}}_1 = \tilde{\mathbf{h}}^{-1}(\tilde{\mathbf{h}}_1) \neq \tilde{\mathbf{h}}^{-1}(\tilde{\mathbf{h}}_2) = \tilde{\mathbf{w}}_2)$$

Since the iro flow function is invertible, for differentiable functions a sufficient condition may be written down as:

$$\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1} - \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})] \quad 0 \qquad \forall \tilde{\mathbf{h}} \in S_{\tilde{h}} \tag{4.7}$$

CONDITION Br-iro. From Banach theorem (THEOREM Br in appendix), uniqueness is guaranteed if the composed function $\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$ is strictly non-expansive for $\tilde{\mathbf{h}} \in S_{\tilde{h}}$. For differentiable functions a sufficient condition is:

$$\|\ \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]\ \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})]\ \|_2 < 1 \qquad \forall \tilde{\mathbf{h}} \in S_{\tilde{h}} \tag{4.8}$$

CONDITION B-iro. Condition Br turns out to be a particular instance of a more general condition requiring that function $\tilde{\mathbf{h}} - \tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$ is strictly monotone increasing for $\tilde{\mathbf{h}} \in S_{\tilde{h}}$:

$$(\tilde{\mathbf{h}}_1 - \tilde{\mathbf{h}}_2)^{\mathsf{T}} (\tilde{\mathbf{h}}_1 - \tilde{\mathbf{h}}_2) > (\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}_1)) - \tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}_2))^{\mathsf{T}} (\tilde{\mathbf{h}}_1 - \tilde{\mathbf{h}}_2) \quad \forall\ \tilde{\mathbf{h}}_1 \neq \tilde{\mathbf{h}}_2 \in S_{\tilde{h}}$$

For differentiable functions a sufficient condition is:

$$(\mathbf{I} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]\ \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})]) \quad 0 \qquad \forall \tilde{\mathbf{h}} \in S_{\tilde{h}} \tag{4.9}$$

CONDITION U-iro. Both conditions A-iro and B-iro are instances of the general uniqueness condition (THEOREM U in appendix) requiring that the composed function $\tilde{\mathbf{h}} - \tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$ is invertible for $\tilde{\mathbf{h}} \in S_{\tilde{h}}$.

CONDITION K-iro (from Gentile, 2003). For differentiable functions, a condition, somehow related to condition U-arc, can be expressed by applying the Kellogg theorem (THEOREM K in appendix) to fixed-point model (3.1). At this aim let us assume that existence conditions hold, thus set $S_{\tilde{h}}$ is non-empty and the iro cost difference function and the iro flow function are continuous. For (invariant) strictly positive choice functions $S_{\tilde{h}}$ is also a bounded open convex set (see sub-section 2.4), therefore the fixed-point model (3.1) is defined over 1) a non-empty set $S_{\tilde{h}}$ and 2) the composed function $\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$ has no fixed-point $\tilde{\mathbf{h}}^*$ on the boundary of $S_{\tilde{h}}$. Thus if 3) the composed function $\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$ is c. differentiable on $S_{\tilde{h}}$ with Jacobian $\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))] = \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]\ \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})]$, uniqueness is guaranteed if 4) matrix $\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]$ has no eigenvalue equal to one, that is:

$$|\ \mathbf{I} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]\ \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})]\ | \neq 0 \quad \forall \tilde{\mathbf{h}} \in S_{\tilde{h}} \tag{4.10}$$

Condition K-arc (4.10) includes as special case condition A (4.7).

Under the assumptions of c. differentiable functions, due to the Bolzano (sign-preserving) theorem, condition (4.10) actually yields either of the following two conditions, but not both:

K1-iro $$|\ \mathbf{I} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]\ \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})]\ | > 0 \quad \forall \mathbf{c} \in S^{\circ} \subseteq R^{n};_{+} \tag{4.10.1}$$

K2-iro $$|\ \mathbf{I} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]\ \mathbf{Jac}[\tilde{\mathbf{w}}(\tilde{\mathbf{h}})]\ | < 0 \quad \forall \mathbf{c} \in S^{\circ} \subseteq R^{n};_{+} \tag{4.10.2}$$

Condition K1-arc (4.5.1) includes as special case conditions Ar (4.6), Br (4.8), B (4.9).

It should be noted that, generally conditions w.r.t. iro variables are not equivalent to those w.r.t. arc variables, even though they may appear formally similar. A relationship between the two groups of conditions can be exploited through the following conditions, which hold under the assumption of invariant strictly positive choice functions.

CONDITION G (from Gentile, 2003). Uniqueness is guaranteed under the following condition about the Jacobian, $\mathbf{Jac}[c(\mathbf{f})]$, of the cost function:

$$(\mathbf{Jac}[c(\mathbf{f})] + \gamma\ \mathbf{I})\ 0 \quad 0\ \forall\ \mathbf{f} \in S_f \tag{4.11}$$

where $0 < \gamma \leq \gamma^* = -\max\{(\mathbf{u} / \|\ \mathbf{B}\ \mathbf{L}\ \mathbf{u}\ \|_2)^{\mathsf{T}}\ \ \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1}\ (\mathbf{u} / \|\ \mathbf{B}\ \mathbf{L}\ \mathbf{u}\ \|_2),$

with $\mathbf{u} \in \{\mathbf{u} : \| \mathbf{u} \| = 1, \| \mathbf{B\,L\,u} \|_2 > 0\} \subseteq R^{\tilde{m}}.\}$

Proof (extended from Gentile, 2003). Jacobian matrix of the composed function: $\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$ is given by:

$\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))] = (\mathbf{L}^{\mathsf{T}}\,\mathbf{B}^{\mathsf{T}}\,\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})]\,\mathbf{B\,L})$.

According to (4.11): $\mathbf{a}^{\mathsf{T}}\,(\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})] + \gamma\,\mathbf{I})\,\mathbf{a} > 0 \quad \forall \mathbf{a} \neq \mathbf{0} \in R^n,\ \forall\,\tilde{\mathbf{h}} \in S_{\tilde{h}}$

It is shown below that this condition implies condtions A-iro (4.7):

$\mathbf{L}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\,\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})]\,\mathbf{B\,L} - \mathbf{Jac}[\tilde{h}(\tilde{\mathbf{w}} = \tilde{w}(\tilde{\mathbf{h}}))]^{-1})\quad 0 \qquad\qquad \forall\,\mathbf{u} \in R^{\tilde{m}},\ \| \mathbf{u} \| = 1$

It is worth to distinguish two cases:

- $\forall\,\mathbf{u} \in R^{\tilde{m}},\ \| \mathbf{u} \| = 1,\ \| \mathbf{B\,L\,u} \|_2 = 0$, assumptions yield

$\mathbf{u}^{\mathsf{T}}\,(\mathbf{L}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\,\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})]\,\mathbf{B\,L} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1})\,\mathbf{u} =$

$\mathbf{u}^{\mathsf{T}}\,(-\,\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1})\,\mathbf{u} > 0 \;\Rightarrow\;$ A-iro (4.7)

- $\forall\,\mathbf{u} \in R^{\tilde{m}},\ \| \mathbf{u} \| = 1,\ \| \mathbf{B\,L\,u} \|_2 > 0$

Let $\mathbf{a} = \mathbf{B\,L\,u} / \| \mathbf{B\,L\,u} \|$. Hence:

$(\mathbf{B\,L\,u} / \| \mathbf{B\,L\,u} \|_2)^{\mathsf{T}}\,(\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})] + \gamma\,\mathbf{I})\,(\mathbf{B\,L\,u} / \| \mathbf{B\,L\,u} \|_2) > 0 \Rightarrow$

$\Rightarrow (\mathbf{B\,L\,u} / \| \mathbf{B\,L\,u} \|_2)^{\mathsf{T}}\,(\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})])\,(\mathbf{B\,L\,u} / \| \mathbf{B\,L\,u} \|_2) > -\gamma \geq -\gamma^* \geq$

$\geq (\mathbf{u} / \| \mathbf{B\,L\,u} \|_2)^{\mathsf{T}}\,\mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1}\,(\mathbf{u} / \| \mathbf{B\,L\,u} \|_2) \Rightarrow$

$\Rightarrow (\mathbf{u} / \| \mathbf{B\,L\,u} \|_2)^{\mathsf{T}}\,(\mathbf{L}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\,\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})]\,\mathbf{B\,L} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1})$ for $(\mathbf{u} / \| \mathbf{B\,L\,u} \|_2) > 0 \Rightarrow$

$\Rightarrow \mathbf{u}^{\mathsf{T}}\,(\mathbf{L}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\,\mathbf{Jac}[\mathbf{c}(\mathbf{f} = \mathbf{B\,L}\,\tilde{\mathbf{h}} + \mathbf{B\,h_b} + \mathbf{f_b})]\,\mathbf{B\,L} - \mathbf{Jac}[\tilde{\mathbf{h}}(\tilde{\mathbf{w}} = \tilde{\mathbf{w}}(\tilde{\mathbf{h}}))]^{-1})\;\mathbf{u} > 0 \;\Rightarrow\;$ A-iro (4.7) ■

Condition G implies condition A-iro, as in the proof. Moreover condition A-arc implies G since the sum of positive definite matrices is a positive definite matrix. It is also noteworthy that condition G does not require features of the composed function $\tilde{\mathbf{h}}(\tilde{\mathbf{w}}(\tilde{\mathbf{h}}))$.

It is should be noted that uniqueness conditions A, B and U allow for non monotone arc cost functions, which may well be the case for functions non-separable w.r.t arc flows; moreover, these conditions allow for non-differentiable arc cost functions, such as piecewise linear functions (as it may occur when equilibrium assignment is embedded within a transportation supply design model). Thus it is worthwhile to introduce some requirements for non monotone arc cost functions (these requirements will be referred to in the examples):

- ~ positivity $\qquad c_a(f_a) \geq 0 \quad \forall f_a \geq 0$
- ~ monotonicity over capacity Q_a $\qquad (c_a(f_a) - c_a(f_a'))\ (f_a - f_a') \geq 0 \quad \forall f_a,\ f_a' \geq Q_a$
- ~ consistency with capacity Q_a $\qquad c_a(f_a = Q_a) = \alpha\, c_a(f_a = 0) \quad$ with $\alpha \geq 1$

Monotone strictly increasing arc cost functions surely meet these requirements if the null-flow cost is strictly positive, $c_a(f_a = 0) \geq 0$, as always the case.

5. Numerical examples

In this section all the uniqueness conditions presented above are compared for a two-link network, where $\mathbf{f} = \mathbf{h}$, $\tilde{h} = h_1$, $\mathbf{c} = \mathbf{w}$, $\tilde{w} = c_1 - c_2$. As regarding the supply model, a separable cost function is associated to each arc a: $c_a = c_a(f_a)$ $\quad a = 1, 2$, with $x_a = \partial c_a(f_a) / \partial f_a \quad a = 1, 2$. Thus the Jacobian of the arc cost vector function is given by the diagonal matrix:

$$\mathbf{Jac}[\mathbf{c}(\mathbf{f})] = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

As regarding the demand model, the choice function is an invariant Logit leading to:

$$p_a(c_1, c_2) = \exp(-c_a/\theta) \,/\, (\exp(-c_1/\theta) + \exp(-c_2/\theta)) \qquad a = 1, 2$$

where $\theta \propto \sigma > 0$, and $\partial\theta / \partial c_a = 0$, $a = 1, 2$. Let $y = (d/\theta)\, p_1(c_1,c_2)\, p_2(c_1,c_2)$, it yields:

$$\mathbf{Jac}[\mathbf{f}(\mathbf{c})] = -(d/\theta) \begin{bmatrix} p_1 \cdot (1-p_1) & -p_1 \cdot p_2 \end{bmatrix} = -y \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$-p_2 \cdot p_1$	$p_2 \cdot (1-p_2)$		-1	1

For differentiable functions, each of the five uniqueness conditions Ar, A, Br, B, K defines a region over the plane $z_1 = x_1 \cdot y$ and $z_2 = x_2 \cdot y$, as shown in figure 1. All the implications reported in section 4 are confirmed. It is noteworthy that conditions w.r.t. arc or iro variables lead to different regions. Moreover, A-arc region is not connected, thus actually only one sub-region may actually be considered due to Bolzano (sign-preserving) theorem. Both A-arc and B-arc conditions imply conditions K1-arc, whilst in this particular case A-iro, B-iro and K1-iro conditions match each other.

Analysis can be further deepened by considering quadratic arc cost functions:

$$c_a(f_a) = c_{o,a}\,(1 + \eta_a\,(f_a / Q_a) + \mu_a\,(f_a / Q_a)^2)$$

where

$c_{o,a}$ is the null-flow cost;

Q_a is the arc capacity;

η_a and $\mu_a > 0$ are two parameters, chosen so that: $c_a(f_a) \geq 0$ and $c_a(Q_a) = \alpha\, c_a(0)$ with $\alpha \geq 1$, leading to $-2\,(1 + (\alpha)^{0.5}) \leq \eta_a \leq (\alpha - 1)$, and $\mu_a = \alpha - 1 - \eta_a$.

Each uniqueness condition defines a region over the plane η_1 and η_2. Figure 2 shows uniqueness regions for $d = 100$, $\theta = 5$, $c_{o,1} = 50$, $c_{o,2} = 30$, $Q_1 = 300$, $Q_2 = 100$, $\alpha = 2$. The square with thick lines shows condition to $-2\,(1 + (\alpha)^{0.5}) \leq \eta_a \leq (\alpha - 1)$, for $\alpha = 2$. Regions in figure 2 are consistent with figure 1 and all the implications reported in section 4. It is noteworthy that conditions w.r.t. arc or iro variables lead to different regions. Moreover, in this particular case A-iro, B-iro and K1-iro conditions match each other.

Other examples, not reported here, show that decreasing the parameter θ of Logit choice function leads to a reduction of uniqueness regions. It is worth recalling that decreasing the parameter θ of Logit choice function leads towards Wardrop user equilibrium.

These results are consistent with condition G, actually in this simple case parameter of condition G can explicitly be computed as $\gamma^* = 2\ \theta\ /\ d$. Conditions G are compared with conditions Ar-arc and A-iro in figure 3.

It also worth noting that each point over plan η_1-η_2 matches to a segment of curve over plan z_1-z_2, as shown in figure 4 for two cases. All pairs (η_1, η_2) analysed in the above examples guarantee arc cost functions that are increasing at least at some points, thus at least a part of the curve must belong to the positive orthant (or N-E quadrant) of plan z_1-z_2.

6. Conclusion

In this paper conditions for SUE uniqueness have been presented. Described conditions are weaker than those existing in literature and indicates that monotonicity of arc cost vector function is an unnecessarily strong assumption.

Condition K-iro seems the most promising one. Nonetheless, for a two-link network, non-uniqueness region can be analytically defined by examining function $\tilde{h}(\tilde{w}(\tilde{h}))/d$ against $\tilde{h}/d$, see figure 5 (left). A comparison with the region for condition K1-iro over plan η_1-η_2 (center) and plan z_1-z_2 (right) shows that uniqueness condition weaker than the condition K-iro may exist.

It may be useful to stress that solution algorithms may be implemented w.r.t. variables (e.g. arc costs) different from those (e.g. iro flows) used for checking uniqueness conditions (see Cantarella and Velonà, 2009, for considerations about conditions A-arc or A-iro).

A future paper will deal with implementation issues such as analysis of existing cost functions and path choice functions as well as large scale applications; analysis of cost functions from Webster method will be also addressed.

Several issues seem worth of further research work, including assignment with variable demand and/or explicit modelling of path choice set.

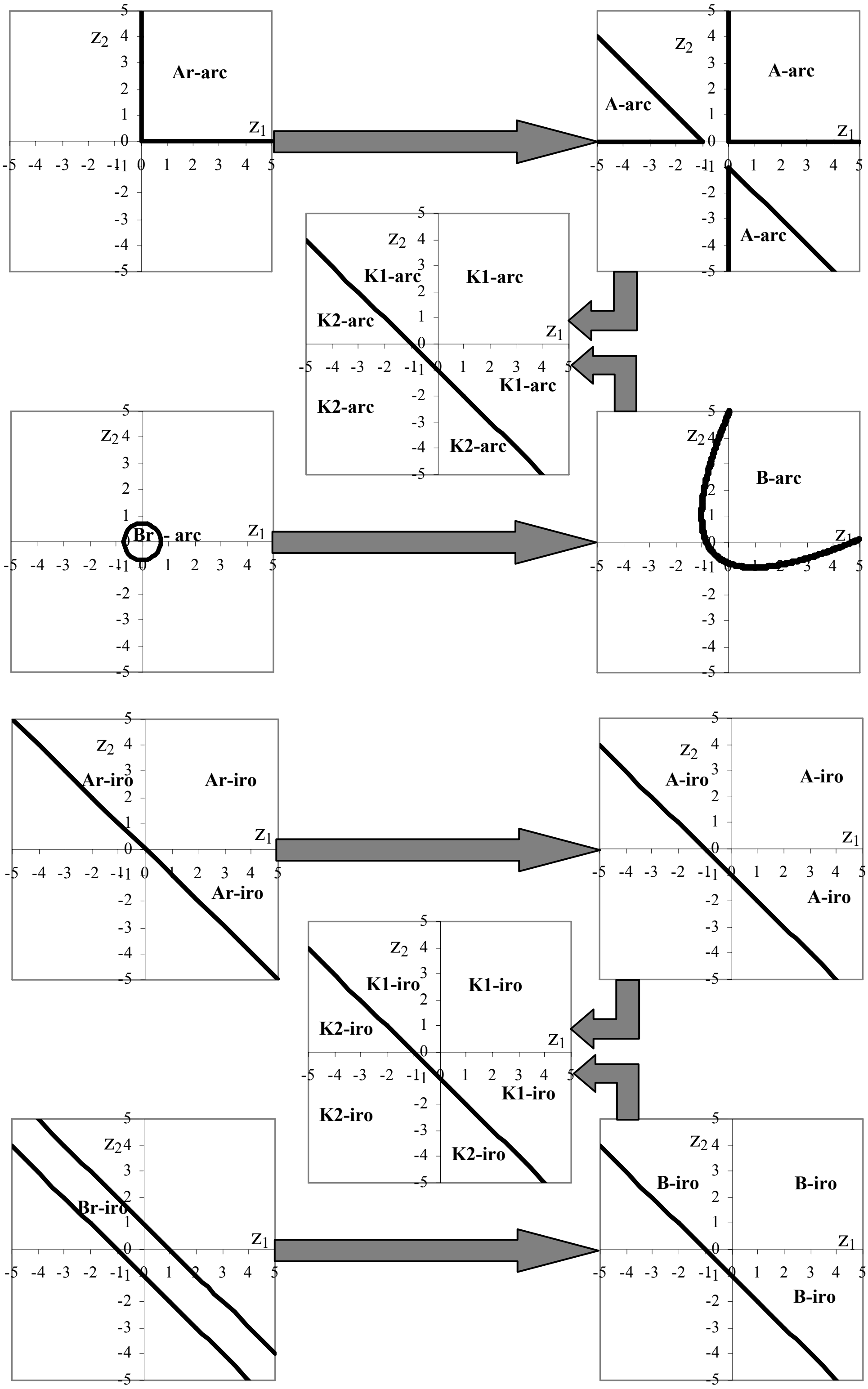

FIGURE 1. Regions for uniqueness conditions w.r.t arc (top) and iro (bottom) variables in z_1-z_2 plan.

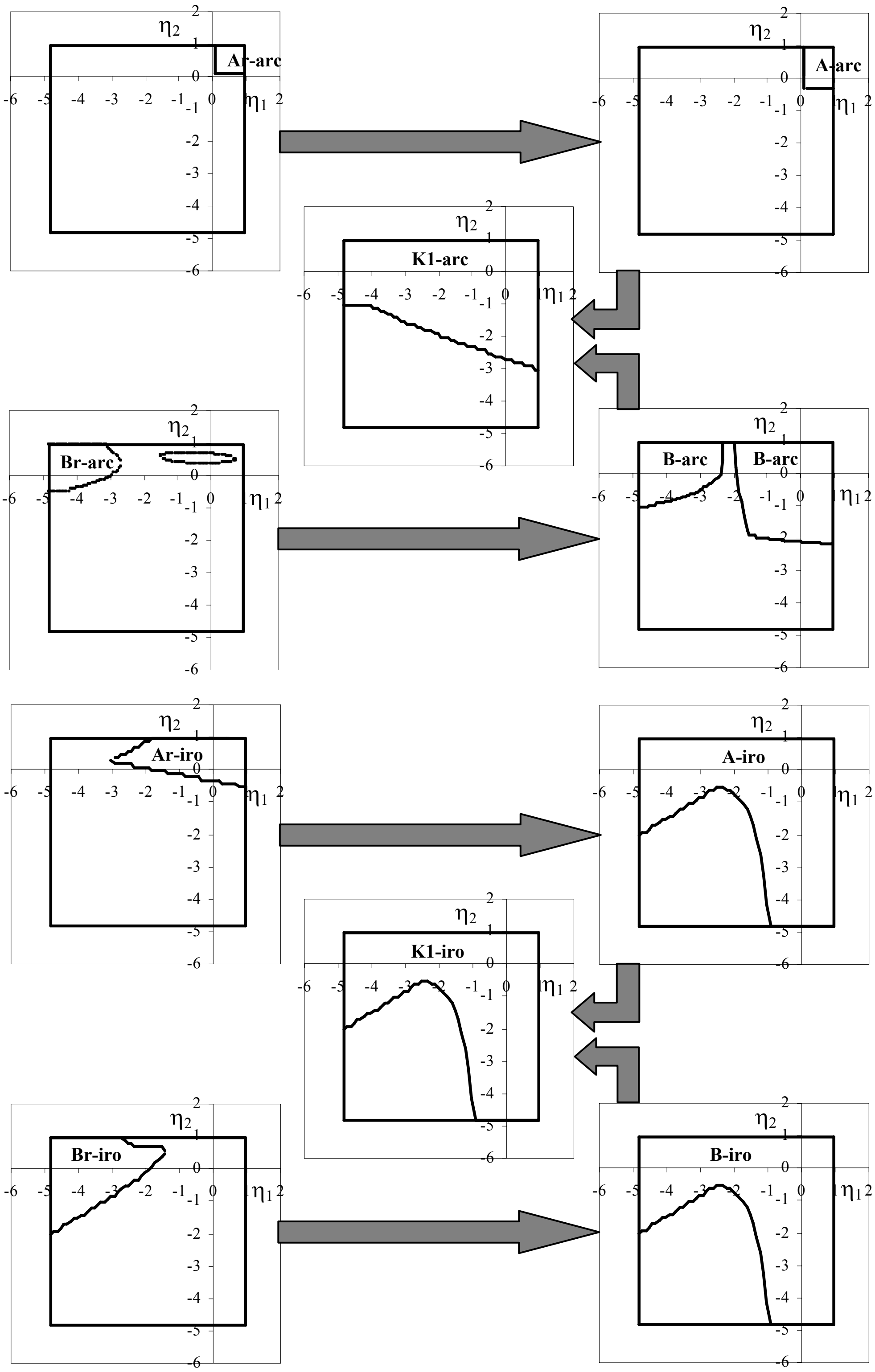

FIGURE 2. Regions for uniqueness conditions w.r.t arc (top) and iro (bottom) variables in η_1-η_2 plan.

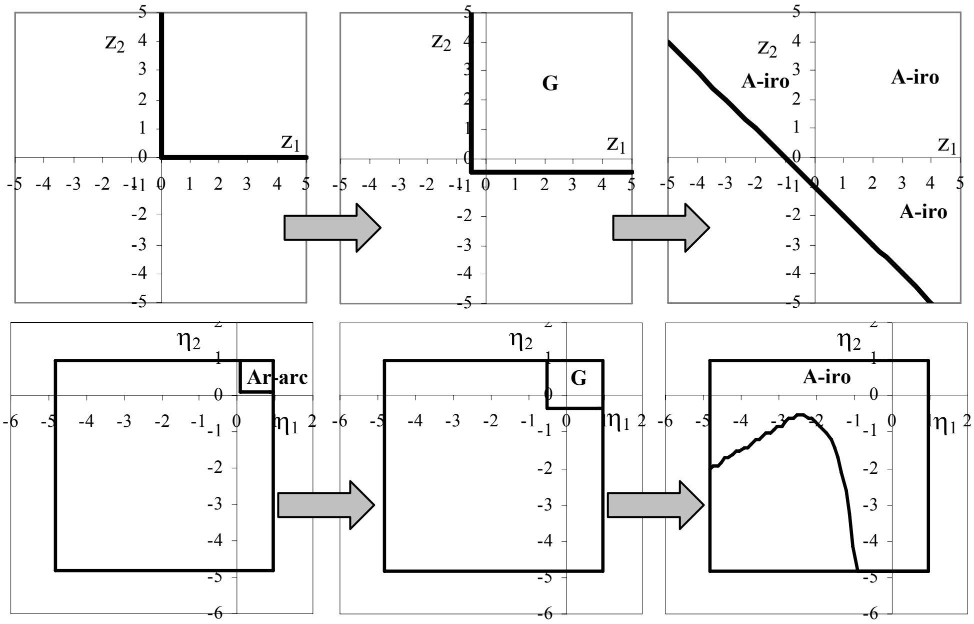

FIGURE 3. Regions for uniqueness condition G and its relationships with Ar-arc and A-iro in z_1-z_2 plan (top) or in η_1-η_2 plan (bottom).

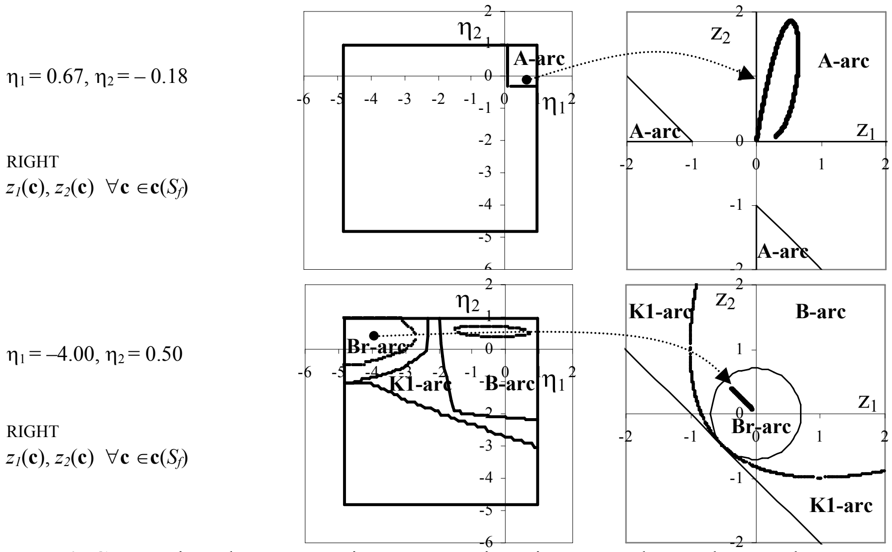

FIGURE 4. Comparison between uniqueness regions in η_1-η_2 plan and z_1-z_2 plan.

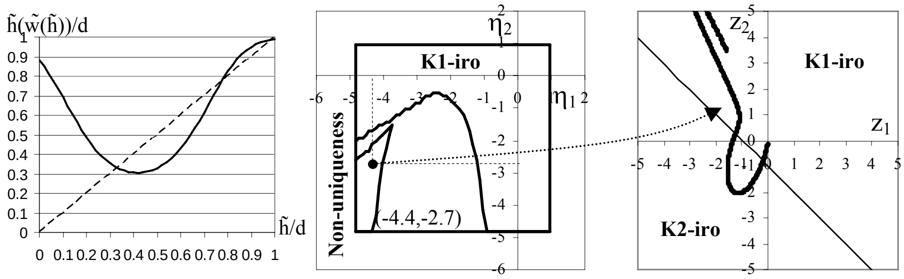

FIGURE 5. Comparison between condition K uniqueness and non-uniqueness regions.

A. Appendix: Theorems for existence and uniqueness of fixed-points.

Let R^n be the set of real vectors with n entries, where Euclidean distance and norm hold. Let $\varphi(\mathbf{x})$ be a function with definition set $S \subseteq R^n$ and image set S, $\varphi(\mathbf{x})$: $S \to S$. This appendix reviews most relevant (sufficient) conditions for existence and/or uniqueness of the fixed-points $\mathbf{x}^* = \varphi(\mathbf{x}^*)$ of function $\varphi(\mathbf{x})$. Reported results may hold in a more general space.

THEOREM E (existence). *If set S is non-empty compact convex and function* $\varphi(\mathbf{x})$ *is continuous, then at least one fixed-point exists. [Proof: it is Brouwer theorem.]*

THEOREM U (uniqueness). *If function* $\mathbf{x} - \varphi(\mathbf{x})$ *is invertible, then at most one fixed-point exists for function* $\varphi(\mathbf{x})$. *[Proof: by reductio ad absurdum.]*

REMARK. Sufficient conditions for c. differentiable functions can be derived through the global inversion theorem, or by applying the theorem below.

THEOREM K (existence & uniqueness). *Let existence conditions of theorem E hold: S is non-empty compact convex and function* $\varphi(\mathbf{x})$ *is continuous; in this case the interior* $S°$ *of set S is a (possibly empty) bounded open convex set. If*

1) *set* $S°$ *is non-empty,*
2) *function* $\varphi(\mathbf{x})$ *has no fixed-point on the boundary of S, say* $\mathbf{x} \neq \varphi(\mathbf{x})\ \forall \mathbf{x} \in \partial S$,
3) *function* $\varphi(\mathbf{x})$ *is continuously differentiable on* $S°$ *with Jacobian* $\mathbf{Jac}[\varphi(\mathbf{x})]$,
4) *matrix* $\mathbf{Jac}[\varphi(\mathbf{x})]$ *has no eigenvalue equal to one, say* $|\,\mathbf{I} - \mathbf{Jac}[\varphi(\mathbf{x})]\,| \neq 0\ \forall \mathbf{x} \in S$,

then exactly one fixed-point exists for function $\varphi(\mathbf{x})$. *[Proof: applying Kellogg theorem to a finite dimensional space, see Kellogg, 1976.]*

Under the assumptions of c. differentiable functions, due to the Bolzano (sign-preserving) theorem, condition K actually yields either of the following two conditions, but not both:

K1 $\quad |\,\mathbf{I} - \mathbf{Jac}[\varphi(\mathbf{x})]\,| > 0\ \forall \mathbf{x} \in S$

K2 $\quad |\,\mathbf{I} - \mathbf{Jac}[\varphi(\mathbf{x})]\,| < 0\ \forall \mathbf{x} \in S$

THEOREM Br (existence & uniqueness). *If set S non-empty compact and function* $\varphi(\mathbf{x})$ *is strictly non-expansive, say* $\|\,\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2)\,\|_2 < \|\,\mathbf{x}_1 - \mathbf{x}_2\,\|_2\ \ \forall \mathbf{x}_1 \neq \mathbf{x}_2 \in S$, *then exactly one fixed-point exists for function* $\varphi(\mathbf{x})$. *[Proof: extension of Banach theorem.]*

REMARK. If function $\varphi(\mathbf{x})$ is continuously differentiable with Jacobian $\mathbf{Jac}[\varphi(\mathbf{x})]$, and the second norm of its Jacobian is less than one, $\|\,\mathbf{Jac}[\varphi(\mathbf{x})]\,\|_2 < 1\ \forall \mathbf{x} \in S$, then function $\varphi(\mathbf{x})$ is strictly non-expansive.

Theorem Br requires a stronger assumption about function $\varphi(\mathbf{x})$ than theorem U (a strictly non-expansive function being surely invertible), but it also guarantee existence (and convergence of the method of successive approximations).

An intermediate condition is given below.

THEOREM B (uniqueness). *If function* $\mathbf{x} - \varphi(\mathbf{x})$ *is strictly monotone increasing, then at most one fixed-point exists for function* $\varphi(\mathbf{x})$. *[Proof: by reductio ad absurdum.]*

REMARK. If function $\varphi(\mathbf{x})$ is c. differentiable with Jacobian $\mathbf{Jac}[\varphi(\mathbf{x})]$, and the difference between the identity matrix and its Jacobian, $(\mathbf{I} - \mathbf{Jac}[\varphi(\mathbf{x})])$, then function $\mathbf{x} - \varphi(\mathbf{x})$ is strictly monotone increasing.

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